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Article

On the Lebesgue Constants

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Abstract: The obtention of simple formulae for the Lebesgue constants arising in the the classical Fourier series approximation is presented. Both even and odd cases are treated enlarging Fejér result. Asymptotic formulae are also obtained.

Keywords: Lebesgue constant; Dirichlet kernel; Fourier series approximation

MSC: 65T40

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1. Introduction

There are many famous important constants in Mathematics, such as e (Napier's constant or Euler's number), π (Archimedes' constant), $\sqrt{2}$ (Pythagoras' constant), γ (Euler's constant) to cite a few [1]. In this article we shall be concerned with the Lebesgue constants which were introduced by H. Lebesgue as best possible upper bound for the approximation of functions through Fourier series [2–4]. These numbers are usually expressed in the form

$$I_n = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin((n+1/2)t)|}{\sin(t/2)} dt \tag{1}$$

where $n \in \mathbb{N}$.

Several famous mathematicians have worked on these constants, established some properties, namely asymptotic and proposed alternative expression. We list some of them.

- Fejér [5] proved the formula

$$I_n = \frac{1}{2n+1} + \frac{2}{\pi}\sum_{k=1}^{n} \frac{1}{k}\tan\left(\frac{k\pi}{2n+1}\right), \quad n \in \mathbb{N}; \tag{2}$$

- Szegö [6] contributed with the formula

$$I_n = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\sum_{j=1}^{(2n+1)k} \frac{1}{4k^2-1}\frac{1}{2j-1}, \quad n \in \mathbb{N}; \tag{3}$$

- Watson [7] established the following asymptotic formula

$$\lim_{n\to\infty}\left(I_n - \frac{4}{\pi^2}\ln(2n+1)\right) = c \tag{4}$$

where

$$c = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\ln(k)}{4k^2-1} - \frac{4}{\pi^2}\psi\left(\frac{1}{2}\right) = \frac{4}{\pi^2}(2.4413238136\ldots), \tag{5}$$

and $\psi(x)$ is the digamma function;

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- Hardy [8] discovered two integral representations

$$\begin{aligned} I_n &= 4\int_0^{\infty} \frac{\tanh((2n+1)x)}{\tanh(x)} \frac{1}{\pi 2 + 4x^2} dx \\ &= \int_0^{\infty} \frac{\sinh((2n+1)x)}{\sinh(x)} \ln\left(\coth\left(\frac{2n+1}{2}x\right)\right) dx. \end{aligned} \tag{6}$$

Later on, other mathematicians have contributed with two-sided estimates. Zhao [9] discovered two-sided inequalities which help to improve the Watson asymptotic expansion formulas. In [3], new inequalities were established for the Lebesgue constants $I_{n/2}$ which allowed obtaining an asymptotic expansion of $I_{n/2}$ in terms of $\frac{1}{n+1}$. More recently, other contributions were published. Shakirov approximated the Lebesgue constant by a logarithmic function [10] and by means of logarithmic-fractional-rational function [4]. The asymptotic behaviour of I_n was also study in [11], although indirectly, since the authors studied the properties of the Dirichlet kernel that is related to the integrand function appearing in (8). It must be remarked that (1) can be rewritten in the form

$$I_n = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin((2n+1)t)|}{\sin(t)} dt \tag{7}$$

where $n \in \mathbb{N}$. As $2n+1,\ n \in \mathbb{N}$, is an odd integer, we are motivated to consider the more general constants

$$I_n = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin(nt)|}{\sin(t)} dt \tag{8}$$

that we continue calling Lebesgue constants and where $n \in \mathbb{N}$.

In the following we will describe the steps involved in the obtention of the Fejér's formula for the Lebesgue odd order numbers. The steps constitute a simple way that allow us to obtain formulae for any positive integer. We are going to consider and treat separately, even $n = 2N$, and odd, $n = 2N+1$, cases ($N \in \mathbb{N}$). Asymptotic formulae are also proposed.

2. The Way to the Lebesgue Constants

Lebesgue studied the approximation of periodic functions by the partial sum of the Fourier series [2] and obtained a formulation that can be stated as [11]

Theorem 1. *Let $f(t)$ be a periodic continuous function on $[-\pi, \pi]$, and*

$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt} dt$$

the Fourier coefficients, where $i = \sqrt{-1}$. If $S_n(f),\ n \in \mathbb{N}$ denotes the n^{th} partial sum of $f(t)$, that is

$$S_n(f) = \sum_{k=-n}^{n} C_k e^{ikt} \tag{9}$$

then

$$S_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\tau) f(t-\tau) d\tau \tag{10}$$

where

$$D_n(t) = \begin{cases} \dfrac{\sin((n+1/2)t)}{\sin(t/2)} & t \neq 0 \\ 2n+1 & t = 0 \end{cases} \tag{11}$$

is the so-called Dirichlet kernel.

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This theorem shows the importance of the Dirichlet kernel and the relation with the Lebesgue constants. Lebesgue showed that,

Corollary 1. *If*

$$\max_{-\pi \le t \le \pi} |f(t)| \le 1$$

then

$$\max_{-\pi \le t \le \pi} |S_n(f)| \le I_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \tag{12}$$

where I_n is the best possible upper bound that can be rewritten as

$$I_n = \frac{1}{\pi} \int_0^{\pi} \frac{|\sin((n+1/2)t)|}{\sin(t/2)} dt \tag{13}$$

The Dirichlet kernel verifies the following relation [11]

$$4 + \frac{8}{\pi} \ln\left(n + \frac{1}{2}\right) \le \int_{-\pi}^{\pi} |\mathcal{D}_n(t)| dt \le 2\pi + 4 + \frac{8}{\pi} \ln(2n),$$

that can be used to find the asymptotic behaviour of the Lebesgue constants. We will consider a general version of (12)

$$I_n = \frac{2}{\pi} \int_0^{\pi/2} \frac{|\sin(nt)|}{\sin(t)} dt \tag{14}$$

where n is any positive integer, even or odd. We will propose an alternative approach that leads to exact closed formulae corresponding any positive values of n. In the following picture we depict the result of numerical integration of (14) for $n = 1, 2, 3, \cdots 2^L$, with $L = 16$. We used a log-scale.

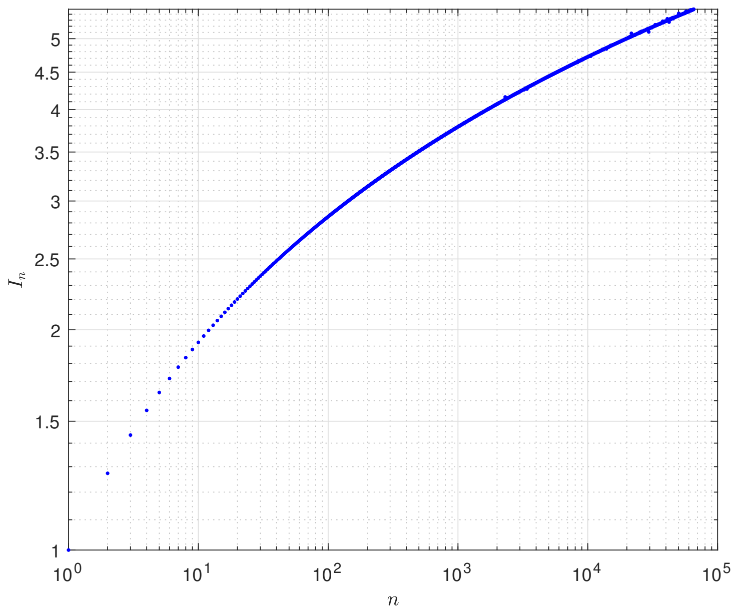

Figure 1. Examples for $n = 1, 2, \cdots$.

We go on by establishing a result that will useful in a later Section. Let us define Dirichlet-like kernel given by

$$\bar{D}_n(t) = \frac{\sin(nt)}{\sin(t)}$$

Theorem 2. *We can show that*

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$$\bar{D}_n(t) = \frac{\sin(nt)}{\sin(t)} = e^{i(n-1)t} + e^{i(n-3)t} + \cdots + e^{-i(n-3)t} + e^{-i(n-1)t} \tag{15}$$

The proof is immediate. We only need to note that

$$\frac{\sin(nt)}{\sin(t)} = \frac{e^{int} - e^{-int}}{e^{it} - e^{-it}} = e^{i(n-1)t}\frac{1 - e^{-i2nt}}{1 - e^{-i2t}}$$

and apply the geometric sum rule.

As observed, the exponents have the generic form $i(n-(2k-1))t,\ k = 1, 2, \cdots$. If n is odd it can assume the value 0. Therefore

Corollary 2. *Let*

$$N = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases},$$

Then

$$\bar{D}_n(t) = \frac{\sin(nt)}{\sin(t)} = \begin{cases} 1 + 2\sum_{k=1}^{N}\cos(2kt) & \text{if } n \text{ is odd} \\ 2\sum_{k=1}^{N}\cos((2k-1)t) & \text{if } n \text{ is even} \end{cases} \tag{16}$$

This expression is suitable to obtain the primitive of $\bar{D}_n(t)$.

3. New Formulation

3.1. Preliminaries

We are going to make a brief study of the kernel that makes easier the way into the solution we search. Consider the function

$$f(t) = |\sin(nt)|, \qquad 0 \le t \le \pi/2,$$

where the sinusoid $\sin(nt)$ has frequency $\frac{n}{2\pi}$ and half-period $T = \frac{\pi}{n}$. Therefore, in the interval $0 \le t \le \pi/2$, there are N half periods. If n is odd, there is another quarter of period. In the half periods with orders $0, 2, 4, \cdots$ the function $\sin(nt)$ is positive. In the others, it is negative.

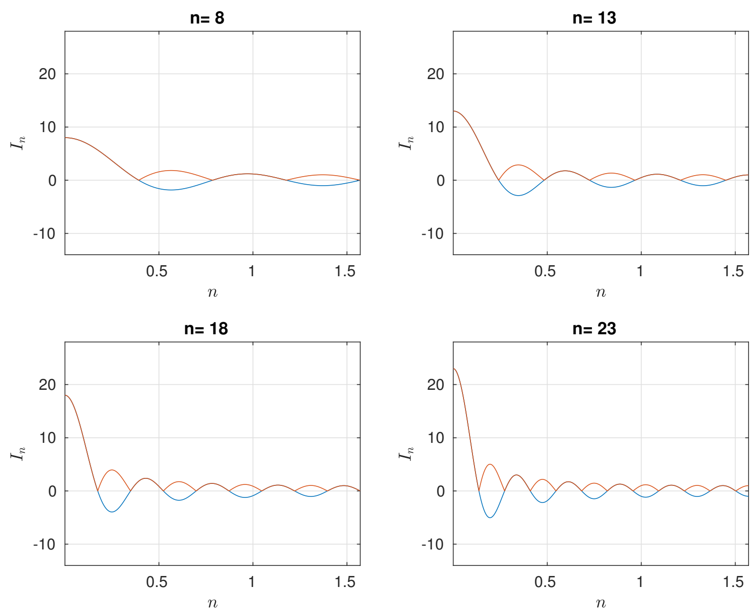

Figure 2. Examples for $n = 8, 13, 18, 23$.

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Therefore, the presence of the absolute value in $f(t)$ allows us to write

$$f(t) = \sum_{m=0}^{N} f_m(t),$$

where

$$f_m(t) = f_{m-1}(t-T),\ mT \leq t < (m+1)T, \qquad m = 1,2,\cdots N-1,$$

with $f_0(t) = \sin(nt),\ 0 \leq t < T$, and the last term $(m = N)$

$$f_N(t) = \begin{cases} f_{N-1}(t-T),\ NT \leq t < NT + T/2, & \textit{if } N \textit{ is odd} \\ 0, & \textit{if } N \textit{ is even} \end{cases}.$$

This means that $f(t)$ is obtained by juxtaposing N positive half periods of $\sin(nt)$. If n is odd, we have to join another one quarter of a period.

3.2. The Even n Case

Theorem 3. *Let $n \in \mathbb{N}$ be an even number. Then*

$$I_n = \frac{4}{\pi} \sum_{k=1}^{n/2} \frac{\tan\left(\frac{(2k-1)\pi}{2n}\right)}{2k-1}. \tag{17}$$

Proof. According to the structure of the numerator of our kernel, we can write We have

$$I_n = \frac{2}{\pi} \sum_{m=0}^{N-1} \int_{mT}^{(m+1)T} \frac{\sin(n(t-mT))}{\sin(t)} dt \tag{18}$$

Attending to

$$\sin(nt) = \sin(n(t + mT - mT)) = \sin(n(t+mT) - m\pi) = (-1)^m \sin(n(t+mT)),$$

we are led to

$$I_n = \frac{2}{\pi} \sum_{m=0}^{N-1} (-1)^m \left[\int_0^T \frac{\sin(n(t+mT))}{\sin(t+mT)} dt \right], \tag{19}$$

that expresses the Lebesgue numbers in a new different way.

To continue, we need to find the primitive of the integrand which is not a big task. In fact, as seen above,

$$\frac{\sin(nx)}{\sin(x)} = 2 \sum_{k=1}^{N} \cos((2k-1)x).$$

It follows that

$$P_n(x) = \int \frac{\sin(nx)}{\sin x} dx = 2 \sum_{k=1}^{N} \frac{\sin((2k-1)x)}{2k-1}. \tag{20}$$

For our application, $x = t + mT$, so that

$$P_n(t) = \int \frac{\sin(n(t+mT))}{\sin(t+mT)} dt = 2 \sum_{k=1}^{n/2} \frac{\sin((2k-1)(t+mT))}{2k-1}. \tag{21}$$

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Let us denote the function in brackets in (19) by $J_m(n)$. We have

$$I_n = \frac{2}{\pi}\sum_{m=0}^{N-1}(-1)^m J_m(n). \tag{22}$$

with

$$J_m(n) = \int_0^T \frac{\sin(n(t+mT))}{\sin(t+mT)}dt = P_n(T) - P_n(0),$$

where $T = \pi/n$. Then $mT = \frac{\pi m}{n}$, $T + mT = \frac{(m+1)\pi}{n}$, and

$$J_m(n) = 2\sum_{k=1}^{N}\frac{\sin((2k-1)(m+1)\pi/n)}{2k-1} - 2\sum_{k=1}^{N}\frac{\sin((2k-1)m\pi/n)}{2k-1}.$$

Simplifying

$$J_m(n) = 2\sum_{k=1}^{N}\frac{\sin((2k-1)(m+1)\pi/n) - \sin((2k-1)m\pi/n)}{2k-1},$$

using the trigonometric identity $\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, we get

$$J_m(n) = 4\sum_{k=1}^{N}\frac{\cos((2k-1)(2m+1)\pi/(2n))\sin((2k-1)\pi/(2n))}{2k-1}. \tag{23}$$

that inserted into (22) gives the expected result. However, we can go ahead and manipulate these formulae trying to obtain any simplification. We proceed to invert the summation order:

$$\sum_{m=0}^{N-1}(-1)^m\left(\sum_{k=1}^{N}\frac{\cos((2k-1)(2m+1)\pi/(2n))\sin((2k-1)\pi/(2n))}{2k-1}\right) =$$
$$\sum_{k=1}^{N}\frac{\sin((2k-1)\pi/(2n))}{2k-1}\left(\sum_{m=0}^{N-1}(-1)^m\cos((2k-1)(2m+1)\pi/(2n))\right)$$

Substituting sinusoid by exponentials and using the rule for summing the geometric sequence, we can show that

$$\sum_{m=0}^{N-1}(-1)^m\cos((2k-1)(2m+1)\pi/(2n)) = \frac{1-(-1)^N\cos\left(\frac{N}{n}(2k-1)\pi\right)}{2\cos\left(\dfrac{(2k-1)\pi}{2n}\right)}$$

Attending to the fact that $2N = n$ and $\cos((2k-1)\pi/2) = 0$, we can write

$$\sum_{m=0}^{N-1}(-1)^m\cos((2k-1)(2m+1)\pi/n) = \frac{1}{2\cos\left(\dfrac{(2k-1)\pi}{2n}\right)}$$

and finally,

$$I_n = \frac{4}{\pi}\sum_{k=1}^{n/2}\frac{\tan\left(\frac{(2k-1)\pi}{2n}\right)}{2k-1}. \tag{24}$$

□

This formula was never proposed. In Figure 3 we compare the value of I_n obtained from (24) with the numerical integration of (14).

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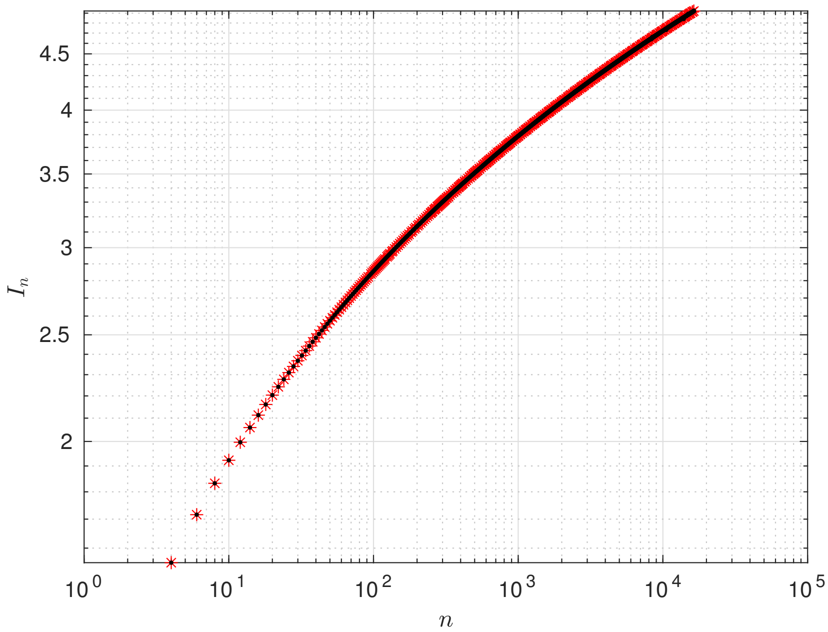

Figure 3. Comparative of (24) (red) with (14) (black).

3.3. The Odd n Case

Theorem 4. *Let $n \in \mathbb{N}$ be an odd number. Then*

$$I_n = \frac{2}{\pi} \sum_{k=1}^{(n-1)/2} \frac{\tan(k\pi/n)}{k} + \frac{1}{n} \tag{25}$$

This formula was proposed first by Fejér but deduced using a completely different procedure [5].

Proof. Let $n = 2N + 1$. Differently from the even n case, we have

$$I_n = \frac{2}{\pi} \sum_{m=0}^{N-1} \int_{mT}^{(m+1)T} \frac{\sin(n(t - mT))}{\sin(t)} dt + \frac{2}{\pi} \int_{NT}^{NT+T/2} \frac{\sin(n(t - NT))}{\sin(t)} dt. \tag{26}$$

We joined an extra quarter of a period.

We can write

$$\sin(nt) = \sin(n(t + mT - mT)) = \sin(n(t + mT) - m\pi) = (-1)^m \sin(n(t + mT)).$$

The second term on the right of the equality is

$$\int_{NT}^{NT+T/2} \frac{\sin(n(t - NT))}{\sin(t)} dt = \int_0^{T/2} \frac{\sin(nt)}{\sin(t + NT)} dt = \int_0^{T/2} \frac{(-1)^N \sin(n(t + NT))}{\sin(t + NT)} dt.$$

$$I_n = \frac{2}{\pi} \left[\sum_{m=0}^{N-1} (-1)^m \int_0^T \frac{\sin(n(t + mT))}{\sin(t + mT)} dt + \int_0^{T/2} (-1)^N \frac{\sin(n(t + NT))}{\sin(t + NT)} dt \right], \tag{27}$$

that re-expresses the Lebesgue numbers in a new different way.

To continue, we need to find the primitive of the integrand which is not a big task attending to (16). In fact, we have

$$\frac{\sin(nx)}{\sin(x)} = 1 + 2 \sum_{k=1}^{(n-1)/2} \cos 2kx,$$

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It follows that

$$P_n(x) = \int \frac{\sin(nx)}{\sin x}dx = \sum_{k=1}^{(n-1)/2} \frac{\sin 2kx}{k} + x.$$

For our application, $x = t + mT$, so that

$$P_n(t) = \int \frac{\sin(n(t+mT))}{\sin(t+mT)}dt = \sum_{k=1}^{N} \frac{\sin(2k(t+mT))}{k} + (t+mT). \tag{28}$$

As above, let us denote the function in brackets in (27) by $J_m(n)$ and the second term by $J_N(n)$ so that

$$I_n = \frac{2}{\pi}\sum_{m=0}^{N-1}(-1)^m J_m(n) + \frac{2}{\pi}J_N(n) \tag{29}$$

with

$$J_m(n) = \int_0^T \frac{\sin(n(t+mT))}{\sin(t+mT)}dt = P_n(T) - P_n(0),$$

and

$$J_N(n) = \int_0^{T/2} \frac{\sin(n(t+mT))}{\sin(t+mT)}dt = P_n(T/2) - P_n(0).$$

As $T = \pi/n$, then $mT = \frac{\pi m}{n}$, $T + mT = \frac{(m+1)\pi}{n}$, and

$$J_m(n) = \sum_{k=1}^{N} \frac{\sin(2k(m+1)\pi/n)}{k} + \frac{(m+1)\pi}{n} - \sum_{k=1}^{N} \frac{\sin(2km\pi/n)}{k} + \frac{m\pi}{n}.$$

We obtain

$$J_m(n) = \sum_{k=1}^{N} \frac{\sin(2k(m+1)\pi/n) - \sin(2km\pi/n)}{k} + \frac{\pi}{n}.$$

Using the trigonometric identity $\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$,

$$J_m(n) = 2\sum_{k=1}^{N} \frac{\cos(k(2m+1)\pi/n)\sin(k\pi/n)}{k} + \frac{\pi}{n}. \tag{30}$$

Concerning the other term, $J_N(n)$ we have:

$$J_N(n) = \sum_{k=1}^{N} \frac{\sin(2k(T/2+NT))}{k} + (T/2+NT) - \sum_{k=1}^{N} \frac{\sin(2kN\pi/n)}{k} - NT.$$

But $n = 2N+1$ and $Tn = \pi$, allowing a simplification of the above expression:

$$J_N(n) = \sum_{k=1}^{N} (-1)^k \frac{\sin(k\pi/n)}{k} + \frac{\pi}{2n}. \tag{31}$$

Let us go ahead and manipulate these formulae trying to obtain simplifications. We turn our attention to (30), (31), and (29). Then

$$\begin{aligned} I_n = {} & \frac{4}{\pi}\sum_{m=0}^{N-1}(-1)^m\left(\sum_{k=1}^{N} \frac{\cos(k(2m+1)\pi/n)\sin(k\pi/n)}{k}\right) + \\ & \frac{2}{\pi}(-1)^N\sum_{k=1}^{N}(-1)^k\frac{\sin(k\pi/n)}{k} + \frac{2}{n}\sum_{m=0}^{N-1}(-1)^m + (-1)^N\frac{1}{n}. \end{aligned}$$

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or

$$
\begin{aligned}
I_n = \frac{4}{\pi} \sum_{m=0}^{N-1} (-1)^m \left(\sum_{k=1}^{N} \frac{\cos(k(2m+1)\pi/n)\sin(k\pi/n)}{k} \right) + \\
\frac{2}{\pi}(-1)^N \sum_{k=1}^{N} (-1)^k \frac{\sin(k\pi/n)}{k} + \frac{1}{n}
\end{aligned}
\tag{32}
$$

Let us change the summation order in the first term on the right in expression (32):

$$
\begin{aligned}
\sum_{m=0}^{N-1} (-1)^m \left(\sum_{k=1}^{N} \frac{\cos(k(2m+1)\pi/n)\sin(k\pi/n)}{k} \right) = \\
\sum_{k=1}^{N} \frac{\sin(k\pi/n)}{k} \left(\sum_{m=0}^{N-1} (-1)^m \cos(k(2m+1)\pi/n) \right).
\end{aligned}
$$

Observe that

$$
\sum_{m=0}^{N-1} (-1)^m \cos(k(2m+1)\pi/n) = \frac{1-(-1)^N \cos\left(\frac{2N}{n}k\pi\right)}{2\cos\left(k\frac{\pi}{n}\right)}. \tag{33}
$$

But, $2N = n-1$, so that

$$
\cos\left(\frac{2N}{n}k\pi\right) = \cos\left(\left(1-\frac{1}{n}\right)k\pi\right) = (-1)^k \cos\left(k\frac{\pi}{n}\right).
$$

Hence

$$
\sum_{m=0}^{N-1} (-1)^m \cos(k(2m+1)\pi/n) = \frac{1-(-1)^{N+k}\cos\left(k\frac{\pi}{n}\right)}{2\cos\left(k\frac{\pi}{n}\right)} = \frac{1}{2\cos\left(k\frac{\pi}{n}\right)} - (-1)^{N+k}\frac{1}{2},
$$

that leads to

$$
\begin{aligned}
\sum_{m=0}^{N-1} (-1)^m \left(\sum_{k=1}^{N} \frac{\cos(k(2m+1)\pi/n)\sin(k\pi/n)}{k} \right) = \\
\frac{1}{2}\sum_{k=1}^{N} \frac{\sin(k\pi/n)}{k} \left(\frac{1}{\cos(k\pi/n)} - (-1)^{N+k} \right) = \\
\frac{1}{2}\sum_{k=1}^{N} \frac{\tan(k\pi/n)}{k} - \frac{(-1)^N}{2} \sum_{k=1}^{N} (-1)^k \frac{\sin(k\pi/n)}{k}
\end{aligned}
$$

Inserting it into (32), we obtain

$$
\begin{aligned}
I_n = \frac{2}{\pi}\sum_{k=1}^{N} \frac{\tan(k\pi/n)}{k} - \frac{2}{\pi}(-1)^N \sum_{k=1}^{N} (-1)^k \frac{\sin(k\pi/n)}{k} + \\
\frac{2}{\pi}(-1)^N \sum_{k=1}^{N} (-1)^k \frac{\sin(k\pi/n)}{k} + \frac{1}{n}
\end{aligned}
\tag{34}
$$

Finally

$$
I_n = \frac{2}{\pi}\sum_{k=1}^{N} \frac{\tan(k\pi/n)}{k} + \frac{1}{n}. \tag{35}
$$

In Figure 4 we compare (35) with the integral representation of I_n (14). □

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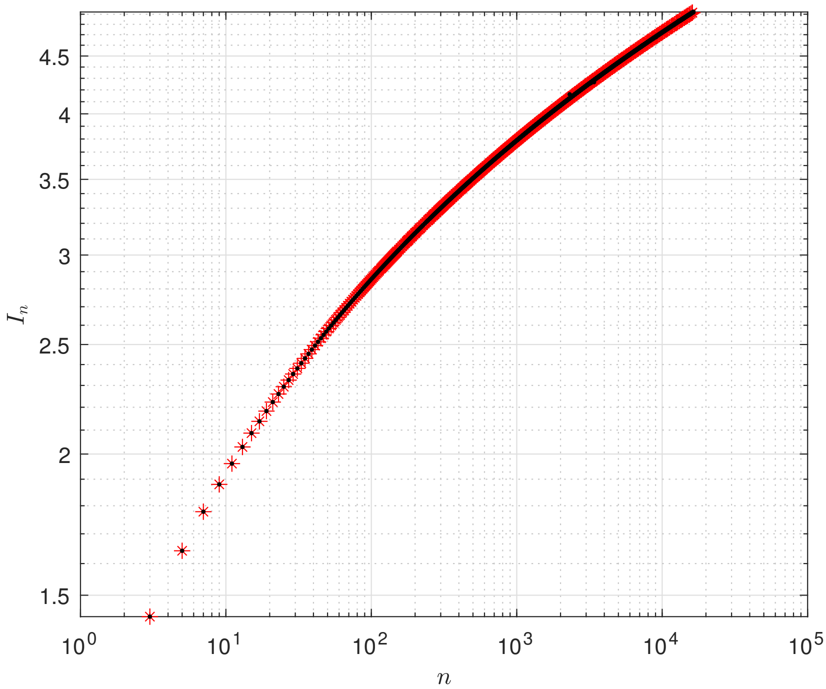

Figure 4. Comparative of (35) (red) with (14) (black).

4. Asymptotic Behavior

We consider the odd n case. The even n case can be solved by relying on the odd case. Let us turn our attention to (29)

$$\begin{aligned} I_n = \frac{2}{\pi}\sum_{m=0}^{N-1}(-1)^m\left[\sum_{k=1}^{N}\frac{\sin(2k(m+1)\pi/n)-\sin(2km\pi/n)}{k}+\frac{\pi}{n}\right]+ \\ \frac{2}{\pi}\left[\sum_{k=1}^{N}(-1)^k\frac{\sin(k\pi/n)}{k}+\frac{\pi}{2n}\right] \end{aligned} \tag{36}$$

Using the formula [12]

$$\sum_{k=1}^{N}\frac{\sin(kx)}{k}=\frac{1}{2}\int_0^x (D_N(t)-1)dt \tag{37}$$

where

$$D_N(t)=\frac{\sin((N+1/2)t)}{\sin(t/2)},$$

we have

$$\sum_{k=1}^{N}\frac{\sin(2k(m+1)\pi/n)-\sin(2km\pi/n)}{k}=\frac{1}{2}\int_0^{2(m+1)\pi/n}(D_N(t)-1)dt-\frac{1}{2}\int_0^{2m\pi/n}(D_N(t)-1)dt.$$

It follows that

$$\begin{aligned} &\sum_{m=0}^{N-1}(-1)^m\sum_{k=1}^{N}\frac{\sin(2k(m+1)\pi/n)-\sin(2km\pi/n)}{k}= \\ &\frac{1}{2}\int_0^{2\pi/n}(D_N(t)-1)dt-\frac{1}{2}\int_{2\pi/n}^{4\pi/n}(D_N(t)-1)dt+\cdots+\frac{1}{2}(-1)^{N-1}\int_{2(N-1)\pi/n}^{2N\pi/n}(D_N(t)-1)dt= \\ &\frac{1}{2}\left(\int_0^{2\pi/n}D_N(t)dt-\int_{2\pi/n}^{4\pi/n}D_N(t)dt+\cdots+(-1)^{N-1}\int_{2(N-1)\pi/n}^{2N\pi/n}D_N(t)dt\right)+ \\ &\quad \min(0,(-1)^{N-1})\frac{\pi}{n}\leq\frac{1}{2}\int_0^{2N\pi/n}|D_N(t)|dt+\min(0,(-1)^{N-1})\frac{\pi}{n} \end{aligned} \tag{38}$$

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It is not difficult to observe that

$$\int_0^{2N\pi/n} |D_N(t)|dt \le \frac{1}{2}\int_{-\pi}^{\pi} |D_N(t)|dt \tag{39}$$

In [11], it is shown that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|dt \le 1+\frac{2}{\pi}+\frac{4}{\pi^2}\ln(2N), \qquad n\ge 3. \tag{40}$$

Hence

$$\begin{aligned}\sum_{m=0}^{N-1}(-1)^m\sum_{k=1}^{N}\frac{\sin(2k(m+1)\pi/n)-\sin(2km\pi/n)}{k} &\le \\ \frac{\pi}{2}\left(1+\frac{2}{\pi}+\frac{4}{\pi^2}\ln(2N)\right)&+\min(0,(-1)^{N-1})\frac{\pi}{n}\end{aligned} \tag{41}$$

On the other hand, when $1\le k\le N$, $0\le \sin(k\pi/n)\le 1$. So

$$\sum_{k=1}^{N}(-1)^k\frac{\sin(k\pi/n)}{k} \le -\sum_{k=1}^{N}\frac{(-1)^{k+1}}{k}. \tag{42}$$

It is well-known that $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}=\ln(2)$ and $\sum_{k=1}^{N}\frac{(-1)^{k+1}}{k}\ge \ln(2)-1/(N+1)$. Therefore

$$\sum_{k=1}^{N}(-1)^k\frac{\sin(k\pi/n)}{k} \le -\ln(2)+1/(N+1). \tag{43}$$

Finally

$$I_n \le 1+\frac{2}{\pi}+\frac{4}{\pi^2}\ln(n-1)+\frac{2}{\pi}\left(-\ln(2)+\frac{1}{N+1}\right)+\frac{1}{n} \tag{44}$$

In Figure 5 we include a comparative illustration of (44) and (14).

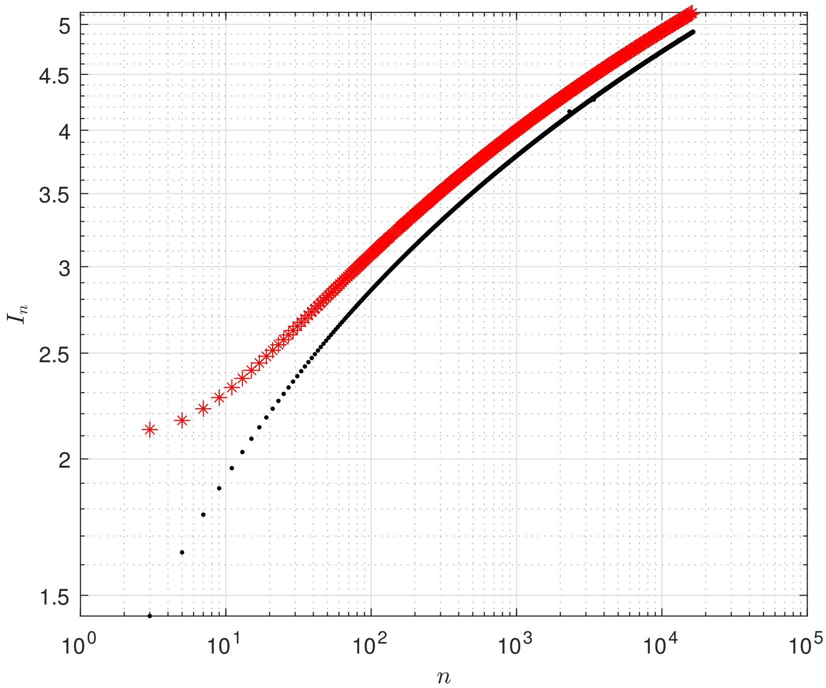

Figure 5. Comparative of (44) (red) with (14) (black).

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For even n case we only need to observe that I_n is a increasing function [6], then $I_n \leq I_{n+1}$ with $n+1$ odd. From (44) we obtain that

$$I_n \leq 1 + \frac{2}{\pi} + \frac{4}{\pi^2}\ln(n) + \frac{2}{\pi}\left(-\ln(2) + \frac{2}{n+2}\right) + \frac{1}{n+1} \tag{45}$$

In Figure 6 a comparative illustration of (45) and (14) is depict.

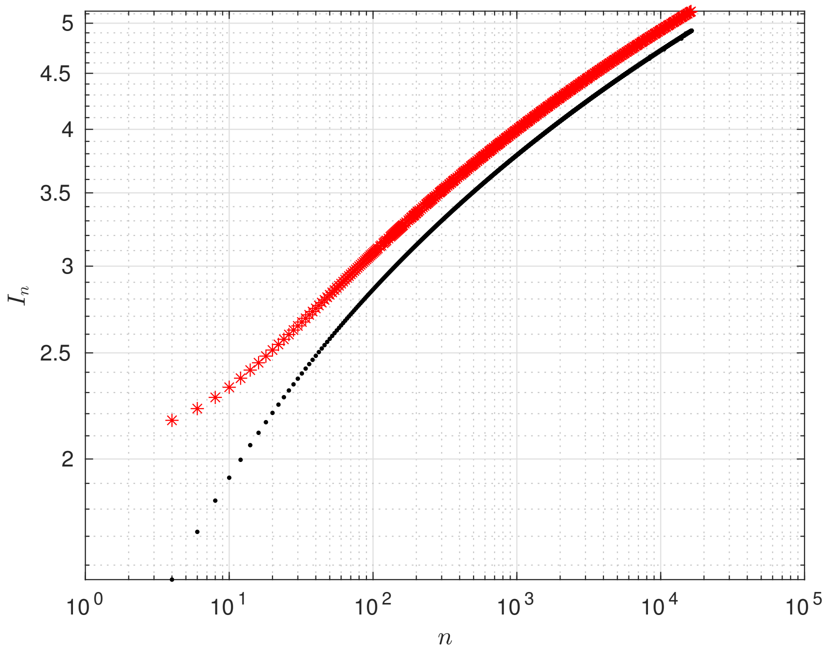

Figure 6. Comparative of (45) (red) with (14) (black).

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