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Differentiation of Solutions of Caputo Boundary Value Problems with respect to Boundary Data

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Abstract: Under suitable continuity and uniqueness conditions, solutions of an α order Caputo fractional boundary value problem are differentiated with respect to boundary values and boundary points. This extends well-known results for nth order boundary value problems. The approach used is a standard technique and makes heavy use of recent results for differentiation of solutions of Caputo fractional intial value problems with respect to initial conditions and continuous dependence for Caputo fractional boundary value problems.

Keywords: continuous dependence; Caputo fractional derivative; fractional differential equation; variational equation

MSC: 26A33, 34A08, 34B15

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1. Introduction

Let $n \in \mathbb{N}$ with $\alpha \in (n-1, n)$ and $a < t_0 < b$ in $\mathbb{R}$. Our concern is characterizing partial derivatives with respect to the boundary data for solutions to the Caputo fractional boundary value problem

$$D^{\alpha}_{*t_0} x(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad a < t_0 < t < b, \tag{1}$$

satisfying conjugate boundary conditions

$$x(t_i) = x_i \tag{2}$$

where $D^{\alpha}_{*t_0} x$ is the Caputo fractional derivative of order α of the function $x(t)$ and $a < t_0 \le t_1 < t_2 < \dots < t_n < b$ and $x_i \in \mathbb{R}$ for $1 \le i \le n$.

Research into fractional differential equations has seen an explosion of results, [1–4,6,15,19,21]. In fact, there seem to be a limitless number of different ways to define a fractional derivative. However, two definitions have become the source of focus amongst a broad range of researchers in the field; namely the Riemann-Liouville and Caputo fractional derivatives. For expository material on fractional differential equations, we refer the reader to [5,13,14,17].

In this paper, we impose suitable continuity and uniqueness hypotheses so that given a solution of (1), (2), one may take the derivative with respect to the boundary data. This derivative solves an associated Caputo fractional boundary value problem called the variational equation with interesting boundary data.

This work is an expansion upon well-known previous work for nth order boundary value problems [8–12,18]. In fact, we use the ideas of these works as a guide to help construct our proofs. To that end, we rely heavily upon two recent results for Caputo fractional differential equations. The first [6] establishes differentiation of solutions of Caputo initial value problems with respect to the initial data, and the second [16] establishes the continuous dependence on boundary conditions for Caputo boundary value problems.

Essentially, with a unique solution to a Caputo boundary value problem, we define a difference quotient with respect to the boundary datum. We then view this difference quotient in terms of an initial value problem. This allows us to apply Theorem 3.2 from [6] to show this difference quotient

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solves the variational equation. Finally, we take a limit by applying the continuous dependence result, Theorem 4.2, from [16] which yields the desired result.

The remainder of the paper is organized as follows. In section 2, one will find brief definitions of fractional integrals and derivatives. For further study, we refer the reader to [5,13,14,17]. Section 3 introduces us to the variational equation and establishes our sufficient hypotheses. For section 4, we present the important recent developments in the field of study that have made this paper possible; namely differentiation of Caputo initial value problems [6] and continuous dependence of boundary data for Caputo boundary value problems [16]. To conclude, we have section 5 that contains the main result and its proof.

2. Fractional Derivatives

Let $\alpha > 0$. The Riemann-Liouville fractional integral of a function x of order α, denoted $I^{\alpha}_{t_0}x$, is defined as

$$I^{\alpha}_{t_0}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}x(s)ds, \quad t_0 \le t,$$

provided the right-hand side exists. Moreover, let $n \in \mathbb{N}$ denote a positive integer and assume $n-1 < \alpha \le n$. The Riemann-Liouville fractional derivative of order α of the function x, denoted $D^{\alpha}_{t_0}x$, is defined as

$$D^{\alpha}_{t_0}x(t) = D^{n}I^{n-\alpha}_{t_0}x(t),$$

provided the right-hand side exists. If a function x is such that

$$D^{\alpha}_{t_0}\left(x(t) - \sum_{i=0}^{n-1}x^{(i)}(t_0)\frac{(t-t_0)^i}{i!}\right)$$

exists, then the Caputo fractional derivative of order α of x is defined by

$$D^{\alpha}_{*t_0}x(t) = D^{\alpha}_{t_0}\left(x(t) - \sum_{i=0}^{n-1}x^{(i)}(t_0)\frac{(t-t_0)^i}{i!}\right).$$

Remark 1. *A sufficient condition to guarantee the existence of the Caputo fractional derivative is the absolute continuity of the $(n-1)$st derivative of $x(t)$. See Theorem 3.1 in [5] and discussion thereafter.*

3. Preliminaries

Throughout this work, we make use of the following assumptions:

(1) $f : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous;
(2) for $1 \le i \le n$, $\partial f(t, x_1, \ldots, x_n)/\partial x_i : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous; and
(3) solutions to initial value problems for (1) are unique on (a,b);

The derivative sought in this manuscript solves a related equation which we define next.

Definition 1. *The α order Caputo fractional variational equation of (1) along a solution $x(t)$ is the differential equation*

$$D^{\alpha}_{*t_0}z(t) = \sum_{j=0}^{n-1}\frac{\partial f}{\partial x_j}(t, x(t), x'(t), \ldots, x^{(n-1)}(t))z^{(j)}. \tag{3}$$

Finally, we present two more hypotheses which establish a uniqueness condition for (1) and (3), respectively.

(4) Given points $a < t_0 \le t_1 < t_2 < \ldots < t_n < b$, if y and z are solutions of (1) such that for $1 \le i \le n$, $y(t_i) = z(t_i)$, then $y(t) = z(t)$ on (a,b); and

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(5) given points $a < t_0 \leq t_1 < t_2 < \ldots < t_n < b$, if u is a solution of (3) along (1) such that for $1 \leq i \leq n$, $u(t_i)$=0, then $u(t) \equiv 0$ on (a, b).

Next, we present two crucial results that make this work possible.
Let $[c, d] \subset \mathbb{R}$ and for $x \in C[c, d]$, define

$$\|x\|_{0,[c,d]} = \max_{t \in [c,d]} |x(t)|.$$

If $k \in \mathbb{N}$, for $x \in C^k[c, d]$, define

$$\|x\|_{k,[c,d]} = \max\{\|x\|_{0,[c,d]}, \|x'\|_{0,[c,d]}, \ldots, \|x^{(k)}\|_{0,[c,d]}\}.$$

We seek a boundary value problem result as an analog of the initial value problem result from Eloe et al [6].

Theorem 1. *Let $f(t, y_0, y_1, \ldots, y_{n-1})$ be continuous and have continuous first partial derivatives with respect to y_j for $0 \leq j \leq n-1$ on an open, connected, convex set $E \subset \mathbb{R} \times \mathbb{R}^n$. Let $(t_0, y_0, \ldots, y_{n-1}) \in E$, and let $y(t) := y(t; t_0, y_0, \ldots, y_{n-1})$ be the unique solution of the initial value problem (1) satisfying*

$$y^{(i)}(t_0) = y_i,\ i = 0, \ldots, n-1, \tag{4}$$

with maximal interval of existence $[t_0, \omega)$. Choose $[c, d] \subset [t_0, \omega)$. Then,

(a) for each $0 \leq j \leq n-1$, $\alpha_j(t) := \partial y(t)/\partial y_j$ exists and is the solution of the variational equation (3) along $y(t)$ on $[c, d]$ and hence, $[t_0, \omega)$ satisfying the initial conditions

$$\alpha_j^{(i)}(t_0) = \delta_{ij},\ 0 \leq i \leq n-1;$$

(b) if, in addition, f has a continuous first derivative with respect to t and

$$f(t_0, y_0, y_1, \ldots, y_{n-1}) = 0,$$

then $\beta(t) := \partial y(t)/\partial t_0$ exists and is the solution of the variational equation (3) along $y(t)$ on $[c, d]$ and hence, $[t_0, \omega)$ satisfying the initial conditions

$$\beta^{(i)}(t_0) = -y^{(i+1)}(t_0),\ 0 \leq i \leq n-1;\ \text{and}$$

(c) $$\beta(t) = -\sum_{i=0}^{n-1} y^{(i+1)}(t_0)\alpha_i(t).$$

We also use recent continuous dependence on boundary conditions results for Caputo fractional differential equations [16]. The first one is when the left-most boundary condition is to the right of the starting point of the Caputo fractional derivative; namely $t_0 < t_1$, and the second is when they are equal; namely $t_0 = t_1$. Note that the second result has an additional condition to establish continuous dependence to the left of t_0.

Theorem 2. *[Case when $t_0 < t_1$] Assume that hypotheses (1), (3), and (4) hold. Let $x(t)$ be a solution of (1) on $[t_0, b)$, $[c, d] \subset [t_0, b)$ with points $c \leq t_0 \leq t_1 < t_2 < \ldots < t_n < d$, and $\epsilon > 0$. Then, there exists a $\delta(\epsilon, [c, d]) > 0$ such that, if for $1 \leq i \leq n$, $|t_i - \tau_i| < \delta$ with $c < \tau_1 < \tau_2 < \tau_3 < \ldots, \tau_n < d$ and $|x(t_i) - y_i| < \delta$ with $y_i \in \mathbb{R}$, then there exists a solution $y(t)$ of (1) satisfying $y(\tau_i) = y_i$. Also,*

$$||x(t) - y(t)||_{n-1,[c,d]} < \epsilon.$$

Theorem 3. *[Case when $t_0 = t_1$] Assume that hypotheses* (1), (3), *and* (4) *hold. Let $x(t)$ be a solution of* (1) *on* $[t_1, b)$, $[c, d] \subset [t_1, b)$ *with points* $c = t_1 < t_2 < \ldots < t_n < d$, *and* $\epsilon > 0$. *Then, there exists a* $\delta(\epsilon, [c, d]) > 0$ *such that, if for* $2 \le i \le n$, $|t_i - \tau_i| < \delta$ *with* $c < \tau_2 < \tau_3 < \ldots, \tau_n < d$ *and for* $1 \le i \le n$, $|x(t_i) - y_i| < \delta$ *with* $y_i \in \mathbb{R}$, *then there exists a solution* $y(t)$ *of* (1) *satisfying* $y(t_1) = y_1$ *and for* $2 \le i \le n$, $y(\tau_i) = y_i$. *Also,*

$$||x(t) - y(t)||_{n-1,[c,d]} < \epsilon.$$

Additionally, if $f_k : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ *is a sequence of continuous functions that converge uniformly to* f *on compact subsets of* $[c, d] \times \mathbb{R}^n$ *and for* $k \ge 1$, t_1^k *is an increasing sequence such that* $t_1^k \uparrow t_1^-$ *as* $k \to \infty$, *then there exists a* K *such that if* $k \ge K$, *then*

$$||x_k(t) - x(t)||_{n-1,[c,d]} \to 0 \quad as \quad k \to \infty.$$

4. Main Results

In this section, we present our boundary value problem analog. This is done under the assumption that $t_0 < t_1$. However, with the additional assumption from Theorem 3, the same result is established for $t_0 = t_1$ and the proof remains the same. Without this additional assumption, the derivative at t_0 would only be a right-hand derivative but the result still holds.

Theorem 4. *Assume conditions* (1)-(5) *are satisfied and that* $t_0 < t_1$. *Let* $x(t) := x(t, t_1, \ldots, t_n, x_1, \ldots, x_n)$ *be a solution of* (1) *satisfying* $x(t_i) = x_i$ *for* $1 \le i \le n$ *on* $[t_0, \omega) \subset (a, b)$. *Then,*

(a) *for each* $1 \le j \le n$, $z_j(t) := \partial x(t)/\partial x_j$ *exists and is the solution of the variational equation* (3) *along* $x(t)$ *on* $[c, d]$ *and hence,* $[t_0, \omega)$ *satisfying the boundary conditions*

$$z_j\left(t_i\right) = \delta_{ij},\ 1 \le i \le n;$$

(b) *if, in addition,* f *has a continuous first derivative with respect to* t *and*

$$f\left(t_j, x(t_j), x'(t_j), \ldots, x^{(n-1)}(t_j)\right) = 0,$$

then $w_j(t) := \partial x(t)/\partial t_j$ *exists and is the solution of the variational equation* (3) *along* $x(t)$ *on* $[c, d]$ *and hence,* $[t_0, \omega)$ *satisfying the initial conditions*

$$w_j\left(t_i\right) = -x'\left(t_i\right)\delta_{ij},\ 1 \le i \le n;\ and$$

(c) *for each* $1 \le j \le n$, $w_j(t) = -x'(t_j) z_j(t)$.

Proof. We will only prove part (a) as the proof of part (b) is similar. Part (c) is immediate consequence from parts (a)and (b) when coupoled with hypothesis (v).

Let $1 \le j \le n$, and consider $\partial x(t)/\partial x_j$. In the interests of conserving space and lessening the tedious notation, we denote $x(t; t_1, \ldots, t_n, x_1, \ldots, x_j, \ldots, x_n)$ by $x(t; x_j)$ as x_j is the boundary value of interest.

Let $\delta > 0$ be as in Theorem 2, $0 < |h| < \delta$ be given, and define

$$z_{jh}(t) = \frac{1}{h}[x(t; x_j + h) - x(t; x_j)].$$

Note that for every $h \neq 0$,

$$\begin{aligned} z_{jh}(t_j) &= \frac{1}{h}[x(t_j, x_j + h) - x(t_j, x_j)] \\ &= \frac{1}{h}[(x_j + h) - x_j] \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{h}[h] \\
&= 1.
\end{aligned}
$$

Also, for every $h \neq 0,\ 1 \leq k \leq n$ with $k \neq j$,

$$
\begin{aligned}
z_{jh}(t_k) &= \frac{1}{h}[x(t_k, x_j + h) - x(t_k, x_j)] \\
&= \frac{1}{h}[x_k - x_k] \\
&= 0.
\end{aligned}
$$

Now that we have established the boundary conditions for $z_{jh}(t)$, we show that $z_{jh}(t)$ solves the variational equation. To that end, for $1 \leq i \leq n-1$, let

$$v_i = x^{(i)}(t_j; x_j)$$

and

$$\epsilon_i = \epsilon_i(h) = x^{(i)}(t_j; x_j + h) - v_i.$$

By Theorem 2, for $1 \leq i \leq n-1,\ \epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 1 for solutions of initial value problems for (1), viewing $x(t)$ as the solution of an initial value problem, and denoting the solution $x(t) = y(t; t_j, x_j, v_1, \ldots, v_{n-1})$, we have

$$z_{jh}(t) = \frac{1}{h}[y(t; t_j, x_j + h, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1}) - y(t; t_j, x_j, v_1, \ldots, v_{n-1})].$$

Then, by utilizing telescoping sums, we have

$$
\begin{aligned}
z_{jh}(t) =& \frac{1}{h}\Big\{[y(t; t_j, x_j + h, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1}) - y(t; t_j, x_j, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1})] \\
&+ [y(t; t_j, x_j, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1}) - y(t; t_j, x_j, v_1, \ldots, v_{n-1} + \epsilon_{n-1})] \\
&+ [y(t; t_j, x_j, v_1, \ldots, v_{n-1} + \epsilon_{n-1})) - \cdots] \\
&+ [y(t; t_j, x_j, v_1, \ldots, v_{n-1} + \epsilon_{n-1})) - y(t; t_j, x_j, v_1, \ldots, v_{n-1})]\Big\}.
\end{aligned}
$$

By Theorem 1 and the Mean Value Theorem, we obtain

$$
\begin{aligned}
z_{jh}(t) =& \frac{1}{h}\Big[\alpha_0(t, y(t; t_j, x_j + \bar{h}, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1}))(x_j + h - x_j) \\
&+ \alpha_1(t, y(t; t_j, x_j, v_1 + \bar{\epsilon}_1, \ldots, v_{n-1} + \epsilon_{n-1}))(v_1 + \epsilon_1 - v_1) + \cdots \\
&+ \alpha_{n-1}(t, y(t; t_j, x_j, v_1, \ldots, v_{n-1} + \bar{\epsilon}_{n-1}))(v_{n-1} + \epsilon_{n-1} - v_{n-1})\Big] \\
=& \alpha_0(t, y(t; t_j, x_j + \bar{h}, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1})) \\
&+ \frac{\epsilon_1}{h}\alpha_1(t, y(t; t_j, x_j, v_1 + \bar{\epsilon}_1, \ldots, v_{n-1} + \epsilon_{n-1})) + \cdots \\
&+ \frac{\epsilon_{n-1}}{h}\alpha_{n-1}(t, y(t; t_j, x_j, v_1, \ldots, v_{n-1} + \bar{\epsilon}_{n-1}))
\end{aligned}
$$

where, for $0 \leq k \leq n-1,\ \alpha_k(t, y(\cdot))$ is the solution of the variational equation (3) along $y(\cdot)$ satisfying

$$\alpha_k^{(i)}(t_j) = \delta_{ik},\ \ 0 \leq i \leq n-1.$$

Furthermore, for $1 \leq i \leq n-1,\ v_i + \bar{\epsilon}_i$ is between v_i and $v_i + \epsilon_i$.

Thus, to show $\lim_{h \to 0} z_{jh}(x)$ exists, it suffices to show, for $1 \leq i \leq n-1,\ \lim_{h \to 0} \epsilon_i / h$ exists.

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Now, from the construction of $z_{jh}(t)$, we have

$$z_{jh}(t_k) = 0,\ 1 \le k \le n \text{ with } k \neq j.$$

Hence, for $1 \le k \le n$ with $k \neq j$, we have a system of $n-1$ linear equations with $n-1$ unknowns:

$$\begin{aligned}&-\alpha_0(t_k, y(t; t_j, x_j + \bar{h}, v_1 + \epsilon_1, \dots, v_{n-1} + \epsilon_{n-1}))\\ &= \frac{\epsilon_1}{h}\alpha_1(t_k, y(t; t_j, x_j, v_1 + \bar{\epsilon}_1, \dots, v_{n-1} + \epsilon_{n-1})) + \cdots\\ &+ \frac{\epsilon_{n-1}}{h}\alpha_{n-1}(t_k, y(t; t_j, x_j, v_1, \dots, v_{n-1} + \bar{\epsilon}_{n-1})).\end{aligned}$$

In the system of equations above, we notice that $y(\cdot)$ is not always the same. Therefore, we consider the coefficient matrix M based on $y(t)$

$$M := \begin{pmatrix} \alpha_1(t_1, y(t)) & \alpha_2(t_1, y(t)) & \cdots & \alpha_{n-1}(t_1, y(t)) \\ \alpha_1(t_2, y(t)) & \alpha_2(t_2, y(t)) & \cdots & \alpha_{n-1}(t_2, y(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1(t_{j-1}, y(t)) & \alpha_2(t_{j-1}, y(t)) & \cdots & \alpha_{n-1}(t_{j-1}, y(t)) \\ \alpha_1(t_{j+1}, y(t)) & \alpha_2(t_{j+1}, y(t)) & \cdots & \alpha_{n-1}(t_{j+1}, y(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1(t_n, y(t)) & \alpha_2(t_n, y(t)) & \cdots & \alpha_{n-1}(t_n, y(t))) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then, there exist $p_i \in \mathbb{R}$ for $1 \le i \le n-1$ not all zero such that

$$p_1 \begin{pmatrix} \alpha_1(t_1, y(t)) \\ \alpha_1(t_2, y(t)) \\ \vdots \\ \alpha_1(t_{j-1}, y(t)) \\ \alpha_1(t_{j+1}, y(t)) \\ \vdots \\ \alpha_1(t_n, y(t)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \alpha_{n-1}(t_1, y(t)) \\ \alpha_{n-1}(t_2, y(t)) \\ \vdots \\ \alpha_{n-1}(t_{j-1}, y(t)) \\ \alpha_{n-1}(t_{j+1}, y(t)) \\ \vdots \\ \alpha_{n-1}(t_n, y(t)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Set

$$w(t, y(t)) := p_1\alpha_1(t, y(t)) + \cdots + p_{n-1}\alpha_{n-1}(t, y(t)).$$

Then, $w(t, y(t))$ is a nontrivial solution of the variational equation (3). However, $w(t, y(t)) = 0$ and, for $1 \le k \le n-1$ with $k \neq j$, $w(x_k, y(t)) = 0$. By hypothesis (v), $w(t, y(t)) = 0$. Thus, $p_1 = p_2 = \cdots = p_{n-1} = 0$ which is a contradiction to the choice of the $p_i's$. Hence, $\det(M) \neq 0$.

Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, for each $1 \le i \le n-1$, we are able to find ϵ_i/h using Cramer's rule.

Note as $h \to 0$, $\det(M(h)) \to \det(M)$, and so for $1 \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)/\det M := B_i$ as $h \to 0$, where M_i is the $n-1 \times n-1$ matrix found by replacing the appropriate column of the matrix defining M by

$$\text{col}\Big[-\alpha_0(t_1, x(t)), -\alpha_0(t_2, x(t)), \dots, -\alpha_0(t_{j-1}, x(t)), -\alpha(t_{j+1}, x(t)), \dots, -\alpha_0(t_k, x(t))\Big].$$

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Now, let $z_j(t) = \lim_{h\to 0} z_{jh}(t)$, and by construction of $z_{jh}(t)$,

$$z_j(t) = \frac{\partial x}{\partial x_j}(t).$$

Furthermore,

$$z_j(t) = \lim_{h\to 0} z_{jh}(t) = \alpha_0(t, x(t)) + \sum_{i=1}^{n-1} B_i \alpha_i(t, x(t))$$

which is a solution of the variational equation (3) along $x(t)$. In addition, for $1 \le j \le n$,

$$z_j(x_k) = \lim_{h\to 0} z_{jh}(x_k) = \delta_{jk}.$$

This completes the argument for $\partial x(t)/\partial x_j$. □

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