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Constructing Fermionic Particle Wave Equation on the Ring in a Novel Way

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Abstract: This research focuses on deriving the equation of motion for fermions on a ring. The paper proposes an alternative equation for fermions on the ring, incorporating a self-rotation angle denoted as Δ, and explores its relationship with the original equation.

Keywords: Quasiparticle; Anyon; fermions on a ring

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1. Introduction

There are two types of particles known as bosons and fermions, each possessing distinct symmetry properties regarding the interchange of identical particles. Particles that have symmetry on identical particle position swaps are called bosons. And this property gives the statistical properties called Bose–Einstein statistics. Particles have antisymmetric on identical particle position swaps called fermions. And this property gives rise to the Pauli exclusion principle, which forbids identical fermions from sharing the same quantum state. [1,2] In this research, we focus on the 2D fermionic ring scenario and propose a novel hypothetical model for the fermionic particle wave function, incorporating its rotational effects. In Chapter III, we point out the problem that the current fermion wave function on the ring fails to account for the rotation effects of the system. In Chapter IV, We start from the Schrödinger equation of fermions on the ring and introduce a self-rotation angle phase to give the same effect of identical particle swaps when the system undergoes rotation at a particular angle.

2. Theoretical Background

When a wave propagates through arbitrary potential, we can use the Schrodinger equation with the scalar potential $V(\mathbf{x})$.

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(\mathbf{x})\right]\Psi(\mathbf{x}) = E\Psi(\mathbf{x}) \tag{1}$$

and we can write a free field equation on the circle with radius R as,

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{x}) = E\Psi(\mathbf{x}) \tag{2}$$

And as the solution of Eq. (2) we can get,

$$\Psi_n(\theta) = \frac{1}{\sqrt{2\pi}}e^{in\theta}, \quad E_n = \frac{n^2\hbar^2}{2mR^2} \tag{3}$$

If we consider the ground state,

$$\Psi_1(\theta) = \frac{1}{\sqrt{2\pi}}e^{i\theta} \tag{4}$$

The fermionic identical particles satisfy the condition that,

$$\Psi(r_2, r_1) = -\Psi(r_1, r_2) \tag{5}$$

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This is also known as the anti-symmetric properties of the fermionic wave function and follows to Pauli Exclusive principle. And if there is N-multiple fermionic particle, by using the Slater determinant, the anti-symmetric wave function can be written as the determinant of a matrix. [3]

$$\Psi(x_1, x_2, ..., x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \chi_1(x_1) & \chi_2(x_1) & ... & \chi_N(x_1) \\ \chi_1(x_2) & \chi_2(x_2) & ... & \chi_N(x_2) \\ ... & ... & ... & ... \\ \chi_1(x_N) & \chi_2(x_N) & ... & \chi_N(x_N) \end{vmatrix} \tag{6}$$

And Eq. (4) simplifies the Eq. (6) that one can use the Vander Monde determinant. [4–6]

$$\begin{aligned} \Psi(x_1, x_2, ..., x_N) &= \frac{1}{\sqrt{2\pi}^N \sqrt{N!}} \begin{vmatrix} e^{i\theta_1} & e^{i2\theta_1} & ... & e^{iN\theta_1} \\ e^{i\theta_2} & e^{i2\theta_2} & ... & e^{iN\theta_2} \\ ... & ... & ... & ... \\ e^{i\theta_N} & e^{i2\theta_N} & ... & e^{iN\theta_N} \end{vmatrix} \\ &= \frac{1}{\sqrt{2\pi}^N \sqrt{N!}} \prod_{i=1}^{N} e^{i\theta_i} \prod_{1 \le i < j \le N} (e^{i\theta_i} - e^{i\theta_j}) \\ &= \frac{1}{\sqrt{2\pi}^N \sqrt{N!}} \prod_{i=1}^{N} e^{iN\theta_i} \prod_{1 \le i < j \le N} (e^{i(\theta_i - \theta_j)} - e^{-i(\theta_i - \theta_j)}) \\ &= \frac{1}{\sqrt{2\pi}^N \sqrt{N!}} (2i)^{N(N-1)/2} \prod_{1 \le i < j \le N} \sin\left(\frac{\theta_i - \theta_j}{2}\right) \end{aligned} \tag{7}$$

3. Problem

Let us consider the four fermion particles on the ring as shown in Figure 1. Then from the left to right figure, we can interpret this process in two different ways. First, this is the process of swapping $x_1 \leftrightarrow x_2$, $x_1 \leftrightarrow x_3$, and $x_1 \leftrightarrow x_4$ which gives a factor of -1 to wave function for each swap. Therefore as a result it gives a -1 sign after all swaps. However, this process is also equivalent to a $\pi/2$ counterclockwise rotation.

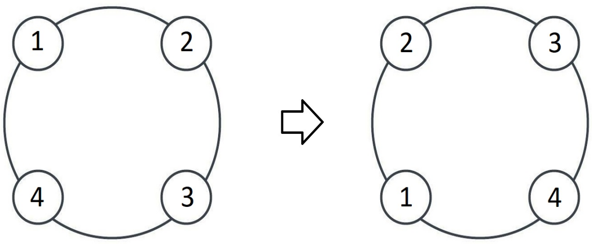

Figure 1. Four fermions on the ring. There are two ways to transform from a left to a right figure. The first method involves swapping (1,2), (2,3), and (3,4). The second method is to rotate the system counterclockwise by $\pi/2$.

Consider three fermion particles on the ring as shown in Figure 2. This configuration can be achieved by swapping $x_1 \leftrightarrow x_2$ and $x_1 \leftrightarrow x_3$, each swap introducing a factor of -1 to the wave function. Therefore as a result it gives a +1 sign after all swaps. However, this process is also the same as $2\pi/3$ counterclockwise rotation.

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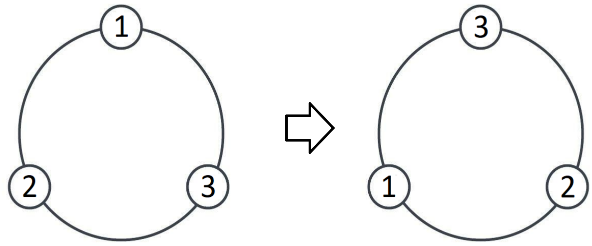

Figure 2. Three fermions on the ring. There are two ways to transform from a left to a right figure. The first method involves swapping (1,2) and (2,3). The second method is to rotate the system counterclockwise by $2\pi/3$.

Thus we need to investigate the relationship between system rotation and the sign of the wave function of fermions on a ring.

4. Hypothetical Model

To solve the problem, we suggest the fermion wave function also depends on the angle rotated from the observer, not only with identical particle position swapping.

Figure 3 shows particle can rotate with its axis, and let us call this angle as **self-rotation angle** and denoted as angle Δ.

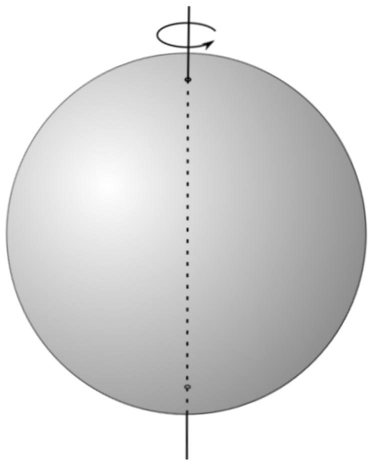

Figure 3. Self-rotation of the axis of a Particle

Next Figure 4 shows particle rotation within its ring system. We designate the angle of rotation as the **system-rotation angle**, denoted by θ. In this case, when the particle transitions from position A to position B, where $\theta = \varphi$, from the observer's perspective, the particle's self-rotation angle is also $\Delta = \varphi$.

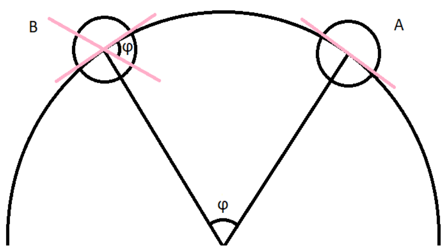

Figure 4. Relation between self-rotation and system-rotation angle. Particle move from position A to B by φ rotation

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We have a hypothetical model that when we rotate the fermionic-ring system by $\theta = \varphi$ angle, the fermionic field equation for a single particle is,

$$\boxed{\psi_i(x_j) = \psi_i(\theta_j, \Delta_j) = e^{i(\Delta_j+2\pi)K/2}\chi_i(\theta_j)} \tag{8}$$

Here, $K \equiv 2S$ (mod 2), and S is total spin of the particle.

Then we can ask how this result could effection on the fermions on the ring. From Eq. (8) we can suppose for single fermion with spin-half as,

$$\psi_i(x_j) = \psi_i(\theta_j, \Delta_j) = e^{i(\frac{\Delta_j}{2}+\pi)}\chi_i(\theta_j) \tag{9}$$

And let us define,

$$\psi_{1,2,...,N}(x_1, x_2, ..., x_N) \equiv \chi_1(x_1)\chi_2(x_2)...\chi_N(x_N) \tag{10}$$

Therefore, as Figure 4 shows when we rotate the $\theta = \varphi$ of the ring, it also gives the effect of self-angle $\Delta = \varphi$, by using Eq. (9) and Eq. (10) one can get

$$\begin{aligned}
&\psi_{2,...,N,1}(x_2, x_3..., x_N, x_1)\\
&= e^{iN(\frac{\Delta}{2}+\pi)}\chi_2(\theta_2)\chi_3(\theta_3)...\chi_N(\theta_N)\chi_1(\theta_1)\\
&= e^{iN(\frac{\Delta}{2}+\pi)}\chi_1(\theta_1)\chi_2(\theta_2)\chi_3(\theta_3)...\chi_N(\theta_N)\\
&= e^{iN(\frac{\Delta}{2}+\pi)}\psi_{1,2,...,N}(x_1, x_2, x_3..., x_N)
\end{aligned} \tag{11}$$

5. Examples

In this chapter, we test the hypothetical model with several examples whether it gives the same result as the original model.

5.1. 2-Particle Case

In the case of two identical particles, $N = 2, \Delta = \pi$

If we describe the wave equation of the right figure of Figure 5 by using Eq. (11). And here $\theta_1 = \pi$, $\theta_2 = 0$

$$\begin{aligned}
\Psi(x_2, x_1) &= \frac{1}{\sqrt{2}}(\psi_{21}(x_2, x_1) - \psi_{12}(x_2, x_1))\\
&= \frac{1}{\sqrt{2}}(\psi_2(x_2)\psi_1(x_1) - \psi_1(x_2)\psi_2(x_1))\\
&= \frac{1}{\sqrt{2}}e^{i(\pi/2+\pi)}e^{i(\pi/2+\pi)} \times\\
&\qquad (\chi_2(0)\chi_1(\pi) - \chi_1(0)\chi_2(\pi))\\
&= \frac{1}{\sqrt{2}}e^{i(3\pi)}(\chi_2(0)\chi_1(\pi) - \chi_1(0)\chi_2(\pi))\\
&= -\frac{1}{\sqrt{2}}(\chi_1(\theta_1)\chi_2(\theta_2) - \chi_2(\theta_1)\chi_1(\theta_2))\\
&= -\Psi(x_1, x_2)
\end{aligned} \tag{12}$$

And this is also the same result of swapping $x_1 \leftrightarrow x_2$ which gives -1 for each swap.

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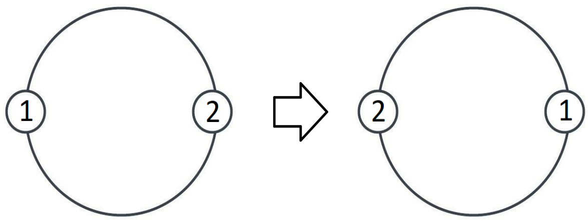

Figure 5. Two fermion on the ring. There is two way to transform from a left to a right figure. The first method is to swap (1,2). The second way is rotating the system anti-clockwise π

5.2. 3-Particle Case

In the case of three identical particles, $N = 3, \Delta = 2\pi/3$

$$\begin{aligned}\psi_{2,3,1}(x_2, x_3, x_1) &= \psi_2(x_2)\psi_3(x_3)\psi_1(x_1) \\ &= (e^{(\pi/3+\pi)i})^3\chi_2(\theta_2)\chi_3(\theta_3)\chi_1(\theta_1) \\ &= e^{4\pi i}\chi_1(\theta_1)\chi_2(\theta_2)\chi_3(\theta_3) \\ &= \psi_{1,2,3}(x_1, x_2, x_3) \end{aligned} \tag{13}$$

$$\begin{aligned}\Psi(x_2, x_3, x_1) &= \frac{1}{\sqrt{6}}(\psi_{1,2,3}(x_2, x_3, x_1) - \psi_{1,3,2}(x_2, x_3, x_1) \\ &\quad +\psi_{2,3,1}(x_2, x_3, x_1) - \psi_{2,1,3}(x_2, x_3, x_1) \\ &\quad +\psi_{3,1,2}(x_2, x_3, x_1) - \psi_{3,2,1}(x_2, x_3, x_1)) \\ &= \Psi(x_1, x_2, x_3) \end{aligned} \tag{14}$$

And this is also the same result of swapping $x_1 \leftrightarrow x_2$, and $x_2 \leftrightarrow x_3$ which gives -1 for each swap.

5.3. n-Particle Case

In the case of four identical particles, $N = n, \Delta = 2\pi/n$

$$\begin{aligned}\psi_{2,3..,n,1}(x_2, x_3, x_4, ..., x_1) &= \psi_2(x_2)\psi_3(x_3)...\psi_1(x_1) \\ &= (e^{(\Delta/2+\pi)i})^n\chi_2(x_2)...\chi_n(x_n)\chi_1(x_1) \\ &= (e^{(n\pi+\pi)i})\chi_1(x_1)\chi_2(x_2)...\chi_n(x_n) \\ &= e^{(n+1)\pi i}\psi_{1,2,3..,n}(x_1, x_2, x_3, .., x_n) \end{aligned} \tag{15}$$

And by using the levi-civita symbol,

$$\varepsilon_{a_1a_2a_3...a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, \ldots, a_n) \text{ is an} \\ \text{even permutation of } (1, 2, \ldots, n) \\ -1 & \text{if } (a_1, a_2, \ldots, a_n) \text{ is an} \\ \text{odd permutation of } (1, 2, \ldots, n) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

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$$\Psi_{2,3..,n,1}(x_2, x_3, x_4, ..., x_1)$$
$$= \frac{1}{\sqrt{n!}} \varepsilon_{a_1 a_2 a_3 \ldots a_n} \psi_{2,3..,n,1}(x_2, x_3, x_4, ..., x_1) \tag{17}$$

And by using this, we can write down the full wave function in n-particle case as,

$$\Psi_{2,3..,n,1}(x_2, x_3, x_4, ..., x_1)$$
$$= e^{(n+1)\pi i} \Psi_{1,2,3..,n}(x_1, x_2, x_3, .., x_n) \tag{18}$$

Therefore, when n is odd, the total wave function remains the same, and when n is even, the total wave function becomes negative compared to the original.

6. Conclusion and Outlook

We review the n-particle fermionic wave function by using the Schrodinger equation and Slater Determinant. We point out the paradox of anti-symmetric properties of system-rotation and position swap inconsistency, and suggest the new hypothetical model with a self-rotation angle. We suggest the new phase factor on the wave equation with self-rotation angle and spin of the particle as Eq. (8). However, further investigation into the relationship between spin and rotational phase factors is necessary, which includes exploring the Anyon interpretation in two dimensions [7,8], and examining the supersymmetric properties.

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