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Article

Exponential Convergence-(t,s)-Weak Tractability of Approximation in Weighted Hilbert Spaces

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Abstract: We study L_2-approximation problems in the weighted Hilbert spaces in the worst case setting. Three interesting weighted Hilbert spaces appear in this paper, whose weights are equipped with two positive parameters γ_j and α_j for $j = 1, 2, \ldots, d$. We consider the worst case error of algorithms that use finitely many arbitrary continuous linear functionals. We discuss the exponential convergence-(t,s)-weak tractability (EC-(t,s)-WT) of these L_2-approximation problems under the absolute or normalized error criterion. In particular, we obtain the sufficient and necessary conditions for EC-(1,1)-WT and EC-$(t,1)$-WT with $t < 1$.

Keywords: L_2-approximation; information complexity; tractability; weighted Hilbert spaces

MSC: 41A81; 47A58; 47B02

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1. Introduction

We study multivariate approximation problems $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ of functions defined over Hilbert spaces with large or huge d in the worst case setting. Such problems appear in statistics (see [1]), computational finance (see [2]) and physics (see [3]). We consider algorithms that use finitely many continuous linear functionals. The information complexity $n(\varepsilon, \mathrm{APP}_d)$ is defined to be the minimal number of linear functionals for which the approximation error of some algorithm is at most ε. Tractability describes the growth rate of the information complexity $n(\varepsilon, \mathrm{APP}_d)$ when the error threshold ε tends to 0 and the dimension d tends to infinity. There are two kinds of tractability, classical tractability based on polynomial convergence and exponential convergence-tractability (EC-tractability) based on exponential convergence. Recently many authors are interested in classical tractability and EC-tractability in weighted Hilbert spaces, such as classical tractability and EC-tractability in analytic Korobov spaces (see [4–9]), classical tractability and EC-tractability in weighted Korobov spaces (see [10–17]), and classical tractability in weighted Gaussian ANOVA spaces (see [18]).

This paper is devoted to discussing EC-tractability of L_2-approximation problems from the weighted Hilbert spaces in the worst case setting. Let $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ be a Hilbert space with weight $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$, where $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ are two positive sequences satisfying $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$. We consider the L_2-approximation problem

$$\mathrm{APP}_d : H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \to L_2([0,1]^d) \ \text{ with } \ \mathrm{APP}_d(f) = f.$$

In the worst case setting the classical tractability of the problem $\mathrm{APP} = \{\mathrm{APP}_d\}$ in weighted Korobov spaces such as strong polynomial tractability and polynomial tractability were discussed in [13,14,18]; quasi-polynomial tractability, uniform weak tractability, weak tractability and (t,s)-weak tractability were investigated in [15,18]. Additionally, [18] also discussed classical tractability in several weighted Hilbert spaces including weighted Korobov spaces and weighted Gaussian ANOVA spaces. The EC-tractability of the problem $\mathrm{APP} = \{\mathrm{APP}_d\}$ in weighted Korobov spaces such as EC-$(t,1)$-weak tractability for $0 < t \leq 1$ were studied in [17]. However, the above weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ satisfy $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 = \alpha_2 = \cdots$.

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In this paper we present three cases of weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ for $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$, that appear in the reference [16]. These weighted Hilbert spaces are similar but also different. [16] studied the strong polynomial tractability, polynomial tractability, weak tractability and (t,s)-weak tractability for $t > 1$ of the problems $\mathrm{APP} = \{\mathrm{APP}_d\}$ in these three weighted Hilbert spaces. However, EC-tractability have not yet been considered for the approximation problems $\mathrm{APP} = \{\mathrm{APP}_d\}$ for the above three weighted Hilbert spaces. We will investigate EC-(t,s)-weak tractability for some $t > 0$, $s > 0$ and get the sufficient and necessary conditions for the EC-$(1,1)$-weak tractability (EC-weak tractability) and EC-$(t,1)$-weak tractability with $t < 1$.

The paper is organized as follows. In Section 2 we present three cases of weighted Hilbert spaces. In Section 3 we give preliminaries about the L_2-approximation problem in the weight Hilbert space. Section 4.1 are devoted to recall some notions about the tractability such as classical tractability and exponential convergence-tractability and state out the main results. In Section 4.2 we give the proof of Theorem 10.

2. Weighted Reproducing Kernel Hilbert Spaces

In this section we introduce multivariate approximation problems in weighted reproducing kernel Hilbert spaces in the worst case setting.

In this paper, let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ be two positive sequences of the reproducing kernel Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weight $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ satisfying

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0, \tag{1}$$

and

$$1 < \alpha_1 \leq \alpha_2 \leq \cdots . \tag{2}$$

Assume that the weighted reproducing kernel function $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}} : [0,1]^d \times [0,1]^d \to \mathbb{C}$ of the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is of product form

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) := \prod_{k=1}^{d} K_{R_{1,\alpha_k,\gamma_k}}(x_k, y_k),$$

where $K_{R_{1,\alpha,\gamma}} : [0,1] \times [0,1] \to \mathbb{C}$ is a universal weighted function,

$$K_{R_{1,\alpha,\gamma}}(x,y) := \sum_{k\in\mathbb{N}_0} R_{\alpha,\gamma}(k) \exp(2\pi i k \cdot (x-y)), \ \ x,\ y \in [0,1].$$

Here, let weight $R_{\alpha,\gamma} : \mathbb{N}_0 \to \mathbb{R}^+$ be a summable function, i.e., $\sum_{k\in\mathbb{N}_0} R_{\alpha,\gamma}(k) < \infty$. Then we have

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k} \cdot (\boldsymbol{x}-\boldsymbol{y})),\ \boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d, \tag{3}$$

and the inner product

$$\langle f,g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sum_{\boldsymbol{k}\in\in\mathbb{N}_0^d} \frac{1}{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})} \widehat{f}(\boldsymbol{k})\overline{\widehat{g}}(\boldsymbol{k}) \tag{4}$$

and

$$||f||_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sqrt{\langle f,f\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})}},$$

where

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(k_j),\ \boldsymbol{k} = (k_1,k_2,\ldots,k_d) \in \mathbb{N}_0^d,$$

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$$\boldsymbol{x}\cdot\boldsymbol{y}:=\sum_{k=1}^{d}x_k\cdot y_k,\ \ \boldsymbol{x}=(x_1,x_2,\ldots,x_d),\ \boldsymbol{y}=(y_1,y_2,\ldots,y_d)\in[0,1]^d,$$

and

$$\widehat{f}(\boldsymbol{k})=\int_{[0,1]^d}f(\boldsymbol{x})\exp(-2\pi i\boldsymbol{k}\cdot\boldsymbol{x})d\boldsymbol{x}.$$

Note that $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ is well defined for $1<\alpha_1\le\alpha_2\le\cdots$ and for all $\boldsymbol{x},\ \boldsymbol{y}\in[0,1]^d$, since

$$|K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})|\le\sum_{\boldsymbol{k}\in\mathbb{N}_0^d}R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})=\prod_{j=1}^{d}(\sum_{k\in\mathbb{N}_0}R_{\alpha_j,\gamma_j}(k))<\infty.$$

If $\gamma_1=\gamma_2=\cdots=1$ and $1<\alpha_1=\alpha_2=\cdots$, then the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is called unweighted space. Here, $\mathbb{N}_0=\{0,\,1,\,\ldots\}$ and $\mathbb{N}=\{1,\,2,\,\ldots\}$.

There are many ways for introducing weighted reproducing kernel Hilbert spaces with weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$. In this paper we consider three weights like the cases in the reference [16].

2.1. A Weighted Korobov Space

Let $\boldsymbol{\alpha}=\{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma}=\{\gamma_j\}_{j\in\mathbb{N}}$ satisfy (1) and (2). We consider a weighted Korobov space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weight

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})=r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}):=\prod_{j=1}^{d}r_{\alpha_j,\gamma_j}(k_j),$$

where

$$r_{\alpha,\gamma}(k):=\begin{cases}1, & \text{for } k=0,\\ \frac{\gamma}{k^{\lceil\alpha\rceil}}, & \text{for } k\ge 1,\end{cases}$$

for $\alpha>1$ and $\gamma\in(0,1]$. We can see the case in the references [16,19]. Then we have the kernel function (3) and the inner product (4) as follows:

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})=K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})=\sum_{\boldsymbol{k}\in\mathbb{N}_0^d}r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})\exp(2\pi i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})),\ \boldsymbol{x},\ \boldsymbol{y}\in[0,1]^d,$$

and

$$\langle f,g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})}=\langle f,g\rangle_{H(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})}=\sum_{\boldsymbol{k}\in\mathbb{N}_0^d}\frac{1}{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})}\widehat{f}(\boldsymbol{k})\overline{\widehat{g}}(\boldsymbol{k}).$$

Remark 1. *Obviously, the kernel $K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ is well defined for $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ satisfying (1) and (2), due to*

$$|K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})|\le\sum_{\boldsymbol{k}\in\mathbb{N}_0^d}r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})=\prod_{j=1}^{d}(1+\zeta(\lceil\alpha_j\rceil)\gamma_j)<\infty,$$

where $\zeta(\cdot)$ is the Riemann zeta function.

2.2. A First Variant of the Weighted Korobov Space

Let $\boldsymbol{\alpha}=\{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma}=\{\gamma_j\}_{j\in\mathbb{N}}$ satisfy (1) and (2), respectively. We consider a first variant of the weighted Korobov space with weight

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})=\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}):=\prod_{j=1}^{d}\psi_{\alpha_j,\gamma_j}(k_j),$$

where

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$$\psi_{\alpha,\gamma}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \frac{\gamma}{k!}, & \text{for } 1 \le k < \lceil \alpha \rceil, \\ \frac{\gamma(k-\lceil \alpha \rceil)!}{k!}, & \text{for } k \ge \lceil \alpha \rceil, \end{cases}$$

for $\alpha > 1$ and $\gamma \in (0,1]$.

Then we have the kernel function (3) and the inner product (4) as follows:

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})),\ \boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d,$$

and

$$\langle f, g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \langle f, g\rangle_{H(K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sum_{\boldsymbol{k}\in\in\mathbb{N}_0^d} \frac{1}{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})} \widehat{f}(\boldsymbol{k})\overline{\widehat{g}}(\boldsymbol{k}).$$

Lemma 2. *([16] Lemma 2) For all j, $k \in \mathbb{N}$ we have*

$$\psi_{\alpha_j,\gamma_j}(k) \le \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k).$$

Remark 3. *From Lemma 2 and $1 < \alpha_1 \le \alpha_2 \le \cdots$ we obtain*

$$\begin{aligned} |K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})| &\le \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{d}(1 + \sum_{k\in\mathbb{N}} \psi_{\alpha_j,\gamma_j}(k)) \\ &\le \prod_{j=1}^{d}(1 + \sum_{k\in\mathbb{N}} \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k)) \\ &= \prod_{j=1}^{d}(1 + \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} \zeta(\lceil \alpha_j \rceil)\gamma_j) \\ &< \infty. \end{aligned}$$

Hence the kernel $K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ is well defined.

2.3. A Second Variant of the Weighted Korobov Space

Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ satisfy (1) and (2), respectively. We consider a second variant of the weighted Korobov space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ (see the references [16,20]) with weight

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{d} \omega_{\alpha_j,\gamma_j}(k_j),$$

where

$$\omega_{\alpha,\gamma}(k) := \left(1 + \frac{1}{\gamma}\sum_{l=1}^{\lceil \alpha \rceil} \theta_l(k)\right)^{-1},$$

for $\alpha > 1$ and $\gamma \in (0,1]$, and

$$\theta_l(k) := \begin{cases} \frac{k!}{(k-l)!}, & \text{for } k \ge l, \\ 0, & \text{for } 0 \le k < l. \end{cases}$$

Then we have the kernel function (3) and the inner product (4) as follows:

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$$K_{R_{d,\boldsymbol{\alpha},\gamma}}(\boldsymbol{x},\boldsymbol{y}) = K_{\omega_{d,\boldsymbol{\alpha},\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \omega_{d,\boldsymbol{\alpha},\gamma}(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})),\ \boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d,$$

and

$$\langle f, g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\gamma}})} = \langle f, g\rangle_{H(K_{\omega_{d,\boldsymbol{\alpha},\gamma}})} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \frac{1}{\omega_{d,\boldsymbol{\alpha},\gamma}(\boldsymbol{k})} \widehat{f}(\boldsymbol{k})\overline{\widehat{g}}(\boldsymbol{k}).$$

Lemma 4. *([16] Lemma 3) For all $j,\ k \in \mathbb{N}$ we have*

$$\omega_{\alpha_j,\gamma_j}(k) \le \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k).$$

Remark 5. *We note that the kernel $K_{R_{d,\boldsymbol{\alpha},\gamma}}(\boldsymbol{x},\boldsymbol{y})$ is also well defined. Indeed, it follows from Lemma 4 and $1 < \alpha_1 \le \alpha_2 \le \cdots$ that*

$$\begin{aligned} |K_{\omega_{d,\boldsymbol{\alpha},\gamma}}(\boldsymbol{x},\boldsymbol{y})| &\le \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \omega_{d,\boldsymbol{\alpha},\gamma}(\boldsymbol{k}) = \prod_{j=1}^d (1 + \sum_{k\in\mathbb{N}} \omega_{\alpha_j,\gamma_j}(k)) \\ &\le \prod_{j=1}^d (1 + \sum_{k\in\mathbb{N}} \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k)) \\ &= \prod_{j=1}^d (1 + \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} \zeta(\lceil \alpha_j \rceil)\gamma_j) \\ &< \infty. \end{aligned}$$

Lemma 6. *Let $R_{\alpha_j,\gamma_j} \in \left\{ r_{\alpha_j,\gamma_j}, \psi_{\alpha_j,\gamma_j}, \omega_{\alpha_j,\gamma_j} \right\}$ for all $j \in \mathbb{N}$. Then we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$,*

$$R_{\alpha_j,\gamma_j}(k) \le \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k).$$

Especially, we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$,

$$R_{\alpha_j,\gamma_j}(k) \le \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} r_{\alpha_1,\gamma_j}(k).$$

Proof. On the one hand, it is obvious from Lemma 2 and Lemma 4 that

$$R_{\alpha_j,\gamma_j}(k) \le \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k) \tag{5}$$

for all $j,\ k \in \mathbb{N}$. Since for all $j \in \mathbb{N}$

$$r_{\alpha_j,\gamma_j}(0) = \psi_{\alpha_j,\gamma_j}(0) = \omega_{\alpha_j,\gamma_j}(0) = 1,$$

we have

$$R_{\alpha_j,\gamma_j}(0) = 1 \le \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} = \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(0).$$

Thus we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ that

$$R_{\alpha_j,\gamma_j}(k) \le \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k).$$

On the other hand, noting for all $j,\ k \in \mathbb{N}$

$$r_{\alpha_j,\gamma_j}(k) \le r_{\alpha_1,\gamma_j}(k),\ \ \psi_{\alpha_j,\gamma_j}(k) \le \psi_{\alpha_1,\gamma_j}(k),\ \ \omega_{\alpha_j,\gamma_j}(k) \le \omega_{\alpha_1,\gamma_j}(k),$$

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and for all $j \in \mathbb{N}$

$$r_{\alpha_j,\gamma_j}(0) = r_{\alpha_1,\gamma_j}(0) = 1, \;\; \psi_{\alpha_j,\gamma_j}(0) = \psi_{\alpha_1,\gamma_j}(0) = 1, \;\; \omega_{\alpha_j,\gamma_j}(0) = \omega_{\alpha_1,\gamma_j}(0) = 1,$$

we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ that

$$R_{\alpha_j,\gamma_j}(k) \le R_{\alpha_1,\gamma_j}(k).$$

Hence by (5) we further get for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ that

$$R_{\alpha_j,\gamma_j}(k) \le R_{\alpha_1,\gamma_j}(k) \le \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} r_{\alpha_1,\gamma_j}(k).$$

□

Remark 7. *Let $R_{\alpha_j,\gamma_j} \in \left\{ r_{\alpha_j,\gamma_j}, \psi_{\alpha_j,\gamma_j}, \omega_{\alpha_j,\gamma_j} \right\}$ for all $j \in \mathbb{N}$. Then we obtain*

$$R_{\alpha_j,\gamma_j}(0) = 1 \quad \textit{and} \quad R_{\alpha_j,\gamma_j}(1) \ge \frac{\gamma_j}{2} \tag{6}$$

for all $j \in \mathbb{N}$. Indeed, for all $j \in \mathbb{N}$ we have

$$\psi_{\alpha_j,\gamma_j}(0) = r_{\alpha_j,\gamma_j}(0) = \omega_{\alpha_j,\gamma_j}(0) = 1,$$

which means $R_{\alpha_j,\gamma_j}(0) = 1$. Due to for all $j \in \mathbb{N}$, we get

$$\psi_{\alpha_j,\gamma_j}(1) = r_{\alpha_j,\gamma_j}(1) = \gamma_j \quad \textit{and} \quad \omega_{\alpha_j,\gamma_j}(1) = \left(1 + \frac{1}{\gamma_j}\right)^{-1} \ge \frac{\gamma_j}{2},$$

which yields $R_{\alpha_j,\gamma_j}(1) \ge \frac{\gamma_j}{2}$.

3. L_2-Approximation in the Weighted Hilbert Spaces

In this paper we investigate the L_2-approximation

$$\mathrm{APP}_d : H(K_{R_{d,\alpha,\gamma}}) \to L_2([0,1]^d) \;\; \text{with} \;\; \mathrm{APP}_d(f) = f,$$

for all $f \in H(K_{R_{d,\alpha,\gamma}})$ in Hilbert space $H(K_{R_{d,\alpha,\gamma}})$ with weight $R_{d,\alpha,\gamma} \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$. It is well known from Remark 1, Remark 3, Remark 5 and [14] that this L_2-approximation is compact for $1 < \alpha_1 \le \alpha_2 \le \cdots$. We approximation APP_d by algorithm $A_{n,d}$ of the form

$$A_{n,d}(f) = \sum_{i=1}^{n} T_i(f) g_i, \text{ for } f \in H(K_{R_{d,\alpha,\gamma}}), \tag{7}$$

where $g_1, g_2, \ldots, g_n$ belong to $L_2([0,1]^d)$ and $T_1, T_2, \ldots, T_n$ are continuous linear functionals on $H(K_{R_{d,\alpha,\gamma}})$. The worst case error for the algorithm $A_{n,d}$ of the form (7) is defined as

$$e(A_{n,d}) := \sup_{||f||_{H(K_{R_{d,\alpha,\gamma}})} \le 1} ||\mathrm{APP}_d(f) - A_{n,d}(f)||_{L_2}.$$

The nth minimal worst-case error, for $n \ge 1$, is defined by

$$e(n, \mathrm{APP}_d) := \inf_{A_{n,d}} e(A_{n,d}),$$

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where the infimum is taken over all linear algorithms of the form (7). For $n = 0$, we use $A_{0,d} = 0$. We call

$$e(0, \mathrm{APP}_d) = \sup_{||f||_{H(K_{R_{d,\alpha,\gamma}})} \leq 1} ||\mathrm{APP}_d(f)||_{L_2}$$

the initial error of the problem APP_d.

We are interested in how the worst case error for the algorithm $A_{n,d}$ depend on the number n and d. To this end, we define the so-called information complexity as

$$n(\varepsilon, \mathrm{APP}_d) := \min\{n \in \mathbb{N}_0 : e(n, \mathrm{APP}_d) \leq \varepsilon\},$$

where $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$. Here, $\mathbb{N}_0 = \{0,1,\ldots\}$ and $\mathbb{N} = \{1,2,\ldots\}$.

It is well known, see e.g., [2,10], that the nth minimal worst case errors $e(n, \mathrm{APP}_d)$ and the information complexity $n(\varepsilon, \mathrm{APP}_d)$ depend on the eigenvalues of the continuously linear operator $W_d = \mathrm{APP}_d^* \mathrm{APP}_d : H(K_{R_{d,\alpha,\gamma}}) \to H(K_{R_{d,\alpha,\gamma}})$. Let $(\lambda_{d,j}, \eta_{d,j})$ be the eigenpairs of W_d, i.e.,

$$W_d \eta_{d,j} = \lambda_{d,j} \eta_{d,j}, \quad \text{for all } j \in \mathbb{N},$$

where the eigenvalues $\lambda_{d,j}$ are ordered,

$$\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq 0,$$

and the eigenvectors $\eta_{d,j}$ are orthonormal,

$$\langle \eta_{d,i}, \eta_{d,j} \rangle_{H(K_{R_{d,\alpha,\gamma}})} = \delta_{i,j}, \quad \text{for all } i, j \in \mathbb{N}.$$

Then the nth minimal worst-case error, $n \geq 1$, is obtained for the algorithm

$$A^\diamond_{n,d} f = \sum_{j=1}^{n} \langle f, \eta_{d,j} \rangle_{H(K_{R_{d,\alpha,\gamma}})} \eta_{d,j}, \quad \text{for all } n \in \mathbb{N}.$$

and

$$e(n, \mathrm{APP}_d) = e(A^\diamond_{n,d}) = \sqrt{\lambda_{d,n+1}}, \quad \text{for all } n \in \mathbb{N}.$$

The initial error $e(0, \mathrm{APP}_d) = \sqrt{\lambda_{d,1}}$. Hence we have $e(n, \mathrm{APP}_d) = \sqrt{\lambda_{d,n+1}}$ for all $n \in \mathbb{N}_0$. The information complexity is

$$n(\varepsilon, \mathrm{APP}_d) = \min\Big\{n \in \mathbb{N}_0 : \sqrt{\lambda_{d,n+1}} \leq \varepsilon\Big\} = \min\Big\{n \in \mathbb{N}_0 : \lambda_{d,n+1} \leq \varepsilon^2\Big\}. \tag{8}$$

Since the eigenvalues $\lambda_{d,j}$ with $j \in \mathbb{N}$ of the operator W_d are $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})$ with $\boldsymbol{k} \in \mathbb{N}_0^d$ (see [10, p. 215]), by (8) the information complexity of APP_d from the space $H(K_{R_{d,\alpha,\gamma}})$ is equal to

$$\begin{aligned} n(\varepsilon, \mathrm{APP}_d) &= \min\Big\{n \in \mathbb{N}_0 : \lambda_{d,n+1} \leq \varepsilon^2\Big\} = \Big|\Big\{n \in \mathbb{N} : \lambda_{d,n} > \varepsilon^2\Big\}\Big| \\ &= \Big|\Big\{\boldsymbol{h} \in \mathbb{N}_0^d : R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) > \varepsilon^2\Big\}\Big| = \left|\left\{\boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2\right\}\right|, \end{aligned} \tag{9}$$

with $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$.

Note that for the L_2-approximation APP_d from the space $H(K_{R_{d,\alpha,\gamma}})$ we do not need to distinguish between the absolute error criterion and the normalized error criterion since the initial error $e(0, \mathrm{APP}_d) = \sqrt{\lambda_{d,1}} = 1$.

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4. Tractability in Weighted Hilbert Spaces and Main Results

In this section we will consider the classical tractability the exponential convergence-tractability (EC-tractability) for the problem $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$.

4.1. Tractability and Main Results

We focus on the behaviours of the information complexity $n(\varepsilon, \mathrm{APP}_d)$ depending on the dimension d and the error threshold ε. Hence we will recall several notions of the classical tractability and exponential convergence-tractability (EC-tractability) notions (see [4,5,7–12,17,21]).

Definition 8. *Let $APP = \{APP_d\}_{d\in\mathbb{N}}$. We say we have:*

- *Strong polynomial tractability (SPT) if there exist non-negative numbers C and p such that*

$$n(\varepsilon, APP_d) \le C(\varepsilon^{-1})^p \ \text{ for all } \ d \in \mathbb{N}, \varepsilon \in (0,1).$$

  In this case we define the exponent p^{str} of SPT as

$$p^{str} := \inf\{p : \exists C > 0 \text{ such that } n(\varepsilon, APP_d) \le C(\varepsilon^{-1})^p, \forall d \in \mathbb{N}, \varepsilon \in (0,1)\}.$$

- *Polynomial tractability (PT) if there exist non-negative numbers C, p and q such that*

$$n(\varepsilon, APP_d) \le Cd^q(\varepsilon^{-1})^p \ \text{ for all } \ d \in \mathbb{N}, \varepsilon \in (0,1).$$

- *Quasi-polynomial tractability (QPT) if there exist two constants $C, t > 0$ such that*

$$n(\varepsilon, APP_d) \le C \exp\left(t(1+\ln \varepsilon^{-1})(1+\ln d)\right) \ \text{ for all } \ d \in \mathbb{N}, \varepsilon \in (0,1).$$

- *Uniform weak tractability (UWT) if for all $t, s > 0$,*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + (\varepsilon^{-1})^s} = 0.$$

- *Weak tractability (WT) if*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \varepsilon^{-1}} = 0.$$

- *(t,s)-weak tractability ((t,s)-WT) for fixed positive t and s if*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + (\varepsilon^{-1})^s} = 0.$$

- *APP_d suffers from the curse of the dimensionality if there exist positive numbers C_1, C_2, ε_0 such that*

$$n(\varepsilon, APP_d) \ge C_1(1+C_2)^d \ \text{ for all } \ 0 < \varepsilon \le \varepsilon_0 \ \text{ and infinitely many } \ d \in \mathbb{N}.$$

We find that (1,1)-WT is the same as WT and

$$\mathrm{SPT} \Longrightarrow \mathrm{PT} \Longrightarrow \mathrm{QPT} \Longrightarrow \mathrm{UWT} \Longrightarrow \mathrm{WT}.$$

In the above definitions about classical tractability, if we replace ε^{-1} by $(1+\ln(\varepsilon^{-1}))$ we will get the following definitions about exponential convergence-tractability (EC-tractability).

Definition 9. *Let $APP = \{APP_d\}_{d\in\mathbb{N}}$. We say we have:*

- *Exponential convergence-strong polynomial tractability (EC-SPT) if there exist non-negative numbers C and p such that*
$$n(\varepsilon, APP_d) \le C\big(1+\ln(\varepsilon^{-1})\big)^p \ \ \textit{for all} \ \ d\in\mathbb{N}, \varepsilon\in(0,1).$$
In this case we define the exponent of EC-SPT as
$$\inf\{p : \exists C>0 \textit{ such that } n(\varepsilon, APP_d) \le C\big(1+\ln(\varepsilon^{-1})\big)^p, \forall d\in\mathbb{N}, \varepsilon\in(0,1)\}.$$
- *Exponential convergence-polynomial tractability (EC-PT) if there exist non-negative numbers C, p and q such that*
$$n(\varepsilon, APP_d) \le Cd^q\big(1+\ln(\varepsilon^{-1})\big)^p \ \ \textit{for all} \ \ d\in\mathbb{N}, \varepsilon\in(0,1).$$
- *Exponential convergence-uniform weak tractability (EC-UWT) if for all $t, s > 0$*
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \big(1+\ln(\varepsilon^{-1})\big)^s} = 0.$$
- *Exponential convergence-weak tractability (EC-WT) if*
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} = 0.$$
- *Exponential convergence-(t,s)-weak tractability (EC-(t,s)-WT) for fixed positive t and s if*
$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \big(1+\ln(\varepsilon^{-1})\big)^s} = 0.$$

We note that EC-$(1,1)$-WT is the same as EC-WT, and
$$\text{EC-SPT} \Longrightarrow \text{EC-PT} \Longrightarrow \text{EC-QPT} \Longrightarrow \text{EC-UWT} \Longrightarrow \text{EC-WT}.$$
If the problem APP has exponential convergence-tractability, then it has classical tractability. Hence we have
$$\text{EC-SPT} \Longrightarrow \text{SPT}, \quad \text{EC-PT} \Longrightarrow \text{PT}, \quad \text{EC-QPT} \Longrightarrow \text{QPT},$$
$$\text{EC-}(t,s)\text{-WT} \Longrightarrow (t,s)\text{-WT}, \quad \text{EC-UWT} \Longrightarrow \text{UWT}, \quad \text{EC-WT} \Longrightarrow \text{WT}.$$

In the worst case setting the classical tractability and EC-tractability of the problem APP $= \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ with $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying
$$1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \ge 0,$$
and
$$1 < \alpha^* = \alpha_1 = \alpha_2 = \cdots.$$
have been solved by [13,15,16] and [17], respectively. The following results have been obtained:

- For $R_{d,\alpha^*,\gamma} \in \{r_{d,\alpha^*,\gamma}, \psi_{d,\alpha^*,\gamma}, \omega_{d,\alpha^*,\gamma}\}$, PT holds iff SPT holds iff
$$s_\gamma := \inf\left\{\kappa > 0 : \sum_{j=1}^{\infty} \gamma_j^\kappa < \infty\right\} < \infty,$$
and the exponent of SPT is
$$p^{\text{str}} = 2\max\Big(s_\gamma, \frac{1}{\alpha}\Big).$$

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- For $R_{d,\alpha^*,\gamma} = r_{d,\alpha^*,\gamma}$, QPT, UWT and WT are equivalent and hold iff
$$\gamma_I := \inf_{j\in\mathbb{N}} \gamma_j < 1.$$
For $R_{d,\alpha^*,\gamma} \in \{\psi_{d,\alpha^*,\gamma}, \omega_{d,\alpha^*,\gamma}\}$,
$$\gamma_I < \infty$$
implies QPT.
- For $R_{d,\alpha^*,\gamma} \in \{r_{d,\alpha^*,\gamma}, \psi_{d,\alpha^*,\gamma}, \omega_{d,\alpha^*,\gamma}\}$ and $t > 1$, (t,s)-WT holds for all $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$.
- For $R_{d,\alpha^*,\gamma} = r_{d,\alpha^*,\gamma}$, EC-WT holds iff
$$\lim_{j\to\infty} \gamma_j = 0.$$
- For $R_{d,\alpha^*,\gamma} = r_{d,\alpha^*,\gamma}$ and $t < 1$, EC-$(t,1)$-WT holds iff
$$\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0.$$

In the worst case setting the classical tractability such as SPT, PT and WT of the problem $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H_{d,\boldsymbol{\alpha},\gamma}$ with $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying (1) and (2), i.e.,
$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0,$$
and
$$1 < \alpha_1 \leq \alpha_2 \leq \cdots.$$
has been solved by [16] as follows:

- For $R_{d,\boldsymbol{\alpha},\gamma} \in \{r_{d,\boldsymbol{\alpha},\gamma}, \psi_{d,\boldsymbol{\alpha},\gamma}, \omega_{d,\boldsymbol{\alpha},\gamma}\}$, SPT and PT are equivalent and hold iff
$$\delta := \liminf_{j\to\infty} \frac{\ln \gamma_j^{-1}}{\ln j} > 0.$$
The exponent of SPT is
$$p^{\mathrm{str}} = 2\max\Big\{\frac{1}{\delta}, \frac{1}{\lceil \alpha_1 \rceil}\Big\}.$$
- For $R_{d,\boldsymbol{\alpha},\gamma} = r_{d,\boldsymbol{\alpha},\gamma}$, WT holds iff
$$\lim_{j\to\infty} \gamma_j < 1.$$
- For $R_{d,\boldsymbol{\alpha},\gamma} \in \{r_{d,\boldsymbol{\alpha},\gamma}, \psi_{d,\boldsymbol{\alpha},\gamma}, \omega_{d,\boldsymbol{\alpha},\gamma}\}$ and $t > 1$, (t,s)-WT holds.

In this paper, we investigate the EC-tractability of the problem $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H_{d,\boldsymbol{\alpha},\gamma}$ with $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying (1) and (2). We obtain sufficient and necessary conditions for EC-$(t,1)$-WT with $0 < t < 1$ and $t = 1$.

Theorem 10. *Let the sequences $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfy (1) and (2). Then the problem $APP = \{APP_d\}_{d\in\mathbb{N}}$ for the weighted Hilbert spaces $H_{R_{d,\boldsymbol{\alpha},\gamma}}$ with $R_{d,\boldsymbol{\alpha},\gamma} \in \{r_{d,\boldsymbol{\alpha},\gamma}, \psi_{d,\boldsymbol{\alpha},\gamma}, \omega_{d,\boldsymbol{\alpha},\gamma}\}$*

(1) is EC-WT, if and only if
$$\lim_{j\to\infty} \gamma_j = 0.$$
(2) is EC-$(t,1)$-WT with $t < 1$, if and only if
$$\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0.$$

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4.2. The Proof

In order to prove Theorem 10 we need the following Lemmas.

Lemma 11. *Let $\eta > 0$, $\varepsilon \in (0,1)$. We have for any $d \in \mathbb{N}$*

$$n(\varepsilon, APP_d) \le \varepsilon^{-2\eta} \prod_{j=1}^{d} \Big(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \zeta(\lceil \alpha_1 \rceil \eta) \gamma_j^{\eta}\Big).$$

Proof. By Lemma 6 we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \lambda_{d,k}^{\eta} &= \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} (R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}))^{\eta} = \prod_{j=1}^{d} \left(1 + \sum_{k=1}^{\infty} \Big(R_{\alpha_j,\gamma_j}(k)\Big)^{\eta}\right) \\
&\le \prod_{j=1}^{d} \left(1 + \sum_{k=1}^{\infty} \Big(\lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} r_{\alpha_1,\gamma_j}(k)\Big)^{\eta}\right) \\
&= \prod_{j=1}^{d} \left(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \sum_{k=1}^{\infty} \Big(r_{\alpha_1,\gamma_j}(k)\Big)^{\eta}\right) \\
&= \prod_{j=1}^{d} \left(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \sum_{k=1}^{\infty} \left(\frac{\gamma_j}{k^{\lceil \alpha_1 \rceil}}\right)^{\eta}\right) \\
&= \prod_{j=1}^{d} \Big(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \zeta(\lceil \alpha_1 \rceil \eta) \gamma_j^{\eta}\Big).
\end{aligned}$$

This yields

$$n \lambda_{d,n}^{\eta} \le \sum_{k=1}^{n} \lambda_{d,k}^{\eta} \le \sum_{k=1}^{\infty} \lambda_{d,k}^{\eta} \le \prod_{j=1}^{d} \Big(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \zeta(\lceil \alpha_1 \rceil \eta) \gamma_j^{\eta}\Big),$$

which means

$$\lambda_{d,n} \le \frac{\prod_{j=1}^{d} \Big(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \zeta(\lceil \alpha_1 \rceil \eta) \gamma_j^{\eta}\Big)^{1/\eta}}{n^{1/\eta}}.$$

It follows from the above inequality and (8)

$$n(\varepsilon, \mathrm{APP}_d) = \min\Big\{n \in \mathbb{N}_0 : \lambda_{d,n+1} \le \varepsilon^2\Big\},$$

that

$$n(\varepsilon, \mathrm{APP}_d) \le \varepsilon^{-2\eta} \prod_{j=1}^{d} \Big(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \zeta(\lceil \alpha_1 \rceil \eta) \gamma_j^{\eta}\Big).$$

This proof is complete. □

Lemma 12. *Let $\varepsilon \in (0,1)$. We have for any $d \ge 2$*

$$n(\varepsilon, APP_d) \ge \left\lceil \Big(\varepsilon^{-2} \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} \gamma_d\Big)^{\frac{1}{\lceil \alpha_1 \rceil}} \right\rceil.$$

Proof. Set

$$H = H(\varepsilon, d, \alpha) := \left\{ h \in \mathbb{N}_0 : h \le \left\lceil \Big(\varepsilon^{-2} \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} \gamma_d\Big)^{\frac{1}{\lceil \alpha_1 \rceil}} \right\rceil - 1 \right\}.$$

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If $h > \left\lceil \left(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\right)^{\frac{1}{\lceil\alpha_1\rceil}} \right\rceil - 1$ and $d \geq 2$, by Lemma 6 we have

$$\prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h) \leq R_{\alpha_d,\gamma_d}(h) \leq \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil} r_{\alpha_1,\gamma_d}(h) = \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil} \frac{\gamma_d}{h^{\lceil\alpha_1\rceil}} \leq \varepsilon^2$$

for any $\{h_1, \cdots, h_{d-1}\} \in \mathbb{N}_0^{d-1}$, which means

$$\left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h) > \varepsilon^2 \right\} = \varnothing \tag{10}$$

for all $h > \left\lceil \left(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\right)^{\frac{1}{\lceil\alpha_1\rceil}} \right\rceil - 1$. It follows from (9) and (10) that

$$\begin{aligned}
n(\varepsilon, \mathrm{APP}_d) &= \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\} \right| \\
&= \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h_d) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in \mathbb{N}_0} \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in H} \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in (H \setminus \{0\})} \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 R^{-1}_{\alpha_d,\gamma_d}(h) \right\} \right| \\
&\quad + \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in (H \setminus \{0\})} n(\varepsilon R^{-1/2}_{\alpha_d,\gamma_d}(h), \mathrm{APP}_{d-1}) + n(\varepsilon, \mathrm{APP}_{d-1}) \\
&= \sum_{h=1}^{\left\lceil \left(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\right)^{\frac{1}{\lceil\alpha_1\rceil}} \right\rceil - 1} n(\varepsilon R^{-1/2}_{\alpha_d,\gamma_d}(h), \mathrm{APP}_{d-1}) + n(\varepsilon, \mathrm{APP}_{d-1}) \\
&\geq \left\lceil \left(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\right)^{\frac{1}{\lceil\alpha_1\rceil}} \right\rceil.
\end{aligned}$$

This finishes the proof. □

Lemma 13. *For* $\prod_{j=1}^{d} \left(\frac{\gamma_j}{2}\right) > \varepsilon^2$ *and* $\varepsilon \in (0,1)$ *we have*

$$n(\varepsilon, APP_d) \geq 2^d.$$

Proof. Set

$$\mathcal{A}(\varepsilon, d) = \left\{ \boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\}.$$

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If $\boldsymbol{h} = \{h_1, h_2, \dots, h_d\} \in \{0,1\}^d$, we have from (6) that

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) = \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) \geq \prod_{j=1}^{d} \left(\frac{\gamma_j}{2}\right).$$

Thus, we have $\{0,1\}^d \in \mathcal{A}(\varepsilon, d)$ for $\prod_{j=1}^{d} \left(\frac{\gamma_j}{2}\right) > \varepsilon^2$. Hence it follows from (9) that

$$n(\varepsilon, \mathrm{APP}_d) = |\mathcal{A}(\varepsilon, d)| \geq \left| \left\{ \boldsymbol{h} \in \{0,1\}^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\} \right| = 2^d$$

for $\prod_{j=1}^{d} \left(\frac{\gamma_j}{2}\right) > \varepsilon^2$. This proof is complete. □

Proof of Theorem 10.

If there are infinitely many $\gamma_j = 0$ for $j \in \mathbb{N}$, the results are obviously true. Without lose of generality we consider only the situation when all the γ_j for $j \in \mathbb{N}$ are positive.

(1) Let $\delta > 0$ and take $\varepsilon = \prod_{j=1}^{d} \left(\frac{\gamma_j}{2}\right)^{\frac{1+\delta}{2}}$, then we have

$$\prod_{j=1}^{d} \left(\frac{\gamma_j}{2}\right) > \varepsilon^2.$$

It follows from Lemma 13 that

$$\begin{aligned} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d + \ln(\varepsilon^{-1})} &\geq \frac{d \ln 2}{d + \frac{1+\delta}{2} \cdot \ln\left(\prod_{j=1}^{d} (2\gamma_j^{-1})\right)} \\ &\geq \frac{d \ln 2}{d + \frac{1+\delta}{2} \cdot d \cdot \ln(2\gamma_d^{-1})} \\ &= \frac{\ln 2}{1 + \frac{1+\delta}{2} \cdot \left(\ln 2 + \ln(\gamma_d^{-1})\right)}. \end{aligned} \tag{11}$$

Assume that App is EC-WT, i.e., for the above fixed ε

$$\lim_{d \to \infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d + \ln(\varepsilon^{-1})} = 0.$$

Combing (11) and the above equality we have

$$0 = \lim_{d \to \infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d + \ln(\varepsilon^{-1})} \geq \lim_{d \to \infty} \frac{\ln 2}{1 + \frac{1+\delta}{2} \cdot \left(\ln 2 + \ln(\gamma_d^{-1})\right)}.$$

This implies $\lim_{d \to \infty} \gamma_d = 0$.

On the other hand, assume that we have $\lim_{d\to\infty} \gamma_d = 0$. For $\eta > 0$ we obtain from the upper bound in Lemma 11 that

$$\begin{aligned}
\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\ln(\varepsilon^{-1})} &\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\left(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\right)}{d+\ln(\varepsilon^{-1})} \\
&\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}}{d+\ln(\varepsilon^{-1})} \\
&\le 2\eta + \limsup_{d\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{d}\gamma_j^{\eta}}{d} \\
&= 2\eta,
\end{aligned}$$

where we used that $\ln(1+x) \le x$ for all $x \ge 0$ and $\limsup_{d\to\infty} \frac{\sum_{j=1}^{d}\gamma_j^{\eta}}{d} = 0$ if $\lim_{d\to\infty}\gamma_d = 0$. Setting $\eta \to 0$, we have

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\ln(\varepsilon^{-1})} = 0,$$

which yields that ET-WT holds.

(2) Assume that EC-$(t,1)$-WT for $t < 1$ holds. First, we note that $\lim_{d\to\infty}\gamma_d = 0$. Indeed, if $\lim_{d\to\infty}\gamma_d \ne 0$, we deduce from Theorem 10 (1) that EC-WT doesn't hold, i.e.,

$$0 < \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\ln(\varepsilon^{-1})} \le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t+\ln(\varepsilon^{-1})}.$$

This means APP is not EC-$(t,1)$-WT for all $t < 1$.

Next, we want to prove $\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0$. Let $\varepsilon = \varepsilon_d \in (0,1)$ such that

$$\ln\left(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\right)^{\frac{1}{\lceil\alpha_1\rceil}} = d^t$$

for large $d \in \mathbb{N}$. From the lower bound in Lemma 12 we obtain

$$\begin{aligned}
\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t+\ln(\varepsilon^{-1})} &\ge \frac{\ln\left\lceil\left(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\right)^{\frac{1}{\lceil\alpha_1\rceil}}\right\rceil}{d^t+\ln(\varepsilon^{-1})} \ge \frac{\ln\left(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\right)^{\frac{1}{\lceil\alpha_1\rceil}}}{d^t+\ln(\varepsilon^{-1})} \\
&= \frac{d^t}{d^t+\ln(\varepsilon^{-1})} = \frac{d^t}{d^t+\lceil\alpha_1\rceil d^t/2+\ln(\gamma_d^{-1})/2-\lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/2} \\
&= \frac{1}{1+\lceil\alpha_1\rceil/2+\ln(\gamma_d^{-1})/(2d^t)-\lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d^t)}.
\end{aligned}$$

It follows from the assumption that

$$\begin{aligned}
0 = \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t+\ln(\varepsilon^{-1})} &\ge \limsup_{d+\varepsilon^{-1}\to\infty} \frac{1}{1+\lceil\alpha_1\rceil/2+\ln(\gamma_d^{-1})/(2d^t)-\lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d^t)} \\
&= \limsup_{d+\varepsilon^{-1}\to\infty} \frac{1}{1+\lceil\alpha_1\rceil/2+\ln(\gamma_d^{-1})/(2d^t)},
\end{aligned}$$

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which implies

$$\lim_{d\to\infty} \frac{d^t}{\ln(\gamma_d^{-1})} = 0.$$

Using the fact that $d^t \geq \ln d^t = t \ln d \geq 0$ for large $d \in \mathbb{N}$, we have

$$0 \leq \lim_{d\to\infty} \frac{\ln d}{\ln(\gamma_d^{-1})} \leq \lim_{d\to\infty} \frac{d^t}{t\ln(\gamma_d^{-1})} = 0,$$

i.e.,

$$\lim_{d\to\infty} \frac{\ln d}{\ln(\gamma_d^{-1})} = 0.$$

On the other hand, assume that $\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0$. Then we obtain that for all $\delta > 0$ there is a number $N_\delta > 0$ such that

$$\gamma_j \leq j^{-\delta} \text{ for all } j \geq N_\delta. \tag{12}$$

Let $\eta > 0$. We get from Lemma 11 that

$$\begin{aligned} \ln n(\varepsilon, \mathrm{APP}_d) &\leq 2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\Big(1 + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta} \zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta\Big) \\ &\leq 2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta} \zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta, \end{aligned} \tag{13}$$

where we used that $\ln(1+x) \leq x$ for all $x > 0$. Choose $\delta = \frac{2}{\eta}$. By (12) and (13) we get

$$\begin{aligned} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t + \ln(\varepsilon^{-1})} &\leq \frac{2\eta\ln(\varepsilon^{-1}) + \sum\limits_{j=1}^{N_{2/\eta}-1} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta + \sum\limits_{j=N_{2/\eta}}^{\max\{d,N_{2/\eta}\}} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta}{d^t + \ln(\varepsilon^{-1})} \\ &\leq 2\eta + \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_1^\eta(N_{2/\eta}-1) + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta) \sum\limits_{j=N_{2/\eta}}^{\max\{d,N_{2/\eta}\}} j^{-2}}{d^t + \ln(\varepsilon^{-1})}. \end{aligned}$$

It follows that

$$\begin{aligned} \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t + \ln(\varepsilon^{-1})} &\leq 2\eta + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta) \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\sum\limits_{j=N_{2/\eta}}^{\max\{d,N_{2/\eta}\}} j^{-2}}{d^t + \ln(\varepsilon^{-1})} \\ &\leq 2\eta + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta) \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\sum\limits_{j=1}^{\infty} j^{-2}}{d^t + \ln(\varepsilon^{-1})} \\ &= 2\eta. \end{aligned}$$

Setting $\eta \to 0$, we have

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t + \ln(\varepsilon^{-1})} = 0.$$

Therefore Theorem 10 is proved. □

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Example 14. *An example for EC-WT.*

Assume that $\gamma_j = j^{-2}$ *and* $\alpha_j = j + 1$ *for all* $j \in \mathbb{N}$*. We consider the above weighted Hilbert spaces* $H_{R_{d,\boldsymbol{\alpha},\gamma}}$*,* $R_{d,\boldsymbol{\alpha},\gamma} \in \{r_{d,\boldsymbol{\alpha},\gamma}, \psi_{d,\boldsymbol{\alpha},\gamma}, \omega_{d,\boldsymbol{\alpha},\gamma}\}$*.*

Obviously, we have $\lim\limits_{j\to\infty} \gamma_j = 0$*. By Lemma 11 we get*

$$\begin{aligned}
\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} &\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\left(1 + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\right)}{d + \ln(\varepsilon^{-1})} \\
&\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}}{d + \ln(\varepsilon^{-1})} \\
&\le 2\eta + \limsup_{d\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta) \sum_{j=1}^{d} \gamma_j^{\eta}}{d} \\
&= 2\eta + \limsup_{d\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta) \sum_{j=1}^{d} j^{-2\eta}}{d} \\
&= 2\eta,
\end{aligned}$$

where we used $\ln(1+x) \le x$ *for all* $x \ge 0$ *and* $\limsup\limits_{d\to\infty} \frac{\sum_{j=1}^{d} j^{-2\eta}}{d} = 0$*. Setting* $\eta \to 0$*, we have*

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} = 0.$$

Hence APP is EC-WT.

Example 15. *An example for EC-*$(t,1)$*-WT for* $t < 1$*.*

Assume that $\gamma_j = 2^{-j}$ *and* $\alpha_j = 2j$ *for all* $j \in \mathbb{N}$*. We consider the above weighted Hilbert spaces* $H_{R_{d,\boldsymbol{\alpha},\gamma}}$*,* $R_{d,\boldsymbol{\alpha},\gamma} \in \{r_{d,\boldsymbol{\alpha},\gamma}, \psi_{d,\boldsymbol{\alpha},\gamma}, \omega_{d,\boldsymbol{\alpha},\gamma}\}$*.*

Note that $\lim\limits_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = \lim\limits_{j\to\infty} \frac{\ln j}{j \ln 2} = 0$*. It follows from Lemma 11 that*

$$\begin{aligned}
\ln n(\varepsilon, APP_d) &\le 2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\left(1 + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\right) \\
&= 2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\left(1 + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)2^{-\eta j}\right) \\
&\le 2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)2^{-\eta j},
\end{aligned}$$

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where we used that $\ln(1+x) \le x$ *for all* $x > 0$*. It yields that*

$$\begin{aligned}
\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} &\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)2^{-\eta j}}{d^t + \ln(\varepsilon^{-1})} \\
&\le 2\eta + \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{d} 2^{-\eta j}}{d^t + \ln(\varepsilon^{-1})} \\
&\le 2\eta + \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{\infty} 2^{-\eta j}}{d^t + \ln(\varepsilon^{-1})} \\
&= 2\eta.
\end{aligned}$$

Setting $\eta \to 0$*, we have*

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} = 0.$$

Hence APP is EC-$(t,1)$*-WT for* $t < 1$*.*

Remark 16. *We note that for Example 14 with* $\gamma_j = j^{-2}$ *and* $\alpha_j = j+1$ *for all* $j \in \mathbb{N}$*, APP is EC-WT, but not EC-*$(t,1)$*-WT for* $t < 1$*. Indeed, let* $\varepsilon = \varepsilon_d \in (0,1)$ *such that*

$$\ln\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}} = d, \quad i.e., \ \varepsilon^{-1} = \frac{d e^{d\lceil\alpha_1\rceil/2}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil/2}},$$

for large $d \in \mathbb{N}$*. From Lemma 12 we have*

$$\begin{aligned}
\frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} &\ge \frac{\ln\left\lceil\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}\right\rceil}{d^t + \ln(\varepsilon^{-1})} \ge \frac{\ln\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}}{d^t + \ln(\varepsilon^{-1})} \\
&= \frac{d}{d^t + \ln(\varepsilon^{-1})} = \frac{d}{d^t + \lceil\alpha_1\rceil d/2 + \ln d - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/2} \\
&= \frac{1}{d^{t-1} + \lceil\alpha_1\rceil/2 + \ln d/d - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d)}.
\end{aligned}$$

For the above fixed ε *and* $t < 1$ *we obtain*

$$\lim_{d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} \ge \lim_{d\to\infty} \frac{1}{d^{t-1} + \lceil\alpha_1\rceil/2 + \ln d/d - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d)} = \frac{2}{\lceil\alpha_1\rceil}.$$

This means that APP is not EC-$(t,1)$*-WT for* $t < 1$*.*

Remark 17. *Obviously, for Example 15 with* $\gamma_j = 2^{-j}$ *and* $\alpha_j = 2j$ *for all* $j \in \mathbb{N}$*, APP is also EC-WT. Indeed, if APP is EC-*$(t,1)$*-WT for* $t < 1$*, then it is EC-WT. Assume that APP is EC-*$(t,1)$*-WT for* $t < 1$*, then we have*

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} = 0.$$

Since

$$0 \le \lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} \le \lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})},$$

we further get

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} = 0,$$

which means that APP is EC-WT.

In this paper we discuss the EC-WT and EC-$(t,1)$-WT with $t < 1$ for the problem APP in weighted Hilbert spaces $H_{R_{d,\alpha,\gamma}}$ for $R_{d,\alpha,\gamma} \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$ with parameters $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$. We obtain the matching necessary and sufficient condition

$$\lim_{j\to\infty} \gamma_j = 0$$

on EC-WT, and the matching necessary and sufficient condition

$$\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0$$

on EC-$(t,1)$-WT with $t < 1$. The weights are used to model the importance of the functions from the weighted Hiblert spaces, so we plan to further study the other EC-tractability notions such as EC-UWT and EC-QWT.

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