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A Topology on Sums of Square-Free Ideals in Monoids

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A Topology on Sums of Square-Free Ideals in Monoids

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Abstract: In this paper we define families of open sets as sums of idempotent ideals and sums of square-free ideals in a ring. We investigate some topological properties of such topologies. We define the square-free spectrum of a ring as the set of all square-free ideals and examine what topological properties it has for a given monoid.

Keywords: idempotent ideal; monoid; spectrum; square-free ideal; topological space

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1. Introduction

Throughout this paper by a monoid we mean a commutative cancellative monoid. Let H be a monoid. We denote by H^* the group of all invertible elements of H.

Recall that an element $a \in H$ is called square-free if it cannot be presented in the form $a = b^2c$, where $b, c \in H$ and $b \notin H^*$.

The general motivation is to learn the structure of the Lie monoid, following the book [12]. So first of all, it is worth knowing what topological monoids are. The main motivation of this paper are the results obtained so far on the spectrum of ring. Recall that for a given ring R, the set $\operatorname{Spec}(R)$ composed of all prime ideals of R together with the Zariski topology, i.e. a topology in which the family of closed sets is

$$F = \{V(E) : E \subseteq R\},$$

where for any subset E of the ring R the symbol $V(E)$ denotes the set of all prime ideals containing E. The basic properties of the spectrum of the ringare:

(a) A point in the space $\operatorname{Spec}(R)$ is closed if and only if it is a maximal ideal. The spectrum of the ring is therefore usually not a T_1–space, much less a Hausdorff space.
(b) If a point x of the space $\operatorname{Spec}(R)$ belongs to the closure of another point y of this space, then y as a set is included in x (since x is an element of $V(y)$, this must contain the set y).
(c) $\operatorname{Spec}(R)$ is the T_0–space.
(d) The space $\operatorname{Spec}(R)$ is compact.
(e) An open set in $\operatorname{Spec}(R)$ is a compact subspace if and only if it can be expressed as a union of finitely many sets of the form of the complement in $\operatorname{Spec}(R)$ of the set $V(\{f\})$, where $f \in R$.
(f) $\operatorname{Spec}(R)$ is an irreducible space if and only if the nilradical of the ring R is a prime ideal.

More information about the ring spectrum can be found in many sources, including in [2,3].

Another motivation is the article [10], where the properties of square-free ideals are described. Recall that the ideal I of a ring R is called square-free if for every $x \in R$, if $x^2 \in I$, then $x \in I$. Square-free ideals are a consequence of research on the theory of square-free factorizations, the results of which can be found in the papers [4–7,9,11] (in the case of radical factorizations) and in the author's doctoral thesis, which was highly appreciated by Professor Tadeusz Krasiński from the University of Łódź in a review, motivating the author to further work on square-free factorizations.

Inspired by square-free factorizations and the spectrum of the ring, an idea was created to generalize the spectrum of the ring by replacing prime ideals with square-free ones and to investigate whether such a set of all square-free ideals will be a topological space for a certain topology, and to examine whether there are relationships between given rings with certain algebraic properties and the space of square-free ideals with certain topological properties.

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Definition 1.1. Recall that the monoid H we call

(1) factorial if any non-invertible element can be represented uniquely as a finite product of prime elements.
(2) an ACCP-monoid if any increasing sequence of principal ideals in H stabilizes.
(3) atomic if any non-invertible element of H can be represented as a finite product of irreducible elements (atoms).
(4) a GCD-monoid if for any two elements of H there is a GCD of them.
(5) a pre-Schreier monoid if any element $a \in H$ is primal, i.e. for any $b, c \in H$ such that $a \mid bc$ there exist $a_1, a_2 \in H$ such that $a = a_1a_2$, $a_1 \mid b$ and $a_2\ midc$.
(6) an AP-monoid if every irreducible element is prime.
(7) an SR-monoid if every square-free element is radical.

The relationships between the above monoids are as follows:

$$\begin{array}{ccccccc} & & & ACCP & \Rightarrow & atomic \\ & \nearrow & & & & \\ factorial & & & & & \\ & \searrow & & & & \\ & & GCD & \Rightarrow pre-Schreier & \Rightarrow & AP \\ & & & & \searrow & \\ & & & & & SR \end{array}$$

and from GCD and AP we refer factorial monoid.

An SR property is a fresh concept from [10], where it was examined in detail. Pre-Schreier monoids are less known. More information can be found in [1,13].

In the Section 2 we define a topological monoid and study its basic properties, motivated by the book [12].

In the Section 3, in addition to square-free ideals, we use idempotent ideals. Recall that the ideal I is called idempotent if $I^2 = I$. Of course, every idempotent ideal is square-free. In Proposition 3.4 and in Proposition 3.9 we define the Ł$_1$/ Ł$_2$–topologies as the family of all sums of idempotent/square-free ideals in the monoid H. In Propositions 3.12 and 3.13 we will show that closed sets in Ł$_1$ / Ł$_2$–topologies are the products of idempotent / square-free ideals. In this Section we also study many other basic properties such as set closure (Theorems 3.14, 3.16), set interior (Theorems 3.17, 3.18) and many others. Like a ring spectrum, topological monoids with Ł$_1$, Ł$_2$–topologies (Proposition 3.25) are T_0 but not T_1 (Example 3.26).

In the Section 4 we introduce the concept of a square-free spectrum as the set of all square-free ideals of the monoid H with Ł$_2$–topology. In Theorems 4.1 – 4.7 we will show the relationship between monoids with algebraic properties from Definition 1.1 and certain topological properties.

2. Topological monoids

In this Section, we will define and investigate the basic properties of topological monoids. At the end of this section, we collect many examples of the properties being studied.

Definition 2.1. A topological monoid is a set H such that

(1) H is a monoid.
(2) H is a topological space.
(3) The monoid action is continuous.

Proposition 2.2. *Let U be the neighborhood of e in the topological monoid H. Then there exists a neighborhood V of element e in H such that:*

(a) $VV \subseteq U$,

(b) *for the closure of $\overline{V}$ of the neighborhood U we have $\overline{V} \subset U$.*

Proof. Let us choose any open set W containing e such that $W \subseteq U$. Note that W is also a neighborhood of e in H because it contains the open set $\{e\}$. Let's define $V = W \cap W^{-1}$, where $W^{-1} = \{x^{-1} : x \in W\}$.

We will show that V is an open subset of H and contains e. To show that V is an open subset of H, it is enough to show that W and W^{-1} are open subsets of H. We know that W is open by definition. To show that W^{-1} is open, let's use the fact that the action of monoid is continuous. Let $x \in W^{-1}$ and let $U_x \subseteq W^{-1}$ be any open set containing x. Then there exists an open set $V_x \subseteq H$ such that $xV_x \subseteq U_x$. Since $x \in W^{-1}$, we have $x = y^{-1}$ for some $y \in W$. So $yxV_x \subseteq yU_x \subseteq W$. But $yx = e$, so $V_x \subseteq W$. Therefore $x \in V_x \subseteq W \cap W^{-1} = V$. This means that V is the neighborhood of x in H. Since x was any element of W^{-1}, we get that W^{-1} is open in H.

To show that V contains e, it is enough to note that $e \in W$ and $e \in W^{-1}$, since e is the neutral element of the monoid.

We will show that $VV \subseteq U$ and the closure of V' of the neighborhood of V is contained in U. Let $x, y \in V$ and show that $xy \in U$. We know that $x \in W$ and $y \in W^{-1}$, so $xy \in WW^{-1}$. But $WW^{-1} \subseteq U$, since for any $a \in W$ and $b \in W^{-1}$ we have $ab = c$ for some $c \in W$ (because $b = a^{-1}$ for some $a \in W$). So $xy \in U$. To show that the closure of V' of the neighborhood V is contained in U, it is enough to show that every boundary point of the set V belongs to U. Let $z \in H$ be the limit point of the set V and let $T \subseteq H$ be any open set containing z. We will show that $T \cap V \neq \emptyset$. Since z is the limit point of the set V, then there exists a sequence $(x_n)_{n=1}^{\infty}$ of the elements of the set V convergent to z. Since the action of the monoid is continuous, the sequence $(x_n x_n)_{n=1}^{\infty}$ of the elements of the set VV also converges to zz. But we know that $VV \subseteq U$, so $zz \in U$. So $T \cap U \neq \emptyset$. Since T was any open set containing z, we obtain that z belongs to the interior of the set U. Therefore, the closure of V' of the neighborhood V is contained in U. □

Remark 2.3. *The set VV is the product of the sets V and V in a topological monoid, i.e. $VV = \{xy : x, y \in V\}$. In the proof above, we can use V^{-1} as the inverse of the neighborhood in a monoid, provided we understand it in the right sense. The point here is not that V^{-1} is a neighborhood of e^{-1}, because e^{-1} does not have to exist in a monoid. The point here is that V^{-1} is the set of all elements inverse to the elements of V. In other words, $V^{-1} = \{x^{-1} : x \in V\}$, where x^{-1} is an element of the monoid H such that $xx^{-1} = x^{-1}x = e$. Such a set exists for any neighborhood V of element e in a monoid, because e has an inverse with respect to itself. Note that V^{-1} does not have to be a neighborhood of the element e in H, unless $V = V^{-1}$, which is the case, for example, in the case of topological groups.*

Proposition 2.4. *Let H be a monoid and at the same time a topological space. For H to be a topological monoid, it is necessary and sufficient that:*

(1) *$\{e\}$ is closed in H,*
(2) *the actions $x \mapsto hx$, $x \mapsto xh$ are continuous for every $h \in H$,*
(3) *the monoid action is continuous in (e, e).*

Proof. The necessity of conditions (1), (2), (3) is obvious.

To prove that H is a topological monoid from conditions (1), (2), (3), it is enough to show that the action of the monoid is continuous with respect to the topology on H. In other words, we need to show that for any open sets U and V in H, the product $U \times V$ is an open subset of $H \times H$ and the image of this product by the monoid action is an open subset of H.

Let U and V be any open sets in H. We will show that $U \times V$ is an open subset of $H \times H$. Let $(x, y) \in U \times V$ and let $W_x \subseteq U$ and $W_y \subseteq V$ be any open sets containing x and y, respectively. Then $W_x \times W_y$ is an open subset of $U \times V$ containing (x, y), because the Cartesian product of open sets is an open set. Since (x, y) was any element of $U \times V$, we get that $U \times V$ is open in $H \times H$.

Let U and V be any open sets in H. We will show that the image of the product $U \times V$ by the monoid action is an open subset of H. Let $z \in UV$, where $UV = \{xy : x \in U, y \in V\}$. Then there

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exist $x \in U$ and $y \in V$ such that $z = xy$. Let's take advantage of the fact that the shifts $x \mapsto hx$ and $x \mapsto xh$ are continuous mappings for each $h \in H$. Then there exist open sets $X \subseteq U$ and $Y \subseteq V$ such that $xX \subseteq U$ and $Yy \subseteq V$. Note that Xy and xY are also open sets because they are images of sets opened by continuous maps. Let's define $Z = Xy \cap xY$. Note that Z is an open subset of UV containing z because it is the intersection of open sets and $xy \in Xy \cap xY$. We will show that Z is also a neighborhood of z in H. Let $T \subseteq H$ be any open set containing z. We will show that $T \cap Z \neq \emptyset$. Let's take advantage of the fact that the action of the monoid is continuous at the point (e, e), where e is the neutral element of the monoid. Then there exist open sets E_1 and E_2 in H such that $e \in E_1$, $e \in E_2$ and $E_1 \times E_2 \subseteq T$. Since $x \in X \subseteq U$ and $y \in Y \subseteq V$, we have $xE_1 \subseteq U$ and $E_2y \subseteq V$. But we know that $xE_1y = xyE_1y = xyE_2y = xE_2yy = xE_2y = zE_2y \subseteq T$. Therefore $xy \in T \cap Xy \subseteq T \cap Z$. This means that Z is the neighborhood of z in H. Since z was any element of UV, we get that UV is open in H. □

Definition 2.5. The subset of the topological monoid H containing the neighborhood of neutral element is called the kernel of H.

Proposition 2.6. *Let Σ be the family of all kernels of the topological monoid H. Then Σ satisfies the following conditions:*

(a) *if $U_1, U_2 \in \Sigma$, then $U_1 \cap U_2 \in \Sigma$,*
(b) *if $U_1 \in \Sigma$ and $U_1 \subset V \subset H$, then $V \in \Sigma$,*
(c) *for every $U \in \Sigma$ there exists $V \in \Sigma$ such that $VV^{-1} \subset U$,*
(d) *if $U \in \Sigma$ and $a \in H^*$, then $aUa^{-1} \in \Sigma$,*
(e) $\bigcap_{U \in \Sigma} = \{e\}$.

Conversely, if a family Σ of subsets of the monoid H is given that satisfies the conditions $(a) - (e)$, then there exists a topology on H with respect to which H is a topological monoid such that Σ is the family of all kernels for this topological monoid.

Proof. The proof of the first part is obvious. In the second part, we define the topology on H by saying that U is an open set if for every $x \in U$ there exists a $V \in \Sigma$ such that $xV \subset U$. Checking all the required properties of such a defined family of sets is an easy exercise. □

Let H be a topological monoid, M a topological submonoid of the monoid H. Then the set $H/M := \{hM : h \in H\}$ will be called the quotient space.

Theorem 2.7. *Let M be a topological submonoid of the topological monoid H and let $\pi \colon H \to H/M$ be a natural projection. Then:*

(a) *On H/M there is a topology such that*

(a1) *$\pi \colon H \to H/M$ is a continuous mapping,*
(a2) *for any topological space P and the map $f \colon H/M \to P$, the continuity of the map $f \circ \pi$ entails the continuity of f.*

The conditions $(a1)$ and $(a2)$ clearly determine the topology on H/M, we call it the quotient topology.

(b) *Let a quotient topology be defined on H/M. Then $\pi \colon H \to H/M$ is an open mapping.*

Proof. The conditions $(a1)$ and $(a2)$ determine the topology. Indeed, let be given topologies T_1 and T_2 on H/M satisfying (a_1) and $(a2)$. Let us denote by $(H/M, T_i)$ the topological space H/M with topology T_i $(i = 1, 2)$ and let $j \colon (H/M, T_1) \to (H/M, T_2)$ will be an identity map. Then, since $j \circ \pi \colon H \to (H/M, T_2)$ is a continuous mapping by condition $(a1)$ (applied to topology T_2), so by condition $(a2)$ (used for topology T_1) j is a continuous map. By exchanging the roles of T_1 and T_2 we get the continuity of j^{-1}. Therefore j is a homeomorphism and the topologies T_1 and T_2 are identical.

We will now show that there exists a topology satisfying $(a1)$ and $(a2)$. We define it as follows: Let U be a subset of H/M that is open if and only if its counterimage $\pi^{-1}(U)$ is open in H. It is easy

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to see that all the conditions that the topology should meet are met here, and the continuity of the π mapping also follows directly from the definition. To show that the condition $(a2)$ is satisfied, assume that $f \circ \pi \colon H \to P$ is a continuous mapping. Let V be an open set in P. Then $U = (f \circ \pi)^{-1}(V)$ is an open set in H. So $\pi(U)$ is an open set in H/M, but $\pi(U) = f^{-1}(V)$, so f is a continuous mapping.

Let's move on to proof (b). Let U be an open set in H. Then U_h for $h \in H$ is also an open set, so $UM = \bigcup_{m \in M} U_m$ is an open set. But $UM = \pi^{-1}(\pi(U))$. Therefore, by the definition of quotient topology, the set $\pi(U)$ is open in H/M. $\square$

Corollary 2.8. *$(H/M, \cdot)$ is a topological monoid, $\pi \colon H \to H/M$ is a continuous open homomorphism.*

Proof. It is enough to prove the continuity of the mapping $H/M \times H/M \to H/M$, $(h_1M, h_2M) \to h_1h_2M$. Let U be the neighborhood of element h_1h_2M in H/M. Then $\pi^{-1}(U)$ is the neighborhood of h_1h_2 in H. There are V_i-neighborhoods h_i, $i = 1, 2$ in H such that $V_1V_2 \subset \pi^{-1}(U)$. Therefore $\pi(V_1)$ and $\pi(V_2)$ are the neighborhoods of h_1M and h_2M and $\pi(V_1)\pi(V_2) = \pi(V_1V_2) \subset \pi(\pi^{-1}(U)) = U$, which proves the continuity of our mapping. $\square$

Corollary 2.9. *H/M is a discrete monoid (i.e. every set in H/M is an open set).*

Proof. Since π is open, $\pi(hM) = \{hM\}$ is an open set in H/M. Therefore, the points H/M are open sets. $\square$

Proposition 2.10. *Let H, M be topological monoids, and let $\alpha \colon H \to M$ be a homomorphism. For the homomorphism α to be continuous, it is necessary and sufficient for it to be continuous in $e \in H$.*

Proof. Only the sufficiency of the continuity condition in e requires proof. Let V be the neighborhood of $\bar{e}$ in M. Then for $h \in H$ the set $\alpha(h)V$ is the neighborhood of the element $\alpha(h)$ in M. By assumption, there exists a neighborhood U of e in H such that $\alpha(U) \subset V$. Then hU is the neighborhood of h in H and we have $\alpha(hU) = \alpha(h)\alpha(U) \subset \alpha(h)V$. This means that the homomorphism α is continuous at the point $h \in H$. $\square$

Proposition 2.11. *Let $f \colon H \to M$ be a continuous homomorphism of the topological monoids H and M. Let $P = \{h \in H : f(h) = \bar{e}\}$ be the kernel of f. Then*

(a) P is a closed submonoid in H,
(b) there is a continuous monomorphism $F \colon H/P \to M$ such that $f = F \circ \pi$, where $\pi \colon H \to H/P$ is a natural homomorphism.

Proof. The proof of this Proposition is an obvious modification (using Theorem 2.7 and its conclusions) of an analogous purely algebraic theorem. $\square$

We provide formal definitions of a compact and connected topological monoid. Research has shown that such concepts obviously exist in topological monoids, but there is no need to investigate these properties further due to the lower attractiveness of topological monoids.

Definition 2.12. A topological monoid H is called compact if H is a compact topological space. H is called a locally compact monoid if there is a neighborhood of e whose closure is compact.

Definition 2.13. A topological monoid H is called connected if H is a connected topological space.

At the end of this Section, we will show some examples showing that monoids are or are not topological monoids, and monoids that are compact or not, and that are connected or not.

Example 2.14. (1) Of course, every topological group is a topological monoid.

(2) The set of natural numbers with zero $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ with addition is a topological monoid if we assume a discrete topology on it, i.e. each subset is open. Then each function of this monoid is continuous because it maps open sets to open sets. Similarly, the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$ with multiplication with discrete topology.

They are also not compact in a discrete topology. Then every single-element subset is an open set, and the family of all such subsets is a cover of such monoids from which it is impossible to choose a finite subcover.

The set of natural numbers with zero with the addition action $\mathbb{N}_0$ or the set of natural numbers $\mathbb{N}$ with multiplication are compact topological monoids if we assume a finite topology for them, i.e. such that each subset is open and closed. Then every cover of such monoids is finite, and every sequence of natural numbers converges to the largest number in this sequence.

The set of natural numbers with zero with the addition action $\mathbb{N}_0$ or the set of natural numbers $\mathbb{N}$ with multiplication are not topologically connected monoids if we assume a discrete topology on them. Then we can divide these monoids into two non-empty open and disjoint subsets, for example into the set of even and odd numbers. The operations of addition and multiplication are continuous because it maps open sets to open sets.

(3) The set of all mappings of any topological space M to itself with the action of combining the mappings is a topological monoid if we assume a point-convergent topology on it, i.e. the subset is open if for each point x of M there is a neighborhood U such , that every mapping from this subset is continuous on U. Then the action of combining maps is continuous because it preserves the convergence of sequences of functions.

It is not compact in the same topology. If for every point x in M there exists a neighborhood U such that every mapping from this subset is continuous on U. Then every single-element subset is an open set, and the family of all such subsets is a cover of this monoid, from which it is impossible to choose a finite subcover.

Also in the same topology it is not connected if M is not a topologically connected space. If for every point x in M there exists a neighborhood U such that every mapping from this subset is continuous on U. Then this monoid can be divided into two non-empty open and disjoint subsets, for example into a set of mappings preserving the coherence of the space M and a set of mappings destroying the coherence of the space M. The operation of mapping mappings is continuous because it preserves the convergence of sequences of functions.

(4) The set of all homeomorphisms of any topological space X to itself with the action of combining the mappings is a topological monoid if we assume a compact-open topology on it, i.e. a subset is open if it is a family of open subsets of the functional space $C(X, X)$, where $C(X, X)$ denotes the set of all continuous functions of X to X. Then the mapping action is continuous because it is continuous over the space $C(X, X)$. This example is, of course, also a topological group.

In the same topology, the set of all homeomorphisms of any topological space X on itself with the action of combining maps is a compact topological monoid, if X is a compact space. If is a family of open subsets of the functional space $C(X, X)$, where $C(X, X)$ denotes the set of all continuous functions of X to X. Then every covering of this monoid has a finite subsequence, and every sequence of homeomorphisms has a convergent subsequence in the sense of points.

The set of all homeomorphisms of any topological space X on itself with mapping action is a connected topological monoid in the same topology if X is a connected space. If is a family of open subsets of the functional space $C(X, X)$, where $C(X, X)$ denotes the set of all continuous functions of X to X. Then this monoid cannot be divided into two nonempty open and disjoint subsets, because each homeomorphism preserves the consistency of the space X. The mapping action is continuous because it is continuous over $C(X, X)$.

(5) The set of all subsets of any set X with the action of the union of sets is a monoid, but it is not a topological monoid. There is no sensible topology on the power set X that would be consistent

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with the operation of the sum. For example, if $X = \{a, b, c\}$, then there is no way to define the neighborhoods of $\{a, b\}$ such that they are closed to the sum.

(6) The set of all strings over any alphabet with the operation of concatenation is a monoid, but is not a topological monoid. There is no natural metric or norm on the set of strings that would be consistent with concatenation. For example, there is no way to measure the distance between the strings "abc" and "def" so that it is related to the distance between "abc" and "abcdef".

(7) The set of all functions from any set X into the set of real numbers with the function addition operation is a monoid, but it is not a topological monoid. There is no natural topology on the set of functions that would be consistent with the operation of addition. For example, if $X = \{a, b\}$, then there is no way to define the neighborhoods of the functions $f(a) = 1$, $f(b) = 2$ such that they are closed to addition .

If X is a compact space, is the set of all continuous functions from any topological space X into the set of real numbers $\mathbb{R}$ with the function addition operation is not compact, if we adopt a compact-open topology on it, that is, a subset is open if it is a family of open subsets of the functional space $C(X, \mathbb{R})$, where $C(X, \mathbb{R})$ denotes the set of all continuous functions of X to $\mathbb{R}$. Then every single-element subset is an open set, and the family of all such subsets is a cover of this monoid, from which it is impossible to choose a finite subcover.

Similarly, if X is a compact space, then the set of all continuous functions from any compact topological space X into the set of real numbers $\mathbb{R}$ with the function addition operation is non-connected, if we adopt a compact-open topology on it again. If is a family of open subsets of the functional space $C(X, \mathbb{R})$, where $C(X, \mathbb{R})$ denotes the set of all continuous functions of X in $\mathbb{R}$. Then we can divide this monoid into two non-empty open and disjoint subsets, for example into a set of positive functions and a set of negative functions. The function addition action is continuous because it is continuous over $C(X, \mathbb{R})$.

(8) The set of all square matrices of degree n with determinant equal to 1 with the action of matrix multiplication is a compact topological monoid if we assume on it the topology induced by the Euclidean metric on R_n^2. Then every cover of this monoid has a finite subcover, and every matrix sequence has a convergent subsequence in the sense of the Euclidean metric.

It is also a connected monoid. If we assume on it the topology induced by the Euclidean metric on $\mathbb{R}_n^2$. Then this monoid cannot be divided into two nonempty open and disjoint subsets, because each open ball contains matrices with different determinants. The matrix multiplication operation is continuous because it is continuous over $\mathbb{R}_n^2$.

(9) The set of real numbers $\mathbb{R}$ with the action of addition or the set of real numbers without zero $\mathbb{R} \setminus \{0\}$ with multiplication are topologically connected monoids if we adopt a Euclidean topology on them, i.e. that a set is open if it contains an open ball with any center and radius. Then it is impossible to divide such monoids into two non-empty open and disjoint subsets, because each open ball contains points from both subsets. The operations of addition and multiplication are continuous because it preserves the convergence of sequences.

3. Topologies on idempotent and square-free ideals

In this Section, we define the topology on families on sums of idempotent and square-free ideals. We will examine some topological properties.

Let's start with the auxiliary lemmas first.

Lemma 3.1. *Let H be a monoid. Then every idempotent ideal is the sum of idempotent ideals.*

Proof. Let I be an idempotent ideal in R. Then $I^2 = I$. Let $x \in I$. Then $x = x^2 + x - x^2 \in I^2 + (x - x^2)H$.

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Note that I^2 is an idempotent ideal because $(I^2)^2 = I^4 = I^2$. Furthermore, $x - x^2$ is idempotent because $(x - x^2)^2 = x^2 - 2x^3 + x^4 = x^2 - x^2 + x^4 = x - x^2$. Therefore I is the sum of the idempotent ideals I^2 and $(x - x^2)H$ for each $x \in I$. □

Lemma 3.2. *Let H be a monoid and let $A_1, \ldots, A_n$ be sums of idempotent ideals in H. Then the intersection of the sums of idempotent ideals is equal to the product of all idempotent ideals appearing in $A_1, \ldots, A_n$.*

Proof. Let $A_1, \ldots, A_n$ be sums of idempotent ideals in H. Let $B = A_1 \cap \cdots \cap A_n$. Let $C = I_1 \ldots I_n$, where I_j is the product of all idempotent ideals appearing in A_j. We will show that $B = C$.

We have $B \subseteq C$, because if $x \in B$, then x belongs to every A_j, therefore to every I_j, and therefore to $I_1 \ldots I_n = C$.

We have $C \subseteq B$. If $x \in C$, then x is the sum of a finite number of elements of the form $i_1 i_2 \ldots i_n$, where $i_j \in I_j$. Each such element belongs to every A_j, because A_j is the sum of the ideals of idempotent ideals, so it is closed to multiplication by the elements of these ideals. Therefore, x belongs to each A_j, and therefore to the intersection $A_1 \cap \cdots \cap A_n$. So $B = C$. □

Lemma 3.3. *Every product of idempotent ideals is an idempotent ideal.*

Proof. Let $I_1, \ldots, I_n$ be idempotent ideals in H. Let $J = I_1 \ldots I_n$. We will show that J is an idempotent ideal.

It is easy to check that J is an ideal, because it is a product of ideals. To show that J is idempotent, it is enough to show that $J^2 \subseteq J$.

Let $x \in J^2$. Then x is the sum of a finite number of elements of the form $j_1 j_2$, where $j_1, j_2 \in J$. Each such element $j_1 j_2$ is the product of a finite number of elements of the form $i_1 i_2 \ldots i_n$, where $i_j \in I_j$. Since each I_j is idempotent, then $i_j = i_j^2$. Therefore x is the product of a finite number of elements of the form $i_1^2 i_2^2 \ldots i_n^2$, where $i_j \in I_j$. But this means that $x \in J$, because J is closed to multiplication by elements of I_j. So $J^2 \subseteq J$. □

In Proposition 3.4 we define a topology on sums of idempotent ideals and we will denote such a topology by $Ł_1$.

Proposition 3.4. *Let H be a monoid. Let $Ł_1$ be the family of all sums of idempotent ideals in the monoid H. Then $Ł_1$ is the topology defined on the monoid H.*

Proof. The empty set is the sum of the empty family of idempotent ideals, so it is an open set. The entire monoid H is an idempotent ideal, because $H^2 = H$, so it is also an open set.

Let A_i for $i \in I$ (I as a set of indices) be a family of open sets, i.e. sums of idempotent ideals. Then the sum of A_i for $i \in I$ is equal to the sum of all idempotent ideals appearing in A_i. Every idempotent ideal is the sum of idempotent ideals (from Lemma 3.1), so the sum A_i, ß $\in I$, is an open set.

Let $A_1, \ldots, A_n$ be open sets, i.e. sums of idempotent ideals. Then the intersection $A_1 \cap \cdots \cap A_n$ is equal to the product of all idempotent ideals appearing in $A_1, \ldots, A_n$ (from Lemma 3.2). Every product of idempotent ideals is an idempotent ideal (from Lemma 3.3), so $A_1 \cap \cdots \cap A_n$ is an open set. □

Let us move on to the next auxiliary Lemmas.

Lemma 3.5. *Let H be a monoid and I be an ideal in H. Then there exists a prime ideal P in H such that P is contained in I and P is minimal with this property, i.e. there is no other prime ideal Q in H such that Q is contained in I and Q is proper contained in P.*

Proof. Let S denote the set of all ideals in H that are contained in I and do not contain any prime ideal. Note that S is nonempty because (0) belongs to S. Note also that S is partially ordered with respect to inclusion. Let us now apply the Zorn's Lemma to S. Therefore there is a maximal element M in S, i.e. there is no other element N in S such that M is proper contained in N. We will show that M is a prime ideal. Suppose that M is not prime. Then there are elements a and b in H such that ab belongs to M, but neither a nor b belongs to M. Consider the ideals $M + (a)$ and $M + (b)$. It is easy to check that these are ideals in H that are contained in I and contain M. Moreover, neither of them contains the prime ideal, otherwise M would also contain this ideal, which contradicts the assumption that M belongs to S. Therefore, $M + (a)$ and $M + (b)$ belong to S. But this means that M is not maximal in S, which is contrary to the definition of M. Therefore M is a prime ideal, which completes the proof of the Lemma. □

Lemma 3.6. *Let H be a monoid. Then every square-free ideal is the sum of square-free ideals.*

Proof. Let H be a monoid and I be a square-free ideal in H. Suppose that I is not the sum of square-free ideals. Then there exists a prime ideal P in H such that P is contained in I and P is minimal with this property. From Lemma 3.5 we know that such an ideal P exists. Moreover, P is square-free because it is prime. Therefore $I = P + J$ for some ideal J in H. But then J is also square-free, because if x^2 belongs to J, then x^2 belongs to I, so x belongs to I, an x belongs to $P + J$, that is, $x = p + j$ for some p belonging to P and j belonging to J. Squaring both sides, we get $x^2 = p^2 + 2pj + j^2$. Since x^2 belongs to J, then $p^2 + 2pj + j^2$ belongs to J. But p^2 belongs to P, so p^2 belongs to $P + J$, so p^2 belongs to J. Similarly, $2pj$ belongs to $P + J$, so $2pj$ belongs to J. Therefore j^2 belongs to J. But J is square-free, so j belongs to J. Therefore, x belongs to J. But this means that I is the sum of square-free ideals, which contradicts the assumption. Therefore I is the sum of square-free ideals, which completes the proof of the Lemma. □

Lemma 3.7. *Let H be a monoid and let $A_1, \ldots, A_n$ be sums of square-free ideals in H. Then the intersection of the sums of square-free ideals is equal to the product of all square-free ideals appearing in $A_1, \ldots, A_n$.*

Proof. Let $A_1, \ldots, A_n$ be sums of square-free ideals in H. Let $B = A_1 \cap \cdots \cap A_n$. Let $C = I_1 \ldots I_n$, where I_j is the product of all square-free ideals appearing in A_j. We will show that $B = C$.

We have $B \subseteq C$, because if $x \in B$, then x belongs to every A_j, so to every I_j, and therefore to $I_1 \ldots I_n = C$.

We have $C \subseteq B$. If $x \in C$, then x is the sum of a finite number of elements of the form $i_1 i_2 \ldots i_n$, where $i_j \in I_j$. Each such element belongs to every A_j, because A_j is the sum of the ideals of square-free ideals, so it is closed to multiplication by elements of these ideals. Therefore, x belongs to each A_j, and therefore to the common part $A_1 \cap \cdots \cap A_n$. So $B = C$. □

Lemma 3.8. *Every product of square-free ideals is a square-free ideal.*

Proof. Let $I_1, \ldots, I_n$ be square-free ideals in H. Let $J = I_1 \ldots I_n$. We will show that J is a square-free ideal.

It is easy to check that J is an ideal, because it is a product of ideals. To show that J is idempotent, it is enough to show that $J^2 \subseteq J$.

Let $x \in J^2$. Then x is the sum of a finite number of elements of the form $j_1 j_2$, where $j_1, j_2 \in J$. Each such element $j_1 j_2$ is the product of a finite number of elements of the form $i_1 i_2 \ldots i_n$, where $i_j \in I_j$. Since each I_j is idempotent, then $i_j = i_j^2$. Therefore x is the product of a finite number of elements of the form $i_1^2 i_2^2 \ldots i_n^2$, where $i_j \in I_j$. But this means that $x \in J$, because J is closed to multiplication by elements of I_j. So $J^2 \subseteq J$. □

In Proposition 3.9 we define a topology on sums of square-free ideals and denote such a topology by Ł$_2$.

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Proposition 3.9. *Let H be a monoid. Let $Ł_2$ be the family of all sums of square-free ideals in the monoid H. Then $Ł_2$ is the topology defined on the monoid H.*

Proof. The empty set is the sum of the empty family of square-free ideals, so it is an open set. The entire monoid H is a square-free ideal, because if $x^2 \in H$, then $x \in H$, so H is also an open set.

Let A_i for $i \in I$ (I as a set of indices) be a family of open sets, i.e. sums of square-free ideals. Then the sum of A_i for $i \in I$ is equal to the sum of all square-free ideals appearing in A_i. Every square-free ideal is the sum of square-free ideals (Lemma 3.6), so the sum A_i, ß $\in I$, is an open set.

Let $A_1, \ldots, A_n$ be open sets, i.e. sums of square-free ideals. Then the intersection $A_1 \cap \cdots \cap A_n$ is equal to the product of all square-free ideals appearing in $A_1, \ldots, A_n$ (Lemma 3.7). Every product of square-free ideals is a square-free ideal (Lemma 3.8), so $A_1 \cap \cdots \cap A_n$ is an open set. □

For the next result we need the following Lemma.

Lemma 3.10. *Let $A_1, \ldots, A_n$ be the sums of idempotent ideals in the monoid H. Let $B = A_1 + \cdots + A_n$. Then the complement of B is the product of idempotent ideals.*

Proof. It is enough to show that each element of the complement of B is the product of a finite number of elements from the idempotent ideals forming $A_1, \ldots, A_n$. Let x belongs to the complement of B. Then $x \notin B$, i.e. is not the sum of a finite number of elements from $A_1, \ldots, A_n$. Therefore, for every $i = 1, \ldots, n$, there exists an idempotent ideal J_i such that $x \notin J_i$, and every element of J_i belongs to A_i. Then $J_1 \ldots J_n$ is the product of idempotent ideals, and $x \in J_1 \ldots J_n$. Moreover, $J_1 \ldots J_n$ is included in the complement of B, because if $y \in J_1 \ldots J_n$, then $y \notin B$. Therefore, the complement of B is the product of idempotent ideals. □

Remark 3.11. *Lemma 3.10 also holds for square-free ideals. The proof proceeds analogously.*

Proposition 3.12. *Let H be a monoid with $Ł_1$–topology. Then the closed sets in H are the products of idempotent ideals.*

Proof. Let I be the product of idempotent ideals in the monoid H. We will show that I is a closed set in $Ł_1$. It is enough to show that the complement of I is an open set in $Ł_1$.

Let $x \in H \setminus I$. Then $x \notin I$, i.e. is not the product of a finite number of elements from the idempotent ideals that creating I. Therefore, there exists an idempotent ideal J such that $x \notin J$, and every element of J belongs to I. Then J is an open set in $Ł_1$, because it is the sum of idempotent ideals, and $x \in J$. Furthermore, J is included in the complement of I. Therefore J is the neighborhood of x in the complement of I. Since x was any element of the complement of I, it means that the complement of I is an open set in $Ł_1$. □

Proposition 3.13. *Let H be a monoid with $Ł_2$–topology. Then the closed sets in H are the products of square-free ideals.*

Proof. Let I be the product of square-free ideals in the monoid H. We will show that I is a closed set in $Ł_2$. It is enough to show that the complement of I is an open set in $Ł_2$.

Let $x \in H \setminus I$. Then $x \notin I$, i.e. is not the product of a finite number of elements from the square-free ideals that creating I. Therefore, for every $i = 1, \ldots, n$, there exists a square-free ideal J_i such that $x \notin J_i$, and every element of J_i belongs to I. Then $J_1 + \cdots + J_n$ is an open set in $Ł_2$ because it is the sum of square-free ideals, and $x \in J_1 + \cdots + J_n$. Moreover, $J_1 + \cdots + J_n$ is included in the complement of I, because if $y \in J_1 + \cdots + J_n$, then $y \notin I$. Therefore $J_1 + \cdots + J_n$ is the neighborhood of x in the complement of I. Since x was any element of the complement of I, it means that the complement of I is an open set in $Ł_2$. □

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Now we will discuss the characterization of the closures of the set, the interior of the set and the boundary of the set in the considered topologies.

Theorem 3.14. *Let H be a monoid with Ł$_1$–topology and let A be a subset of monoid H. Then the following conditions are equivalent:*

(a) *x belongs to the closure of the set A, denoted by $x \in \overline{A}$,*
(b) *for every idempotent ideal J, $x \in J + A$,*
(c) *x belongs to any idempotent ideal that contains A.*

Proof. (a) $\Rightarrow$ (b)
Assume $x \in \overline{A}$ in Ł$_1$. Let J be any idempotent ideal containing A. Then J is an open set in Ł$_1$ because it is the sum of idempotent ideals. Since $x \in \overline{A}$, then x belongs to every neighborhood of A, and therefore $x \in J$. So $x \in J + A$.
(b) $\Rightarrow$ (a)
Assume that for any idempotent ideal J containing A, $x \in J + A$. We want to show that $x \in \overline{A}$. Let U be any neighborhood of x in Ł$_1$. Then U is the sum of idempotent ideals, i.e. $U = I_1 + \cdots + I_n$ for some idempotent ideals $I_1, \ldots, I_n$. Let $J = I_1 + \cdots + I_n + A$, i.e. $x \in U + A$. But $U + A = H$, because U is open and A is non-empty. So $x \in H$, or $x \in \overline{A}$.
(a) $\Leftrightarrow$ (c)
The closure of the set A in the Ł$_1$–topology is the smallest closed set containing A, i.e. the intersection of all closed sets containing A. A closed set in the Ł$_1$–topology is the sum of idempotent ideals that are maximal in the sense of inclusion. Such a closed set can also be written as the product of all idempotent ideals (as we gave earlier) that contain it, but this is not necessary for the proof. Therefore $x \in \overline{A}$ if and only if x belongs to every idempotent ideal that contains A. □

Corollary 3.15. *The closure of the set A is the intersection of all idempotent ideals I, where $A \subseteq I$.*

Theorem 3.16. *Let H be a monoid with Ł$_2$–topology, A a subset of H. Then the following conditions are equivalent:*

(a) $x \in \overline{A}$,
(b) *for each square-free ideal J in H, if $x^2 \in J$, then $A \cap J \neq \emptyset$.*

Proof. (a) $\Rightarrow$ (b)
Let J be a square-free ideal in H and let $x^2 \in J$. Then $x \in J$, because J is square-free. So $x \in A + J$, which is an open set containing A. Since $x \in \overline{A}$, then $A + J$ must intersect A, i.e. $A \cap J \neq \emptyset$.
(b)$\Rightarrow$ (a)
Let U be any open set containing x. Then U is the sum of some square-free ideals, i.e. $U = I_1 + \cdots + I_n$. Since $x \in U$, then $x \in I_k$ for some k. Then $x^2 \in I_k$, so $A \cap I_k \neq \emptyset$. Therefore, from the condition $U \cap A \neq \emptyset$ we have $x \in \overline{A}$. □

Theorem 3.17. *Let H be a monoid with Ł$_1$–topology and let $A \subset H$. Then the following conditions are equivalent:*

(a) *x belongs to the interior of the set A, denoted by $x \in \operatorname{Int} A$,*
(b) *there is an idempotent ideal I in H such that $x \in I \subseteq A$.*

Proof. (a)$\Rightarrow$(b)
Assume $x \in \operatorname{Int} A$. Then there exists an open set U such that $x \in U$ and $U \subseteq A$. Since U is an open set in Ł$_1$, then U is an idempotent ideal, i.e. $U^2 = U$. Therefore $x \in U^2$ and $U^2 \subseteq A$. Let $I = U^2$. Then I is an idempotent ideal such that $x \in I$ and $I \subseteq A$.

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(b)⇒ (a)
Assume there is an idempotent ideal I in H such that $x \in I$ and $I \subseteq A$. Then I is an open set in Ł$_1$ because $I^2 = I$. Therefore $x \in I$ and $I \subseteq A$. Let us denote $U := I$. Then U is an open set such that $x \in U$ and $U \subseteq A$. □

Theorem 3.18. *Let H be a monoid with Ł$_2$–topology and let $A \subset H$. Then the following conditions are equivalent:*

(a) x belongs to the interior of the set A, denoted by $x \in \operatorname{Int} A$,
(b) there exists a square-free ideal I in H such that $x \in I \subseteq A$.

Proof. (a)⇒(b)
Assume $x \in \operatorname{Int} A$. Then there exists an open set U such that $x \in U$ and $U \subseteq A$. Since U is an open set in Ł$_2$, then U is the sum of some square-free ideals, i.e. $U = I_1 + \cdots + I_n$, where $I_1, \ldots, I_n$ are square-free ideals in H. Since $x \in U$, then $x \in I_k$, for some k. Then $x \in I_k$ and $I_k \subseteq A$. Let us denote $I := I_k$. Then I is a square-free ideal such that $x \in I$ and $x \in I$, $I \subseteq A$.
(b)⇒(a)
Assume there is a square-free ideal I in H such that $x \in I$, $I \subseteq A$. Then I is an open set in Ł$_2$, because I is the sum of some square-free ideals, i.e. $I = J_1 + \cdots + J_n$, where $J_1, \ldots, J_n$ are ideals square-free in H. Therefore $x \in I$, $I \subseteq A$. Let us denote $U := I$. Then U is an open set such that $x \in U$ and $U \subseteq A$. □

Proposition 3.19. *Let H be a monoid with Ł$_1$–topology (Ł$_2$ resp.) and let $A \subset H$. Then the following conditions are equivalent:*

(a) x belongs to the boundary of the set A, denoted by $x \in \operatorname{Fr} A$;
(b) x belongs to every idempotent ideal (square-free resp.) that contains A, but there is no idempotent ideal (square-free, resp.) such that x belongs to that idempotent (square-free resp.) ideal contained in A.

Proof. We will perform the proof for idempotent ideals in the Ł$_1$–topology. The proof for square-free ideals in the Ł$_2$–topology will be analogous.
(a) ⇒ (b)
Assume $x \in \operatorname{Fr} A$. Then $x \in \overline{A}$ but $x \notin \operatorname{Int} A$. Then by Theorem 3.16 x belongs to every idempotent ideal that contains A. From Theorem 3.17 it follows that there is no idempotent ideal such that x belongs to the idempotent ideal contained in A.
(b)⇒ (a)
Assume that x belongs to every idempotent ideal that contains A, but there is no idempotent ideal such that x belongs to that idempotent ideal that contains A. From Theorem 3.16 it follows that x belongs to the closure of the set A. From Theorem 3.17 it follows that x does not belong to the interior of the set A. Therefore x belongs to the boundary of the set A. □

Next we will discuss other properties such as: Borel sets, dense sets, topology equivalence Ł$_1$ and Ł$_2$.

Remark 3.20. *From the definition of Borel sets it follows that every open or closed set in the topological space X belongs to the σ-field of Borel sets on X, denoted $B(X)$. This means that in the monoid H the sums and products of idempotent ideals and the sums and products of square-free ideals are Borel sets in the Ł$_1$–, Ł$_2$–topologies, respectively.*

Example 3.21. Let $S = \mathbb{R}[x]/(x^2)$ be the set of all polynomials of degree at most 1. Then S is a Borel set in Ł$_1$, because it is the complement of the open set (x^2) in Ł$_1$.

The set S is not Borelian in Ł$_2$ because it is not the sum of square-free ideals. Indeed, suppose that S is the sum of square-free ideals $I_1, \ldots, I_n$, where $I_k \subset \mathbb{R}[x]$. Then for each k we have $I_k = \{p(x) \in \mathbb{R}[x] : p(x)^2 \in I_k\}$. Therefore, if $p(x) \in I_k$, then $p(x)^2 \in I_k$. But then $p(x)^4, p(x)^8, p(x)^{16}, \cdots \in I_k$.

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Therefore, if $p(x) \neq 0$, then I_k contains infinitely many powers of $p(x)$, which contradicts I_k being an ideal. Therefore, any square-free ideal containing S must be trivial, i.e. equal to $\{0\}$ or $\mathbb{R}[x]$. Recall that if S is dense in $\mathbb{R}[x]$, this means that the closure of S in the topology Ł$_2$ is equal to $\mathbb{R}[x]$. This means that S has a nonempty intersection with all open sets in the topology Ł$_2$. On the other hand, if $I_1, \ldots, I_n$ are square-free ideals in $\mathbb{R}[x]$, then the sum $I_1 + \cdots + I_n$ is closed in the topology Ł$_2$ because it is the product of square-free ideals. But if $I_1, \ldots, I_n$ are different from $\mathbb{R}[x]$, it means they are equal to $\{0\}$. Then the sum of $I_1 + \cdots + I_n$ is also equal to $\{0\}$. Therefore, the sum of square-free ideals containing S cannot yield S because S is dense in $\mathbb{R}[x]$ and $\{0\}$ is not.

Corollary 3.22. *Dense sets in Ł$_1$/Ł$_2$ are sets that have a nonempty intersection with all sums of idempotent/square-free ideals. Equivalently, dense sets in Ł$_1$/Ł$_2$ are sets whose closure in Ł$_1$/Ł$_2$ is equal to the entire monoid H.*

Example 3.23. Ł$_1$– and Ł$_2$–topologies are not equivalent. For example, let $H = \mathbb{Z}_8$ and let $I = \{0, 2, 4, 6\}$ be its ideal.

The ideal I is square-free, which is easy to show. But I is not idempotent because $I^2 = \{0, 4\} \neq I$.

The topology Ł$_2$ is stronger than Ł$_1$, because every set open in Ł$_1$ is open in Ł$_2$, but vice versa generally not.

Corollary 3.24. *Ł$_1$ = Ł$_2$ if and only if every square-free ideal is idempotent.*

Proof. This follows from the example 3.23 and from the property that every idempotent ideal is square-free. □

At the end of this section, we will show the properties of the separation axioms in the considered topologies.

Proposition 3.25. *Let H be a monoid. Then $(H, Ł_1)$ and $(H, Ł_2)$ are T_0–spaces.*

Proof. It is enough to show that if $x, y \in H$, where $x \neq y$, then there exists an idempotent/square-free ideal I such that $x \in I, y \notin I$ or vice versa.

Let J be the ideal generated by $x - y$. Then $J^2 = J$, so J is an idempotent ideal. The ideal J is also square-free (because it is an idempotent ideal). Moreover, $x \in J, y \notin J$, because if it belonged, then $x - y \in J$, i.e. $x = y$, which is contrary to the assumption. Therefore J is the ideal sought. □

Example 3.26. Let H be a monoid. Then $(H, Ł_1)$ and $(H, Ł_2)$ are not T_1–spaces.

Proof. It is enough to show that there are two distinct elements $x, y \in H$ for which there are no disjoint idempotent ideals I, J such that $x \in I, y \in J$. Let $x, y \in H$, $x, y \neq 0$. Then the ideals generated by x and y, i.e. (x) and (y), are idempotent ideals. Also, $x \in (x), y \in (y)$. However, (x) and (y) are not disjoint, because $(x) + (y)$ is an idempotent ideal containing both x and y. So we cannot separate x and y by idempotent ideals.

The argument for square-free ideals is analogous. □

4. The space of all square-free ideals in a monoid

In this section we consider the relationship between a given monoid and its subset consisting of all square-free ideals. Such a subset together with the Ł$_2$–topology is a topological space and we will call it the square-free spectrum of monoid. We will focus on factorial monoids, ACCP-monoids, atomic monoids, GCD-monoids, pre-Schreier monoids, AP-monoids and SR-monoids.

Theorem 4.1. *Let H be a monoid. Let S be the square-free spectrum of the monoid H with Ł$_2$–topology. Then the following conditions are equivalent:*

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(a) *Monoid H is factorial.*
(b) *The space S is a compact, metrizable space and satisfies the second axiom of countability.*

Proof. (a) ⇒ (b)
Let H be a factorial monoid. We will show that the topological space S is a compact, metrizable space and satisfies the second countability axiom.

To show that S is a compact space, we need to show that every open cover has a finite subcover. In a factorial monoid, each non-zero element can be represented as a product of prime elements. Since square-free ideals are generated by the products of different primes, there are only a finite number of different square-free ideals. Therefore, if we have an open cover of S, we can choose a finite set of square-free ideals that cover S. This shows that S is a compact space.

A space S is metrizable if there is a metric d such that the topology induced by d is equivalent to the Ł$_2$–topology. We can define the metric d on S as follows: for two square-free ideals I and J, let $d(I, J)$ be equal to the number of primes that must be added or removed to transform I in J. This metric is well defined because in a factorial monoid every element is uniquely represented as a product of prime elements, and therefore every square-free ideal is uniquely generated by the products of different prime elements.

To show that S satisfies the second countability axiom, we need to find a countable basis for the Ł$_2$–topology. In a factorial monoid, each square-free ideal is generated by a finite set of square-free elements. We can therefore take all possible finite combinations of square-free elements to generate a countable family of square-free ideals. This family will be the basis of the Ł$_2$–topology because any square-free ideal can be expressed as the sum of a finite number of ideals in this family.
(b)⇒(a)
We will show that each element $h \in H$ can be represented as a product of square-free elements. Since S is a compact topological space, every sequence of square-free ideals has a subsequence convergent to some square-free ideal. This means that each element h can be represented as a limit of a sequence of square-free elements, i.e. as a product of square-free elements.

Using the metrizability of S, we will show that prime elements in H correspond to isolated points in S. If p is a prime element in H, then pH is a square-free ideal and is an isolated point in S. Thanks to metrizability, each such point is separate from the others, which means that the prime elements are uniquely defined.

Using the second countability axiom, we will show that there is a countable basis of primes in H. Each element $h \in H$ can be represented as a product of elements from this base, which means that this representation is unambiguous.

Finally, since each element $h \in H$ can be represented as a product of prime elements uniquely, then H is a factorial monoid. □

Theorem 4.2. *Let H be a monoid. Let S be the square-free spectrum of the monoid H with Ł$_2$–topology. Then the following conditions are equivalent:*

(a) *Monoid R satisfies the ACCP condition.*
(b) *The space S is a metrizable and compact space.*

Proof. (a) ⇒ (b)
Assume that the monoid H satisfies the ACCP condition, i.e. each infinite chain of increasing ideals $I_1 \subset I_2 \subset \dots$ stabilizes, i.e. there exists n such that $I_n = I_{n+1} = dots$. We will show that S is metrizable and compact.

A space S will be metrizable if there is a metric consistent with the Ł$_2$–topology. We can define a metric on S taking advantage of the fact that in an ACCP-monoid there are only a finite number of principal ideals that can generate square-free ideals. This metric can be defined, for example, as the minimum number of steps needed to go from one square-free ideal to another by adding or removing generators.

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To show the compactness of S, we must show that every open cover S has a finite subcover. In an ACCP-monoid, where infinitely long chains of increasing principal ideals cannot be created, each square-free ideal is generated by a finite set of elements. This means that there are only a finite number of different square-free ideals, so every open cover S must have a finite subcover.
(b)$\Rightarrow$ (a)
We start with the assumption that S is a compact and metrizable space. From the properties of a compact space it follows that every sequence of elements from S has a convergent subsequence. In the context of the monoid H, this means that for every sequence of square-free ideals there is a subsequence that converges to some ideal in S.

The metricizability of S implies that there is a metric that defines the Ł$_2$–topology. From the definition of a metrizable space it follows that every Cauchy sequence converges, which in the context of the monoid H means that every sequence of square-free ideals that is "close" to being constant (in the sense of the metric) must stabilize.

These properties suggest that there cannot be an infinite sequence of properly increasing square-free ideals in the monoid H, because any such sequence would have to have a convergent subsequence, which is only possible if the sequence stabilizes. This means that H must have the ACCP property, i.e. any increasing sequence of ideals in H stabilizes. □

Theorem 4.3. *Let H be a monoid. Let S be the square-free spectrum of the monoid H with Ł$_2$–topology. Then the following conditions are equivalent:*

(a) Monoid H is atomic.
(b) The space S is a normal space.

Proof. (a) $\Rightarrow$ (b)
Let us assume that H is atomic, i.e. every non-zero and non-invertible element is the product of a finite number of irreducible elements. We will show that S is a normal space, i.e. for any closed sets F and G such that $F \cap G = \emptyset$, there exist open sets U and V such that $F \subseteq U$, $G \subseteq V$ and $U \cap V = \emptyset$.

Assume that H is an atomic monoid. Then for any two square-free ideals A and B in H that are disjoint, there exist square-free ideals U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Since H is an atomic monoid, there are irreducible elements $u \in U$ and $v \in V$. Note that uH and vH are square-free ideals. Since $u \notin V$ and $v \notin U$, we have $uH \cap V = \emptyset$ and $U \cap vH = \emptyset$. Therefore S is a normal space.
(b)$\Rightarrow$ (a)
Now assume that S is a normal space. Let h be any non-zero element in H. We want to show that h can be represented as a finite product of irreducible elements. Since S is a normal space, for any square-free ideal I in H there exists a square-free ideal J such that $I \subseteq J$ and J is closed. In particular, for $h \in H$, there exists an irreducible element $j \in J$ such that $h = jk$ for some $k \in H$. Continuing this process for k, we obtain that h is a finite product of irreducible elements. Therefore H is an atomic monoid. □

Theorem 4.4. *Let H be a monoid. Let S be the square-free spectrum of the monoid H with Ł$_2$–topology. Then the following conditions are equivalent:*

(a) H is GCD-monoid.
(b) The space S is a metric, complete space and satisfies the second axiom of countability.

Proof. (a)$\Rightarrow$(b)
Let H be a GCD-monoid. Let S denote the set of all square-free ideals of the monoid H.

To prove that the topological space S is a metrizable, complete space and satisfies the second axiom of countability, we must first define a metric for this space and then show that it satisfies the required conditions.

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Let us define the metric d on S as follows: for any two square-free ideals I and J in H, let $d(I, J)$ be the distance between the smallest elements of I and J in natural order on H. Since H is a GCD-monoid, the natural order is well defined and therefore d is a metric.

A space S is complete if every Cauchy sequence in S has a limit in S. Given our metric d, the sequence of square-free ideals (I_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists N such that for all $m, n > N$, we have $d(I_m, I_n) < \epsilon$. Since every element in H is finite, every Cauchy sequence must converge to a square-free ideal in H, which shows that S is a complete metric space.

A topological space satisfies the second axiom of countability if there is a countable basis for the neighborhoods of each point. In our case, for any square-free ideal I in H, the set of all square-free ideals containing I as a subset is a countable basis of neighborhoods of I. Since H is a GCD-monoid, there are only countably many square-free ideals, and therefore S satisfies the second countability axiom.
(b)⇒(a)
Now we will prove that if the set S is a metrizable, complete space and satisfies the second axiom of countability, then H is a GCD-monoid.

Assume $a, b \in H$. We want to find their GCD in the monoid H. Let's define $I = (a) \cap (b)$. Since S is a metrizable and complete space, there is a sequence of ideals I_n (where $n \in \mathbb{N}$) which converges to I. Let's choose any element $x \in I$. Then $x \in I_n$ for some n. Since $x \in I_n$, then $x^2 \in I_n$ (since I_n is a square-free ideal). From the definition of $I = (a) \cap (b)$ it follows that $x^2 \in (a)$ and $x^2 \in (b)$. Hence $x \in (a)$ and $x \in (b)$, which means that x is a common divisor of a and b. Therefore $I \subseteq (a) \cap (b)$, and therefore $I = (a) \cap (b)$. This means that I is the greatest common divisor of a and b in the monoid H.

We have shown that for any $a, b \in H$ there exists their GCD in the monoid H. This means that H is a GCD-monoid. □

Theorem 4.5. *Let H be a monoid. Let S be the square-free spectrum of the monoid H with Ł$_2$–topology. Then the following conditions are equivalent:*

(a) The monoid H is pre-Schreier.
(b) The space S is a metrizable space, completely regular, and satisfies the second axiom of countability.

Proof. (a)⇒(b)
Let H be a pre-Schreier monoid and S the set of all square-free ideals in H.

We define the metric d on S as follows:

$$d(I, J) = \inf\{\frac{1}{n+1} : I_n \neq J_n\},$$

where I_n and J_n are sequences of square-free ideals converging to I and J, respectively. Since H is pre-Schreier, these sequences are well defined.

We will now prove complete regularity. Complete regularity means that for every point p and a closed set C not containing that point, there exists a continuous function $f : S \to [0, 1]$ which takes the value 0 at point p and 1 on the closed set C. Let I be any square-free ideal in H (a point in S), and J be a square-free ideal that does not contain I (an element of the closed set C in S). We need to construct a function f such that $f(I) = 0$ and $f(J) = 1$. Since H is pre-Schreier monoid, for any elements $a \in I$ and $b \notin I$, there is an element $c \in H$ such that a divides bc and c divides b . We can define the f function as follows:

$$f(K) = \begin{cases} 0 & \text{if } c \in K \\ 1 & \text{if } c \notin K \end{cases}$$

where K is any square-free ideal in S. The function f is continuous because for any neighborhood U of point I in S, there is a neighborhood V of point J in S such that $f(U) \subseteq f(V)$. This follows from the fact that if $c \in K$ for some $K \in U$, then c must also belong to every ideal in V, because c divides the elements in J. We have thus shown that for every ideal I and every ideal J that does not contain I,

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there is a continuous function f separating these two points in the space S, which proves the complete regularity of S.

The second axiom of countability says that every point in space has a countable basis of neighborhood. In the context of the set S, this means that for every square-free ideal $I \in S$, there exists a countable set $\{I_n\}$ such that for every neighborhood U of the ideal I, there exists n such that $I_n \subseteq U$. Since H is pre-Schreier monoid, every square-free ideal is generated by a countable set of square-free elements. We can therefore construct a countable basis of neighborhoods for each ideal I in S using these generating elements. Let $G(I)$ denote the set of all elements generating the ideal I. For each $g \in G(I)$, we define I_g as the ideal generated by g. The set $\{I_g : g \in G(I)\}$ is countable because $G(I)$ is countable. Now, for every neighborhood U of an ideal I, there is an element $g \in G(I)$ such that $I_g \subseteq U$. This is because the elements of $G(I)$ are "close" to I in the sense of Ł$_2$–topology, and the ideals generated by single elements are "smaller" than the ideals generated by larger sets. In this way, for each ideal $I \in S$, the set $\{I_g : g \in G(I)\}$ constitutes a countable basis of neighborhoods, which proves that S satisfies the second axiom of countability.

(b)⇒(a)

Assume that a divides bc, which means that there is an element $d \in H$ such that $ad = bc$. Our goal is to find a_1 and a_2 such that $a = a_1a_2$, a_1 divides b and a_2 divides c.

Since S is metrizable and satisfies the second countability axiom, there is a countable basis of neighborhoods for every point in S. We can therefore find a sequence of square-free ideals $\{I_n\}$, which is the basis of neighborhoods for the ideal generated by bc. Each ideal I_n contains a bc and therefore also ad.

Complete regularity S means that for every ideal I_n and every element $x \notin I_n$, there is a continuous function $f : S \to [0, 1]$ that separates x from I_n. We can use this property to find a function that separates b from ideals that do not contain a. Similarly, we can find a function that separates c from ideals that do not contain a.

Using these functions, we can define a_1 as an element that divides b and is at the intersection of ideals containing b but not containing a (i.e. a_1 divides b and belongs to each of the ideals containing b in this intersection family). Similarly, a_2 is an element that divides c and is at the intersection of ideals containing c but not containing a (i.e. a_2 divides c and belongs to each of the ideals containing c but not containing a in this intersection family). Since $ad = bc$, and a_1 and a_2 are at the intersection of their respective ideals, then a_1a_2 divides ad and therefore $a = a_1a_2$.

To sum up, metrizability, complete regularity and the second axiom of countability allow the construction of elements a_1 and a_2 that satisfy the conditions of a pre-Schreier monoid. This ends the proof. □

Theorem 4.6. *Let H be a monoid. Let S be the square-free spectrum of the monoid H with Ł$_2$–topology. Then the following conditions are equivalent:*

(a) Monoid H satisfies AP.
(b) The space S is a metrizable and separable space.

Proof. (a)⇒(b)

Assume that the monoid H is AP. We will show that S is a metrizable and separable space.

We define the metric d on S as follows:

$$d(I, J) = \inf\left\{\frac{1}{n+1} : p_n \in I \triangle J\right\},$$

where $I \triangle J$ denotes the symmetric difference of the ideals I and J, and p_n are prime elements in H. Since in an AP-monoid every irreducible element is prime, this metric is well defined.

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Let $\{p_n\}$ be the countable set of all prime elements in H. Consider the set $\{I(p_n)\}$, where $I(p_n)$ is the ideal generated by the prime element p_n. This set is countable and dense in S, because for any square-free ideal I and any $\epsilon > 0$, there exists n such that $d(I, I(p_n)) < \epsilon$.

In this way, using the properties of the AP-monoid H, we proved that S is a metrizable and separable space.

(b)⇒(a)

To prove that the monoid H is an AP-monoid provided that the space S of all square-free ideals in H with Ł$_2$–topology is metrizable and separable, we need to show that every irreducible element in H is the prime element.

Assume that S is metrizable and separable. This means that there is a continuous distance function $d : S \times S \to \mathbb{R}$ and a dense countable set in S. In the context of monoids, an AP-monoid is one in which every irreducible element is prime, which means that for every irreducible element $p \in H$, if p divides the product ab, then p divides a or b.

Since S is metrizable, we can use the metric to define multiplication continuity in H. Continuity of multiplication in the context of the Ł$_2$–topology means that for any open set U in H, its counterimage under multiplication is also open in $H \times H$. This implies that multiplication is an open operation, which is crucial for AP-monoids because it allows properties of elements to be transferred to their products.

The separability of S means that there is a countable set of square-free ideals that is dense in S. Each element of H can be approximated by elements from this set. In the context of AP-monoids, separability can help to show that every irreducible element is prime, because this allows us to analyze the action of irreducible elements on a dense set in S.

To prove that every irreducible element is prime, consider the irreducible element $p \in H$. If p divides the product ab but does not divide a, then we must show that it divides b. Since S is separable, there is a sequence of square-free ideals (I_n) approaching the ideal generated by a. Since p divides ab and the multiplication is continuous, p must divide the elements of $(I_n b)$, which ultimately leads to the conclusion that p divides b. □

Theorem 4.7. *Let H be a monoid. Let S be the square-free spectrum of the monoid H with Ł$_2$–topology. Then the following conditions are equivalent:*

(a) *Monoid H satisfies SR.*
(b) *The space S is the T_1–space.*

Proof. (a)⇒(b)

Assume that H is an SR-monoid. Take any two different square-free ideals I and J of S. Because they are different, there is an element $s \in I$ that does not belong to J. Since s is square-free and H is SR-monoid, s is also radical. This means that any ideal containing s must also contain every element that s divides. In particular, any ideal containing s cannot be a subset of J. Hence, the set of all ideals containing s is a neighborhood of I not containing J, which shows that S is a T_1–space.

(b)⇒(a)

Now assume that S is the T_1–space. Let us take any square-free element $s \in H$. The ideal generated by s, denoted by (s), is a square-free ideal. Since S is the T_1–space, the set $\{(s)\}$ is closed. This means that there is no other square-free ideal that contains s and is different from (s). This implies that s must be radical, because every element that s divides must belong to (s). Otherwise, we would have another square-free ideal containing s, which is impossible in the T_1–space. Hence H is an SR-monoid. □

In the examples below, we present selected examples to confirm that the above theorems are sufficient for there to be a relationship between a given monoid and its square-free spectrum of a given monoid with the Ł$_2$–topology.

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Example 4.8. 1. We will show an example of a monoid H, which is ACCP, is not a factorial monoid, and its square-free spectrum S is metrizable, compact, but does not satisfy the second countability axiom.

Let $H = \mathbb{Z}[\sqrt{-5}] \setminus \{0\}$ with multiplication be a monoid.

(a) To show that H is ACCP, it is enough to note that each principal ideal in H is of the form $(a + b\sqrt{-5})$, where a and b are integers. The norm of such an ideal is $N(a + b\sqrt{-5}) = a^2 + 5b^2$, which is a positive integer. Therefore, each prime ideal in H is finite because it contains at most $N(a + b\sqrt{-5})$ elements. Since every prime ideal is finite, then every ideal in H is finite, because every ideal is the sum of prime ideals. It follows that every increasing sequence of ideals in H stabilizes, i.e. H is ACCP-monoid.

(b) To show that H is not factorial, we just need to find an element that has more than one irreducible factorization. For example, we can show that 6 has two different factorizations: $6 = 2 \cdot 3$ and $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. One can check that $2, 3, 1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreducible in H, but they are not primes because they do not divide by themselves.

(c) To show that S is metrizable, we just need to find a metric on S that induces the Ł$_2$–topology. Just use the metric from the proof of Theorem 4.2.

(d) To show that S is compact, it is enough to show that every open cover of S has a finite subcover. For example, let $\mathcal{U}$ be any open cover of S. Let I be a square-free ideal in S, e.g. (2). Then there is an open set $V \in \mathcal{U}$ that contains I. Since V is the sum of square-free ideals, there is a finite family of square-free ideals $\{J_1, \ldots, J_n\}$ such that $V = J_1 + \cdots + J_n$. Let J be the largest square-free ideal in this family. Then $J \subseteq V$. Since J is maximal in S, J is a prime ideal. So J "divides" every other square-free ideal in S. It follows that for every $K \in A$, there is an open set $W \in \mathcal{U}$ that contains K and such that $W \subseteq V$. Therefore $\{V\}$ is a finite subcover of S, so S is compact.

(e) To show that S does not satisfy the second countability axiom, it is enough to find an uncountable family of closed and disjoint subsets in S. For example, let $\mathcal{F}$ denote the set of all prime ideals in S. It can be shown that $\mathcal{F}$ is uncountable because every prime ideal in S is of the form (p), where p is a prime of $\mathbb{Z}$ that is not a sum two squares. Moreover, each element of $\mathcal{F}$ is a closed subset of S because it is a product of square-free ideals (only itself). Also, any two different elements of $\mathcal{F}$ are disjoint because they have no common divisors. Therefore $\mathcal{F}$ is an uncountable family of closed and disjoint subsets in S, so S does not satisfy the second countability axiom.

2. We will show an example of a monoid H that is atomic, not ACCP, but its square-free spectrum S is a normal space, not metrizable and not compact.

Let $H = (\mathbb{Q} \setminus \{0\}) + X\mathbb{R}[X]$ with multiplication be a monoid.

(a) To show that H is atomic, simply apply the Theorem in Theorem 2.1 of [8].

(b) To show that H is not ACCP-monoid, it is enough to find an increasing sequence of ideals in H that does not stabilize. Note that the sequence $(x) \subset (x/2) \subset (x/4) \subset \ldots$ is an increasing sequence of principal ideals in H, but does not stabilize.

(c) To show that the set S with Ł$_2$–topology is normal, one can use the fact that there are square-free ideals in the monoid $(\mathbb{Q} \setminus \{0\}) + X\mathbb{R}[X]$ which are generated by an element of the form $q + Xr$, where $q \in \mathbb{Q} \setminus \{0\}$ and $r \in \mathbb{R}$. It can be shown that for any two such elements $a = q_1 + Xr_1$ and $b = q_2 + Xr_2$, there exists a continuous function $f : S \to [0,1]$ such that $f((a)) = 0$ and $f((b)) = 1$. This can be done by taking $f((c)) = \frac{1}{1+|r_1 - r_2|}$, where $c = q + Xr$ is any ideal generator of (c). It can be checked that f is well-defined, continuous and satisfies the conditions of Urysohn's lemma. Intuitively, the function f "measures" the distance between the ideals (a) and (b) using the coefficient r with the variable X. The greater this distance, the smaller the value of f. So f takes the value 0 on (a) and the value 1 on (b), and takes intermediate values on the remaining ideals. The function f thus separates the ideals (a) and (b) using open sets $f^{-1}([0, 1/2))$ and $f^{-1}((1/2.1])$.

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(d) To show that the set S is not metrizable, we can use the fact that a metrizable space satisfies the first axiom of countability, i.e. it has a countable subfamily of the family of all open sets. It can be shown that S does not satisfy this condition by considering a family of all sets of the form $(q + Xr)$, where $q \in \mathbb{Q} \setminus \{0\}$ and $r \in \mathbb{R}$. This family is uncountable and consists of open sets that are pairwise disjoint. Therefore, any subfamily of the family of all open sets must contain a subset of every element of that family, and therefore cannot be countable. Intuitively, the sets $(q + Xr)$ are so small that they cannot be covered by a finite number of metric balls. Moreover, these sets are so different that they cannot be immersed in Euclidean space by a continuous function.

(e) To show that the set S is not compact, we can use the fact that a compact space has a finite subcover for every open cover. It can be shown that S does not satisfy this condition by considering S to be covered by sets of the form $(q + Xr)$, where $q \in \mathbb{Q} \setminus \{0\}$ and $r\ in\mathbb{R}$. It can be proven that such sets are open and non-empty, and that they do not have a non-empty intersection if r is different. Therefore, any subcover of this cover must contain all its elements, and therefore cannot be finite. Intuitively, the sets $(q + Xr)$ are so numerous that they cannot be covered by a finite number of open sets. Moreover, these sets are so scattered that they cannot be glued together using a continuous function.

3. We will give an example of the monoid H, which is GCD-monoid but is not a factorial monoid. And its square-free spectrum S is metrizable, satisfies the second countability axiom, is complete, but not compact.

Let $H = \mathbb{Z}[i] \setminus \{0\}$ with multiplication be a monoid.

(a) The monoid H is a GCD-monoid. To show this, we need to show that for any two elements a, $b \in \mathbb{Z}[i] \setminus \{0\}$ there exists an element $d \in \mathbb{Z}[i] \setminus \{0\}$ such that d divides both a and b and if some other element of d' divides both a and b, then d' divides d. In $\mathbb{Z}[i]$, the norm of each element $z = a + bi$ is defined as $N(z) = a^2 + b^2$. The norm has an important property: if z divides w in $\mathbb{Z}[i]$, then $N(z)$ divides $N(w)$ in $\mathbb{Z}$. Moreover, the norm is multiplicative, which means that $N(zw) = N(z)N(w)$. For any $a, b \in \mathbb{Z}[i] \setminus \{0\}$, we can find their GCD as follows: We calculate $N(a)$ and $N(b)$. We find the GCD for $N(a)$ and $N(b)$ in $\mathbb{Z}$, which is well defined because $\mathbb{Z}$ is a Euclidean ring. Using Euclid's algorithm in $\mathbb{Z}[i]$, we find $d \in \mathbb{Z}[i] \setminus \{0\}$ which is a common divisor of a and b and whose norm is equal to the GCD of $N(a)$ and $N(b)$. We show that any other common divisor d' of elements a and b divides d. Since $\mathbb{Z}[i]$ is a Euclidean ring with respect to the norm, the Euclidean algorithm can be used to find the GCD, which makes GCD-monoid $\mathbb{Z}[i]$ without zero.

(b) H is not a factorial monoid because not every non-invertible element in it is a unique product of prime elements. An example of such an element is the number 6, which has two different ambiguous prime factors: $5 = 1 \cdot 5 = (1 + 2i)(1 - 2i)$.

(c) To show that the space S is metrizable, it is enough to find a metric on S that induces the Ł$_2$–topology. We can take from Theorem 4.4.

(d) The space S of all square-free ideals satisfies the second axiom of countability, because every point in this space has a countable basis of neighborhoods. This is a result of the fact that $\mathbb{Z}[i]$ is a Euclidean ring, so each ideal is principal and generated by a single element. For each element $z \in \mathbb{Z}[i]$, neighborhoods can be defined using an element norm. The norm in $\mathbb{Z}[i]$ is a function of $N : \mathbb{Z}[i] \to \mathbb{N}$, where $N(a + bi) = a^2 + b^2$. Since norm values are integers, for each element z there are only a finite number of elements with a smaller norm. This means that we can create a countable basis of neighborhoods for each point in S, taking advantage of the fact that the ideals generated by the lower norm elements form the neighborhoods of the point generated by z. Due to the fact that each ideal in $\mathbb{Z}[i]$ is principal and each element has a countable number of divisors (due to the countability of $\mathbb{Z}[i]$), the space S satisfies the second axiom countability.

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(e) To show that the set S is complete, i.e. every convergent sequence in S has a limit in S, one can use the fact that S is homeomorphic to the space $\mathbb{R}^2$ using function $\varphi : S \to \mathbb{R}^2$ given by $\varphi((a+bi)) = (a,b)$, where $(a+bi)$ is the generator of the square-free ideal. We can use this homeomorphism to show that if $(a_n + ib_n)$ is a sequence of square-free ideals convergent to $(a+bi) \in \mathbb{R}^2$, then $a+bi$ is a square-free element, i.e. $(a+bi) \in A$. This can be done by taking advantage of the fact that $x \in H$ is square-free if and only if for every $t \in R$, if t^2 divides x, then t is invertible. Then, if $t \in H$ is such that t^2 divides $a+bi$, then t^2 also divides $a_n + b_n i$ for every $n \in \mathbb{N}$, because $a_n + b_n i \to a + bi$. Therefore t is invertible because H is monoid. Therefore $a+bi$ is a square-free element, i.e. $(a+bi) \in H$.

(f) To show that the set S is not compact, i.e. not every open cover S has a finite subcover, we can use the fact that S is homeomorphic to the space $\mathbb{R}^2$ using the function $\varphi : S \to \mathbb{R}^2$ given by $\varphi((a+bi)) = (a,b)$, where $(a+bi)$ is the generator of the square-free ideal. We can use this homeomorphism to show that there is an open cover S that has no finite subcover. For example, we can take the coverage of S by sets of the form $(n, n+1) \times (m, m+1)$, where $n, m \in \mathbb{Z}$. It can be proven that such sets are open and non-empty, and that they do not have a non-empty intersection, if n or m are different. Therefore, any subcover of this cover must contain all its elements, and therefore cannot be finite.

4. We will now show an example where H is a pre-Schreier monoid, it is not GCD-monoid, but its square-free spectrum S is metrizable, satisfies the second countability axiom, is completely regular, but not complete.

From [1] Example 2.10, let A be an integer closure of $\mathbb{C}[X]$ in $\overline{\mathbb{C}[X]}$, let M be a maximal ideal in A and let $H = \overline{\mathbb{Q}} + MA_M$. Then H is a pre-Schreier monoid, but not GCD-monoid.

(a) The proof that R is pre-Schreier but not GCD is provided in [1] Example 2.10.

(b) To show that S is metrizable, we can use the proof of Theorem 4.5, where the metric is proposed.

(c) The space S satisfies the second countability axiom, it is enough to also use Theorem 4.5.

(d) The space S is completely regular. Just use the function from the proof of Theorem 4.5. Namely, the characteristic function of ideals: $f : S \to [0,1]$, which takes the value 0 at p and 1 on the closed set C. Let I be any square-free ideal in H (a point in S), and J be a square-free ideal that does not contain I (an element of the closed set C in S). Since H is pre-Schreier, for any elements $a \in I$ and $b \notin I$, there is an element $c \in H$ such that a divides bc and c divides b . We can define the f function as follows:

$$f(K) = \begin{cases} 0 & \text{if } c \in K \\ 1 & \text{if } c \notin K \end{cases}$$

where K is any square-free ideal in S.

(e) The space S is not complete. Let us suppose that the topological space S is a complete space. Then for any continuous function $f : S \to S$ there is a fixed point, i.e. there is $x \in S$ for which $f(x) = x$. This is a consequence of Brouwer's fixed point theorem for complete spaces. Now consider the function $f : S \to S$ given by $f(I) = I^2$ for any ideal $I \in S$. This function is continuous because for any ideals $I, J \in S$ we have $I \subseteq J$ entails $I^2 \subseteq J^2$. Note that for any square-free ideal $I \in S$ we have $f(I) = I^2 \neq I$ because I is square-free. Therefore, the function f has no fixed point. We obtained a contradiction with the assumption that S is a complete space. Therefore, the topological space S of all square-free ideals in H is not a complete space.

5. We will now show an example of the monoid H, which is an AP-monoid and is not pre-Schreier. However, its square-free spectrum S is metrizable, separable, but not completely regular.

Let R be a Dedekind ring that is not a field. Let's define $H := R \setminus \{0\}$ with the multiplication operation. Then H is a monoid.

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(a) The H monoid satisfies the AP condition directly from the definition of a Dedekind ring.

(b) A Dedekind ring that is not a field is not pre-Schreier. A Dedekind ring is defined as an integral domain in which every non-zero proper ideal decomposes into a product of prime ideals. This distribution is unambiguous up to the order of the factors. An important feature of Dedekind rings is that each ideal can be expressed as a product of prime ideals, which is crucial for their structure. In pre-Schreier rings, if an element a divides the product bc, then there must be elements a_1 and a_2 in R such that $a = a_1a_2$, where a_1 divides b and a_2 divides c. However, in a Dedekind ring, the fact that the ideal generated by a divides the ideal generated by bc does not imply that there are such elements a_1 and a_2. This is because ideals in Dedekind rings can have complex structures and it is not always possible to find such specific divisors of elements. Furthermore, a Dedekind ring may contain ideals that are non-principal (i.e., not generated by a single element), which is contrary to the requirements for pre-Schreier rings. In pre-Schreier rings, any ideal generated by an element dividing a product must also be generated by elements dividing the components of that product, which is not always the case in Dedekind rings. Therefore, in the above considerations, there is nothing to prevent removing zero from the Dedekind ring and giving it only a multiplication operation, in order to similarly conclude that H is not a pre-Schreier monoid.

(c) We will show that S is a metrizable space. We define the metric d on S as follows:

$$d(I, J) = \inf\left\{\frac{1}{n+1} : p_n \in I \triangle J\right\},$$

where $I \triangle J$ denotes the symmetric difference of the ideals I and J, and p_n are prime elements in H. Since every irreducible element in H is prime, this metric is well-defined.

(d) Since the space S is countable, it is also separable. The center can be formed from the minimal prime ideals that generate all other square-free ideals.

(e) Complete regularity requires that for every closed set F and a point x not in F there exists a continuous function $f : S \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in F$. In a Dedekind ring, which is not a field, the maximal ideals are also prime ideals, which means that every square-free ideal is the intersection of maximal ideals (also in the H monoid). However, such a continuous function f cannot be found because there is no "distance" between maximal ideals in the topological sense.

6. We will show an example of the monoid H, which is an SR-monoid, not an AP-monoid. However, its square-free spectrum S is T_1, it is not separable.

Let $H = \mathbb{Z}[\sqrt{-5}] \setminus \{0\}$ with multiplication be a monoid.

(a) The monoid H satisfies SR: If $x \in H$ is square-free, it means that there is no $t \in H$ such that t^2 divides x and t is not invertible. Let us suppose that x is not radical, i.e. there exists $r \in H$ and $n \in \mathbb{N}$ such that x divides r^n but does not divide r . Then $r^n = xs$ for some $s \in H$, but r is not of the form $r = xt$ for any $t \in H$. Note that $N : H \to \mathbb{N}$ defined as $N(a + b\sqrt{-5}) = a^2 + 5b^2$ is a norm on H that behaves like absolute value function, i.e. $N(xy) = N(x)N(y)$ for any $x, y \in H$. Therefore we have $N(r^n) = N(x)N(s)$, i.e. $N(r)^n = N(x)N(s)$. Since $N(r)$ is not divisible by $N(x)$, it must be divisible by $N(s)$. Let $N(s) = N(r)^k$ for some $k \in \mathbb{N}$. Then $N(r)^{n-k} = N(x)$, i.e. $N(x)$ is a power of a certain natural number. But then x is the square of some element of H, which contradicts the assumption that x is square-free. So x must be radical.

(b) The monoid H is not AP: Note that $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ is a factorization irreducible in H. None of these factors is prime because they are not divisible by either 2 or 3.

(c) The space S is T_1: To show that the topological space S of the monoid H is a T_1–space, we need to show that for every pair of distinct points in S, each of them has a neighborhood that does not contains the second point. In the T_1–space, each singleton, i.e. a one-point set, is a closed set. It is a Dedekind monoid (a Dedekind ring without zero with multiplication), which means that any non-zero prime ideal is maximal. In the Dedekind monoid, each maximal ideal corresponds to a point in the space S. In topological spaces associated with Dedekind monoids, closed sets are associated with ideals of the monoid. In particular, the points in the space S correspond to the maximal ideals in $\mathbb{Z}[\sqrt{-5}] \setminus \{0\}$, and closed sets correspond to the monoid ideals. Since maximal ideals are closed and every prime ideal is maximal, every point in S is closed. To show this formally, consider two different points in S that correspond to two different maximum ideals M_1 and M_2 in $\mathbb{Z}[\sqrt{-5}] \setminus \{0\}$. Since $M_1 \neq M_2$, there is an element $a \in M_1$ that does not belong to M_2. The set $\{a\}$ is closed in S because it corresponds to the ideal generated by a, which is a subset of M_1. Similarly, for every element $b \in M_2$ that does not belong to M_1, the set $\{b\}$ is closed in S. Since every point in S is closed, the space S satisfies the definition of the T_1–space. This means that for every pair of distinct points in S, there are neighborhoods (in this case the points themselves) that are disjoint, which is required in the T_1–space.

(d) The space S is not separable because there is no countable dense subspace in it. In the context of a $Ł_2$–topology, where open sets are sums of square-free ideals and closed sets are products of square-free ideals, separability would require that there exists a countable family of square-free ideals that is dense throughout the space S. The monoid H consists of elements of the form $a + b\sqrt{-5}$, where a and b are integers, excluding zero. Square-free ideals in this monoid are generated by elements that are not squares of other elements in H. However, due to the infinite number of primes in $\mathbb{Z}$, there are an infinite number of different square-free ideals in H, which makes it impossible to "cover" them with a countable family of ideals. This further complicates the possibility of finding a countable basis of neighborhoods for each point in A, which is required for the separability of metric spaces.

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