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Existence and Multiplicity of Nontrivial Solutions for Semilinear Elliptic Equations Involving Hardy-Sobolev Critical Exponents

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Abstract: A class of semi-linear elliptic equations with critical Hardy-Sobolev exponent has been considered. This model is widely used in hydrodynamics and glaciology, gas combustion in thermodynamics, quantum field theory and statistical mechanics, as well as gravity balance problems in galaxies. The $(PS)_c$ sequence of energy functional has been investigated, and then the mountain pass lemma was used to prove the existence of at least one nontrivial solution. Also a multiplicity result has been obtained. Some known results have been generalized.

Keywords: semilinear elliptic equation; Hardy-Sobolev critical exponent; mountain pass lemma; $(PS)_c$ condition

1. INTRODUCTION

Consider the problem

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u + f(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

Here Ω is an open bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^N (N \geq 3)$, and $0 \in \Omega$. $0 \leq \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$, $2^*(s) = 2(N-s)/(N-2)(0 < s < 2)$ is the Hardy-Sobolev critical exponent and $2^* = 2^*(0)$ is the Sobolev critical exponent. $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $F(x,t) = \int_0^t f(x,s)ds$. We point out that (1) is related to the application of hydrodynamics and glaciology ([1]). And it is also used in some physical or mathematical problems, such as the theory of gas combustion in thermodynamics ([2]), quantum field theory and statistical mechanics ([3–5]), as well as gravity balance problems in galaxies ([2,6]). For more investigations on solutions for nonlinear equations with Hardy potential, one can see [7–9] etc.

The modern variational method ([10–13]) plays a significant role in studying PDEs (see [14–18]). In 1973, the mountain pass lemma was proposed by A. Ambrosetti and P. Rabinwitz in [14], it is a milestone in the history of the development of critical point theory. However, in the process of studying the properties for certain equations, there are a lot of phenomena that lose compactness conditions, such as semilinear elliptic equations that involving Sobolev critical exponent or Hardy-Sobolev critical exponent on bounded domain. In 1983, H. Brezis and L. Nirenberg first chose special mountain pass and selected energy estimates to prove the existence of a critical point if the energy functional satisfies the local (PS) condition (see [15]), they investigated the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

and obtained that there exists a $\lambda_* \in (0, \lambda_1)$ such that for any $\lambda \in (\lambda_*, \lambda_1)$, problem (2) admits a positive solution. It is a special case of equation (1) ($s = 0$, $\mu = 0$ and $f(x,u) = \lambda u$). Since then, many excellent results based on the above methods (see [11,19–21]) appeared.

In the past decades, the semilinear elliptic equation with Hardy term and Sobolev critical exponent (i.e. when $s=0$ and $\mu\neq 0$) has been investigated by many mathematicians, one can refer to [22–25] etc. For example, the following elliptic problem

$$\begin{cases} -\Delta u-\mu\frac{u}{|x|^2}=|u|^{2^*-2}u+\lambda u, & x\in\Omega,\\ u=0, & x\in\partial\Omega. \end{cases} \tag{3}$$

is considered in [22–24].

For simplicity, in the following, we denote the condition (H1) and (H2) as follows:

(H1) $0<\lambda<\lambda_1(\mu)$ and $0\leq\mu\leq\overline{\mu}-1$;

(H2) $\overline{\mu}-1<\mu<\overline{\mu}$ and $\lambda_*(\mu)<\lambda<\lambda_1(\mu)$.

In [24], by the variational method, E. Jannelli proved that: If (H1) or (H2) holds, then (3) has at least one positive solution in $H_0^1(\Omega)$. Later in [26], the authors investigated problem (1) with $f(x,u)=\lambda|u|^{q-2}u$ or $f(x,u)=\lambda u$. And obtained the following conclusion.

Theorem A ([26]). Assume $0\leq s<2$, $q=2$, and $\beta=2(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})$. If (H1) or (H2) holds, then problem (1) has a positive solution u in $H_0^1(\Omega)$.

Also there are some results dealing with the case $\mu\neq 0$, $s\neq 0$ and the general form $f(x,u)$ (see [27,28]). In [27], M.C. Wang and Q. Zhang showed that problem (1) has at least one nonnegative solution. In [28], L. Ding and C.L. Tang also investigated problem (1) and obtained the existence result. Inspired by [26–28], we study the existence of nontrivial solutions for problem (1). Our main conclusions are

Theorem 1. Suppose that $N\geq 3$, $0\leq\mu<\overline{\mu}$, $\beta=\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}$, $0<s<2$, $f(x,t)$ satisfies

(f_1) $f\in C(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R}^+)$ and $\lim\limits_{t\to 0^+}\frac{f(x,t)}{t}=\lambda$, $\lim\limits_{t\to+\infty}\frac{f(x,t)}{t^{2^*(s)-1}}=\eta$, uniformly for $x\in\overline{\Omega}$, where $\lambda,\eta>0$.

(f_2) There exists $2<\rho\leq 2^*(s)$, such that $\frac{1}{\rho}f(x,t)t-F(x,t)\geq-\left(\frac{1}{2}-\frac{1}{\rho}\right)\lambda t^2$, for any $x\in\overline{\Omega}$, $t\in\mathbb{R}^+$.

If (H1) or (H2) holds, then (1) has a positive solution u in $H_0^1(\Omega)$, where

$$\lambda_*(\mu)=\min_{\varphi\in H_0^1(\Omega)\backslash\{0\}}\frac{\int_\Omega|\nabla\varphi|^2/|x|^{2\beta}dx}{\int_\Omega\varphi^2/|x|^{2\beta}dx},$$

and

$$\lambda_1(\mu)=\inf_{u\in H_0^1(\Omega)\backslash\{0\}}\frac{\int_\Omega\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)dx}{\int_\Omega|u|^2dx}.$$

Theorem 2. Suppose that $N\geq 3$, $0\leq\mu<\overline{\mu}$, $\beta=\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}$, $0<s<2$. $f(x,t)$ satisfies (f_2) and

(f_3) $f\in C(\overline{\Omega}\times\mathbb{R},\mathbb{R}^+)$, $\lim\limits_{|t|\to 0^+}\frac{f(x,t)}{t}=\lambda$, $\lim\limits_{|t|\to+\infty}\frac{f(x,t)}{t^{2^*(s)-1}}=\eta$, uniformly for $x\in\overline{\Omega}$, where λ, $\eta>0$.

If (H1) or (H2) holds, then (1) has at least two distinct nontrivial solutions in $H_0^1(\Omega)$.

Remark 1.

(i) Let $f(x,u)=\lambda u$, we can get $\frac{1}{\rho}f(x,u)u-F(x,u)=-\left(\frac{1}{2}-\frac{1}{\rho}\right)\lambda u^2$, thus $-\left(\frac{1}{2}-\frac{1}{\rho}\right)\lambda$ is the best constant.

(ii) Comparing with [27] and [28], the restrictions on the nonlinear term $f(x,u)$ are weaken.

(iii) If $f(x,u)=\lambda u+\eta u^{2^*(s)-1}$, then it is easy to verify that $f(x,u)$ satisfies (f_1)-(f_3).

Remark 2.

To prove Theorem A, when (H1) holds, the authors used the analytical techniques as that in [24]. In this paper, by accurate estimates of $\|u_\varepsilon\|^2$ and $\int_\Omega|u_\varepsilon|^{2^*(s)}/|x|^s dx$, we obtain $c<\frac{2-s}{2(N-s)}A_{\mu,s}^{(N-s)/(2-s)}$, thus in this case, the mountain pass lemma could also be used. We unify the methods for proving the

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existence of solutions of equation (1) for both cases (H1) and (H2). The results in this paper integrally contain all the cases of Theorem A in [26].

2. PROOF OF THEOREMS

Obviously, in Theorem 1, the values of $f(x,t)$ are irrelevant for $t<0$, therefore, we define

$$f(x,t)=0 \text{ for } x\in\overline{\Omega} \text{ and } t\leq 0.$$

By Hardy inequality and Hardy-Sobolev inequality (see [29]), we define equivalent norm and inner product in $H_0^1(\Omega)$

$$\|u\|:=\left[\int_\Omega\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)dx\right]^{\frac{1}{2}},\ (u,v):=\int_\Omega\left(\nabla u\nabla v-\mu\frac{uv}{|x|^2}\right)dx,\ \forall u,v\in H_0^1(\Omega).$$

Let

$$u^+:=\max\{0,u\},\ F^+(x,t):=\int_0^t f^+(x,s)ds,\ f^+(x,t):=\begin{cases} f(x,t), & t\geq 0,\\ 0, & t<0.\end{cases}$$

The energy functional $J:H_0^1(\Omega)\to\mathbb{R}$ to (1) is given by

$$J(u)=\frac{1}{2}\|u\|^2-\frac{1}{2^*(s)}\int_\Omega\frac{(u^+)^{2^*(s)}}{|x|^s}dx-\int_\Omega F^+(x,u)dx,\ u\in H_0^1(\Omega).$$

We can easily obtain that $J(u)$ is well defined with $J\in C^1(H_0^1(\Omega),\mathbb{R})$ and

$$\langle J'(u),v\rangle=(u,v)-\int_\Omega\frac{(u^+)^{2^*(s)-1}}{|x|^s}vdx-\int_\Omega f^+(x,u)vdx,\ u,v\in H_0^1(\Omega).$$

When $0\leq\mu<\bar{\mu}$, the best constant can be defined as follows (see [30])

$$A_{\mu,s}:=\inf_{u\in H_0^1(\Omega)\backslash\{0\}}\frac{\int_\Omega\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)dx}{\left(\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}}. \tag{4}$$

Lemma 1. Suppose (f_1) hold. For any $0<\varepsilon_1<\min\{\lambda,\eta\}$ and $\alpha_1,\alpha_2\in(1,2^*(s)-1)$, there exists $\xi>0$ such that

$$f(x,t)\geq(\lambda-\varepsilon_1)t+(\eta-\varepsilon_1)t^{2^*(s)-1}-\xi t^{\alpha_1},\ \text{for } t\geq 0 \text{ and } x\in\overline{\Omega}, \tag{5}$$

and

$$f(x,t)\leq(\lambda+\varepsilon_1)t+(\eta+\varepsilon_1)t^{2^*(s)-1}+\xi t^{\alpha_2},\ \text{for } t\geq 0 \text{ and } x\in\overline{\Omega}. \tag{6}$$

Proof.

It follows from (f_1) that $\forall\varepsilon_1>0$, $\exists\delta>0$ and $M_1>0$,

$$\left|\frac{f(x,t)}{t}-\lambda\right|\leq\varepsilon_1,\quad \text{for } (t,x)\in[0,\delta]\times\overline{\Omega}, \tag{7}$$

and

$$\left|\frac{f(x,t)}{t^{2^*(s)-1}}-\eta\right|\leq\varepsilon_1,\quad \text{for } (t,x)\in[M_1,+\infty)\times\overline{\Omega}, \tag{8}$$

from (7), we get

$$f(x,t)\geq(\lambda-\varepsilon_1)t,\quad \text{for } (t,x)\in[0,\delta]\times\overline{\Omega},$$

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for $\alpha_1 \in (1, 2^*(s)-1)$, if we take $\xi \geq \max\left\{0, (\eta-\varepsilon_1)\delta^{2^*(s)-1-\alpha_1}\right\}$, then for any $t \in [0, \delta]$, we have $(\eta-\varepsilon_1)t^{2^*(s)-1} - \xi t^{\alpha_1} \leq 0$, thus

$$f(x,t) \geq (\lambda-\varepsilon_1)t + (\eta-\varepsilon_1)t^{2^*(s)-1} - \xi t^{\alpha_1}, \quad \text{for } (t,x) \in [0,\delta] \times \overline{\Omega}.$$

From (8), we know

$$f(x,t) \geq (\eta-\varepsilon_1)t, \quad \text{for } (t,x) \in [M_1, +\infty) \times \overline{\Omega},$$

for $\alpha_2 \in (1, 2^*(s)-1)$, if we take $\xi \geq \max\left\{0, (\lambda-\varepsilon_1)M_1^{1-\alpha_1}\right\}$, then for any $t \in [M_1, +\infty)$, we have $(\lambda-\varepsilon_1)t - \xi t^{\alpha_1} \leq 0$, thus

$$f(x,t) \geq (\lambda-\varepsilon_1)t + (\eta-\varepsilon_1)t^{2^*(s)-1} - \xi t^{\alpha_1}, \ (t,x) \in [M_1, +\infty) \times \overline{\Omega}.$$

When $t \in [\delta, M_1]$, taking $\xi \geq \max\left\{0, \max\limits_{t\in[\delta,M_1]}\{(\lambda-\varepsilon_1)t^{1-\alpha_1} + (\eta-\varepsilon_1)t^{2^*(s)-1-\alpha_1}\}\right\}$, we have

$$f(x,t) \geq (\lambda-\varepsilon_1)t + (\eta-\varepsilon_1)t^{2^*(s)-1} - \xi t^{\alpha_1}, \ (t,x) \in [\delta, M_1] \times \overline{\Omega}.$$

As mentioned above, if we take

$$\xi \geq \max\left\{0, (\lambda-\varepsilon_1)M_1^{1-\alpha_1}, (\eta-\varepsilon_1)\delta^{2^*(s)-1-\alpha_1}, \max_{t\in[\delta,M_1]}\{(\lambda-\varepsilon_1)t^{1-\alpha_1} + (\eta-\varepsilon_1)t^{2^*(s)-1-\alpha_1}\}\right\},$$

then

$$f(x,t) \geq (\lambda-\varepsilon_1)t + (\eta-\varepsilon_1)t^{2^*(s)-1} - \xi t^{\alpha_1}, \ \text{for } t \geq 0 \text{ and } x \in \overline{\Omega}.$$

Similarly, we may obtain that there exists $\xi > 0$ such that

$$f(x,t) \leq (\lambda+\varepsilon_1)t + (\eta+\varepsilon_1)t^{2^*(s)-1} + \xi t^{\alpha_2}, \ \text{for } t \geq 0 \text{ and } x \in \overline{\Omega}.$$

The conclusion is proved.

Now we introduced the extremal functions. Let

$$C_\varepsilon = \left(\frac{2\varepsilon(\bar{\mu}-\mu)(N-s)}{\sqrt{\bar{\mu}}}\right)^{\sqrt{\bar{\mu}}/(2-s)}, \ U_\varepsilon(x) = \frac{y_\varepsilon(x)}{C_\varepsilon}, \tag{9}$$

define a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that

$$\varphi(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| \geq 2R, \end{cases}$$

where $B_{2R}(0) \subset \Omega$, $0 \leq \varphi(x) \leq 1$, for $R < |x| < 2R$, set

$$u_\varepsilon(x) = \varphi(x)U_\varepsilon(x), \ v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\left(\int_\Omega |u_\varepsilon(x)|^{2^*(s)}|x|^{-s}dx\right)^{\frac{1}{2^*(s)}}}. \tag{10}$$

Lemma 2 ([26]). Let $v_\varepsilon(x)$ be defined as above, then $v_\varepsilon(x)$ satisfies

$$\|v_\varepsilon(x)\|^2 = A_{\mu,s} + O\left(\varepsilon^{\frac{N-2}{2-s}}\right), \tag{11}$$

$$\int_{\Omega}|v_\varepsilon(x)|^q dx = \begin{cases} O\left(\varepsilon^{\frac{\sqrt{\bar{\mu}}q}{2-s}}\right), & 1 \leq q < \frac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}, \\ O\left(\varepsilon^{\frac{\sqrt{\bar{\mu}}q}{2-s}}|\ln\varepsilon|\right), & q = \frac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}, \\ O\left(\varepsilon^{\frac{\sqrt{\bar{\mu}}(N-q\sqrt{\bar{\mu}})}{(2-s)\sqrt{\bar{\mu}-\mu}}}\right), & \frac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} < q < 2^*. \end{cases} \tag{12}$$

Lemma 3 ([31]). Let $u_\varepsilon(x)$, $U_\varepsilon(x)$, $A_{\mu,s}$, C_ε be defined as above, then the exact estimates of $\|u_\varepsilon\|^2$ and $\int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx$ are as follows:

$$\|u_\varepsilon(x)\|^2 = C_\varepsilon^{-2}A_{\mu,s}^{\frac{N-s}{2-s}} + D, \quad \int_\Omega \frac{|u_\varepsilon(x)|^{2^*(s)}}{|x|^s}dx = C_\varepsilon^{-2^*(s)}A_{\mu,s}^{\frac{N-s}{2-s}} + E, \tag{13}$$

where

$$D = \int_{R\leq|x|\leq 2R}\left(|\nabla u_\varepsilon(x)|^2 - \mu\frac{u_\varepsilon^2(x)}{|x|^2}\right)dx - \int_{|x|\geq R}\left(|\nabla U_\varepsilon(x)|^2 - \mu\frac{U_\varepsilon^2(x)}{|x|^2}\right)dx,$$

$$E = -\int_{|x|\geq R}\frac{|U_\varepsilon(x)|^{2^*(s)}}{|x|^s}dx + \int_{R\leq|x|\leq 2R}\frac{|u_\varepsilon(x)|^{2^*(s)}}{|x|^s}dx.$$

Moreover, $\exists\, R_0 > 0$ such that for any $R \leq R_0$,

$$\lim_{\varepsilon\to 0^+} D < \int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}}dx. \tag{14}$$

Lemma 4. Suppose (f_1), (f_2) and $\lambda < \lambda_1(\mu)$ hold. Assume $\{u_n\} \subset H_0^1(\Omega)$ is a $(\mathrm{PS})_c$ sequences, that is,

$$J(u_n) \to c \in \left(0, \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right),$$

and

$$J'(u_n) \to 0, \text{ in } \left(H_0^1(\Omega)\right)^{-1}.$$

Then there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, or a subsequence $u_{n_k} \rightharpoonup u$ weakly in $H_0^1(\Omega)$, moreover, $J'(u) = 0$ and u is a nontrivial solution of (1).

Proof.

First, we claim that if (f_1), (f_2) and $\lambda < \lambda_1(\mu)$ hold, then any $(\mathrm{PS})_c$ sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Otherwise, suppose that $\|u_n\| \to \infty$, since $J(u_n) \to c$, there exists N_1 such that when $n > N_1$, $J(u_n) < c + 1$. $J'(u_n) \to 0$ implies $-\frac{1}{\rho}\langle J'(u_n), u_n\rangle < o(1)\|u_n\|$, thus for any $\varepsilon_1 \in (0, \lambda_1(\mu) - \lambda)$, when $n > N_1$,

$$\begin{aligned} c + 1 + o(1)\|u_n\| &\geq J(u_n) - \frac{1}{\rho}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\rho}\right)\|u_n\|^2 + \int_\Omega\left(\frac{1}{\rho}f^+(x,u_n)u_n - F^+(x,u_n)\right)dx \\ &\quad + \left(\frac{1}{\rho} - \frac{1}{2^*(s)}\right)\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\rho}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\rho}\right)(\lambda + \varepsilon_1)\int_\Omega u_n^2 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\rho}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\rho}\right)\frac{\lambda + \varepsilon_1}{\lambda_1(\mu)}\|u_n\|^2 \\ &= \left(\frac{1}{2} - \frac{1}{\rho}\right)\left(1 - \frac{\lambda + \varepsilon_1}{\lambda_1(\mu)}\right)\|u_n\|^2. \end{aligned}$$

Which shows that $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega)$. By the reflexivity of $H_0^1(\Omega)$, we know that there exists u such that $u_n \rightharpoonup u$ (or a subsequence of u_n convergence to u). Furthermore, $J'(u) = 0$ by the weak continuity of J'. From $u_n \in H_0^1(\Omega)$, $u_n \rightharpoonup u$, by the compactness of the embedding, we have $u_n \to u$ in $L^\gamma(\Omega)$ for any $1 < \gamma < 2^*(s)$. Let $f_1(x,u) = f(x,u)u$, from (f_1), we have $|f_1(x,u_n)| \le a + b|u_n|^{2^*(s)}$, by the definition of Урысон operator, we know $f_1 : L^{2^*(s)}(\Omega) \to L^1(\Omega)$ is a continuous operator. Thus

$$\lim_{n\to\infty} \int_\Omega (f_1(x,u_n) - f_1(x,u))dx = 0,$$

that is,

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)u_n dx = \int_\Omega f(x,u)u dx. \tag{15}$$

Similarly,

$$\lim_{n\to\infty} \int_\Omega F(x,u_n) dx = \int_\Omega F(x,u) dx.$$

In addition, by the convergence of $\|u_n\|$, $u_n \to u$ in $H_0^1(\Omega)$.

Assume that $u \equiv 0$ in Ω, from $\langle J'(u_n), u_n\rangle = o(1)$ and (15) we know

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s} dx = o(1). \tag{16}$$

By (4),

$$\|u_n\|^2 \ge A_{\mu,s}\left(\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}}. \tag{17}$$

From (16) and (17), we have

$$o(1) \ge \|u_n\|^2\left(1 - A_{\mu,s}^{-\frac{2^*(s)}{2}} \|u_n\|^{2^*(s)-2}\right).$$

If $\|u_n\| \to 0$, then (16) implies that $J(u_n) \to 0$, while $J(u_n) \to c$, which contradicts $c > 0$. Hence

$$\|u_n\|^2 \ge A_{\mu,s}^{\frac{N-s}{2-s}} + o(1). \tag{18}$$

By (12), (16) and (18), we obtain

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s} dx + o(1) \\ &= \frac{2-s}{2(N-s)}\|u_n\|^2 + o(1) \\ &\ge \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}} + o(1), \end{aligned}$$

which contradicts $c < \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}$. Thus, u is not constantly equal to 0 and u is a nontrivial solution of problem (1).

Lemma 5. If (f_1), (f_2) and $\lambda < \lambda_1(\mu)$ hold, then the functional J admits a (PS) sequence at level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \left\{\gamma \in C\Big([0,1],\ H_0^1(\Omega)\Big);\ \gamma(0) = 0,\ J(\gamma(1)) < 0\right\}.$$

Proof.

We need to prove J satisfy all assumptions of the mountain pass lemma except for the (PS) condition. Obviously, $J(0) = 0$. Moreover, from the Hardy-Sobolev inequality and the Hardy inequality, we can easily get

$$\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \le C_1 \|u\|^{2^*(s)}, \ \|u\|_q^q \le C_2 \|u\|^q \text{ for } 1 \le q \le 2^*, \ u \in H_0^1(\Omega). \tag{19}$$

Then, by (6), (19) and Lemma 1, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)} \int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s} dx - \int_\Omega F^+(x,u) dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{C_1}{2^*(s)}\|u\|^{2^*(s)} - \frac{\eta + \varepsilon_1}{2^*(s)}\|u\|_{2^*(s)}^{2^*(s)} - \frac{\lambda + \varepsilon_1}{2}\|u\|_2^2 - \frac{\xi}{\alpha_2 + 1}\|u\|_{\alpha_2+1}^{\alpha_2+1} \\ &\ge \frac{1 - (\lambda + \varepsilon_1)/\lambda_1(\mu)}{2}\|u\|^2 - \frac{C_1}{2^*(s)}\|u\|^{2^*(s)} - \frac{C_2}{2^*(s)}\|u\|^{2^*(s)} - \frac{\xi}{\alpha_2 + 1}\|u\|^{\alpha_2+1}, \end{aligned}$$

which implies that $\exists \alpha, \ \rho > 0$ such that

$$J(u) \ge \alpha > 0, \ \forall u \in \Big\{ u \in H_0^1(\Omega) | \|u\| = \rho \Big\}.$$

Taking $u_0 \in H_0^1(\Omega) \setminus \{0\}$, such that $\int_\Omega \frac{(u_0^+)^{2^*(s)}}{|x|^s} dx \ge C_3 > 0$, for any $t > 0$, we have

$$\begin{aligned} J(tu_0) &= \frac{t^2}{2}\|u_0\|^2 - \frac{t^{2^*(s)}}{2^*(s)} \int_\Omega \frac{(u_0^+)^{2^*(s)}}{|x|^s} dx - \int_\Omega F^+(x, tu_0) dx \\ &\le \frac{t^2}{2}\|u_0\|^2 - \frac{t^{2^*(s)}}{2^*(s)} C_3, \end{aligned}$$

notice $\lim_{t \to +\infty} J(tu_0) = -\infty$, then there exists $t_0 > 0$ such that $\|t_0 u_0\| > \rho$ and $J(t_0 u_0) \le 0$. By mountain pass theorem without the (PS) condition (see Theorem 2.2 in [15]), we know that J admits a PS sequence at the c level.

Lemma 6. Assume (f_1), (f_2) and $0 < s < 2$, if (H1) or (H2) holds, then

$$0 < c < \frac{2 - s}{2(N - s)} A_{\mu,s}^{\frac{N-s}{2-s}}. \tag{20}$$

Proof.

Define

$$g(t) := J(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \int_\Omega F^+(x, tv_\varepsilon) dx$$

and

$$\bar{g}(t) := \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)}.$$

It is easy to see that $\lim_{t \to +\infty} g(t) = -\infty$, $g(0) = 0$ and $g(t) > 0$ when t is small enough, so there exists some $t_\varepsilon > 0$, such that $g(t_\varepsilon) = \sup_{t \ge 0} g(t) > 0$, which shows that $c > 0$. Obviously $g'(t_\varepsilon) = 0$, that is,

$$0 = g'(t_\varepsilon) = t_\varepsilon \|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-1} - \int_\Omega f^+(x, t_\varepsilon v_\varepsilon) v_\varepsilon dx,$$

thus

$$\|v_\varepsilon\|^2 = t_\varepsilon^{2^*(s)-2} + \frac{1}{t_\varepsilon}\int_\Omega f^+(x, t_\varepsilon v_\varepsilon)v_\varepsilon dx \geq t_\varepsilon^{2^*(s)-2},$$

therefore,

$$\bar{t}_\varepsilon := \|v_\varepsilon\|^{\frac{2}{2^*(s)-2}} \geq t_\varepsilon. \tag{21}$$

From (7), we know

$$\int_\Omega f(x, t_\varepsilon v_\varepsilon)v_\varepsilon dx \leq (\lambda+\varepsilon_1)t_\varepsilon \int_\Omega v_\varepsilon^2 dx + (\eta+\varepsilon_1)t_\varepsilon^{2^*(s)-1}\int_\Omega v_\varepsilon^{2^*(s)}dx + \xi t_\varepsilon^{\alpha_2}\int_\Omega v_\varepsilon^{\alpha_2+1}dx.$$

Hence

$$\|v_\varepsilon\|^2 \leq t_\varepsilon^{2^*(s)-2} + (\lambda+\varepsilon_1)\int_\Omega |v_\varepsilon|^2 dx + (\eta+\varepsilon_1)|t_\varepsilon|^{2^*(s)-2}\int_\Omega |v_\varepsilon|^{2^*(s)}dx + \xi|t_\varepsilon|^{\alpha_2-1}\int_\Omega |v_\varepsilon|^{\alpha_2+1}dx.$$

Moreover, from Lemma 2, we have

$$t_\varepsilon^{2^*(s)-2} \geq A_{\mu,s} - \varepsilon_2. \tag{22}$$

On the other hand, $\bar{g}(t) \leq \bar{g}(\bar{t}_\varepsilon)$ for any $t \in [0, \bar{t}_\varepsilon]$. From (5), (11), (12), (21), (22) and Lemma 2, we get

$$\begin{aligned} g(t_\varepsilon) &= \bar{g}(\bar{t}_\varepsilon) - \int_\Omega F(x, t_\varepsilon v_\varepsilon)dx \\ &\leq \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda-\varepsilon_1}{2}t_\varepsilon^2\int_\Omega |v_\varepsilon|^2 dx + \frac{\xi}{\alpha+1}|t_\varepsilon|^{\alpha_1+1}\int_\Omega |v_\varepsilon|^{\alpha_1+1}dx \\ &\quad - \frac{\eta-\varepsilon_1}{2^*(s)}|t_\varepsilon|^{2^*(s)}\int_\Omega |v_\varepsilon|^{2^*(s)}dx \\ &\leq \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \frac{(\lambda-\varepsilon_1)(A_{\mu,s}-\varepsilon_2)^{\frac{2}{2^*(s)-2}}}{2}\int_\Omega |v_\varepsilon|^2 dx \\ &\quad + \frac{\xi}{\alpha_1+1}(A_{\mu,s}-\varepsilon_2)^{\frac{\alpha_1+1}{2^*(s)-2}}\int_\Omega |v_\varepsilon|^{\alpha_1+1}dx - \frac{\eta-\varepsilon_1}{2^*(s)}(A_{\mu,s}-\varepsilon_2)^{\frac{2^*(s)}{2^*(s)-2}}\int_\Omega |v_\varepsilon|^{2^*(s)}dx. \end{aligned} \tag{23}$$

If (H1) holds, notice that $2 \geq \frac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}$, then when ε is sufficiently small, the sign of $-\frac{\lambda-\varepsilon_1}{2}t_\varepsilon^2\int_\Omega |v_\varepsilon|^2 dx + \frac{\xi}{\alpha+1}|t_\varepsilon|^{\alpha_1+1}\int_\Omega |v_\varepsilon|^{\alpha_1+1}dx - \frac{\eta-\varepsilon_1}{2^*(s)}|t_\varepsilon|^{2^*(s)}\int_\Omega |v_\varepsilon|^{2^*(s)}dx$ is decided by the sign of $-\int_\Omega |v_\varepsilon|^{2^*(s)}dx$.Thus, when ε is small enough, (20) holds true.

If (H2) holds, since $\alpha_1 > 1$ is arbitrary, we can choose $\alpha_1 > \frac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}$, then by Lemma 2, we know that when ε is sufficiently small, the sign of $\frac{\xi}{\alpha+1}|t_\varepsilon|^{\alpha_1+1}\int_\Omega |v_\varepsilon|^{\alpha_1+1}dx - \frac{\eta-\varepsilon_1}{2^*(s)}|t_\varepsilon|^{2^*(s)}\int_\Omega |v_\varepsilon|^{2^*(s)}dx$ is decided by the sign of $-\int_\Omega |v_\varepsilon|^{2^*(s)}dx$. Thus from (23), when ε, ε_1 and ε_2 are small enough,

$$g(t_\varepsilon) < \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda_*(\mu)}{2}A_{\mu,s}^{\frac{2}{2^*(s)-2}}\int_\Omega |v_\varepsilon|^2 dx.$$

From (13) we know

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}}-A_{\mu,s}^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\int_\Omega|v_\varepsilon|^2dx}\\
=&\lim_{\varepsilon\to 0^+}\frac{\|u_\varepsilon\|^{\frac{2(N-s)}{2-s}}-A_{\mu,s}^{\frac{N-s}{2-s}}\left(\int_\Omega\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\left(\int_\Omega\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{\frac{N-2}{2-s}\frac{2}{2^*(s)}}\int_\Omega|u_\varepsilon|^2dx}\\
=&\lim_{\varepsilon\to 0^+}\frac{\left(D+C_\varepsilon^{-2}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-s}{2-s}}-A_{\mu,s}^{\frac{N-s}{2-s}}\left(E+C_\varepsilon^{-2^*(s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\left(E+C_\varepsilon^{-2^*(s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}\frac{2}{2^*(s)}}\int_\Omega|u_\varepsilon|^2dx}\\
=&\lim_{\varepsilon\to 0^+}\frac{\left(C_\varepsilon^2D+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-s}{2-s}}-A_{\mu,s}^{\frac{N-s}{2-s}}\left(EC_\varepsilon^{2^*(s)}+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}}{C_\varepsilon^2A_{\mu,s}^{\frac{N-2}{2-s}}\left(EC_\varepsilon^{2^*(s)}+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}\frac{2}{2^*(s)}}\int_\Omega|u_\varepsilon|^2dx} \qquad (24)\\
=&\lim_{\varepsilon_0\to 0}\frac{\left(C^2D\varepsilon_0+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-s}{2-s}}-A_{\mu,s}^{\frac{N-s}{2-s}}\left(EC^{2^*(s)}\varepsilon_0^{\frac{N-s}{N-2}}+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}}{C^2\varepsilon_0A_{\mu,s}^{\frac{(N-s)(N-2)}{(2-s)^2}}\int_\Omega|u_\varepsilon|^2dx}\\
=&\frac{N-s}{2-s}\lim_{\varepsilon_0\to 0^+}\frac{PC^2\left(D+\varepsilon_0\frac{\partial D}{\partial\varepsilon_0}\right)-A_{\mu,s}^{\frac{N-s}{2-s}}QC^{2^*(s)}\left(E\varepsilon_0^{\frac{2-s}{N-2}}+\frac{\partial E}{\partial\varepsilon_0}\varepsilon_0^{\frac{N-s}{N-2}}\right)}{C^2A_{\mu,s}^{\frac{(N-s)(N-2)}{(2-s)^2}}\int_\Omega|u_\varepsilon|^2dx}\\
=&\frac{N-s}{2-s}\frac{D}{\int_\Omega\frac{|\varphi(x)|^2}{|x|^{2\beta}}dx},
\end{aligned}
$$

where $C=\left(\frac{2(\bar{\mu}-\mu)(N-s)}{\sqrt{\bar{\mu}}}\right)^{\frac{\sqrt{\bar{\mu}}}{2-s}}$, $\varepsilon_0=\varepsilon^{\frac{\sqrt{\bar{\mu}}}{2-s}}$, $P=\left(C^2D\varepsilon_0+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}$, $Q=\left(EC^{2^*(s)}\varepsilon_0^{\frac{N-s}{N-2}}+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-4+s}{2-s}}$.
By (14), (13) and (24), we have

$$
\lim_{\varepsilon\to 0^+}\frac{\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}}-A_{\mu,s}^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\int_\Omega|v_\varepsilon|^2dx}<\frac{N-s}{2-s}\frac{\int_\Omega\frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}}dx}{\int_\Omega\frac{|\varphi(x)|^2}{|x|^{2\beta}}dx},
$$

so, if ε is small enough, then $c\le g(t_\varepsilon)<\frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}$.

Proof of Theorem 1.

By Lemmas 4, 5 and 6, we can get that equation (1) has a nonnegative solution $u\in H_0^1(\Omega)$, by the maximum principle, this solution is positive. Which completes the proof.

Proof of Theorem 2.

Since (f_3) contains (f_1), Theorem 1 implies the existence of a positive solution u_1 for equation (1). Let $f(x,t)=-h(x,-t)$ for $t\in\mathbb{R}$, $h(x,u)$ satisfies (f_1) and (f_2), then

$$
-\Delta u-\mu\frac{u}{|x|^2}=\frac{|u|^{2^*(s)-2}}{|x|^s}u+h(x,u)
$$

has at least one nonnegative solution v. Let $u_2 = -v$, then u_2 is a solution of

$$-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u + f(x,u).$$

Clearly, $u_1, u_2 \neq 0$. So problem (1) has at least two distinct nontrivial solutions.

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References

1. R. Dautray, J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, in: Physical Origins and Classical Methods, Springer-Verlag, Berlin Heidelberg, 1990.
2. D.D. Joseph, T.S. Lundgren, Quasilinear Dirichlet problem driven by positive sources, Arch. Ration. Mech. Anal. 49 (4) (1973) 241-269.
3. H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (3) (1983) 486-490.
4. S. Coleman, V. Glazer, A. Martin, Action minima among solutions to a class of Euclidean scalar field equations, Commun. Math. Phys. 58 (2) (1978) 211-221.
5. W. Strauss, Existence of solitary waves in higher dimensions, Commun. Math. Phys. 55 (2) (1977) 149-162.
6. P.L. Lions, Minimization problems in $L^1(\mathbb{R}^3)$, J. Funct. Anal. 41 (2) (1981) 236-275.
7. A. Kassymov, D. Suragan, Multiplicity of positive solutions for a nonlinear equation with a Hardy potential on the Heisenberg group, Bulletin des Sciences Mathatiques 165 (2020) 102916.
8. J. Liu, Z.Q. Zhao, Leray–Lions type p(x)-biharmonic equations involving Hardy potentials, Appl. Math. Lett. 149 (2024) 108907.
9. M. Fărcăşeanu, Isolated singularities for semilinear elliptic systems with Hardy potential, J. Math. Anal. Appl. 527(1) Part 2, (2023) 127415.
10. P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, Slam Rev. 24 (4) (1982) 441-467.
11. A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problem, J. Funct. Anal. 122 (2) (1994) 519-543.
12. P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applictions to Differential Equations, in: GCBMS Reg. Conf. Series. Math., Vol 65, Amer. Math. Soc., Providence, RI, 1986.
13. M. Struwe, Variational Methods: Applications to Non-linear Partial Differential Equations and Hamiltonian Systems, fourth ed., Springer-Verlag, Berlin Heidelberg, 1990.
14. A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (4) (1973) 349-381.
15. H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Commun. Pure Appl. Math. 36 (4) (1983) 437-477.
16. I. Ekeland, N. Ghoussoub, Selected new aspects of the calculus of variations in the large, Bull. Amer. Math. Soc. 39 (2) (2002) 207-265.
17. R.S. Palais, S. Smale, A generalized Morse theory, Bull. Amer. Math. Soc. 70 (1) (1964) 165-172.
18. J.P.G. Azorero, I.P. Alonso, Hardy inequalities and some critical elliptic and parabolic problems, J. Differ. Equations 144 (2) (1998) 441-476.
19. D.M. Cao, S.J. Peng, A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms, J. Differ. Equations 193 (2) (2003) 424-434.
20. J.P.G. Azorero, I.P. Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, Trans. Amer. Math. Soc. 323 (2) (1991) 877-895.
21. P. Padilla, The effect of the shape of the domain on the existence of solutions of an equation involving the critical Sobolev exponent, J. Differ. Equations 124 (2) (1996) 449-471.
22. A. Ferrero, F. Gazzola, Existence of solutions for singular critical growth semilinear elliptic equations, J. Differ. Equations 177 (2) (2001) 494-522.

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23. D.M. Cao, P.G. Han, Solutions for semilinear elliptic equations with critical exponents and Hardy potential, J. Differ. Equations 205 (2) (2004) 521-537.
24. E. Jannelli, The role played by space dimension in elliptic critical problems, J. Differ. Equations 156 (2) (1999) 407-426.
25. D.S. Kang, Y.B. Deng, Existence of solution for a singular critical elliptic equation, J. Math. Anal. Appl. 284 (2) (2003) 724-732.
26. D.S. Kang, S.J. Peng, Positive solutions for singular critical elliptic problems, Appl. Math. Lett. 17 (4) (2004) 411-416.
27. M.C. Wang, Q. Zhang, Existence of solutions for singular critical semilinear elliptic equation, Appl. Math. Lett. 94 (2) (2019) 217-223.
28. L. Ding, C.L. Tang, Existence and multiplicity of solutions for semilinear elliptic equations with Hardy terms and Hardy-Sobolev critical exponents, Appl. Math. Lett. 20 (12) (2007) 1175-1183.
29. N. Ghoussoub, C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, Trans. Amer. Math. Soc. 352 (12) (2000) 5703-5743.
30. K.S. Chou, C.W. Chu, On the best constant for a weighted Sobolev-Hardy inequality, J. Lond. Math. Soc. 48 (2) (1993) 137-151.
31. L.L. Wang, Y.H. Fan, Existence and nonexistence of positive solutions for semilinear elliptic equations involving Hardy–Sobolev critical exponents, Mathematics, 12 (2024) 1616.