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Approaching the Conformal Limit of Quark Matter with Different Chemical Potentials

Connor Brown [1], Veronica Dexheimer [1,*], Rafael Bán Jacobsen [2] and Ricardo Luciano Sonego Farias [2]

[1] Kent State University, Kent, Ohio 44243 USA
[2] Universidade Federal de Santa Maria, Santa Maria RS, Brazil
* Correspondence: vdexheim@kent.edu

Abstract: We study in detail the influence of different chemical potentials (baryon, charged, strange, and neutrino) on how and how fast a free gas of quarks in the zero-temperature limit reaches the conformal limit. We discuss the influence of non-zero masses, the inclusion of leptons, and different constraints, such as charge neutrality, zero-net strangeness, and fixed lepton fraction. We also investigate for the first time how the symmetry energy of the system under some of these conditions approaches the conformal limit. Finally, we briefly discuss what kind of corrections are expected from perturbative QCD as one goes away from the conformal limit.

Keywords: conformal limit; quark matter; chemical potential; symmetry energy

1. Introduction and Formalism

In the zero temperature limit, baryons start to overlap at a few times saturation density and, through some mechanism that is not yet understood, quarks become effectively deconfined [1]. In this work we discuss dense matter in terms of baryon chemical potential μ_B, instead of baryon (number) density n_B, as the former (together with other chemical potentials, such as charged μ_Q or strange μ_S) is the fixed or independent quantity in the grand canonical ensemble. The correspondence between n_B and μ_B is model dependent, but, at finite temperature, the μ_B at which deconfinement takes place is expected to be even lower (see e.g., [2]), which highlights the importance of studying quark matter. We are particularly interested in understanding the conformal limit, the asymptotically high μ_B at which matter can be described by a free (non-interacting) gas of massless quarks. For this reason, in the present work, we focus on modelling quark matter only and for the time being restrict ourselves to the zero-temperature limit.

To describe the quarks, we make use of a free Fermi gas under different assumptions. To start, we describe them simply by a massless gas, then introduce different non-zero quark masses, and vary independently the baryon, charged, and strange chemical potentials. We further link the chemical potentials by imposing charge neutrality and/or zero net strangeness. We also discuss the role played by leptons, discussing β equilibrium and the role played by neutrinos (with chemical potential μ_ν). We investigate large μ_B and different μ_Q and μ_ν, as these are important for astrophysical scenarios, such as neutron stars and neutron-star mergers. On the other hand, we investigate the effects of μ_S, which is important for discussions related to relativistic heavy-ion collisions and the early universe [3].

We also discuss the symmetry energy of quark matter for some of the constraints we study and investigate how it changes as we approach the conformal limit. Several works have addressed the symmetry energy of quark matter [4–7]. This physical quantity is defined as the difference of energy per baryon E/N_B (or energy density per baryon density ε/n_B) of fully isospin asymmetric matter $\delta = 1$ and isospin-symmetric matter $\delta = 0$:

$$E_{\rm sym} = \frac{E_{\delta=1}}{N_B} - \frac{E_{\delta=0}}{N_B} = \frac{\varepsilon_{\delta=1}}{n_B} - \frac{\varepsilon_{\delta=0}}{n_B}\,, \tag{1}$$

where δ was originally defined for matter with neutrons and protons in terms of densities n_i as

$$\delta = \frac{n_n - n_p}{n_n + n_p} \,. \tag{2}$$

In this case and also when one is considering up and down quarks, δ can also be written as

$$\delta = -2Y_I = 1 - 2Y_Q \;(\mathrm{non-strange\ matter})\,, \tag{3}$$

with fractions Y_i summing over $i =$ baryons and/or quarks and defined in terms of particle isospin Q_{I_i} and electric charge Q_i

$$Y_I = \frac{\sum_i Q_{I_i} n_i}{\sum_i n_i}\,, \quad Y_Q = \frac{\sum_i Q_i n_i}{\sum_i n_i}\,, \tag{4}$$

with baryon (number) density $n_B = \sum n_i$, where the quark densities are divided by 3.

However, it is important to note that, as discussed in Ref. [8] and Appendix A of Ref. [9], in the presence of hyperons (or in our case strange quarks), Equation (3) does not apply. For this reason, we restrain to the discussion of symmetry energy for the 2-flavor case (with up and down quarks).

When leptons are included, we assume β equilibrium, in which case electrons and muons have chemical potential $\mu_e = \mu_\mu = -\mu_Q$. In the special case that (electron and muon) neutrinos are trapped, μ_ν is determined by fixing the lepton fraction

$$Y_l = \frac{\sum_{lep} n_{lep}}{\sum_i n_i}\,, \tag{5}$$

usually hold equal to the canonical value 0.4, to simulate conditions created in supernova explosions [10].

Finally, we briefly discuss the effects of interactions in the case that they are week enough to be discussed perturbatively, i.e., using perturbative Quantum Chromodynamics, pQCD). At large temperatures and/or quark chemical potentials, the strong coupling becomes small enough to allow an infinite number of terms to be approximated by a finite number of terms to describe interactions [11]. At zero temperature, pQCD corrections have been calculated up to next-to-next-to-next-to-leading order ($\mathrm{N^3LO}$) [12,13] with non-zero quark masses included until next-to-next-to-leading order ($\mathrm{N^2LO}$) [14–16] .

2. Results

We describe in detail the free Fermi gas formalism we use in this work (for quarks and leptons) in Appendix A. We begin our discussion by ignoring the contribution of leptons to the thermodynamical quantities (later we include different possibilities and discuss them). In the figures that follow, the pressure P and baryon density n_B are normalized by respective values of a free gas with the same number of quark flavors included, but with quark masses $m_i = 0$ and $\mu_Q = \mu_S = 0$. Simple analytical equations for the pressure of all the massless cases discussed in this work are derived in Appendix B. We start our discussion considering only one chemical potential, and then expand our discussion to two and three chemical potentials.

2.1. One Chemical Potential μ_B

We start by comparing the quark mass effect on n_B versus μ_B in the left upper panel of Figure 1. Because in this case μ_Q and μ_S are zero, all quarks present the same chemical potential $\mu_i = \mu_\mathrm{u} = \mu_\mathrm{d} = \mu_\mathrm{s} = \frac{1}{3}\mu_B$. Due to our normalization (thermodynamical quantities divided by the massless case with the respective number of flavors), all massless cases have constant value 1. Nevertheless, this does not mean that they are the same (if not normalized). To discuss the effect of quark masses, we start with 1 flavor with mass corresponding to the Particle Data Group (PDG [17]) mass of the up ($m = 2.3$ MeV) or down ($m = 4.8$ MeV) quarks, then we look at the 2-flavor case with PDG masses for both light quarks. After that, we look at 3-flavors and use first only non-zero mass for the strange quark (with PDG value of $m = 95$ MeV) and then the PDG (from hereon "realistic") masses for the 3 quarks.

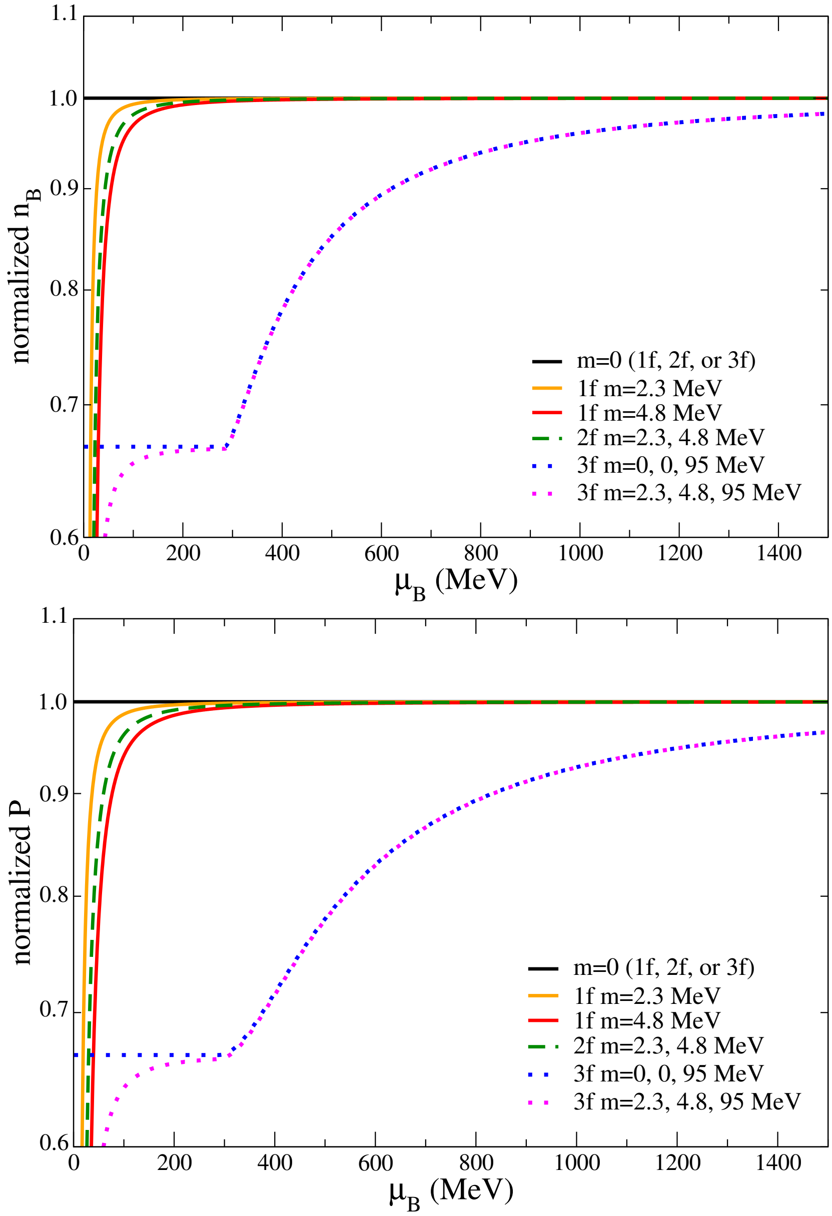

Figure 1. Baryon density (upper panel) and pressure (lower panel) of quarks with different number of flavors and different masses normalized by the respective massless cases.

We find that the introduction of realistic quark masses decreases the density for low μ_B, with the s-quark mass affecting the density until larger μ_B (up to 621 MeV) than the two light quarks (up to 55 MeV). To calculate these thresholds, we use throughout this paper the criteria of a deviation of 10% from the black line with value 1. For P versus μ_B, shown in the lower panel of Figure 1, the lines are very similar in shape (to the ones in the upper panel of the figure). The introduction of realistic quark masses decreases again P for low μ_B, with the s-quark mass affecting the pressure until larger μ_B (up to 834 MeV) than the two light quarks (up to 77 MeV).

2.2. *Two Chemical Potentials μ_B and μ_Q*

Now, we abandon the unphysical 1-flavor case, and continue with 2- and 3-flavor cases. The 2-flavor case has recently become more relevant for dense matter because it has been shown that the core of neutron stars can harbor 3-, as well as 2-flavor quark matter [18]. For this case we add another (charged) chemical potential, breaking some of the degeneracy in the quark chemical potentials: $\mu_{\rm up} = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q$, $\mu_{\rm down} = \mu_{\rm strange} = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q$. Once more, we normalize thermodynamical quantities dividing by the respective values of the same quantity for a free gas with the same number of quark flavors included, but with $m_i = 0$, in addition to $\mu_Q = 0$. Following this procedure, we aim at determining how the conformal limit and its deviation depend on μ_Q.

When μ_Q is determined by charge neutrality, the results even for the massless case depend on the number of flavors. In this case, only the 3-flavor case is coincidentally equal to the $\mu_Q = 0$ case (see the explanation following Equations (A29)–(A32) in Appendix B). For 2-flavor, this is not the case, and the pressure is lower than in the $\mu_Q = 0$ case, establishing a new lower conformal limit (see upper panel of Figure 2). Expressions for the pressure for each particular chemical potential case (always keeping $m_i = 0$ for simplicity) can be found in Appendix B. Compare e.g., Equations (A17) and (A26).

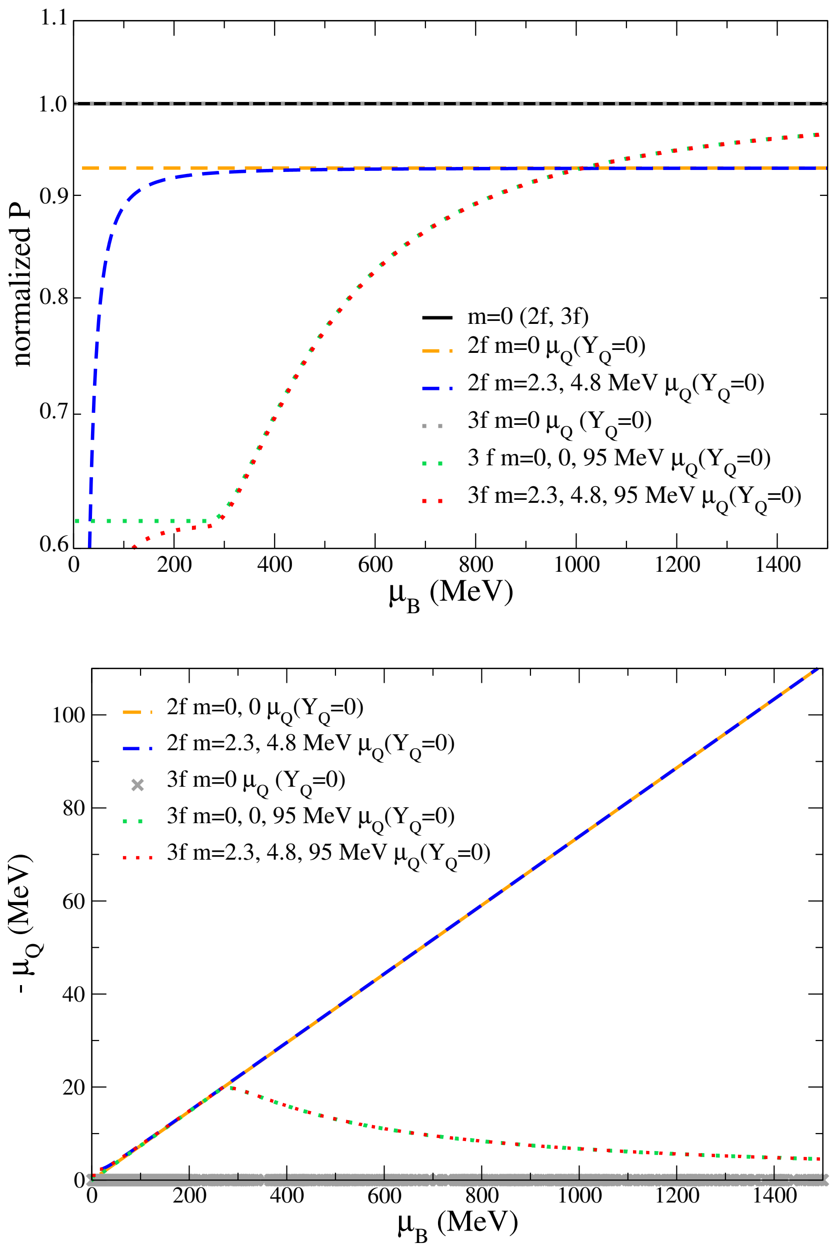

Figure 2. *Cont.*

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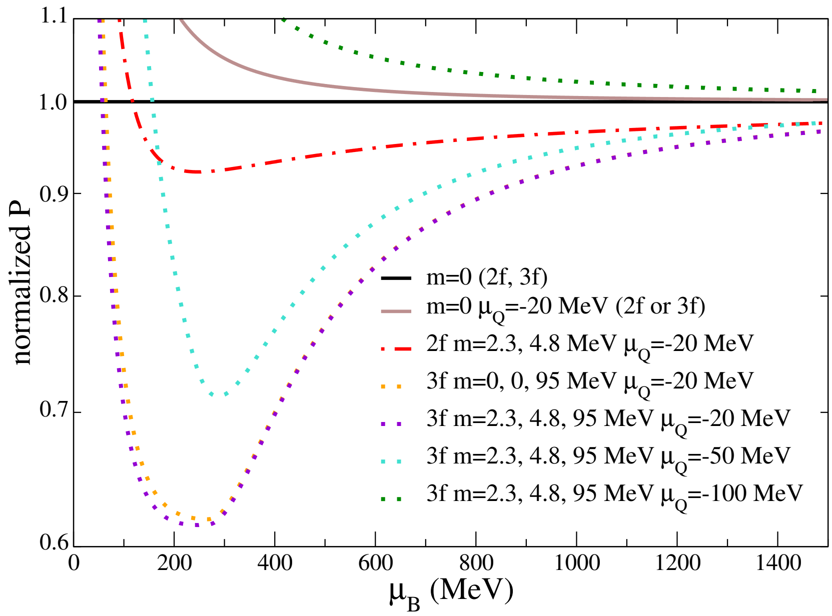

Figure 2. Pressure (upper panel) and charged chemical potential (middle panel) of quarks with 2 chemical potentials normalized by the respective massless case with one chemical potential, μ_B. The charged chemical potential is determined by charge neutrality. For massless 3-flavor quarks, the cases with and without μ_Q coincide. Lower panel: Pressure of quarks with 2 chemical potentials, being μ_Q fixed to different values, normalized by the respective massless case with one chemical potential, μ_B.

When adding quark masses, μ_Q determined by charge neutrality lowers the pressure (in comparison to the respective massless case and to the massless case with $\mu_Q = 0$) such that it goes to the respective conformal limit at larger μ_B. Using again the criteria of 10% deviations from the respective conformal limit, the s-quark mass affects pressure until $\mu_B = 839$ MeV and the two light quark masses until $\mu_B = 118$ MeV.

Nevertheless, one issue about this approach should be noted: we are comparing very small values of μ_Q with very large values of μ_B. See the middle panel of Figure 2 for a comparison. This is particularly the case for 3-flavors of quarks, and (except for extremely low μ_B) this behavior is independent of the quark masses. For small values of μ_B, both for 2 and 3-flavors, the dependence of μ_Q and μ_B can be predicted in fair agreement with Equation (A25). For this reason, next, we add a fixed charged chemical potential to study how it affects the conformal limit, which translates into an increase in pressure (see e.g., the different lines for 3-flavor quark matter with realistic masses in the lower panel of Figure 2), specially at low values of μ_B. For massless quarks and $\mu_Q = -20$ MeV, the pressure is always above the conformal limit for $\mu_Q = 0$, independently of the number of flavors. Once the quark masses are finite, the pressure decreases, specially in the 3-flavor case. For larger absolute values of μ_Q, the pressure becomes larger, even going above the conformal case (with and without μ_Q). For example, for the 3-flavor case with realistic quark masses and $\mu_Q = -50$ MeV, the pressure deviates 10% (of the $\mu_Q = 0$ conformal limit) at $\mu_B = 698$ MeV and for $\mu_Q = -100$ MeV at $\mu_B = 415$ MeV (the latter one from above). Finally, there is one important remark regarding the behavior of the normalized pressure: in the lower panel of Figure 2, it is shown that this physical quantity decreases for small values of μ_B; however, this behavior doesn't mean that the pressure itself (not normalized) is not a monotonically increasing function of μ_B. Here, we must remember that our normalization is carried out by dividing the thermodynamical quantities (such as pressure) by the massless case with the respective number of flavors, and the free Fermi pressure of this system of massless quarks used for normalization scales as μ_B^4; therefore, in those ranges of μ_B where P for massive quarks increases at a lower rate than μ_B^4, the normalized pressure decreases without implying any thermodynamical inconsistency.

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2.3. Three Chemical Potentials μ_B, μ_Q, and μ_S or μ_ν

Going further, we can add another (strange) chemical potential and constrain it, e.g., to strangeness neutrality. The issue is that at zero temperature strangeness neutrality means that there are no strange quarks, and the 3-flavor reduces to the 2-flavor case. For this reason, we fixed μ_S instead to specific values. μ_S breaks the degeneracy in the remaining quark chemical potentials: $\mu_{\rm up} = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q$, $\mu_{\rm down} = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q$, $\mu_{\rm strange} = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q + \mu_S$. Once more, we normalize thermodynamical quantities dividing by the respective values of the same quantity for a free gas with the same number of quark flavors included, but with $m_i = 0$, in addition to $\mu_Q = 0$.

Fixing μ_S increases the pressure, similar to fixing μ_Q. Compare, for example, the massless 3-flavor case in the upper panel in Figure 3 and lower panel in Figure 2 and note that the pressure for a given μ_B is now much higher. When quark masses are added, the similarity disappears, because μ_S only affects the strange quarks, which do not appear for low values of μ_B, unless the μ_S value is larger than the strange quark mass, which corresponds to our case of $\mu_S = 100$ MeV. For $\mu_S = 50$ and $\mu_S = 100$ MeV, the 10% deviation from the conformal limit takes place at $\mu_B = 1743$ and $\mu_B = 4227$ MeV, respectively (both from above).

Now we consider the case in which additionally $\mu_Q \neq 0$, determined to reproduce charge neutrality (middle panel of Figure 3). For massless 3-flavor quarks, the cases with charge neutrality and without μ_Q coincide. When masses are introduced, the curves are still very similar (to the upper panel for the $\mu_Q = 0$ case), except at very small μ_B, where the quark masses are comparable to both μ_B and μ_Q. For $\mu_S = 50$ and $\mu_S = 100$ MeV, the 10% deviation from the conformal limit takes place at $\mu_B = 1743$ and $\mu_B = 4227$ MeV, respectively (both from above).

When a fixed value of μ_Q is used, it increases the pressure further, specifically at low μ_B (see lower panel of Figure 3). For $\mu_Q = \mu_S = 50$ and $\mu_Q = \mu_S = 100$ MeV, the 10% deviation from the conformal limit takes place at $\mu_B = 2070$ and $\mu_B = 4723$ MeV, respectively (both from above).

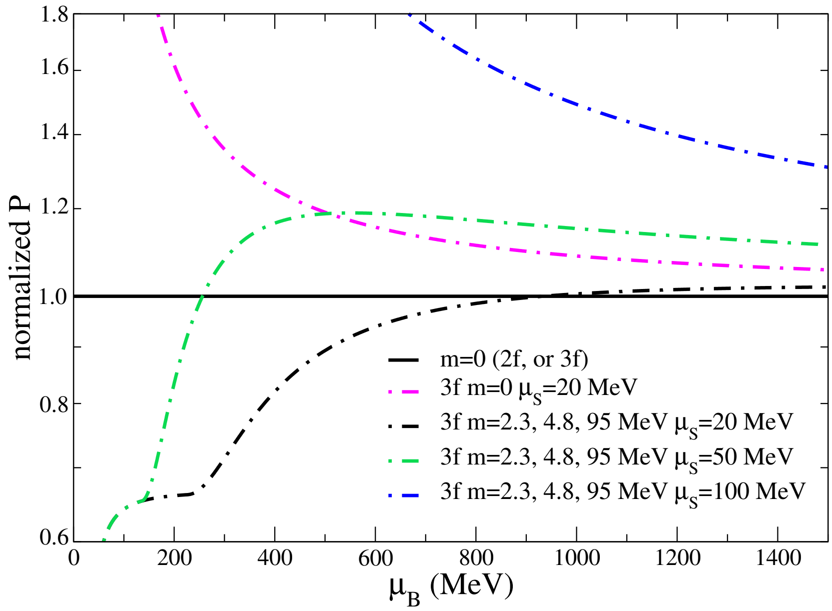

Figure 3. *Cont.*

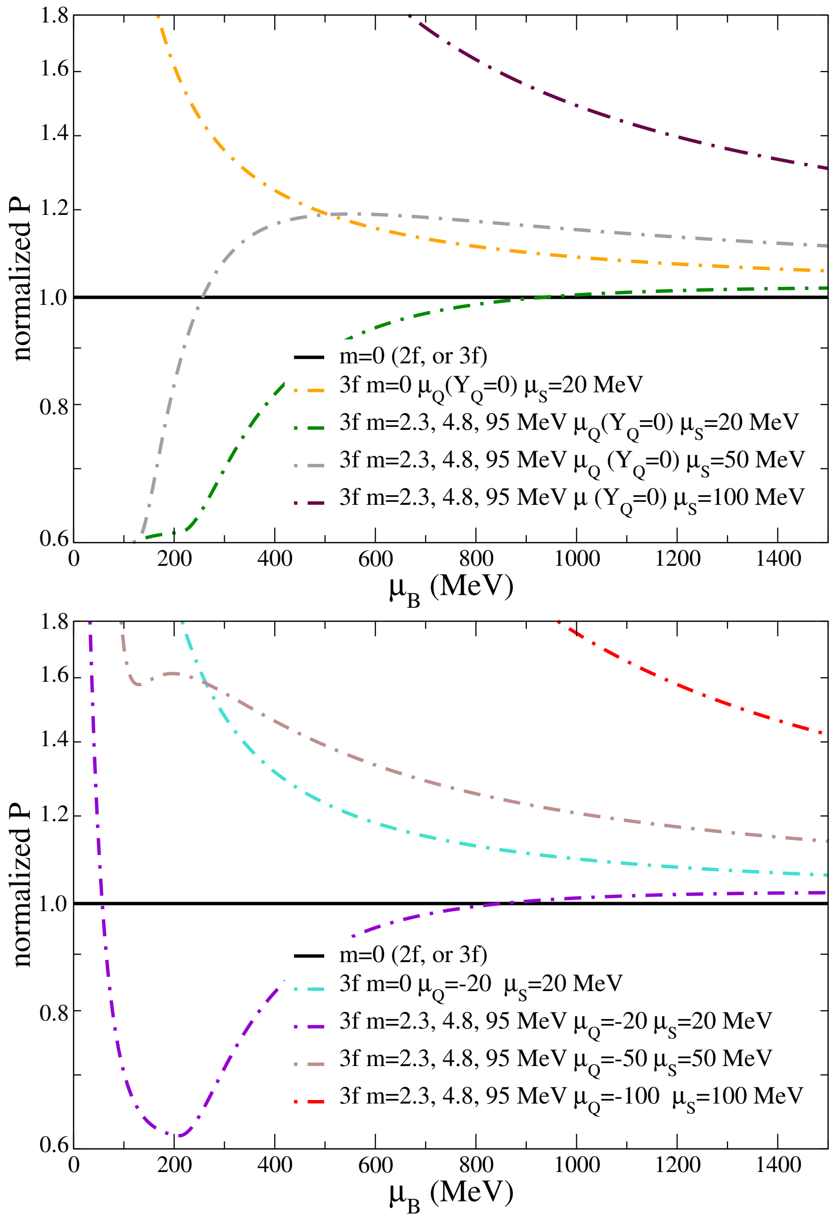

Figure 3. Pressure and charged chemical potential of quarks with 2 or 3 chemical potentials, including the strange chemical potential, normalized by the respective massless case (with one chemical potential, μ_B). The charged chemical potential is either zero (upper panel), determined by charge neutrality (middle panel), or fixed (lower panel). For massless 3-flavor quarks, the cases with charge neutrality and without μ_Q coincide.

Next, we investigate the effects of having much larger values of μ_Q and μ_S, comparable to μ_B, for 3 flavors of quarks in the upper panel of Figure 4. As expected, the changes due to the additional chemical potentials take place at much lower μ_B (notice the different scale in the y-axis of the figure) and practically all the curves are above the one chemical potential (μ_B) conformal limit. An exception is the case with large (negative) μ_Q (and $\mu_S = 0$) because, according to Equations (A1) and (A5), quarks can only exist after a given $\mu_B = 381$ MeV, at which the momentum k_i and P become finite (see Equation (A28) for the massless case). In this case, the pressure differs from the one chemical potential conformal limit by more than 10% until $\mu_B = 10\,583$ MeV. In the case of large μ_S, quarks can exist at any μ_B and the pressure differs from the one chemical potential conformal limit by more than 10% until $\mu_B = 44\,237$ MeV. When we combine large μ_S and (absolute value of) μ_Q, the pressure differs from the one chemical potential conformal limit by more than 10% until $\mu_B = 48\,897$ MeV. In this case,

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the curve in the upper panel of Figure 4 begins only at $\mu_B = 1000$ MeV. This can be understood once more from Equations (A1) and (A5). The same effect can also be seen (although more subtle) in the bottom panel of Figure 2, where the fixed μ_Q cases start at $\mu_B = -\mu_Q$.

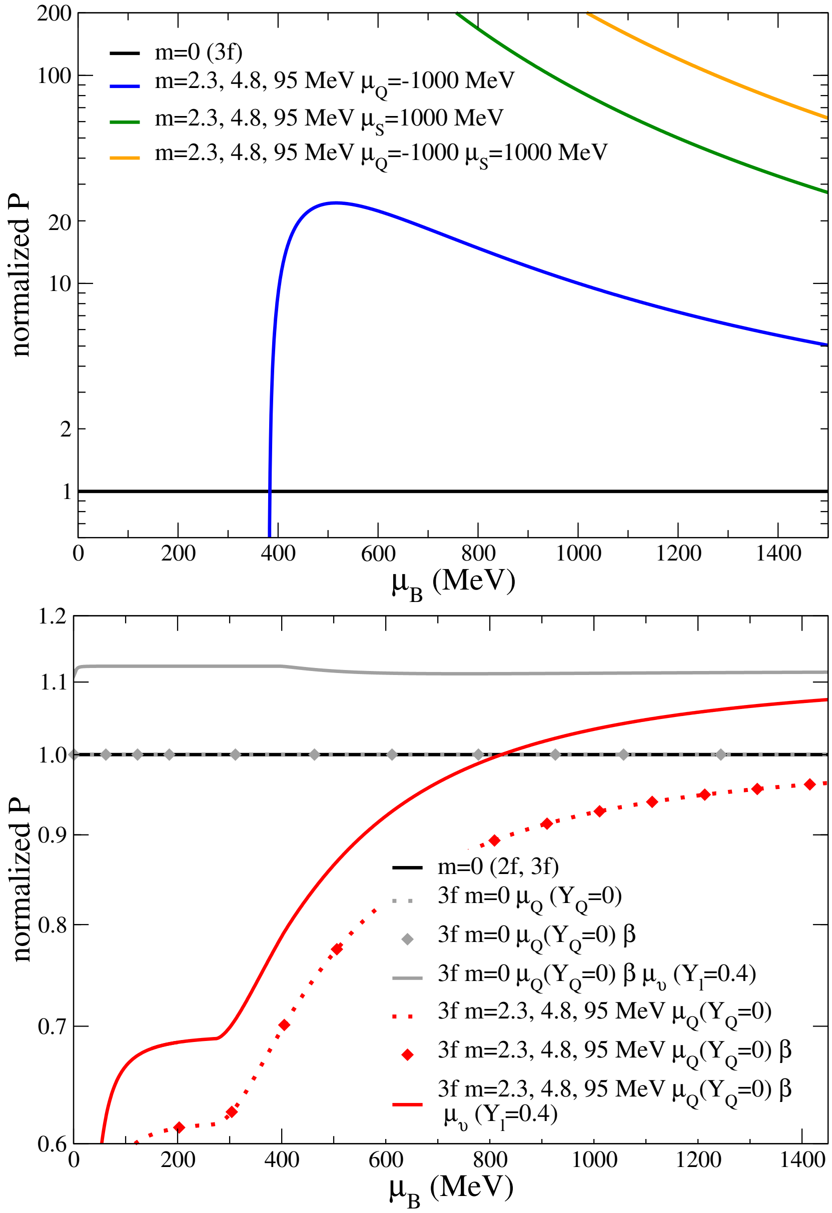

Figure 4. Upper panel: Pressure of quarks with 2 or 3 large chemical potentials, normalized by the respective massless case with one chemical potential, μ_B. Lower panel: Pressure of quarks and leptons with 2 or 3 chemical potentials, normalized by the respective massless case with one chemical potential, μ_B. For β-equilibrium with leptons, μ_Q is determined by charge neutrality. When neutrinos are present, their chemical potential μ_ν is determined by fixing the lepton fraction, Y_l.

Finally, we investigate changes due to the inclusion of a free gas of leptons (electrons and muons) in β equilibrium (and participating in the fulfillment of charge neutrality). As it can be seen in the lower panel of Figure 4, the inclusion of leptons doesn't change the pressure. The picture changes though when lepton number is fixed. In this case, which also includes neutrinos, the pressure is considerably higher because the large amount of negative leptons forces the appearance of a large amount of up quarks, changing considerably the quark composition of the system. The grey full line

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shows a kink for $\mu_B \sim 400$ MeV, when the muons appear. Note that the difference in massless versus massive quarks is still very pronounced when Y_l is fixed.

2.4. Symmetry Energy

As already discussed, we calculate the symmetry energy only for the 2-flavor case, for which it was originally defined. We fix n_B in this case (instead of μ_B as we have been doing) because the symmetry energy is defined for a given n_B, but limit the x-axis to approximately the corresponding range from the previous figures. Figure 5 shows that the curves are a monotonically increasing function of density. The light quark masses don't affect the results. Notice that the latter statement applies to every thermodynamical quantity that is not normalized by the respective conformal limit (and does not include derivatives). Numerically, we define $\delta = 0$ as the 2-flavor $\mu_Q = 0$ case (corresponding to the 2-flavor lines in Figure 1) and $\delta = 1$ as the 2-flavor $Y_Q = 0$ case (with $\mu_Q \neq 0$ corresponding to the 2-flavor lines in the top and middle panels of Figure 2).

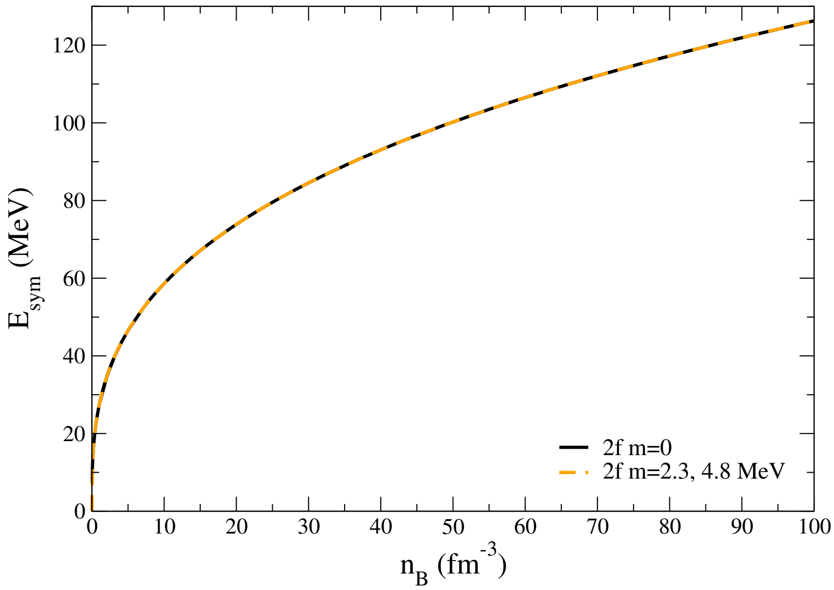

Figure 5. Symmetry energy of quarks with 1 or 2 chemical potentials as a function of baryon density for different masses. The two curves overlap.

3. Discussion and Conclusions

Perturbative corrections to a free gas of quarks due to interactions always bring down the pressure to lower values. Although these corrections have been calculated to higher orders for massless and massive (strange) quarks, they cannot accurately be carried out to low baryon chemical potentials μ_B (or, interchangeably, low baryon densities n_B in the zero-temperature limit). For example, for the relevant regime of densities inside neutron stars, $\mu_B \leq 1500$ MeV, pQCD predicts that the pressure is lower than 80% of the free gas value (see for example Figure 1 of Ref. [15]) but with a very large band going all the way to $P = 0$.

In this work, we have investigated the equation of state of a free gas of quarks focusing on how the conformal limit is reached when different chemical potentials are varied and different constraints (e.g., , for laboratory vs. astrophysics) are considered. This is done by using combinations of 1, 2, or 3 chemical potentials out of the 4 we consider, each related to a possible conserved quantity: baryon number B (μ_B), electric charge (μ_Q), strangeness (μ_S), and lepton number (μ_ν). We have also derived expressions for massless quarks under different conditions to illustrate our discussion.

We have studied the effects of using different quark masses (including PDG values), number of flavors, and different ways to fix the various chemical potentials considered. The latter procedure implies enforcing charge neutrality and, when leptons were included, β equilibrium. When leptons (electrons, muons, and their respective neutrinos) are present, the pressure in not altered. An exception is the case in which the lepton fraction is fixed. For different cases, we have quantified the deviation from the one-chemical potential (massless) conformal limit by verifying at which μ_B the pressure deviates by more than 10%. This value varied from $\mu_B = 77$ to 48 897 MeV. This shows that one must be careful about making statements concerning comparisons with "the" conformal limit. Finally, we have shown that the conformal limit of the symmetry energy is monotonically increasing and does not depend on quark masses.

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Abbreviations

The following abbreviations are used in this manuscript:

perturbative Quantum Chromodynamics (pQCD), Particle Data Group (PDG).

Appendix A. General Expressions

For each quark flavor i, we can write

$$\mu_i = \frac{1}{3}\mu_B + Q_i\mu_Q + Q_{S_i}\mu_S, \tag{A1}$$

where $1/3$ has been used as the baryon number and Q_i and Q_{S_i} are the electric charge and strangeness of each quark. μ_B, μ_Q, and μ_S are the baryon, charged, and strange independent chemical potentials of the system. In our formalism, the isospin chemical potential $\mu_I = \mu_Q$ [8].

The general expressions for (number) density, energy density and pressure of a relativistic free Fermi gas of particles i using thee natural system of units are

$$n_i = \frac{g_i}{2\pi^2}\int_0^\infty dk_i\, k_i^2(f_{i+} - f_{i_-}), \tag{A2}$$

$$\varepsilon_i = \frac{g_i}{2\pi^2}\int_0^\infty dk_i E_i k_i^2(f_{i_+} + f_{i_-}), \tag{A3}$$

$$P_i = \frac{1}{3}\frac{g_i}{2\pi^2}\int_0^\infty dk_i \frac{k_i^4}{E_i}(f_{i_+} + f_{i_-}), \tag{A4}$$

where $g_i = 6$ is the spin and color degeneracy factor, k_i is the momentum,

$$E_i = \sqrt{k_i^2 + m_i^2} \geq 0, \tag{A5}$$

is the energy of the state, m_i the mass, $f_\pm$ the distribution function of particles and antiparticles $f_{i_\pm} = (e^{(E_i \mp \mu_i)/T} + 1)^{-1}$, with μ_i being the particle chemical potential, and T the temperature.

In the $T = 0$ limit, antiparticles provide no contribution, $f_- = 0$, and $f_+ = 1$ up to the Fermi momentum, $k_i = k_{F_i}$, $E_i = \mu_i$ and the integrals for the above thermodynamic quantities are evaluable analytically

$$n_i = \frac{g_i}{6\pi^2}k_{F_i}^3, \tag{A6}$$

$$\varepsilon_i = \frac{g_i}{2\pi^2}\left[\left(\frac{1}{8}m_i^2 k_{F_i} + \frac{1}{4}k_{F_i}^3\right)\sqrt{m_i^2 + k_{F_i}^2} - \frac{1}{8}m_i^4 \ln\frac{k_{F_i} + \sqrt{m_i^2 + k_{F_i}^2}}{m_i}\right], \tag{A7}$$

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$$P_i = \frac{1}{3}\frac{g_i}{2\pi^2}\left[\left(\frac{1}{4}k_{F_i}^3 - \frac{3}{8}m_i^2 k_{F_i}\right)\sqrt{m_i^2 + k_{F_i}^2} + \frac{3}{8}m_i^4 \ln\frac{k_{F_i} + \sqrt{m_i^2 + k_{F_i}^2}}{m_i}\right]. \tag{A8}$$

Appendix B. Massless Quarks

For the massless particle case, the expressions above further reduce to

$$n_i = \frac{g_i}{6\pi^2}k_{F_i}^3 = \frac{g_i}{6\pi^2}\mu_i^3, \tag{A9}$$

$$\varepsilon_i = \frac{g_i}{8\pi^2}k_{F_i}^4 = \frac{g_i}{8\pi^2}\mu_i^4, \tag{A10}$$

$$P_i = \frac{1}{3}\frac{g_i}{8\pi^2}k_{F_i}^4 = \frac{1}{3}\frac{g_i}{8\pi^2}\mu_i^4, \tag{A11}$$

reproducing $\varepsilon_i = 3P_i$.

Note that, in the case of massless free quarks, we can also write $\mu_i = k_i$. Therefore, we can write the chemical potential for each quark flavor using Equation (A1)

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q = k_u\ , \tag{A12}$$

$$\mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q = k_d\ , \tag{A13}$$

$$\mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q + \mu_S = k_s\ . \tag{A14}$$

We use the convention that both the strangeness and μ_S are positive. Alternatively, one could use both as negative without changing the results. Equations (A12) and (A13) are equal if $\mu_Q = 0$. Equations (A12)–(A14) are equal if $\mu_Q = 0$ and $\mu_S = 0$. The density and pressure of each quark flavor can be written further as

$$n_i = \frac{\mu_i^3}{\pi^2} = \frac{k_i^3}{\pi^2}\ , \tag{A15}$$

$$P_i = \frac{\mu_i^4}{4\pi^2} = \frac{k_i^4}{4\pi^2}\ . \tag{A16}$$

Next, we discuss the pressure for specific conditions concerning number of flavors and chemical potential constraints (not including leptons):

- 2-flavor, $\mu_Q = 0$

$$P = P_u + P_d = 2P_u = \frac{2\mu_u^4}{4\pi^2} = \frac{\mu_B^4}{162\pi^2} = \frac{\mu_B^4}{1598.88}\ . \tag{A17}$$

- 3-flavor, $\mu_Q = 0$, $\mu_S = 0$

$$P = P_u + P_d + P_s = 3P_u = \frac{3\mu_u^4}{4\pi^2} = \frac{\mu_B^4}{108\pi^2} = \frac{\mu_B^4}{1065.92}\ . \tag{A18}$$

- 2-flavor, μ_Q fixed

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$$
\begin{aligned}
P = P_u + P_d &= \frac{1}{4\pi^2}\left(\mu_u^4 + \mu_d^4\right) = \frac{1}{4\pi^2}\left[\left(\frac{1}{3}\mu_B + \frac{2}{3}\mu_Q\right)^4 + \left(\frac{1}{3}\mu_B - \frac{1}{3}\mu_Q\right)^4\right] \\
&= \frac{1}{4\pi^2}\left(\frac{\mu_B^4}{81} + \frac{4\mu_B^3}{27}\frac{2\mu_Q}{3} + \frac{6\mu_B^2}{9}\frac{4\mu_Q^2}{9} + \frac{4\mu_B}{3}\frac{8\mu_Q^3}{27} + \frac{16\mu_Q^4}{81}\right. \\
&+ \left.\frac{\mu_B^4}{81} - \frac{4\mu_B^3}{27}\frac{\mu_Q}{3} + \frac{6\mu_B^2}{9}\frac{\mu_Q^2}{9} - \frac{4\mu_B}{3}\frac{\mu_Q^3}{27} + \frac{\mu_Q^4}{81}\right) \\
&= \frac{1}{324\pi^2}\left[2\mu_B^4 + 4\mu_B^3\mu_Q + 30\mu_B^2\mu_Q^2 + 28\mu_B\mu_Q^3 + 17\mu_Q^4\right]. \qquad \text{(A19)}
\end{aligned}
$$

- 2-flavor, μ_Q from charge neutrality

  Starting from $\sum_i Q_i n_i = 0$

$$\frac{2}{3}n_u - \frac{1}{3}n_d = 0\,, \qquad \text{(A20)}$$

$$\frac{2}{3}\frac{\mu_u^3}{\pi^2} - \frac{1}{3}\frac{\mu_d^3}{\pi^2} = 0\,, \qquad \text{(A21)}$$

$$2\mu_u^3 = \mu_d^3\,, \qquad \text{(A22)}$$

$$2\left(\frac{1}{3}\mu_B + \frac{2}{3}\mu_Q\right)^3 = \left(\frac{1}{3}\mu_B - \frac{1}{3}\mu_Q\right)^3, \qquad \text{(A23)}$$

$$2^{\frac{1}{3}}\frac{1}{3}\mu_B - \frac{1}{3}\mu_B = -2^{\frac{1}{3}}\frac{2}{3}\mu_Q - \frac{1}{3}\mu_Q\,, \qquad \text{(A24)}$$

$$\mu_Q = \frac{-\left(2^{\frac{1}{3}} - 1\right)\mu_B}{2^{\frac{4}{3}} + 1} = -0.07\,\mu_B\,. \qquad \text{(A25)}$$

We can then use Equations (A22) and (A25) to calculate the pressure

$$
\begin{aligned}
P &= P_u + P_d = \frac{1}{4\pi^2}\left(\mu_u^4 + \mu_d^4\right) = \frac{1}{4\pi^2}\left(\mu_u^4 + 2^{\frac{4}{3}}\mu_u^4\right) \\
&= \frac{1}{4\pi^2}\left(1 + 2^{\frac{4}{3}}\right)\mu_u^4 = \frac{1}{4\pi^2}\left(1 + 2^{\frac{4}{3}}\right)\left(\frac{1}{3}\mu_B + \frac{2}{3}\mu_Q\right)^4 \\
&= \frac{1}{4\pi^2}\left(1 + 2^{\frac{4}{3}}\right)\left[\frac{1}{3}\mu_B - \frac{2}{3}\left(\frac{2^{\frac{1}{3}} - 1}{2^{\frac{4}{3}} + 1}\mu_B\right)\right]^4 \\
&= \frac{1}{4\pi^2}\left(1 + 2^{\frac{4}{3}}\right)\left[\frac{2^{\frac{4}{3}} + 1 - 2^{\frac{4}{3}} + 2}{3(2^{\frac{4}{3}} + 1)}\right]^4 \mu_B^4 \\
&= \frac{1}{4\pi^2(2^{\frac{4}{3}} + 1)^3}\mu_B^4 = \frac{\mu_B^4}{1721.59}\,. \qquad \text{(A26)}
\end{aligned}
$$

- 3-flavor, μ_Q fixed, $\mu_S = 0$

$$P = P_u + P_d + P_s = \frac{1}{4\pi^2}(\mu_u^4 + \mu_d^4 + \mu_s^4) = \frac{1}{4\pi^2}(\mu_u^4 + 2\mu_d^4)\,, \qquad \text{(A27)}$$

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because $\mu_d = \mu_s$ are equal, resulting in

$$\begin{aligned} P &= \frac{1}{4\pi^2}\left[\left(\frac{1}{3}\mu_B + \frac{2}{3}\mu_Q\right)^4 + 2\left(\frac{1}{3}\mu_B - \frac{1}{3}\mu_Q\right)^4\right] \\ &= \frac{1}{4\pi^2}\left(\frac{\mu_B^4}{81} + \frac{4\mu_B^3}{27}\frac{2\mu_Q}{3} + \frac{6\mu_B^2}{9}\frac{4\mu_Q^2}{9} + \frac{4\mu_B}{3}\frac{8\mu_Q^3}{27} + \frac{16\mu_Q^4}{81}\right. \\ &+ \left. 2\frac{\mu_B^4}{81} - 2\frac{4\mu_B^3}{27}\frac{\mu_Q}{3} + 2\frac{6\mu_B^2}{9}\frac{\mu_Q^2}{9} - 2\frac{4\mu_B}{3}\frac{\mu_Q^3}{27} + 2\frac{\mu_Q^4}{81}\right) \\ &= \frac{1}{324\pi^2}\left(3\mu_B^4 + 36\mu_B^2\mu_Q^2 + 24\mu_B\mu_Q^3 + 18\mu_Q^4\right). \end{aligned} \tag{A28}$$

- 3-flavor, μ_Q from charge neutrality, $\mu_S = 0$

  Starting again from $\sum_i Q_i n_i = 0$

$$\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s = 0\,, \tag{A29}$$

$$\frac{2}{3}\left(\frac{\mu_u^3}{\pi^2}\right) - \frac{1}{3}\left(\frac{\mu_d^3}{\pi^2}\right)\frac{1}{3} - \left(\frac{\mu_s^3}{\pi^2}\right) = 0\,, \tag{A30}$$

$$2\mu_u^3 - \mu_d^3 - \mu_s^3 = 0\,, \tag{A31}$$

but, since in this case $\mu_d = \mu_s$, we have:

$$\mu_u^3 = \mu_d^3\,, \tag{A32}$$

which implies (from Equations (A12) and (A13)) $\mu_Q = 0$ and reproduces the 3-flavor case with $\mu_Q = 0$, $\mu_S = 0$.

- 3-flavor, zero net strangeness

  Starting from $\sum Q_{S_i} n_i = 0$, at $T = 0$ this implies $n_s = 0$, no matter if $\mu_Q = 0$ or $\mu_Q \neq 0$. As a consequence, this case reproduces the respective 2-flavor case.

- 3-flavor, μ_Q fixed, μ_S fixed

$$\begin{aligned} P &= P_u + P_d + P_s = \frac{1}{4\pi^2}(\mu_u^4 + \mu_d^4 + \mu_s^4) \\ &= \frac{1}{4\pi^2}\left[\left(\frac{1}{3}\mu_B + \frac{2}{3}\mu_Q\right)^4 + \left(\frac{1}{3}\mu_B - \frac{1}{3}\mu_Q\right)^4 + \left(\frac{1}{3}\mu_B - \frac{1}{3}\mu_Q + \mu_S\right)^4\right]. \end{aligned} \tag{A33}$$

Using the result from Equation (A28)

$$\begin{aligned} P &= \frac{1}{324\pi^2}\left(3\mu_B^4 + 36\mu_B^2\mu_Q^2 + 24\mu_B\mu_Q^3 + 18\mu_Q^4\right) \\ &+ \frac{1}{4\pi^2}\left(\mu_S^4 - \frac{4}{27}\mu_Q^3\mu_S - \frac{4}{3}\mu_Q\mu_S^3 + \frac{4}{3}\mu_B\mu_S^3 + \frac{4}{27}\mu_B^3\mu_S\right. \\ &+ \left.\frac{6}{9}\mu_Q^2\mu_S^2 + \frac{6}{9}\mu_B^2\mu_S^2 - \frac{12}{27}\mu_B^2\mu_Q\mu_S + \frac{12}{27}\mu_B\mu_Q^2\mu_S - \frac{12}{9}\mu_B\mu_Q\mu_S^2\right). \end{aligned} \tag{A34}$$

- 3-flavor $\mu_Q = 0$, μ_S fixed

Using Equation (A34) with $\mu_Q = 0$

$$P = \frac{1}{\pi^2}\left(\frac{1}{108}\mu_B^4 + \frac{\mu_S^4}{4} + \frac{1}{3}\mu_B\mu_s^3 + \frac{1}{27}\mu_B^3\mu_S + \frac{1}{6}\mu_B^2\mu_S^2\right). \tag{A35}$$

- 3-flavor, μ_Q from charge neutrality, μ_S fixed

Starting from $\sum Q_i n_i = 0$

$$\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s = 0\,, \tag{A36}$$

$$2\mu_u^3 - \mu_d^3 - \mu_s^3 = 0\,, \tag{A37}$$

$$2\left(\frac{1}{3}\mu_B + \frac{2}{3}\mu_Q\right)^3 - \left(\frac{1}{3}\mu_B - \frac{1}{3}\mu_Q\right)^3 - \left(\frac{1}{3}\mu_B - \frac{1}{3}\mu_Q + \mu_S\right)^3 = 0\,, \tag{A38}$$

$$\begin{aligned}&\frac{2\mu_B^3}{27} + \frac{12\mu_B^2\mu_Q}{27} + \frac{24\mu_B\mu_Q^2}{27} + \frac{16\mu_Q^3}{27} - \frac{2\mu_B^3}{27} + \frac{6\mu_B^2\mu_Q}{27} - \frac{6\mu_B\mu_Q^2}{27}\\ &+ \frac{2\mu_Q^3}{27} - \mu_S^3 - \frac{3\mu_B^2\mu_S}{9} + \frac{6\mu_B\mu_Q\mu_S}{9} - \frac{3\mu_Q^2\mu_S}{9} - \frac{3\mu_B\mu_S^2}{3} + \frac{3}{3}\mu_Q\mu_S^2 = 0\,,\end{aligned} \tag{A39}$$

$$\frac{2\mu_B^2\mu_Q}{3} + \frac{2\mu_B\mu_Q^2}{3} + \frac{2\mu_Q^3}{3} - \mu_S^3 - \frac{\mu_B^2\mu_S}{3} + \frac{2\mu_B\mu_Q\mu_S}{3} - \frac{\mu_Q^2\mu_S}{3} - \mu_B\mu_S^2 + \mu_Q\mu_S^2 = 0\,. \tag{A40}$$

In the above expression, we still need to isolate μ_Q and replace in Equation (A34).

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