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Abstract: A one-dimensional model for fluid and solute transport in poroelastic materials (PEM) is studied. Although the model was recently derived and some exact solutions, in particular steady-state solutions and their applications, were studied, special cases occurring when some parameters vanish were not analysed earlier. Since the governing equations are nonintegrable in nonstationary case, the Lie symmetry method and modern tools for solving ODE systems are applied in order to construct time-dependent exact solutions. Depending on parameters arising in the governing equations, several special cases with new Lie symmetries are identified. Some of them have a highly nontrivial structure that cannot be predicted from a physical point of view or using Lie symmetries of other real-world models. Applying the symmetries obtained, multiparameter families of exact solutions are constructed, including those in terms of elementary and special functions (hypergeometric, Whittaker, Bessel and modified Bessel functions). A possible application of the solutions obtained is demonstrated and it is shown that some exact solutions can describe (at least qualitatively) the solute transport in PEM. The obtained exact solutions can also be used as test problems for estimating the accuracy of approximate analytical and numerical methods for solving relevant boundary value problems.

Keywords: poroelastic material; nonlinear differential equation; Lie symmetry; exact solution

MSC: 35B06; 35C05; 74L15

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1. Introduction

The poroelastic theory considers the system formed by an elastic material with pores that can be penetrated by a fluid and/or dissolved solutes. A typical biological example is the glucose solute penetrating a tissue layer. The basic theory of poroelasticity was developed by Biot [1] and its state-of-the-art can be found in the well-known books [1–3] (see also recent studies, e.g., [4–8]). A poroelastic material is considered as the superposition of two continuous media: the matrix (skeleton), occupying the fractional volume θ_M, and the system of pores saturated by a fluid, occupying the fractional volume θ_F ($\theta_F + \theta_M = 1$). The deformation of the system under the fluid pressure is described by a deformation vector, and the dynamics of the deformation under the forces needs in general to be described by second order tensors. The relationship between stress and strain is usually assumed to be linear. The flux of fluid depends on the hydrostatic pressure and solute gradient (called osmotic pressure). The diffusive and convective transport mechanisms should be taking into account as well. The general three-dimensional theory describing transport in poroelastic materials (PEM) is very complex because relevant mathematical models involve 3D nonlinear partial differential equations (PDEs). As a result, in order to obtain analytical results, one-dimensional versions are usually discussed [9–11]. Here we study a one-dimensional model for fluid and solute transport in poroelastic materials(PEM) that was derived in [10] and generalized [11]. The governing equations of the model read as

$$2u_{tx} = k(p_{xx} - \sigma_1 c_{xx}), \tag{1}$$

$$\rho u_{tt} + \rho_t u_t + \rho u_t u_{tx} = (\lambda + 2\mu)u_{xx} - (p_x - \sigma_1 c_x), \tag{2}$$

$$\rho_t + \rho_x u_t = k(\rho_F^0 - \rho)(p_{xx} - \sigma_1 c_{xx}), \tag{3}$$

$$\theta_{Ft} + \theta_{Fx} u_t = k(1 - \theta_F)(p_{xx} - \sigma_1 c_{xx}), \tag{4}$$

$$(c\theta_F)_t + (c\theta_F)_x u_t + 2c\theta_F u_{tx} = kS\Big(c(p_x - \sigma_1 c_x)\Big)_x + Dc_{xx}, \tag{5}$$

where $\sigma_1 = \sigma RT$ and $\rho_F = \rho_F^0$ are positive constant and the lower subscripts t and x denote differentiation with respect to these variables. The physical/biological meanings of the notations used above are presented in Table 1.

Table 1. Description of the symbols used in equations (1)–(5).

Symbol	Description
u	deformation vector
ρ	mass density
θ_F	fractional volume of fluid phase F
θ_M	fractional volume of matrix phase M
ρ_F	mass density of fluid phase F
c	solute concentration in PEM
p	mechanical pressure in PEM
σ	reflection coefficient of PEM
RT	gas constant times temperature
$\lambda + 2\mu$	elastic modulus with Lame constants λ and μ
k	hydraulic conductivity
D	solute diffusivity in PEM
$S = 1 - \sigma$	sieving coefficient of solute in the PEM

The nonlinear system of PDEs (1)–(5) was integrated in the stationary case. As a result, all steady-state solutions were identified and examples of their application for the glucose fluid transport in a biological tissue was provided [10]. In the nonstationary case, system (1)–(5) is not integrable, therefore the classical Lie method [12–16] was adopted for search exact solutions. Nowadays this method is widely used for construction of exact solutions of nonlinear PDEs arising in real-world applications and the most remarkable works are cited in the above cited books. However, it can be easily noted that there are not many studies devoted to *multicomponent systems of PDEs* because essential technical difficulties occur if one intends to find exact solutions for such systems by applying Lie symmetries. Taking into account the above observation, we refer the reader to the recent works [17–22], devoted to applications of the classical Lie method to the nonlinear three-component systems of PDEs.

Although several nontrivial Lie symmetries were identified and successfully applied for finding exact solutions in [10] (see also generalisations in [11]), some limiting cases have been not analysed therein. In particular, the special cases $D = 0$ and/or $S = 0$ were not examined in [10,11]. It is proved here that the special cases listed above lead to a rich Lie symmetry involving symmetry operators that do not occur for system (1)–(5) with $DS \neq 0$. It should be stressed that the governing equations with $D = 0$ and/or $S = 0$ can still be used as a real-world model in some cases. For example, it is known that the diffusion term in equation (5) is negligible (i.e. $D = 0$) if molecules of large size (because of high atomic weight) are dissolved. A typical example is albumin, for which the diffusivity is circa 50 times smaller than the glucose diffusivity [23].

This paper is organised as follows. In Section 2, all possible extensions of the Lie algebra of invariance of the nonlinear system of PDEs (1)–(5) are derived. It is proved that there are four inequivalent cases when the system admits additional Lie symmetries depending on the values of the parameters D and S. Section 3 is devoted to the constructions of exact solutions of system (1)–(5) with $S = 0$. Because the system admits an additional Lie symmetry, new reductions to systems of ODEs and, as a result, new exact solutions of the nonlinear model in question are derived. In Section 4, an exact solution is analysed in order to show its applicability for modeling solute transport in PEM. The analysis is supported by 3D plots of the solution. In Section 5, new exact solutions of the nonlinear

system (1)–(5) with arbitrary D and S are constructed. The solution obtained were not identified in the previous study [10]. In Section 6, we present conclusions highlighting the main results obtained.

2. Lie Symmetries

Here we start from the governing equations (1)–(5) of the model for fluid and solute transport in poroelastic materials that was derived in [10]. By the application of the substitution

$$p_x^* = p_x - \sigma_1 c_x, \tag{6}$$

where p^* is a so-called effective pressure, the above system takes the form

$$
\begin{aligned}
&2u_{tx} = kp_{xx}^*, \\
&\rho u_{tt} + \rho_t u_t + \rho u_t u_{tx} = \lambda^* u_{xx} - p_x^*, \\
&\rho_t + \rho_x u_t = k(\rho_F^0 - \rho)p_{xx}^* \\
&\theta_{Ft} + \theta_{Fx} u_t = k(1 - \theta_F)p_{xx}^*, \\
&(c\theta_F)_t + (c\theta_F)_x u_t + 2c\theta_F u_{tx} = Dc_{xx} + kS(cp_x^*)_x.
\end{aligned}
\tag{7}
$$

In (7), parameters k, λ^*, D, and S are constants satisfying the restrictions

$$k > 0,\ \lambda^* > 0,\ \rho_F^0 > 0,\ D \geq 0,\ 0 \leq S \leq 1. \tag{8}$$

Theorem 1. *[10] System (7) with arbitrary given parameters k, λ^*, ρ_F^0, D, and S is invariant under an infinity-dimensional Lie algebra generated by the Lie symmetries :*

$$\partial_t,\ \partial_x,\ \partial_u,\ x\partial_u,\ c\partial_c,\ g(t)\partial_{p^*}, \tag{9}$$

where $g(t)$ is an arbitrary smooth function (hereafter, the notations $\partial_z \equiv \frac{\partial}{\partial_z}$, $(z = t, x, u, \dots)$ are used).

According to the standard terminology, Lie algebra (9) is called the principal algebra of system (7) (see, e.g., Chapter 1 in [16]). Because the latter involves several parameters, a Lie symmetry classification problem arises that was not solved in [10].

We remind the reader that Lie symmetry classification (LSC) problems (group classification problems) are the most difficult those in Lie symmetry analysis. One may claim that Sophus Lie carried out some theoretical foundations for solving LSC problems, in particular he has done classification of Lie symmetries for a class of linear two-dimensional PDEs. During the last few decades, a vast number of papers were published devoted to theoretical foundations and applications of algorithms for classification of Lie symmetries of specific PDEs (systems of PDEs). Unfortunately, a misleading terminology is used in many of them because instead of a complete solving of the LSC problem for a given PDE, only examples of extensions of the principal algebra are presented. The current state-of-the-art and the relevant list of most important publications can be found in [16] (see Chapter 2 therein).

Theorem 2. *System (7) with restrictions (8) admits the extensions of the principal algebra (9) only in the cases presented in Table 2.*

Table 2. Lie symmetries of system (7).

	Restrictions	Lie symmetries extending algebra (9)
1	$S = 1, \; D \neq 0$	∂_c
2	$S = 0, \; D \neq 0$	$x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u$
3	$S = 1, \; D = 0$	$h(c)\,\partial_c$
4	$S = 0, \; D = 0$	$x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u, \; cH^2\left(\frac{\theta_F - 1}{\rho_F^0 - \rho}, \frac{c\theta_F}{\rho_F^0 - \rho}\right)\partial_c,$ $H^1\left(\frac{\theta_F - 1}{\rho_F^0 - \rho}, \frac{c\theta_F}{\rho_F^0 - \rho}\right)\left((\theta_F - 1)\partial_{\theta_F} + \frac{c}{\theta_F}\partial_c\right)$

In Table 2, H^1, H^2 and h are arbitrary smooth functions.

Proof of this theorem is based on the infinitesimal criteria of invariance, which was formulated by S. Lie in his pioneering works [12,13]. In the case of a system of PDEs of arbitrary order, this criteria can be found, e.g., in [14] (see Section 1.2.5). Note, system (7) consists of five PDEs of the second order. So, a linear first-order operator of the form

$$X = \xi^0 \partial_t + \xi^1 \partial_x + \eta^1 \partial_u + \eta^2 \partial_\rho + \eta^3 \partial_{p^*} + \eta^4 \partial_{\theta_F} + \eta^5 \partial_c$$

(here the coefficients ξ^0, ξ^1 and η^i, $i = 1, \ldots, 5$, to-be-determined functions of independent and dependent variables) is a Lie symmetry (operator of Lie's invariance, point symmetry) of system (7) provided the following equalities are simultaneously satisfied

$$X^{(2)}\left(2u_{tx} - kp^*_{xx}\right) = 0,$$

$$X^{(2)}\left(\rho u_{tt} + \rho_t u_t + \rho u_t u_{tx} - \lambda^* u_{xx} + p^*_x\right) = 0,$$

$$X^{(2)}\left(\rho_t + \rho_x u_t - k(\rho_F^0 - \rho)p^*_{xx}\right) = 0, \qquad (10)$$

$$X^{(2)}\left(\theta_{Ft} + \theta_{Fx}u_t - k(1 - \theta_F)p^*_{xx}\right) = 0,$$

$$X^{(2)}\left((c\theta_F)_t + (c\theta_F)_x u_t + 2c\theta_F u_{tx} - kS(cp^*_x)_x - Dc_{xx}\right) = 0,$$

for each solution $\left(u(t,x), \; \rho(t,x), \; p^*(t,x), \; \theta_F(t,x), \; c(t,x)\right)$ of the PDE system (7). Here $X^{(2)}$ is the second-order prolongation of the operator X, which is again the first-order operator with coefficients defined by the well-known formulae via the first- and second-order derivatives of unknown coefficients ξ^0, ξ^1 and η^i (see, e.g., [14], Section 1.2.1).

After relevant calculations, formulae (10) are reducible to a linear system of PDEs, called the system of determining equations (DEs), for finding the functions ξ^0, ξ^1 and η^i. Some of the equations belonging to the system of DEs, do not contain parameters S and D, therefore solving them, we obtain the most general form of the Lie symmetry operator

$$X = \alpha_0 \partial_t + \alpha_1 \partial_x + \left(\frac{\alpha_4}{2\lambda^*}x^2 + \alpha_3 x + \alpha_2\right)\partial_u + (\alpha_4 x + g(t))\partial_{p^*} + \eta^4(\rho, \theta_F, c)\partial_{\theta_F} + \eta^5(\rho, \theta_F, c)\partial_c, \quad (11)$$

where α_i $(i = 0, \ldots, 3)$ are arbitrary constants and $g(t)$ ia an arbitrary smooth function, while the constant α_4 and the functions η^4 and η^5 must satisfy the restriction $S\alpha_4 = 0$ and the remaining DEs

$$S\eta^4 = 0, \; S\eta_\rho^5 = 0, \; S\eta_{\theta_F}^5 = 0, \tag{12}$$

$$D\eta_c^4 = 0, \; D\eta_\rho^5 = 0, \; D\eta_{\theta_F}^5 = 0, \; D\eta_{cc}^5 = 0, \tag{13}$$

$$D(1-S)\left(c\eta_c^5 - \eta^5\right) = 0, \; S(1-S)\left(c\eta_c^5 - \eta^5\right) = 0, \tag{14}$$

$$\theta_F^2\left((\rho_F^0 - \rho)\eta_\rho^5 + (1-\theta_F)\eta_{\theta_F}^5\right) - (1-S)\,\theta_F\left(c\eta_c^5 - \eta^5\right) - c\eta^4 = 0, \tag{15}$$

$$\theta_F\left((\rho_F^0 - \rho)\eta_\rho^4 + (1-\theta_F)\eta_{\theta_F}^4 + \eta^4\right) - c\eta_c^4 = 0. \tag{16}$$

If one assumes that all parameters arising in (7) are arbitrary constants then the result presented in Theorem 1 in a straightforward way is derived. However, here we need to determine all possible values of parameters (under restrictions (8)) when a wider Lie symmetry occurs. Looking at the system of DEs (12)–(15), one notes (see equations (12)) that two different cases, $S \neq 0$ and $S = 0$, should be separately examined.

In the case $S \neq 0$, we immediately obtain $\alpha_4 = 0$, $\eta^4 = 0$, $\eta_\rho^5 = \eta_{\theta_F}^5 = 0$. So, the system of determining equations (12)–(16) is reduced to the form

$$D\eta_{cc}^5 = 0, \; (1-S)\left(c\eta_c^5 - \eta^5\right) = 0.$$

If $S \neq 1$ then one readily finds $\eta^5 = \alpha_5 c$, hence the principal algebra (9) is only obtained. However, assuming $S = 1$, Cases 1 and 3 of Table 2 are obtained when $D \neq 0$ and $D = 0$, respectively.

In the case $S = 0$, an extension of the principal algebra (9) via the operator $x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u$ can be easily identified. If $D \neq 0$ then Case 2 in Table 2 is obtained.

Finally, special values $S = 0$ and $D = 0$ lead to the widest Lie symmetry. In fact, equations (12)–(14) simply disappear and the remaining equations are

$$\theta_F^2\left((\rho_F^0 - \rho)\eta_\rho^5 + (1-\theta_F)\eta_{\theta_F}^5\right) - \theta_F\left(c\eta_c^5 - \eta^5\right) - c\eta^4 = 0, \tag{17}$$

$$\theta_F(\rho_F^0 - \rho)\eta_\rho^4 + \theta_F(1-\theta_F)\eta_{\theta_F}^4 - c\eta_c^4 + \theta_F\eta^4 = 0. \tag{18}$$

Solving equation (18), one finds

$$\eta^4 = (\theta_F - 1)H^1\left(\frac{\theta_F - 1}{\rho_F^0 - \rho}, \frac{c\theta_F}{\rho_F^0 - \rho}\right).$$

Substituting the above function into equation (17), the function

$$\eta^5 = \frac{c}{\theta_F}H^1\left(\frac{\theta_F - 1}{\rho_F^0 - \rho}, \frac{c\theta_F}{\rho_F^0 - \rho}\right) + cH^2\left(\frac{\theta_F - 1}{\rho_F^0 - \rho}, \frac{c\theta_F}{\rho_F^0 - \rho}\right)$$

is obtained. Thus, the Lie symmetry operator (11) with the above functions η^4 and η^5 produce exactly the Lie symmetries

$$cH^2\left(\frac{\theta_F - 1}{\rho_F^0 - \rho}, \frac{c\theta_F}{\rho_F^0 - \rho}\right)\partial_c, \; H^1\left(\frac{\theta_F - 1}{\rho_F^0 - \rho}, \frac{c\theta_F}{\rho_F^0 - \rho}\right)\left((\theta_F - 1)\partial_{\theta_F} + \frac{c}{\theta_F}\partial_c\right),$$

listed in Case 4 of Table 2.

The proof is completed.

Although the cases listed in Table 2 are very special and are questionable from the applicability point of view, new Lie symmetries are very interesting. In particular, system (7) with $S = 0$ admits the

symmetry $x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u$ that can be not predicted using any physical laws. Here we study this case in detail and our goal is to construct a wide range of time-dependent solution.

3. Ansätze, Reductions and Exact Solutions in the Case $S = 0$

As it was noted above, the most interesting cases from the symmetry point of view are the second and the fourth cases (see Table 2) because they involve the Lie symmetry $Y = x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u$ that cannot be predicted by any physical/biological consideration. The fourth case is rather artificial from applicability point of view. Indeed, the restrictions $D = S = 0$ means that both diffusive and convective transports are neglected what is very strong assumption (however, one cannot claim that such a special case is absolutely unrealistic). Thus, we study the second case, $S = 0$ in what follows.

The general technique for constructing exact solution using Lie symmetries is well-known from the theoretical point of view. There are two different approaches described, e.g., in Section 1.3 [16]. One of them is based on construction of optimal systems of inequivalent (nonconjugated) subalgebras of the given Lie algebra of symmetries. Although this technique is very popular, one successfully works only in the case of Lie algebras of low dimensionality (up to 4). In fact, all optimal systems for such algebras were described in a seminal work [24]. However, a pure algebraic problem occurs if one deals with a Lie algebra of the dimensionality 5 and higher. To the best of our knowledge, there is no generalisations of the results presented in [24] on higher-dimensional Lie algebras. Notably, the Lie algebra of symmetries of system (7) is infinity-dimensional (see the operator $g(t)\partial_{p^*}$ in Theorem 1).

The second technique for constructing exact solution via Lie symmetries is based on the most general linear combination of basic operaintors of the Lie algebra in question. Of course, many technical difficulties arise in the case of higher-dimensional Lie algebras, however, these difficulties can be overcome by a careful analysis of the relevant invariant surface condition. Here this is demonstrated in the case of Lie algebra with the basic operators

$$\partial_t, \; \partial_x, \; \partial_u, \; x\partial_u, \; c\partial_c, \; g(t)\partial_{p^*}, \; Y = x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u.$$

Let us consider the most general linear combination of Lie symmetries listed above

$$X = \alpha_1\partial_t + \alpha_2\partial_x + \left(\beta_3\frac{x^2}{2} + \beta_1 x + \beta_0\right)\partial_u + \beta_2 c\partial_c + (\beta_3\lambda^* x + g(t))\partial_{p^*}, \qquad (19)$$

where α_1, α_2 and β_i ($i = 0, 1, 2, 3$) are arbitrary constants, $\alpha_1^2 + \alpha_2^2 \neq 0$ (if $\alpha_1 = \alpha_2 = 0$ then there is no reduction to an ODE system).

Depending on the value of the constants α_1 and α_2, operator (19) produce three inequivalent ansätze for reduction of system (7) to ODE systems. Thus, one need to consider such cases:
I. $\alpha_1\alpha_2 \neq 0 \Rightarrow \alpha_1 = 1, \; \alpha_2 = \alpha \neq 0$;
II. $\alpha_1 \neq 0 \Rightarrow \alpha_1 = 1, \; \alpha_2 = 0$;
III. $\alpha_1 = 0, \; \alpha_2 \neq 0 \Rightarrow \alpha_2 = 1$.

From the very beginning, it can be noted that the function $g(t)$ does not play role in the first two cases and we may set $g(t) = 0$ without losing a generality. In fact, ansätze obtained contain the added term $g^*(t)$ in expressions for the function p^* and this term vanishes after substituting into system (7). So, one always can generalise an arbitrary solution by inserting into the pressure component the additional term $g^*(t)$ that simply follows from the Lie group generated by the operator $g(t)\partial_{p^*}$:

$$(p^*)' = p^* + \varepsilon g(t).$$

In the third case, the function $g(t)$ does play a role.

In the case I operator (19) leads to the ansatz

$$
\begin{aligned}
&u(t,x) = f_1(\omega) + \tfrac{\beta_3}{6\alpha}x^3 + \tfrac{\beta_1}{2\alpha}x^2 + \tfrac{\beta_0}{\alpha}x, \ \omega = x - \alpha t,\\
&\rho(t,x) = f_2(\omega),\\
&p^*(t,x) = f_3(\omega) + \tfrac{\lambda^*\beta_3}{2\alpha}x^2,\\
&\theta_F(t,x) = f_4(\omega),\\
&c(t,x) = f_5(\omega)e^{\tfrac{\beta_2}{\alpha}x},
\end{aligned}
\tag{20}
$$

where $f_i(\omega)$ $(i = 1,\ldots,5)$ are new smooth functions.

Substituting ansatz (20) into PDE system (7), one arrives at the ODE system

$$
\begin{aligned}
&F' = -\tfrac{k}{2\alpha}\left(\phi' + \tfrac{\lambda^*\beta_3}{\alpha}\right),\\
&(1+F)f_2 F' + f_2' F = \tfrac{\lambda^*}{\alpha^2}F' - \tfrac{\phi}{\alpha^2} + \tfrac{\lambda^*\beta_1}{\alpha^3},\\
&(1+F)f_2' = \tfrac{k}{\alpha}(f_2 - \rho_F^0)\left(\phi' + \tfrac{\lambda^*\beta_3}{\alpha}\right),\\
&(1+F)f_4' = \tfrac{k}{\alpha}(f_4 - 1)\left(\phi' + \tfrac{\lambda^*\beta_3}{\alpha}\right),\\
&(1+F)(f_4 f_5)' + f_4 f_5\left(\tfrac{\beta_2}{\alpha}F + 2F'\right) = -\tfrac{D}{\alpha}\left(f_5'' + \tfrac{2\beta_2}{\alpha}f_5' + \left(\tfrac{\beta_2}{\alpha}\right)^2 f_5\right),
\end{aligned}
\tag{21}
$$

where new notations $F = f_1'$ and $\phi = f_3'$ are used.

We are looking for exact solutions of the ODE system (21) in the two essentially different subcases *(i)* $1 + F = 0$ and *(ii)* $1 + F \neq 0$.

Subcase (i). System (21) after the relevant calculations leads to the result

$$
\begin{aligned}
&f_1 = -\omega + C_1,\\
&f_2 = -\tfrac{\lambda^*\beta_3}{2\alpha^3}\omega^2 + \left(\tfrac{C_2}{\alpha^2} - \tfrac{\lambda^*\beta_1}{\alpha^3}\right)\omega + C_4,\\
&f_3 = -\tfrac{\lambda^*\beta_3}{2\alpha}\omega^2 + C_2\omega + C_3,\\
&f_4 = f_4(\omega),\\
&f_5'' + \tfrac{2\beta_2}{\alpha}f_5' + \left(\left(\tfrac{\beta_2}{\alpha}\right)^2 - \tfrac{\beta_2}{D}f_4\right)f_5 = 0,
\end{aligned}
\tag{22}
$$

where C_i are arbitrary constants and constant $D > 0$.

If $\beta_2 = 0$, then the general solution of equation (22) has the form $f_5 = C_5 + C_6\omega$. Thus, taking into account ansatz (20) and the above formulae, the exact solution of the PDE system (7) with $S = 0$

$$
u(t,x) = \tfrac{\beta_3}{6\alpha}x^3 + \tfrac{\beta_1}{2\alpha}x^2 + \left(\tfrac{\beta_0}{\alpha} - 1\right)x + \alpha t + C_1,
\tag{23}
$$

$$
\rho(t,x) = -\tfrac{\lambda^*\beta_3}{2\alpha^3}(x - \alpha t)^2 + \left(\tfrac{C_2}{\alpha^2} - \tfrac{\lambda^*\beta_1}{\alpha^3}\right)(x - \alpha t) + C_4,
\tag{24}
$$

$$
p^*(t,x) = \tfrac{\lambda^*\beta_3}{2}(2x - \alpha t)t + C_2(x - \alpha t) + C_3,
\tag{25}
$$

$$
\theta_F(t,x) = \theta_F(x - \alpha t),
$$

$$
c(t,x) = C_5 + C_6(x - \alpha t)
$$

(here θ_F is an arbitrary smooth function) is obtained.

If $\beta_2 \neq 0$, then the general solution of equation (22) can be found in an explicit form only for some fixed function f_4. Let us solve equation (22) assuming $f_4(\omega) = \theta_0 \omega^n$ (here $\theta_0 > 0$ and n are arbitrary constants). So, one need to construct exact solutions of the equation

$$f_5'' + \frac{2\beta_2}{\alpha} f_5' + \left(\left(\frac{\beta_2}{\alpha} \right)^2 - \frac{\beta_2 \theta_0}{D} \omega^n \right) f_5 = 0. \tag{26}$$

Equation (26) has essentially different general solutions depending on the value of the constants β_2 and n. In the simplest case when $n = 0$ the general solution of equation (26) has the form

$$f_5(\omega) = \begin{cases} C_5 \exp\left[\left(-\frac{\beta_2}{\alpha} + \sqrt{\frac{\theta_0 \beta_2}{D}} \right) \omega \right] + C_6 \exp\left[\left(-\frac{\beta_2}{\alpha} - \sqrt{\frac{\theta_0 \beta_2}{D}} \right) \omega \right], & \text{if } \beta_2 > 0, \\[3mm] \exp\left(-\frac{\beta_2}{\alpha} \omega \right) \left[C_5 \cos\left(\sqrt{-\frac{\theta_0 \beta_2}{D}} \, \omega \right) + C_6 \sin\left(\sqrt{-\frac{\theta_0 \beta_2}{D}} \, \omega \right) \right], & \text{if } \beta_2 < 0. \end{cases} \tag{27}$$

The obtained functions $f_4 = \theta_0$ and f_5 from (27) lead to the components

$$\theta_F(t, x) = \theta_0,$$

$$c(t, x) = \begin{cases} \left[C_5 \exp\left(\sqrt{\frac{\theta_0 \beta_2}{D}} (x - \alpha t) \right) + C_6 \exp\left(-\sqrt{\frac{\theta_0 \beta_2}{D}} (x - \alpha t) \right) \right] e^{\beta_2 t}, & \text{if } \beta_2 > 0, \\[3mm] \left[C_5 \cos\left(\sqrt{-\frac{\theta_0 \beta_2}{D}} (x - \alpha t) \right) + C_6 \cos\left(\sqrt{-\frac{\theta_0 \beta_2}{D}} (x - \alpha t) \right) \right] e^{\beta_2 t}, & \text{if } \beta_2 < 0 \end{cases}$$

of the exact solution of the PDE system (7) with $S = 0$ and $D > 0$, while other components of the corresponding solution are presented in formulae (23)–(25).

Now let us return to equation (26). In the case $n \neq -2$, this equation is reduced to the Bessel equation

$$\tau^2 g''(\tau) + \tau g'(\tau) + \left(\tau^2 - \frac{1}{(2+n)^2} \right) g(\tau) = 0,$$

if $\beta_2 < 0$, and to the modified Bessel equation

$$\tau^2 g''(\tau) + \tau g'(\tau) - \left(\tau^2 + \frac{1}{(2+n)^2} \right) g(\tau) = 0$$

if $\beta_2 > 0$, by the substitution

$$g(\tau) = \frac{e^{\frac{\beta_2}{\alpha} \omega}}{\sqrt{\omega}} f_5, \ \tau = \frac{2}{n+2} \sqrt{\frac{|\beta_2| \theta_0}{D}} \, \omega^{\frac{n}{2}+1}.$$

Thus, the general solution of equation (26) with $n \neq -2$ can be expressed in the following forms

$$f_5(\omega) = \sqrt{\omega} e^{-\frac{\beta_2}{\alpha} \omega} \left(C_5 J_{\frac{1}{n+2}} \left[\frac{2}{n+2} \sqrt{-\frac{\beta_2 \theta_0}{D}} \, \omega^{\frac{n}{2}+1} \right] + C_6 Y_{\frac{1}{n+2}} \left[\frac{2}{n+2} \sqrt{-\frac{\beta_2 \theta_0}{D}} \, \omega^{\frac{n}{2}+1} \right] \right), \tag{28}$$

if $\beta_2 < 0$, and

$$f_5(\omega) = \sqrt{\omega} e^{-\frac{\beta_2}{\alpha} \omega} \left(C_5 I_{\frac{1}{n+2}} \left[\frac{2}{n+2} \sqrt{\frac{\beta_2 \theta_0}{D}} \, \omega^{\frac{n}{2}+1} \right] + C_6 K_{\frac{1}{n+2}} \left[\frac{2}{n+2} \sqrt{\frac{\beta_2 \theta_0}{D}} \, \omega^{\frac{n}{2}+1} \right] \right), \tag{29}$$

if $\beta_2 > 0$. Here J and Y are the Bessel functions, while I and K are the modified Bessel functions.

Note that solution (27) can be derived from (28)–(29) by setting $n = 0$ in the last ones.

In the case $n = -2$, the general solution of equation (26) has the form

$$
f_5(\omega) = \begin{cases}
\sqrt{\omega}\, e^{-\frac{\beta_2}{\alpha}\omega}\left(C_5 \omega^{\frac{\sqrt{4\beta_2\theta_0 + D}}{2\sqrt{D}}} + C_6 \omega^{-\frac{\sqrt{4\beta_2\theta_0 + D}}{2\sqrt{D}}} \right), & \text{if} \quad 4\beta_2\theta_0 + D > 0, \\[4mm]
\sqrt{\omega}\, e^{-\frac{\beta_2}{\alpha}\omega}(C_5 + C_6 \ln \omega), & \text{if} \quad 4\beta_2\theta_0 + D = 0.
\end{cases}
\tag{30}
$$

Note that in the case $n = -2$ and $4\beta_2\theta_0 + D < 0$ equation (26) has only complex solutions, not real ones.

The functions f_5 (from (28)–(30)) and $f_4 = \theta_0 \omega^n$ produce the components

$$
\theta_F(t, x) = \theta_0 (x - \alpha t)^n, \ n \neq -2,
$$

$$
c(t, x) = \begin{cases}
\sqrt{x - \alpha t}\left(C_5 I_{\frac{1}{n+2}}[\phi] + C_6 K_{\frac{1}{n+2}}[\phi] \right)e^{\beta_2 t}, \ \phi = \frac{2\sqrt{|\beta_2|\theta_0}}{(n+2)\sqrt{D}}(x - \alpha t)^{\frac{n}{2}+1}, & \text{if} \quad \beta_2 > 0, \\[4mm]
\sqrt{x - \alpha t}\left(C_5 J_{\frac{1}{n+2}}[\phi] + C_6 Y_{\frac{1}{n+2}}[\phi] \right)e^{\beta_2 t}, & \text{if} \quad \beta_2 < 0
\end{cases}
$$

and

$$
\theta_F(t, x) = \theta_0 (x - \alpha t)^{-2},
$$

$$
c(t, x) = \begin{cases}
\sqrt{x - \alpha t}\left(C_5 (x - \alpha t)^{\frac{\sqrt{4\beta_2\theta_0 + D}}{2\sqrt{D}}} + C_6 (x - \alpha t)^{-\frac{\sqrt{4\beta_2\theta_0 + D}}{2\sqrt{D}}} \right) e^{\beta_2 t}, & \text{if} \quad 4\beta_2\theta_0 + D > 0, \\[4mm]
\sqrt{x - \alpha t}(C_5 + C_6 \ln(x - \alpha t))\, e^{\beta_2 t}, & \text{if} \quad 4\beta_2\theta_0 + D = 0
\end{cases}
$$

of the solutions of the PDE system (7) with $S = 0$. The remaining components of the corresponding solutions are still given by formulae (23)–(25).

Subcase (ii). In this case, the functions f_1, f_2, f_3 and f_4 can be expressed via the function F, namely:

$$
\begin{aligned}
&f_1(\omega) = \int F(\omega)d\omega, \\
&f_2(\omega) = C_3(1 + F(\omega))^{-2} + \rho_F^0, \\
&f_3(\omega) = -\frac{\lambda^* \beta_3}{2\alpha}\omega^2 + C_1 \omega + C_2 - \frac{2\alpha}{k}\int F(\omega)d\omega, \\
&f_4(\omega) = C_4(1 + F(\omega))^{-2} + 1,
\end{aligned}
\tag{31}
$$

where the function F is a solution of the equation

$$
\left(\frac{C_3(1 + F^2)}{(1 + F)^3} + \rho_F^0(1 + F) - \frac{\lambda^*}{\alpha^2} \right)F' - \frac{2}{\alpha k}F - \frac{\lambda^* \beta_3}{\alpha^3}\omega + \frac{C_1}{\alpha^2} - \frac{\lambda^* \beta_1}{\alpha^3} = 0,
\tag{32}
$$

while the function f_5 must be founded from the equation

$$
\begin{aligned}
&(1 + F)\left(\left(\frac{C_4}{(1+F)^2} + 1 \right)f_5 \right)' + \left(\frac{C_4}{(1+F)^2} + 1 \right)\left(\frac{\beta_2}{\alpha}F + 2F' \right)f_5 \\
&= -\frac{D}{\alpha}\left(f_5'' + \frac{2\beta_2}{\alpha}f_5' + \left(\frac{\beta_2}{\alpha} \right)^2 f_5 \right).
\end{aligned}
\tag{33}
$$

ODE (32) is a nonlinear Abel type equations and its full integration is questionable, moreover, even particular solutions are unknown. However, assuming that the function F is linear, one can find the exact solution

$$
F(\omega) = -1 + C_0 \omega,
\tag{34}
$$

provided the following restrictions are satisfied

$$C_3 = 0, \ C_0 = \frac{2\alpha^2 + k\alpha C_1 - k\lambda^* \beta_1}{k\lambda^* \alpha} \neq 0, \ \beta_3 = \frac{\alpha^2 C_0 (C_0 \alpha k \rho_F^0 - 2)}{k\lambda^*}.$$

Another possibility to integrate ODE (32) arises if one sets

$$C_3 = 0, \ C_1 = \frac{\lambda^* \beta_1}{\alpha}, \ \beta_3 = 0. \tag{35}$$

In this case ODE (32) takes the form

$$\left(\rho_F^0 (1 + F) - \frac{\lambda^*}{\alpha^2} \right) F' - \frac{2}{k\alpha} F = 0.$$

The latter is integrable via the Lambert function:

$$F(\omega) = \exp\left(C(\omega) - \text{LambertW}\left(\frac{\rho_F^\theta \alpha^2}{\rho_F^\theta \alpha^2 - \lambda^*} \, e^{C(\omega)} \right) \right), \tag{36}$$

where

$$C(\omega) = \frac{2\alpha(\omega + C_5)}{k(\rho_F^\theta \alpha^2 - \lambda^*)}.$$

The exact solution (36) can also be presented in the implicit form

$$(F(\omega))^{\left(1 - \frac{\lambda^*}{\alpha^2 \rho_F^0} \right)} = \exp\left(\frac{2(\omega + C_5)}{\alpha k \rho_F^0} - F(\omega) \right).$$

It can be easily seen that a special case occurs when $\frac{\lambda^*}{\alpha^2 \rho_F^0} = 1$. In this case, one again arrives at the linear function

$$F(\omega) = \frac{2\alpha}{k\lambda^*} \, \omega + C_5. \tag{37}$$

Each of the exact solutions (34)–(37) can be used for finding the function f_5 by solving the linear ODE (33). Because the formulae obtained are very cumbersome (in particular, the Heun functions are obtained if one applies (37)), here we present only the details concerning solution (34). In this case, ODE (33) can be rewritten in the form

$$D f_5'' + \left(C_0 \alpha \omega + \frac{C_4 \alpha}{C_0 \omega} + \frac{2D\beta_2}{\alpha} \right) f_5' + \left(C_0 \beta_2 \omega - \frac{C_4 \beta_2}{C_0^2 \omega^2} + \frac{C_4 \beta_2}{C_0 \omega} + 2C_0 \alpha - \beta_2 + \frac{D\beta_2^2}{\alpha^2} \right) f_5 = 0. \tag{38}$$

Although this equation is very awkward, one may find the substitution

$$g(\tau) = \exp\left(\frac{C_0 \alpha}{4D} \, \omega^2 + \frac{\beta_2}{\alpha} \, \omega \right) \omega^{\frac{DC_0 + C_4 \alpha}{2DC_0}} f_5, \ \tau = \frac{C_0 \alpha}{2D} \, \omega^2, \tag{39}$$

transforming ODE (38) into the well-known Whittaker equation (see, e.g., [25])

$$g'' + \left(-\frac{1}{4} + \frac{\mu}{\tau} + \frac{\frac{1}{4} - \nu^2}{\tau^2} \right) g = 0, \tag{40}$$

where

$$\mu = \frac{3C_0 D\alpha - C_4 \alpha^2 - 2D\beta_2}{4C_0 D\alpha}, \ \nu = \frac{\sqrt{C_4^2 \alpha^2 + 2C_4 D(2\beta_2 - C_0 \alpha) + C_0^2 D^2}}{4C_0 D}.$$

The general solution of equation (40) has the form

$$g(\tau) = C_5 M_{\mu,\nu}(\tau) + C_6 W_{\mu,\nu}(\tau), \tag{41}$$

where M and W are the Whittaker functions. In the case $\mu = \frac{1}{2} + \nu$ (that leads to the conditions $\beta_2 = 0$ or $\beta_2 = C_0\alpha$), the function M takes the form $M_{\frac{1}{2}+\nu,\nu}(\tau) = e^{-\frac{\tau}{2}}\tau^{\frac{1}{2}+\nu}$. Thus, the general solution of the Whittaker equation in this special case has the form

$$g(\tau) = e^{-\frac{\tau}{2}}\tau^{\frac{1}{2}+\nu}\left(C_5 + C_6 \int e^{\tau}\tau^{-1-2\nu}d\tau\right). \tag{42}$$

Taking into account (39) and (42), we arrive at the solution of ODE (38) under restrictions $\beta_2 = 0$ and $\beta_2 = C_0\alpha$, namely:

$$f_5(\omega) = \exp\left(-\frac{C_0\alpha}{2D}\omega^2\right)\omega^{1-\frac{C_4\alpha}{DC_0}}\left(C_5 + C_6\int e^{\frac{C_0\alpha\omega^2}{2D}}\omega^{-2+\frac{C_4\alpha}{DC_0}}d\omega\right), \quad \beta_2 = 0; \tag{43}$$

$$f_5(\omega) = \exp\left(-C_0\omega - \frac{C_0\alpha}{2D}\omega^2\right)\omega^{-\frac{C_4\alpha}{DC_0}}\left(C_5 + C_6\int e^{\frac{C_0\alpha\omega^2}{2D}}\omega^{\frac{C_4\alpha}{DC_0}}d\omega\right), \quad \beta_2 = C_0\alpha. \tag{44}$$

Using the obtained functions f_5 and the above formulae, one can construct exact solutions of the PDE system (7). In particular, setting $C_6 = 0$ in (43) for simplicity, an exact solution of the system in question can be written down in the form

$$
\begin{aligned}
u(t,x) &= \frac{\beta_3}{6\alpha}x^3 + \frac{\beta_1}{2\alpha}x^2 + \left(\frac{\beta_0}{\alpha} - 1\right)x + \alpha t + \frac{C_0}{2}(x - \alpha t)^2, \\
\rho(t,x) &= \rho_F^0, \\
p^*(t,x) &= \frac{\lambda^*\beta_3}{2}(2x - \alpha t)t + \left(C_1 - \frac{C_0\alpha}{k}(x - \alpha t)\right)(x - \alpha t) + C_2, \\
\theta_F(t,x) &= \frac{C_4}{C_0^2(x-\alpha t)^2} + 1, \\
c(t,x) &= C_5 \exp\left[-\frac{C_0\alpha}{2D}(x - \alpha t)^2\right](x - \alpha t)^{1-\frac{C_4\alpha}{DC_0}},
\end{aligned}
\tag{45}
$$

where

$$C_0 = \frac{C_1\alpha - \lambda^*\beta_1}{\lambda^*\alpha} \neq 0, \quad \beta_3 = \frac{\alpha^2 C_0(k\alpha\rho_F^0 C_0 - 2)}{k\lambda^*}.$$

Similarly, setting $C_6 = 0$ in (44), another solution of the PDE system (7) with $S = 0$ can be derived in the form

$$c(t,x) = C_5 \exp\left[C_0\alpha t - \frac{C_0\alpha}{2D}(x - \alpha t)^2\right](x - \alpha t)^{-\frac{C_4\alpha}{DC_0}},$$

while other components have the same form as in (45).

In the case *II* operator (19) leads to the ansatz

$$
\begin{aligned}
u(t,x) &= f_1(x) + P_2(x)t, \\
\rho(t,x) &= f_2(x), \\
p^*(t,x) &= f_3(x) + \lambda^*\beta_3 tx, \\
\theta_F(t,x) &= f_4(x), \\
c(t,x) &= f_5(x)e^{\beta_2 t},
\end{aligned}
\tag{46}
$$

where $f_i(x)$ ($i = 1,\ldots,5$) are new smooth functions, while $P_2(x) = \frac{\beta_3}{2}x^2 + \beta_1 x + \beta_0$.

The special case is $P_2(x) = 0$, it means that the operator $x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u$ is not taken into account, hence one may apply ansatz (46) to system (7) with an arbitrary parameter S (see Section 5).

Here we substitute ansatz (20) with $P_2 \neq 0$ into PDE system (7) with $S = 0$ to obtain a reduced ODE system. As a result, one arrives at the system

$$
\begin{aligned}
f_3'' &= \tfrac{2}{k} P_2', \\
f_1'' &= \tfrac{1}{\lambda^*}\left(f_3' + P_2' P_2 f_2\right), \\
f_2' &= \tfrac{k(\rho_F^0 - f_2)}{P_2} f_3'', \\
f_4' &= \tfrac{k(1 - f_4)}{P_2} f_3'', \\
D f_5'' &- P_2(f_4 f_5)' - \left(\beta_2 + 2P_2'\right) f_4 f_5 = 0.
\end{aligned}
\tag{47}
$$

The first four equations of the ODE system (47) can be easily integrated, producing the functions

$$
\begin{aligned}
f_1(x) &= \tfrac{2}{\lambda^* k} \int \left(\int P_2 \, dx\right) dx + \tfrac{C_3}{\lambda^*} \int \ln P_2 \, dx + \tfrac{\rho_F^0}{2\lambda^*} \int P_2{}^2 dx + \tfrac{C_1}{2\lambda^*} x^2 + C_5 x + C_6, \\
f_2(x) &= \tfrac{C_3}{P_2{}^2} + \rho_F^0, \\
f_3(x) &= \tfrac{2}{k} \int P_2 \, dx + C_1 x + C_2, \\
f_4(x) &= \tfrac{C_4}{P_2{}^2} + 1,
\end{aligned}
\tag{48}
$$

while the last equation on the function $f_5(x)$ takes the form

$$
D f_5'' - \left(P_2 + \frac{C_4}{P_2}\right) f_5' - \left(\beta_2 + 2P_2' + \frac{\beta_2 C_4}{P_2{}^2}\right) f_5 = 0.
\tag{49}
$$

It is difficult to construct exact solutions of equation (49) without additional restrictions. Note that equation (49) in the case $C_4 = 0$ is the Heun equation and its general solution can be constructed using software such as Maple. Since this case leads to very cumbersome formulae, we omit those here.

Let us set $\beta_0 = \beta_1 = \beta_2 = C_4 = 0$, $\beta_3 \neq 0$, then equation (49) takes the form

$$
f_5'' - \frac{\beta_3}{2D} x^2 f_5' - \frac{2\beta_3}{D} x f_5 = 0.
$$

The latter is integrable in the terms of modified Bessel functions. Thus, we arrive at the exact solution of the PDE system (7) with $S = 0$:

$$
\begin{aligned}
u(t,x) &= \tfrac{\beta_3}{2} t x^2 + \tfrac{2C_3}{\lambda^*} x \ln x + \tfrac{\beta_3^2 \rho_F^0}{40\lambda^*} x^5 + \tfrac{\beta_3}{12\lambda^* k} x^4 + \tfrac{C_1}{2\lambda^*} x^2 + C_5 x + C_6, \\
\rho(t,x) &= \tfrac{4C_3}{\beta_3^2 x^4} + \rho_F^0, \\
p^*(t,x) &= \lambda^* \beta_3 \, t x + \tfrac{\beta_3}{3k} x^3 + C_1 x + C_2, \\
\theta_F(t,x) &= 1, \\
c(t,x) &= \sqrt{x}\, e^{\beta x^3} \Big[\left(\beta_3 x^3 + 4D\right)\left(C_7 I_{\frac{1}{6}}\left[\beta x^3\right] + C_8 K_{\frac{1}{6}}\left[\beta x^3\right]\right) + \\
&\qquad \beta_3 x^3 \left(C_7 I_{-\frac{5}{6}}\left[\beta x^3\right] - C_8 K_{\frac{5}{6}}\left[\beta x^3\right]\right)\Big], \quad \beta = \tfrac{\beta_3}{12D}.
\end{aligned}
$$

Let us set $\beta_0 = \beta_3 = 0$, $\beta_1 \neq 0$, then equation (49) takes the form

$$
D f_5'' - \left(\beta_1 x + \frac{C_4}{\beta_1 x}\right) f_5' - \left(\beta_2 + 2\beta_1 + \frac{\beta_2 C_4}{\beta_1^2 x^2}\right) f_5 = 0.
\tag{50}
$$

Applying the substitution

$$g(\tau) = e^{-\frac{\beta_1}{4D}\omega^2} \omega^{\frac{D\beta_1 - C_4}{2D\beta_1}} f_5, \ \tau = \frac{\beta_1}{2D}\omega^2$$

to equation (50), we again arrive at the Whittaker equation (40) with the parameters

$$\mu = -\frac{C_4 + D(3\beta_1 + 2\beta_2)}{4D\beta_1}, \ \nu = \frac{\sqrt{C_4^2 + 2C_4 D(\beta_1 + 2\beta_2) + \beta_1^2 D^2}}{4D\beta_1}.$$

Note that we consider only the special case $\mu = \frac{1}{2} + \nu$ that leads to the condition

$$C_4 = -\frac{D(6\beta_1^2 + 5\beta_1\beta_2 + \beta_2^2)}{2\beta_1}.$$

Now making the same calculations as in *subcase (ii)*, we obtain the general solution of equation (50)

$$f_5(x) = x^{-2-\frac{\beta_2}{\beta_1}}\left(C_7 + C_8 \int e^{\frac{\beta_1}{2D}x^2} x^{\frac{(\beta_1 - \beta_2)(2\beta_1 + \beta_2)}{2\beta_1^2}} dx\right). \tag{51}$$

Setting for simplicity $C_8 = 0$ and using formulae (46), (48) and (51), we arrive at the exact solution of the PDE system (7)

$$u(t,x) = \beta_1 tx + \frac{\beta_1(2 + \beta_1 \rho_F^0 k)}{6\lambda^* k}x^3 + \frac{C_1}{2\lambda^*}x^2 + \frac{C_3}{\lambda^*}x\ln x + C_5 x + C_6,$$

$$\rho(t,x) = \frac{C_3}{\beta_1^2 x^2} + \rho_F^0,$$

$$p^*(t,x) = \frac{\beta_1}{k}x^2 + C_1 x + C_2,$$

$$\theta_F(t,x) = 1 - \frac{D(6\beta_1^2 + 5\beta_1\beta_2 + \beta_2^2)}{2\beta_1^3}\frac{1}{x^2},$$

$$c(t,x) = C_7 x^{-2-\frac{\beta_2}{\beta_1}} e^{\beta_2 t}.$$

In the case *III* operator (19) produces the ansatz (we remind the reader that the operator $g(t)\partial_{p^*}$ should be taken into account in this case):

$$\begin{aligned}
u(t,x) &= f_1(t) + \frac{\beta_3}{6}x^3 + \frac{\beta_1}{2}x^2 + \beta_0 x,\\
\rho(t,x) &= f_2(t),\\
p^*(t,x) &= f_3(t) + g(t)x + \frac{\lambda^*\beta_3}{2}x^2,\\
\theta_F(t,x) &= f_4(t),\\
c(t,x) &= f_5(t)e^{\beta_2 x},
\end{aligned} \tag{52}$$

where $f_i(t)$ $(i = 1,\ldots,5)$ are new smooth functions. Substituting ansatz (52) into the PDE system (7) with $S = 0$, we arrive at the condition $\beta_3 = 0$. In this case the corresponding ODE system has the form

$$\begin{aligned}
f_2 f_1'' &= \lambda^*\beta_1 - g(t),\\
f_2' &= 0, \ f_4' = 0,\\
\beta_2 f_4 f_5 f_1' &+ f_4 f_5' - \beta_2^2 D f_5 = 0.
\end{aligned} \tag{53}$$

Solving system (53), we obtain the exact solution of the PDE system (7) with $S = 0$ as follows

$$u(t,x) = \frac{\lambda^* \beta_1}{2\rho_0} t^2 - G(t) + \frac{\beta_1}{2} x^2 + \beta_0 x,$$
$$\rho(t,x) = \rho_0,$$
$$p^*(t,x) = p^*(t) + g(t)x,$$
$$\theta_F(t,x) = \theta_0,$$
$$c(t,x) = C_3 \exp\left(\beta_2 x + \frac{\beta_2^2 D}{\theta_0} t - \frac{\lambda^* \beta_1 \beta_2}{2\rho_0} t^2 + \beta_2 G(t) \right),$$

where $G'' = \frac{g(t)}{\rho_0}$ and $p^*(t)$ are arbitrary smooth functions.

Remark 1. *In order to find exact solutions, the condition $\beta_3 = 0$ was used in several case. It means that the operator $x\partial_{p^*} + \frac{x^2}{2\lambda^*}\partial_u$ is not taken into account, hence one may apply the relevant ansätze to system (7) with a nonzero parameter S. This is done in Section 5.*

4. An Example of the Exact Solution Describing Solute Transport in PEM

In order to show that a given exact solution describes solute transport in PEM, one should check some properties from the very beginning. In particular, the components $\rho(t,x)$, $\theta_F(t,x)$, $p(t,x)$ and $c(t,x)$ must be nonnegative, moreover $\theta_F(t,x) \leq 1$. The displacement function $u(t,x)$ should be either nonnegative (this means that PEM is expanding), or nonpositive (PEM is shrinking). There is also a natural initial condition for the displacement, $u(0,x) = 0$, i.e., no deformation of PEM in the initial time moment $t = 0$. Obviously, only some exact solutions satisfy the above requirements on a given space interval $[0, L]$ and for $t \leq T$, $T > 0$.

Here we look in detail at the exact solution (45). By using the space translation $x \rightarrow x + x_0$ ($x_0 > 0$) and setting $\beta_0 = \alpha$, $\beta_1 = -\alpha C_0 \Rightarrow C_1 = 0$, $C_4 \rightarrow -C_4 C_0^2$, and $\beta_3 = 0 \Rightarrow C_0 = \frac{2}{\alpha k \rho_F^0}$, the solution takes the form

$$u(t,x) = t\left(\alpha + \frac{\alpha t - 2x - 2x_0}{k\rho_F^0} \right),$$

$$\rho(t,x) = \rho_F^0,$$

$$p^*(t,x) = C_2 - \frac{2}{k^2 \rho_F^0}(x + x_0 - \alpha t)^2, \tag{54}$$

$$\theta_F(t,x) = 1 - \frac{C_4}{(x + x_0 - \alpha t)^2},$$

$$c(t,x) = C_5 \exp\left[-\frac{(x + x_0 - \alpha t)^2}{Dk\rho_F^0} \right] (x + x_0 - \alpha t)^{1 + \frac{2C_4}{Dk\rho_F^0}}.$$

Obviously, the initial condition $u(0,x) = 0$ is satisfied. It can easily be shown that all components of the solution (54) are bounded and nonnegative in the domain $\Omega = \{(t,x) \in (0, T) \times (0, L)\}$ and $0 \leq \theta_F(t,x) < 1$ if the following restrictions hold :

$$\alpha > 0, \ \rho_F^0 \geq \frac{2(L + x_0)}{\alpha k}, \ C_2 \geq \frac{2}{k^2 \rho_F^0}(L + x_0)^2,$$

$$0 < C_4 \leq (x_0 - \alpha T)^2, \ x_0 > \alpha T, \ C_5 \geq 0. \tag{55}$$

Using the function p^* from (54) and formula (6), one can derive the hydrostatic pressure of the poroelastic materials by the formula

$$p(t,x) = C_2 - \frac{2}{k^2 \rho_F^0}(x + x_0 - \alpha t)^2 + RT C_5 \exp\left[-\frac{(x + x_0 - \alpha t)^2}{Dk\rho_F^0} \right](x + x_0 - \alpha t)^{1 + \frac{2C_4}{Dk\rho_F^0}}. \tag{56}$$

Plots of the functions u, θ_F, c and p defined in the domain $\Omega = \{(t, x) \in (0, 1) \times (0, 1)\}$ with the parameter restrictions (55) are presented in Figures 1 and 2. The above restrictions (55) guarantee positive values of displacement, i.e. a given layer of PEM is expanding.

In order to get negative values of displacement, i.e. a given layer of PEM is shrinking, one needs the restrictions that are listed below. An example is presented in Figures 3 and 4.

If $\alpha > 0$ then

$$k\rho_F^0 - 2T > 0, \quad \frac{\alpha\left(k\rho_F^0 + T\right)}{2} \le x_0 \le \frac{L(k\rho_F^0 - 2T)}{2T} + \frac{\alpha\left(k\rho_F^0 + T\right)}{2}, \quad C_2 \ge \frac{2}{k^2\rho_F^0}\left(L + x_0\right)^2,$$

$$0 < C_4 \le (x_0 - \alpha T)^2, \quad C_5 \ge 0.$$

If $\alpha < 0$ then

$$0 < x_0 < \frac{\alpha\left(k\rho_F^0 + T\right)}{2} - L + \frac{k\rho_F^0 L}{2T},$$

$$0 < C_4 \le x_0^2, \quad C_2 \ge \frac{2(L + x_0 - \alpha T)^2}{k^2\rho_F^0}.$$

Thus, we have demonstrated that the exact solution (45) with correctly-specified coefficients can describe (at least qualitatively) the solute transport in PEM.
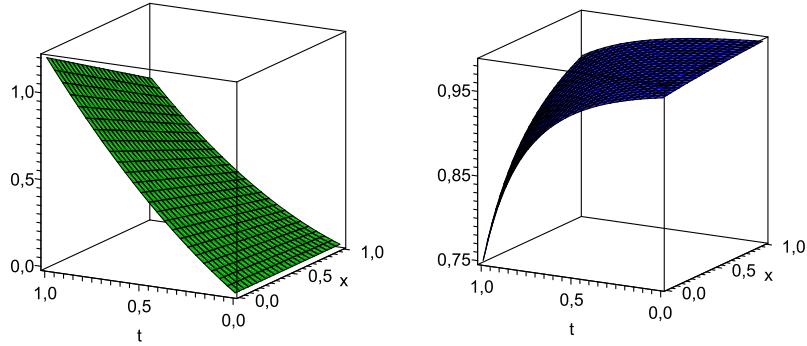

Figure 1. Surfaces representing the components u (left) and θ_F (right) of the exact solution (54) with the parameters $D = 1$, $C_2 = 13$, $C_4 = 1/4$, $C_5 = 3$, $\alpha = 2$, $\rho_F^0 = 10$, $k = 1/2$, $x_0 = 3$.
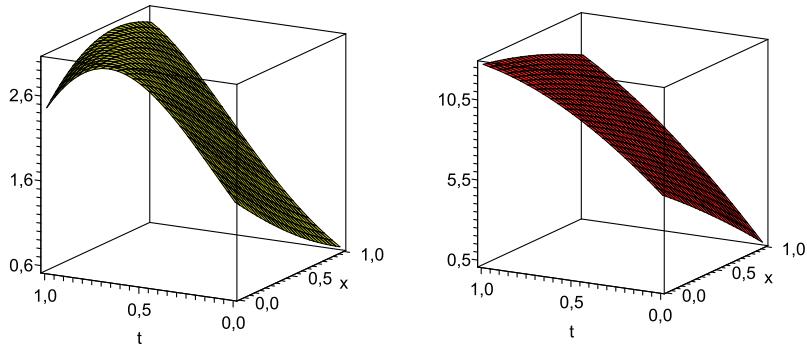

Figure 2. Surfaces representing the components c (left) and p (right) of the exact solution (54) and (56) with the parameters $D = 1$, $C_2 = 13$, $C_4 = 1/4$, $C_5 = 3$, $\alpha = 2$, $\rho_F^0 = 1$, $k = 1/2$, $x_0 = 3$, $RT = 1/5$.
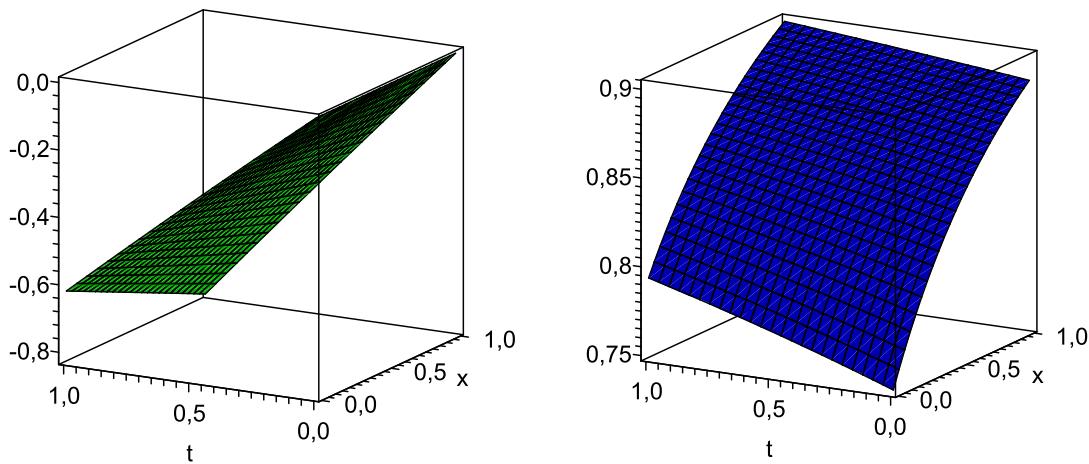
Figure 3. Surfaces representing the components u (left) and θ_F (right) of the exact solution (54) with the parameters $D = 1$, $C_2 = 2$, $C_4 = 1$, $C_5 = 3$, $\alpha = -1/5$, $\rho_F^0 = 10$, $k = 1$, $x_0 = 2$, $RT = 1/5$.
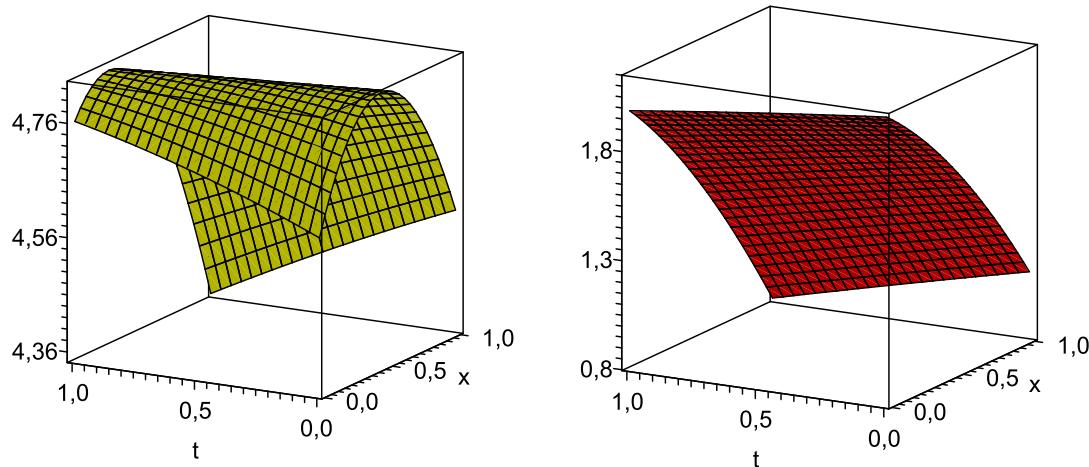

Figure 4. Surfaces representing the components c (left) and p (right) of the exact solution (54) and (56) with the parameters $D = 1$, $C_2 = 2$, $C_4 = 1$, $C_5 = 3$, $\alpha = -1/5$, $\rho_F^0 = 10$, $k = 1$, $x_0 = 2$, $RT = 1/5$.

5. New Exact Solutions of System (7) with $S \neq 0$

In [10], some simplest exact solutions of system (7) with an arbitrary parameter S are constructed. Here we show how new solutions can be derived. As noted above in Remark 1, the condition $\beta_3 = 0$ in the ansätze constructed in Section 3 means that those are applicable for (7) with nonzero S as well.

The simplest case occurs in the case of ansatz (46). Substituting the latter into system (7) with an arbitrary parameter S, we arrive at the ODE system

$$f_3'' = 0,$$
$$\lambda^* f_1'' - f_3' = 0, \tag{57}$$
$$Df_5'' + kSf_3'f_5' - \beta_2 f_4 f_5 = 0.$$

Thus, solving ODE system (57) and taking into account (46) with $P_2 = 0$, we obtain the solution of system (7)

$$u(t, x) = \frac{p_1}{2\lambda^*} x^2 + u_1 x + u_0,$$
$$\rho(t, x) = \rho_0(x),$$
$$p^*(t, x) = p_1 x + p_0,$$
$$\theta_F(t, x) = \theta_0(x),$$
$$c(t, x) = g(x) e^{\beta_2 t},$$

where $\rho_0(x)$ and $\theta_0(x)$ are arbitrary smooth functions, while $g(x)$ is an arbitrary solution of the linear ODE

$$g'' + \frac{k p_1 S}{D} g' - \frac{\beta_2}{D} \theta_0(x) g = 0.$$

If $\beta_2 \neq 0$ then the solutions of the above ODE can be written down in an explicit form provided the function $\theta_0(x)$ is correctly-specified (typical examples are $(x + x_0)^k$ and e^{kx}).

If $\beta_2 = 0$ then

$$c(t, x) = \begin{cases} C_0 + C_1 \exp\left(-\frac{k p_1 S}{D} x\right), & \text{if } S \neq 0, \\[2mm] C_0 + C_1 x, & \text{if } S = 0 \end{cases}$$

(here C_0 and C_1 are arbitrary constants).

Let us construct solutions using restrictions (35). In this case, the function F from (37) can be used to specify the functions f_i, $i = 1, ..., 4$ in (31). Setting $C_5 = -1$ for simplicity and using the restriction $\frac{\lambda^*}{\alpha^2 \rho_F^0} = 1$, we arrive at the formulae

$$f_1(\omega) = \omega\left(\frac{\omega}{\alpha k \rho_F^0} - 1\right),$$

$$f_2(\omega) = \rho_F^0,$$

$$f_3(\omega) = C_2 + \left(\alpha \beta_1 \rho_F^0 + \frac{2\alpha}{k}\right)\omega - \frac{2}{k^2 \rho_F^0} \omega^2,$$

$$f_4(\omega) = \frac{C_4 \alpha^2 k^2 (\rho_F^0)^2}{4\omega^2} + 1,$$

while the equation for finding the function f_5 takes the form

$$D f_5'' + \left(A_0 + A_1 \omega + \frac{A_2}{\omega}\right) f_5' + \left(B_0 + B_1 \omega + \frac{B_2}{\omega} + \frac{B_3}{\omega^2}\right) f_5 = 0. \tag{58}$$

In (58), coefficients A_i and B_i are defined by the formulae

$$A_0 = \frac{2D\beta_2}{\alpha} + \alpha S\left(2 + \beta_1 k \rho_F^0\right), \quad A_1 = \frac{2(1 - 2S)}{k \rho_F^0}, \quad A_2 = \frac{C_4 \alpha^2 k \rho_F^0}{2},$$

$$B_0 = \frac{4(1 - S)}{k \rho_F^0} + \frac{D\beta_2^2}{\alpha^2} + \beta_2\left(2S - 1 + \beta_1 k S \rho_F^0\right), \tag{59}$$

$$B_1 = \frac{2\beta_2(1 - 2S)}{\alpha k \rho_F^0}, \quad B_2 = \frac{C_4 \alpha \beta_2 k \rho_F^0}{2}, \quad B_3 = -\frac{C_4 \alpha^2 \beta_2 k^2 (\rho_F^0)^2}{4}.$$

The general solution of equation (58) can be constructed via the Heun functions. To avoid cumbersome formulae, we consider only some cases in which equation (58) can be reduced to known equations, in particular, the Whittaker equation and the Bessel equation.

In the case of the the additional restrictions

$$B_1 = \frac{A_0 A_1}{2D}, \quad B_2 = \frac{A_0 A_2}{2D}, \tag{60}$$

equation (58) is transformed into the Whittaker equation (40) with the parameters

$$\mu = -\frac{A_0^2 - 4B_0 D + 2A_1(A_2 + D)}{8A_1 D}, \quad \nu = \frac{\sqrt{(A_2 - D)^2 - 4B_3 D}}{4D}$$

by the substitution

$$g(\tau) = \exp\left(\frac{A_1}{4D}\omega^2 + \frac{A_0}{2D}\omega\right)\omega^{\frac{A_2+D}{2D}} f_5, \quad \tau = \frac{A_1}{2D}\omega^2.$$

Taking into account (59), restrictions (60) lead to the condition $\beta_1 = -\frac{2}{k\rho_F^0}$. Thus, using formula (41) and the above substitution, one obtains the function f_5 in an explicit form. Finally, applying ansatz (20) with the above specified beta-s and renaming $C_4 \to -\frac{2C_4}{\alpha^2 k\rho_F^0}$, the following solution of the nonlinear system (7) was constructed:

$$
\begin{aligned}
u(t,x) &= \frac{1}{\alpha k\rho_F^0}(x-\alpha t)^2 - (x-\alpha t) - \frac{1}{\alpha k\rho_F^0}x^2 + \frac{\beta_0}{\alpha}x, \\
\rho(t,x) &= \rho_F^0, \\
p^*(t,x) &= p_0 - \frac{2(x-\alpha t)^2}{k^2\rho_F^0}, \\
\theta_F(t,x) &= 1 - \frac{C_4 k\rho_F^0}{2(x-\alpha t)^2}, \\
c(t,x) &= \exp\left(\beta_2 t + \frac{2S-1}{2Dk\rho_F^0}(x-\alpha t)^2\right)(x-\alpha t)^{-\frac{1}{2}+\frac{C_4}{2D}} \times \\
&\qquad \left(C_5 M_{\mu,\nu}\left[\frac{1-2S}{Dk\rho_F^0}(x-\alpha t)^2\right] + C_6 W_{\mu,\nu}\left[\frac{1-2S}{Dk\rho_F^0}(x-\alpha t)^2\right]\right), \\
\mu &= \frac{C_4(1-2S) - D(2S + \beta_2 k\rho_F^0 - 3)}{4D(1-2S)}, \quad \nu = \frac{\sqrt{(C_4+D)^2 - 2\beta_2 C_4 Dk\rho_F^0}}{4D},
\end{aligned}
\tag{61}
$$

where $\alpha \neq 0$, β_0, β_2, p_0, C_4 and C_5 are arbitrary constants

To construct exact solutions of equation (58) in terms of elementary functions, we consider the special case $\mu = \frac{1}{2} + \nu$, which leads to the conditions

$$\beta_2 = \frac{2}{k\rho_F^0} \text{ or } \beta_2 = \frac{4S(C_4(1-2S) + D)}{Dk\rho_F^0}.$$

Thus, the general solution of equation (58) with $\beta_1 = -\frac{2}{k\rho_F^0}$ has the forms

$$f_5(\omega) = \exp\left(\frac{2S-1}{Dk\rho_F^0}\omega^2 - \frac{2}{\alpha k\rho_F^0}\omega\right)\omega^{\frac{C_4}{D}}\left(C_5 + C_6 \int e^{\frac{(1-2S)\omega^2}{Dk\rho_F^0}}\omega^{-\frac{C_4}{D}}d\omega\right) \tag{62}$$

in the case $\beta_2 = \frac{2}{k\rho_F^0}$, and

$$f_5(\omega) = \exp\left(\frac{2S-1}{Dk\rho_F^0}\omega^2 - \frac{\beta_2}{\alpha}\omega\right)\omega^{\frac{\beta_2 k\rho_F^0}{4S}}\left(C_5 + C_6 \int e^{\frac{(1-2S)\omega^2}{Dk\rho_F^0}}\omega^{-2+\frac{C_4(4S-1)}{D}}d\omega\right) \tag{63}$$

in the case $\beta_2 = \frac{4S(C_4(1-2S)+D)}{Dk\rho_F^0}$.

Thus, the component $c(t, x)$ of the solution (61) of the PDE system (7), corresponding to solution (62) with $C_6 = 0$, has the form

$$c(t, x) = C_5 \exp\left[\frac{2}{k\rho_F^0} t + \frac{2S-1}{Dk\rho_F^0}(x - \alpha t)^2\right](x - \alpha t)^{\frac{C_4}{D}}.$$

Setting $C_6 = 0$ in (63), another solution of the PDE system (7) can be derived in the form

$$c(t, x) = C_5 \exp\left[\frac{4S(C_4(1-2S)+D)}{Dk\rho_F^0} t + \frac{(2S-1)}{Dk\rho_F^0}(x - \alpha t)^2\right](x - \alpha t)^{\frac{C_4(1-2S)+D}{D}},$$

while other components have the same form as in (61).

In the case when the additional restrictions

$$A_0 = A_1 = B_0 = B_1 = 0$$

take place, i.e. (see (59))

$$S = \frac{1}{2}, \ \beta_1 = \pm\frac{2}{\alpha k\rho_F^0}\sqrt{\alpha^2 + \frac{8D}{k\rho_F^0}}, \ \beta_2 = -\frac{\alpha}{2D}\left(\alpha \pm \sqrt{\alpha^2 + \frac{8D}{k\rho_F^0}}\right), \tag{64}$$

equation (58) is transformed into the Bessel equation

$$\tau^2 g''(\tau) + \tau g'(\tau) + \left(\tau^2 - \frac{(A_2 - D)^2 - 4B_3 D}{D^2}\right)g(\tau) = 0,$$

if $B_2 > 0$, and to the modified Bessel equation

$$\tau^2 g''(\tau) + \tau g'(\tau) - \left(\tau^2 + \frac{(A_2 - D)^2 - 4B_3 D}{D^2}\right)g(\tau) = 0$$

if $B_2 < 0$, by the substitution

$$g(\tau) = \omega^{\frac{A_2 - D}{2D}} f_5, \ \tau = 2\sqrt{\frac{|B_2|}{D}}\,\omega.$$

Thus, the general solution of equation (58) under restrictions (64) takes the form

$$f_5(\omega) = \omega^{\frac{1}{2} - \frac{C_4 \alpha^2 k\rho_F^0}{4D}}\left(C_5 J_\nu\left[\sqrt{\frac{2\alpha\beta_2 C_4 k\rho_F^0}{D}}\,\sqrt{\omega}\right] + C_6 Y_\nu\left[\sqrt{\frac{2\alpha\beta_2 C_4 k\rho_F^0}{D}}\,\sqrt{\omega}\right]\right),$$

if $\alpha\beta_2 C_4 > 0$, and

$$f_5(\omega) = \omega^{\frac{1}{2} - \frac{C_4 \alpha^2 k\rho_F^0}{4D}}\left(C_5 I_\nu\left[\sqrt{-\frac{2\alpha\beta_2 C_4 k\rho_F^0}{D}}\,\sqrt{\omega}\right] + C_6 K_\nu\left[\sqrt{-\frac{2\alpha\beta_2 C_4 k\rho_F^0}{D}}\,\sqrt{\omega}\right]\right),$$

if $\alpha\beta_2 C_4 < 0$. Here $\nu = \frac{1}{2D}\sqrt{(\alpha^2 C_4 k\rho_F^0 - 2D)^2 + 4\alpha^2\beta_2 C_4 Dk^2(\rho_F^0)^2}$, J_ν and Y_ν are the Bessel functions, I_ν and K_ν are the modified Bessel functions.

Taking into account the above formulae and renaming $C_4 \to -\frac{2C_4}{\alpha^2 k \rho_F^0}$, the exact solution of the PDE system (7) with $S = \frac{1}{2}$ takes the form

$$u(t,x) = \frac{1}{\alpha k \rho_F^0}(x - \alpha t)^2 - (x - \alpha t) + \left(\frac{\beta}{\lambda^* k} - \frac{1}{\alpha k \rho_F^0}\right)x^2 + \frac{\beta_0}{\alpha}x,$$

$$\rho(t,x) = \rho_F^0,$$

$$p^*(t,x) = p_0 - \frac{1}{2k^2 \rho_F^0}\left(2(x - \alpha t) - \beta k \rho_F^0\right)^2,$$

$$\theta_F(t,x) = 1 - \frac{C_4 k \rho_F^0}{2(x - \alpha t)^2},$$

$$c(t,x) = \begin{cases} (x - \alpha t)^{\frac{C_4 + D}{2D}} e^{-\frac{\beta}{2D}x}\left(C_5 J_\nu\left[\frac{\sqrt{\beta C_4}}{D}\sqrt{x - \alpha t}\right] + C_6 Y_\nu\left[\frac{\sqrt{\beta C_4}}{D}\sqrt{x - \alpha t}\right]\right), & \beta C_4 > 0, \\ (x - \alpha t)^{\frac{C_4 + D}{2D}} e^{-\frac{\beta}{2D}x}\left(C_5 I_\nu\left[\frac{\sqrt{-\beta C_4}}{D}\sqrt{x - \alpha t}\right] + C_6 K_\nu\left[\frac{\sqrt{-\beta C_4}}{D}\sqrt{x - \alpha t}\right]\right), & \beta C_4 < 0, \end{cases}$$

where β_0, p_0, C_4, C_5 and C_6 are arbitrary constants, while

$$\alpha = \pm\sqrt{\frac{\lambda^*}{\rho_F^0}}, \quad \beta = \alpha \pm \sqrt{\alpha^2 + \frac{8D}{k\rho_F^0}}, \quad \nu = \frac{\sqrt{(C_4 + D)^2 + \alpha \beta C_4 k \rho_F^0}}{D}.$$

Finally, using ansatz (52) and making similar calculations to those in the end of Section 3, one obtains the following exact solution of the PDE system (7):

$$u(t,x) = \frac{\lambda^* \beta_1}{2\rho_0}t^2 - G(t) + \frac{\beta_1}{2}x^2 + \beta_0 x,$$
$$\rho(t,x) = \rho_0,$$
$$p^*(t,x) = p^*(t) + \rho_0 G''(t)x,$$
$$\theta_F(t,x) = \theta_0,$$
$$c(t,x) = C_3 \exp\left(\beta_2 x + \frac{\beta_2^2 D}{\theta_0}t - \frac{\lambda^* \beta_1 \beta_2}{2\rho_0}t^2 + \beta_2 G(t) + \frac{k\beta_2 \rho_0 S}{\theta_0}G'(t)\right).$$

6. Conclusions

Here the one-dimensional model for fluid and solute transport in PEM based on the nonlinear PDE system (7) is studied. Although the model was recently derived and some exact solutions, in particular steady-state solutions and their applications, were studied in [10] and [11], special cases occurring when some parameters vanish were not analysed therein. Since the governing equations are nonintegrable in nonstationary case, the Lie symmetry method and modern tools for solving ODE systems are applied in order to construct time-dependent exact solutions. Depending on parameters arising in the governing equations, several special cases with new Lie symmetries are identified. Some of them have a highly nontrivial structure, see Cases 2 and 4 in Table 2, that cannot be predicted from a physical point of view or using Lie symmetries of other real-world models. Applying the symmetries obtained for reduction of the governing PDEs, multiparameter families of exact solutions are constructed, including those in terms of elementary and special functions (hypergeometric, Whittaker, Bessel and modified Bessel functions). A possible application of the solutions obtained is demonstrated in Section 4. It is shown that the exact solution obtained in Section 3 can describe (at least qualitatively) the solute transport in PEM provided relevant parameters are specified correctly. The obtained exact solutions can also be used as test problems for estimating the accuracy of approximate analytical and numerical methods for solving relevant boundary value problems for the nonlinear PDE system (7).

In Section 5, new exact solutions are constructed for the nonlinear PDE system (7) with arbitrary parameters. These solutions essentially generalize the results derived earlier in [10].

This work can be continued in different directions. For example, a natural generalisation of the model has been developed in [26] by taking into account internal sources/sinks. One may expect that the generalised model also admits new Lie symmetries provided some parameters vanish. Another work could be done in order to generalize the model on higher-dimensional cases and study properties of the model derived, in particular, applying the Lie symmetry method.

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