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Some Results on Coefficient Estimate Problems for Four-Leaf-Type Bounded Turning Functions

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Abstract: Let $\mathcal{B}T_4 l$ denotes a subclass of bounded turning functions connected with a four-leaf-type domain. The goal of the study is to probe into coefficients of $|b_6|, |b_7|, |b_8|$, the bounds of the logarithmic coefficients and the third-order determinants of $H_{3,1}, H_{3,2}, H_{3,3}$ for functions in this class.

Keywords: Hankel determinant; Schwarz functions; bounded turning functions; four-leaf-type domain

MSC: 30C45; 30C80

1. Introduction and Definitions

Let $\mathcal{H}$ be a representation for a family of mappings of the following kind, denoted by g

$$g(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n \tag{1}$$

in the unit disc $\mathcal{U} = \{\xi \in \mathbb{C} : |z| < 1\}$. We refer to $\mathcal{S}$ as a subfamily of $\mathcal{H}$, which consists of univalent functions in $\mathcal{U}$.

For functions $g \in \mathcal{H}$ of the form $g(\xi) = \xi + b_2\xi^2 + b_3\xi^3 + \cdots$ and positive integers i and n, the Hankel determinant $H_{i,n}(g)$ is defined by

$$H_{i,n}(g) := \begin{vmatrix} b_n & a_{n+1} & \cdots & b_{n+i-1} \\ b_{n+1} & a_{n+2} & \cdots & b_{n+i} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n+i-1} & b_{n+i} & \cdots & b_{n+2i-2} \end{vmatrix} \quad (b_1 = 1).$$

The Hankel determinant $H_{i,n}(g)$ was introduced by Pommerrenke [1,2]. The third Hankel determinants are follows:

$$H_{3,1}(g) = b_3b_5 + 2b_2b_3b_4 - b_3^3 - b_4^2 - b_2^2b_5, \tag{2}$$

$$H_{3,2}(g) = b_4(b_3b_5 - b_4^2) - b_5(b_2b_5 - b_3b_4) + b_6(b_2b_4 - b_3^2), \tag{3}$$

$$H_{3,3}(g) = b_5(b_4b_6 - b_5^2) - b_6(b_3b_6 - b_4b_5) + b_7(b_3b_5 - b_4^2). \tag{4}$$

For the first time, the bounds of the third-order Hankel determinant $H_{3,1}(g)$ for the families of $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$ were investigated by Babalola[3]. Recently, the sharp bounds of the Hankel determinant $H_{3,1}(g)$ of subclasses of analytic functions were obtained by many authors [4–10].

For $g \in \mathcal{S}$, Let $F_g(\xi)$ be defined as the logarithmic coefficients of $g(\xi)$,

$$F_g(\xi) = log\frac{g(\xi)}{\xi} = 2\sum_{n=1}^{\infty} \delta_n \xi^n. \tag{5}$$

The $\delta_n = \delta_n(g)$ are referred to as the the logarithmic coefficients of g. In the theory of univalent functions, These coefficients play an important role for different estimates. The problem of the best upper bounds for δ_n is still open. In fact even the proper order of magnitude is still not known. It is known, however, for the starlike functions that the best bound is $|\delta_n| \leq \frac{1}{n}(n \geq 1)$ and that this is not true in general [11].

Using (1) and differentiating (5), we have

$$\begin{cases} \delta_1 = \frac{1}{2}b_2, \\ \delta_2 = \frac{1}{2}(b_3 - \frac{1}{2}b_2^2), \\ \delta_3 = \frac{1}{2}(b_4 - b_2b_3 + \frac{1}{3}b_2^3), \\ \delta_4 = \frac{1}{2}(b_5 - b_4b_2 + b_3b_2^2 - \frac{1}{2}b_3^2 - \frac{1}{4}b_2^4), \\ \delta_5 = \frac{1}{2}(b_6 - b_2b_5 - b_3b_4 + b_4b_2^2 + b_2b_3^2 - b_2^3b_3 + \frac{1}{5}b_2^5). \end{cases} \tag{6}$$

In 2022, Sunthrayuth et al. [12] introduced a subclass of bounded turning functions associated with a four-leaf function defined by

$$\mathcal{B}T_{4l} = \left\{ g \in \mathcal{S} : g'(\xi) \prec 1 + \frac{5}{6}\xi + \frac{1}{6}\xi^5, (\xi \in \mathcal{U}) \right\}.$$

Sunthrayuth et al. [12] obtained Kruskal inequality, the bounds of the coefficient inequalities and the two-order Hankel determinant of bounded turning class $\mathcal{B}T_{4l}$.

Utilizing the estimates of the coefficients of the Schwartz function, we study the third-order Hankel determinants $H_{3,1}(g)$, $H_{3,2}(g)$ and $H_{3,3}(g)$ for the class $\mathcal{B}T_{4l}$, also, we obtain the bounds of the logarithmic coefficients for $g \in \mathcal{B}T_{4l}$.

Let ω_0 be the family of Schwarz functions. Thus, the function $w \in \omega_0$ may be expressed as a power series

$$w(\xi) = \sum_{n=1}^{\infty} c_n \xi^n. \tag{7}$$

Lemma 1 (see [13]). *Suppose a function w is member of ω_0. Then*

$$|c_2| \leq 1 - |c_1|^2, \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}, \quad |c_4| \leq 1 - |c_1|^2 - |c_2|^2,$$

$$|c_5| \leq 1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1+|c_1|}, \quad |c_6| \leq 1 - |c_1|^2 - |c_2|^2 - |c_3|^2,$$

$$|c_7| \leq 1 - |c_1|^2 - |c_2|^2 - |c_3|^2 - \frac{|c_4|^2}{1+|c_1|}.$$

Lemma 2 (see [14,15]). *Suppose a function w is member of ω_0. then*

$$|c_1c_3 - c_2^2| \leq 1 - |c_1|^2, \quad |c_2c_4 - c_3^2| \leq (1 - c_1^2)^2, \quad |c_3c_5 - c_4^2| \leq 1 - c_1^2.$$

Lemma 3 (see [12]). *If $g \in \mathcal{B}T_{4l}$, then*

$$|b_2| \leq \frac{5}{12}, \quad |b_3| \leq \frac{5}{18}, \quad |b_4| \leq \frac{5}{24}, \quad |b_5| \leq \frac{1}{6}.$$

Lemma 4 (see [11]). *If $w \in \omega_0$, then $|c_n| \leq 1 \quad (n \geq 1)$.*

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Lemma 5 (see [16]). *Let $w \in \omega_0$, then for $\gamma \in \mathbb{C}$, we have*

$$\left|c_2 + \gamma c_1^2\right| \leq max\left\{1, |\gamma|\right\}$$

Lemma 6 (see [17]). *If $w \in \omega_0$ is in the form (7). Then, we get*

$$|c_3 + \gamma c_1 c_2 + \varsigma c_1^3| \leq 1,$$

where $(\gamma, \varsigma) \in D_0 \cup D_1$, with

$$D_1 = \{(\gamma, \varsigma) \in \mathbb{R}^2 : |\gamma| \leq \frac{1}{2}, -1 \leq \varsigma \leq 1\},$$

$$D_2 = \{(\gamma, \varsigma) \in \mathbb{R}^2 : \frac{1}{2} \leq |\gamma| \leq 2, \frac{4}{27}(|\gamma| + 1)^3 - (|\gamma| + 1) \leq \varsigma \leq 1\},$$

Lemma 7 (see [18]). *Let $w \in \omega_0$ and $\gamma \in \mathbb{C}$, we obtain*

$$\left|c_4 + 2\gamma c_1 c_3 + \gamma c_2^2 + 3\gamma^2 c_1^2 c_2 + \gamma^3 c_1^4\right| \leq max\{1, |\gamma|^3\}.$$

Lemma 8 (see [18]). *If $w \in \omega_0$, then for all $\gamma \in \mathbb{C}$, we have*

$$|c_5 + 2\gamma c_1 c_4 + 2\gamma c_2 c_3 + 3\gamma^2 c_1 c_2^2 + 3\gamma^2 c_1^2 c_3 + 4\gamma^3 c_1^3 c_2 + \gamma^4 c_1^5| \leq 1.$$

2. The bounds of the third Hankel determinant for $g \in \mathcal{B}T_{4l}$

Theorem 1. *If $g \in \mathcal{B}T_{4l}$, then*

$$|b_6| \leq \frac{5}{36}, \quad |b_7| \leq \frac{5}{42}, \quad |b_8| \leq \frac{5}{48}.$$

The bounds are sharp.

Proof. For a function $g \in \mathcal{B}T_{4l}$, there exists a Schwarz function $w(\xi)$, such that

$$g'(\xi) = 1 + \frac{5}{6}w(\xi) + \frac{1}{6}w^5(\xi),$$

Comparing the coefficients, we yield

$$b_2 = \frac{5}{12}c_1, \quad b_3 = \frac{5}{18}c_2, \quad b_4 = \frac{5}{24}c_3, \quad b_5 = \frac{1}{6}c_4, \tag{8}$$

$$b_6 = \frac{5c_5 + c_1^5}{36} \tag{9}$$

$$b_7 = \frac{5c_6 + 5c_1^4 c_2}{42}, \tag{10}$$

$$b_8 = \frac{5c_7 + 5c_1^4 c_3 + 10c_1^3 c_2^2}{48}. \tag{11}$$

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From (9) and Lemma 1, we have

$$\begin{aligned}
|b_6| &\leq \frac{1}{36}[5|c_5| + |c_1|^5] \\
&\leq \frac{1}{36}[5(1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1+|c_1|}) + |c_1|^5] \\
&= \frac{1}{36}(5 - 5|c_1|^2 + |c_1|^5 - 5|c_2|^2 - \frac{5|c_3|^2}{1+|c_1|}) \\
&\leq \frac{1}{36}(5 - 5|c_1|^2 + |c_1|^5) \\
&\leq \frac{5}{36}.
\end{aligned}$$

From (10) and Lemma 1, we achieve

$$\begin{aligned}
|b_7| &\leq \frac{1}{42}[5|c_6| + 5|c_1|^4|c_2|] \\
&\leq \frac{1}{42}[5(1 - |c_1|^2 - |c_2|^2 - |c_3|^2) + 5|c_1|^4|c_2|] \\
&= \frac{1}{42}(5 - 5|c_1|^2 + 5|c_1|^4|c_2| - 5|c_2|^2 - 5|c_3|^2) \\
&\leq \frac{1}{42}[5 - 5|c_1|^2 + 5|c_1|^4(1 - |c_1|^2)] \\
&= \frac{1}{42}(5 - 5|c_1|^2 + 5|c_1|^4 - 5|c_1|^6) \\
&\leq \frac{5}{42}.
\end{aligned}$$

From (11) and Lemma 1, we get

$$\begin{aligned}
|b_8| &\leq \frac{1}{48}[5|c_7| + 5|c_1|^4|c_3| + 10|c_1|^3|c_2|^2] \\
&\leq \frac{1}{48}[5(1 - |c_1|^2 - |c_2|^2 - |c_3|^2 - \frac{|c_4|^2}{1+|c_1|}) + 5|c_1|^4|c_3| + 10|c_1|^3|c_2|^2] \\
&= \frac{1}{48}[5 - 5|c_1|^2 + (-5 + 10|c_1|^3)|c_2|^2 + 5|c_1|^4|c_3| - 5|c_3|^2 - 5\frac{|c_4|^2}{1+|c_1|}] \\
&\leq \frac{1}{48}[5 - 5|c_1|^2 + (-5 + 10|c_1|^3)|c_2|^2 + 5|c_1|^4(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|})] \\
&= \frac{1}{48}[5 - 5|c_1|^2 + 5|c_1|^4 - 5|c_1|^6 + \frac{-5 - 5|c_1| + 10|c_1|^3 + 5|c_1|^4}{1+|c_1|}|c_2|^2].
\end{aligned}$$

Setting $c = |c_1|$ and $d = |c_2|$, we get

$$|b_8| \leq \frac{1}{48}\epsilon_1(c, d)$$

where

$$\epsilon_1(c, d) = 5 - 5c^2 + 5c^4 - 5c^6 + \frac{-5 - 5c + 10c^3 + 5c^4}{1+c}d^2.$$

The critical points of ϵ_1 satisfy

$$\begin{cases} \frac{\partial \epsilon_1}{\partial c} = -10c + 20c^3 - 30c^5 + \frac{15c^4+40c^3+30c^2}{(1+c)^2} d^2, \\ \frac{\partial \epsilon_1}{\partial d} = \frac{-10-10c+20c^3+10c^4}{1+c} d = 0. \end{cases}$$

Applying numerical computations, we have

$$\begin{cases} c_0 = 0, \\ d_0 = 0, \end{cases} \quad \begin{cases} c_1 = 0.8668, \\ d_1 = 0.7940, \end{cases} \quad \begin{cases} c_2 = 0.8668, \\ d_2 = -0.7940. \end{cases}$$

Thus, in $(0,1) \times (0, 1-c^2)$, there is no critical points which satisfies $0 \le d \le 1 - c^2$.
For $c = 0$,

$$\epsilon_1(0, d) = 5 - 5d^2 \le 5.$$

For $d = 0$,

$$\epsilon_1(c, 0) = 5 - 5c^2 + 5c^4 - 5c^6 \le 5.$$

For $d = 1 - c^2$,

$$\epsilon_1(c, 1-c^2) = 5c^2 + 10c^3 - 5c^4 - 15c^5 + 5c^7 = \eta_1(c) \le \eta_1(0.7202) = 2.5799.$$

Thus, we have

$$|b_8| \le \frac{5}{48}.$$

The bounds hold for $c = 0$. The proof of Theorem 1 is completed. □

Theorem 2. *If $g \in \mathcal{B}T_{4l}$, then*

$$|H_{2,3}(g)| \quad \le \quad 0.044516.$$

Proof. Let $g \in \mathcal{B}T_{4l}$. From (8), we receive

$$|H_{2,3}(g)| = |b_3 b_5 - b_4^2| = \frac{5}{1728} \left| 16c_2c_4 - 15c_3^2 \right| = \frac{5}{1728} |15(c_2c_4 - c_3^2) + c_2c_4|. \tag{12}$$

Utilizing inequality, Lemma 1 and Lemma 2 in (12), we get

$$\begin{aligned} |H_{2,3}(g)| \quad & \le \frac{5}{1728}[15|c_2c_4 - c_3^2| + |c_2||c_4|] \\ & \le \frac{5}{1728}[15(1-|c_1|^2)^2 + |c_2|(1-|c_1|^2 - |c_2|^2)] \\ & = \frac{5}{1728}[15 - 30|c_1|^2 + 15|c_1|^4 + (1-|c_1|^2)|c_2| - |c_2|^3)] \\ & \le \frac{5}{1728}[15 - 30|c_1|^2 + 15|c_1|^4 + (1-|c_1|^2)|c_2| - |c_2|^3]. \end{aligned}$$

Let $|c_1| = c$ and $|c_2| = d$, we obtain

$$|H_{2,3}(g)| \quad \le \frac{5}{1728}\epsilon_2(c, d).$$

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Consider

$$\begin{cases} \frac{\partial \epsilon_2}{\partial c} = -60c + 60c^3 - 2cd = 0, \\ \frac{\partial \epsilon_2}{\partial d} = 1 - c^2 - 3d^2 = 0. \end{cases}$$

Applying numerical computations, we get

$$\begin{cases} c_1 = 0, \\ d_1 = -0.5774, \end{cases} \quad \begin{cases} c_2 = 0, \\ d_2 = 0.5774, \end{cases} \quad \begin{cases} c_3 = -1, \\ d_3 = 0, \end{cases} \quad \begin{cases} c_4 = 1, \\ d_4 = 0, \end{cases} \quad \begin{cases} c_5 = 0.9998, \\ d_5 = -0.0111, \end{cases}$$

$$\begin{cases} c_6 = -0.9998, \\ d_6 = -0.0111. \end{cases}$$

Thus, there is no critical point in $(0,1) \times (0, 1-c^2)$.

(1)For $d = 0$,

$$\epsilon_2(c, 0) = 15 - 30c^2 + 15c^4 \leq 15.$$

(2)For $c = 0$,

$$\epsilon_2(0, d) = 15 + d - d^3 \leq \epsilon_2(0, \frac{\sqrt{3}}{3}) = \frac{2\sqrt{3} + 135}{9} = 15.384888.$$

(3)For $d = 1 - c^2$,

$$\epsilon_2(c, 1-c^2) = 15 - 29c^2 + 13c^4 + c^6 \leq 15.$$

Therefore, we yield

$$|H_{2,3}(g)| \leq \frac{10\sqrt{3} + 675}{15552} = 0.044516.$$

The proof of Theorem 2 is completed. $\square$

Theorem 3. *If $g \in \mathcal{B}T_{4l}$, then*

$$|H_{3,1}(g)| \quad \leq \quad \frac{3025}{46656} = 0.064836.$$

Proof. Assume that $g \in \mathcal{B}T_{4l}$. From (8) and (2), we achieve

$$\begin{aligned} |H_{3,1}(g)| &= \frac{1}{46656} \left| 2250c_1c_2c_3 - 1000c_2^3 - 2025c_3^2 + 2160c_2c_4 - 1350c_1^2c_4 \right| \\ &= \frac{1}{46656} \left| 1000c_2(c_1c_3 - c_2^2) + 1250c_1c_2c_3 + 2025(c_2c_4 - c_3^2) + 135c_2c_4 - 1350c_1^2c_4 \right| \end{aligned} \tag{13}$$

By applying the triangle inequality, Lemma 1 and Lemma 2 in (13), we receive

$$\begin{aligned}46656\,|H_{3,1}(g)| \quad &\le 1000|c_2||c_1c_3-c_2^2|+1250|c_1||c_2||c_3|+2025|c_2c_4-c_3^2|+135|c_2||c_4|+1350|c_1|^2|c_4| \\ &\le 1000|c_2|(1-|c_1|^2)+1250|c_1||c_2|(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|})+2025(1-|c_1|^2)^2 \\ &+135|c_2|(1-|c_1|^2-|c_2|^2)+1350|c_1|^2(1-|c_1|^2-|c_2|^2) \\ &= 2025-2700|c_1|^2+675|c_1|^4+(1135+1250|c_1|-1135|c_1|^2-1250|c_1|^3)|c_2| \\ &-1350|c_1|^2|c_2|^2-\frac{135+1385|c_1|}{1+|c_1|}|c_2|^3.\end{aligned}$$

Setting $|c_1|=c$ and $|c_2|=d$, we have

$$\begin{aligned}46656\,|H_{3,1}(g)| \quad \le \quad & 2025-2700c^2+675c^4+(1135+1250c-1135c^2-1250c^3)d \\ & -1350c^2d^2-\frac{135+1385c}{1+c}d^3=\epsilon_3(c,d)\end{aligned}$$

where $c\in[0,1]$ and $d\in[0,1-c^2]$.

Taking the partial derivative with respect to c and y respectively, and we have

$$\frac{\partial\epsilon_3}{\partial c}=-5400c+2700c^3+(1250-2270c-3405c^2)d-2700cd^2-\frac{1250}{(1+c)^2}d^3$$

and

$$\frac{\partial\epsilon_3}{\partial d}=1135+1250c-1135c^2-1250c^3-2700c^2d-\frac{405+4155c}{1+c}d^2.$$

Setting $\frac{\partial\epsilon_3}{\partial c}=\frac{\partial\epsilon_3}{\partial d}=0$ and simplifying, we yield

$$\begin{cases}-5400c-10800c^2-2700c^3+5400c^4+2700c^5+(1250+230c-6695c^2-9080c^3-3405c^4)d \\ -2700c(1+c)^2d^2-1250d^3=0, \\ 1135+2358c+115c^2-2385c^3-1250c^4-2700c^2(1+c)d-(405+4155c)d^2=0.\end{cases}$$

Applying Newton's methods, we yield

$$\begin{cases}c_1=-1, \\ d_1=0,\end{cases}\quad \begin{cases}c_2=0.1018, \\ d_2=-1.3080,\end{cases}\quad \begin{cases}c_3=1.0726, \\ d_3=-1.1909,\end{cases}\quad \begin{cases}c_4=1.7878 \\ d_4=-0.3706,\end{cases}\quad \begin{cases}c_5=3.3060, \\ d_5=-6.5565,\end{cases}$$

$$\begin{cases}c_6=-1.3977, \\ d_6=0.0899.\end{cases}$$

Thus, there is no critical point in $(0,1)\times(0,1-c^2)$.

For $c=0$,

$$\epsilon_3(0,d)=2025+1135d-135d^3\le\epsilon_3(0,1)=3025.$$

For $d=0$,

$$\epsilon_3(c,0)=2025-2700c^2+675c^4\le 2025.$$

For $d=1-c^2$,

$$\epsilon_3(c,0)=3025-4665c^2+1605c^4+35c^6\le\epsilon_3(0,0)=3025.$$

Thus, we obtain

$$|H_{3,1}(g)|\le\frac{3025}{46656}=0.064836.$$

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The proof of Theorem 3 is completed. □

Theorem 4. *If $g \in \mathcal{B}T_{4l}$, then*

$$|b_2b_5 - b_3b_4| \leq \frac{15.503407}{432} = 0.035888.$$

Proof. Let $g \in \mathcal{B}T_{4l}$. From (8), we obtain

$$|b_2b_5 - b_3b_4| = \frac{1}{432}|30c_1c_4 - 25c_2c_3|. \tag{14}$$

Applying Lemma 1 and the triangle inequality in (14), we get

$$\begin{aligned}|b_2b_5 - b_3b_4| &\leq \frac{1}{432}[30|c_1|(1-|c_1|^2-|c_2|^2)+25|c_2|(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|})]\\ &= \frac{1}{432}\left[30|c_1|-30|c_1|^3+(25-25|c_1|^2)|c_2|-30|c_1||c_2|^2-\frac{25}{1+|c_1|}|c_2|^3\right].\end{aligned}$$

Setting $|c_1| = c$ and$|c_2| = d$, we yield

$$|b_2b_5 - b_3b_4| \leq \frac{1}{432}\epsilon_4(c,d).$$

Taking the partial derivative with respect to c, and d respectively, we get

$$\frac{\partial\epsilon_4}{\partial c} = 30 - 90c^2 - 50cd - 30d^2 + \frac{25}{(1+c)^2}d^3$$

and

$$\frac{\partial\epsilon_4}{\partial d} = 25 - 25c^2 - 60cd - \frac{75}{1+c}d^2.$$

Setting $\frac{\partial\epsilon_4}{\partial c} = \frac{\partial\epsilon_4}{\partial d} = 0$,and simplifying, we receive

$$\begin{cases} -90c^4 - 180c^3 - 60c^2 + 60c + 30 - 50c(1+c)^2d - 30(1+c)^2d^2 + 25d^3 = 0,\\ -25c^3 - 25c^2 + 25c + 25 - 60(c^2+c)d - 75d^2 = 0.\end{cases}$$

Applying Newton's methods, we have $\begin{cases} c_1 = -1,\\ d_1 = 0,\end{cases}$ $\begin{cases} c_2 = 0.4098,\\ d_2 = -0.8978,\end{cases}$ $\begin{cases} c_3 = 0.4271,\\ d_3 = 0.4258,\end{cases}$

$\begin{cases} c_4 = -0.4550\\ d_4 = -0.2931,\end{cases}$ $\begin{cases} c_5 = -0.7258,\\ d_5 = 0.3023.\end{cases}$

Therefore, we get $\varepsilon_4(c,d) \leq \varepsilon_4(0.4271, 0.4258) = 15.503407$.

(a)For $c = 0$,

$$\varepsilon_4(0,d) = 25d - 25d^3 \leq \varepsilon_4(0, \frac{\sqrt{3}}{3}) = \frac{50\sqrt{3}}{9}.$$

(b)For $d = 0$,

$$\varepsilon_4(c,0) = 30c - 30c^3 \leq \varepsilon_4(\frac{\sqrt{3}}{3}, 0) = \frac{20\sqrt{3}}{3} = 11.546666.$$

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(b)For $d = 1 - c^2$,

$$\varepsilon_4(c, 1-c^2) = 25c - 20c^3 - 5c^5 = \eta_2(c) = \eta_2(0.6017) = 10.2913.$$

Thus, we get

$$|b_2b_5 - b_3b_4| \le \frac{15.503407}{432} = 0.035888.$$

□

Theorem 5. *If $g \in \mathcal{B}T_{4l}$, then*

$$|b_2b_4 - b_3^2| \le \frac{200.27}{2592} = 0.077265.$$

Proof. Let $g \in \mathcal{B}T_{4l}$. From (8), we get

$$\begin{aligned}|b_2b_4 - b_3^2| &= \frac{1}{2592}|225c_1c_3 - 200c_2^2| \\ &= \frac{1}{2592}|200(c_1c_3 - c_2^2) + 25c_1c_3|.\end{aligned}$$

Applying Lemma 1, Lemma 2 and the triangle inequality, we receive

$$\begin{aligned}|b_2b_4 - b_3^2| &\le \frac{1}{2592}[200(1-|c_1|^2) + 25|c_1|(1-|c_1|^2 - \frac{|c_2|^2}{1+|c_1|})] \\ &= \frac{1}{2592}[200 + 25|c_1| - 200|c_1|^2 - 25|c_1|^3 - \frac{|c_1|}{1+|c_1|}|c_2|^2] \\ &\le \frac{1}{2592}(200 + 25|c_1| - 200|c_1|^2 - 25|c_1|^3) \\ &= \frac{1}{2592}\eta_3(|c_1|) \le \frac{1}{2592}\eta_3(0.0618) = \frac{200.2700}{2592} = 0.077265.\end{aligned}$$

□

Theorem 6. *If $g \in \mathcal{B}T_{4l}$, then*

$$|H_{3,2}(g)| \le 0.025986.$$

Proof. Let $g \in \mathcal{B}T_{4l}$. From Lemma 3, Theorem 1, Theorem 2, Theorem 4 and Theorem 5, we yield

$$\begin{aligned}|H_{3,2}(g)| &\le |b_4||b_3b_5 - b_4^2| + |b_5||b_2b_5 - b_3b_4| + |b_6||b_2b_4 - b_3^2| \\ &= \frac{5}{24} \times 0.044516 + \frac{1}{6} \times 0.035888 + \frac{5}{36} \times 0.077265 \\ &= 0.025986.\end{aligned}$$

□

Theorem 7. *If $g \in \mathcal{B}T_{4l}$, then*

$$|b_4b_6 - b_5^2| \le \frac{24.385252}{864} = 0.028224.$$

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Proof. Let $f \in \mathcal{B}T_{4l}$. From (8) and (9), we have

$$|b_4b_6 - b_5^2| = \frac{1}{864}|25c_3c_5 - 24c_4^2 + 5c_3c_1^5| = \frac{1}{864}|24(c_3c_5 - c_4^2) + c_3c_5 + 5c_3c_1^5|.$$

Using Lemma 1 and 2, we have

$$\begin{aligned}
|b_4b_6 - b_5^2| &\leq \frac{1}{864}(24|c_3c_5 - c_4^2| + |c_3||c_5| + 5|c_3||c_1^5|) \\
&\leq \frac{1}{864}[24(1-|c_1|^2) + |c_3|(1-|c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1+||c_1}) + 5|c_1|^5|c_3|] \\
&= \frac{1}{864}[24 - 24|c_1|^2 + (1-|c_1|^2 + 5|c_1|^5 - |c_2|^2)|c_3| - \frac{1}{1+|c_1|}|c_3|^3].
\end{aligned}$$

Setting $c = |c_1|, d = |c_2|$ and $e = |c_3|$, we yield

$$|b_4b_6 - b_5^2| \qquad \leq \frac{1}{864}\epsilon_5(c,d,e)$$

where $\epsilon_5(c,d,e) = 24 - 24c^2 + (1 - c^2 + 5c^5 - d^2)e - \frac{1}{1+c}e^3$ and $(c,d,e) \in \Omega = \{(c,d,e) : 0 \leq 1, 0 \leq d \leq 1 - c^2, 0 \leq e \leq 1 - c^2 - \frac{d^2}{1+c}\}$. Consider

$$\frac{\partial \epsilon_5}{\partial d} = -2de \leq 0,$$

thus there are no points in Ω.

(1)For $c = 0$,

$$\epsilon_5(0,d,e) = 24 + (1-d^2)e - e^3 = \eta_4(d,e).$$

It is evident that there is on point in $(0,1) \times (0, 1-d^2)$.

(2)For $d = 0$,

$$\epsilon_5(c,0,e) = 24 - 24c^2 + (1 - c^2 + 5c^5)e - \frac{1}{1+c}e^3 = \eta_5(c,e).$$

Partial derivative of η_5 with respect to c, and then with respect to e, we achieve

$$\frac{\partial \eta_5}{\partial c} = -48c + (-2c + 25c^4)e + \frac{1}{(1+c)^2}e^3,$$

and

$$\frac{\partial \eta_5}{\partial e} = 1 - c^2 + 5c^5 - \frac{3}{1+c}e^2.$$

Setting $\frac{\partial \eta_5}{\partial c} = \frac{\partial \eta_5}{\partial e} = 0$ and simplifying, we yield

$$\begin{cases} -48c(1+c)^2 + (25c^6 + 50c^5 + 25c^4 - 2c^3 - 4c^2 - 2c)e + e^3 = 0, \\ 5c^6 + 5c^5 - c^3 - c^2 + c + 1 - 3e^2 = 0. \end{cases}$$

We obtain a critical point $(0.0039, 0.5785)$, thus, we have $\eta_5(c,e) \leq \eta_5(0.0039, 0.5785) = 24.385252$.

(3)For $e = 0$,

$$\epsilon_5(c,d,0) = 24 - 24c^2 \leq 24.$$

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(4)For $e = 1 - c^2 - \frac{d^2}{1+c}$,

$$\begin{aligned}\epsilon_5(c, d, 1 - c^2 - \frac{d^2}{1+c}) \quad &= 24 + c - 24c^2 - 2c^3 + 6c^5 - 5c^7 + \frac{-5c^5 + 4c^3 - c^2 - 4c + 1}{1+c}d^2 \\ &+ \frac{-2+4c}{(1+c)^2}d^4 + \frac{1}{(1+c)^4}d^6 = \eta_6(c,d) \leq \eta_6(0.0016, 0.5767) = 24.148169.\end{aligned}$$

(5)For $c = d = 0$

$$\epsilon_5(0, 0, e) = 24 + e - e^3 = \eta_7(e) \leq \eta_7(\frac{\sqrt{3}}{3}) = 24 + \frac{2\sqrt{3}}{9} = 24.384889.$$

(6)For $c = e = 0$,

$$\epsilon_5(0, d, 0) = 24.$$

(7)For $d = e = 0$,

$$\epsilon_5(c, 0, 0) = 24 + e - e^3 \leq 24.384889.$$

(8)For $e = 0, d = 1 - c^2$,

$$\epsilon_5(c, 1 - c^2, 0) = 24 - 24c^2 \leq 24.$$

(9)For $d = 0$ and $e = 1 - c^2$,

$$\epsilon_5(c, 0, 1 - c^2) = 24 + c - 24c^2 - 2c^3 + 6c^5 - 5c^7 = \eta_8(c) \leq \eta_8(0.0206) = 24.0104.$$

(10)For $c = 0, e = 1 - d^2$

$$\epsilon_5(0, d, 1 - d^2) = 24 + d^2 - 2d^4 + d^6 = \eta_9(d) \leq \eta_9(\frac{\sqrt{3}}{3}) = 24.148148.$$

Thus, we get

$$|b_4 b_6 - b_5^2| \leq \frac{24.385252}{864} = 0.035888.$$

□

Theorem 8. *If $g \in \mathcal{B}T_{4l}$, then*

$$|b_3 b_6 - b_4 b_5| \leq \frac{5}{144}.$$

The bound is sharp.

Proof. Let $g \in \mathcal{B}T_{4l}$. From (8) and (9), we receive

$$|b_3 b_6 - b_4 b_5| = \frac{1}{1296}|50c_2c_5 - 45c_3c_4 + 50c_2c_1^5|.$$

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Using Lemma 1, we get

$$\begin{aligned}|b_3b_6 - b_4b_5| &\le \frac{1}{1296}(50|c_2||c_5| + 45|c_3||c_4| + 50|c_2||c_1^5|)\\ &\le \frac{1}{1296}[50|c_2|(1-|c_1|^2-|c_2|^2-\frac{|c_3|^2}{1+|c_1|}) + 45|c_3|(1-|c_1|^2-|c_2|^2) + 50|c_1|^5|c_2|]\\ &= \frac{1}{1296}[(50-50|c_1|^2+50|c_1|^5)|c_2| - 50|c_2|^3 + (45-45|c_1|^2-45|c_2|^2)|c_3| - \frac{50|c_2|}{1+|c_1|}|c_3|^2].\end{aligned}$$

By setting $|c_1| = c, |c_2| = d$ and $|c_3| = e$, we have

$$|b_3b_6 - b_4b_5| \quad \le \frac{1}{1296}\Psi(c,d,e)$$

where $\Psi(c,d,e) = (50-50c^2+50c^5)d - 50d^3 + (45-45c^2-45d^2)e - \frac{50d}{1+c}e^2$, $(c,d,e) \in \Omega$.

Differentiating partially with respect to c, d and e, respectively, we get

$$\frac{\partial\Psi}{\partial c} = (-100c + 250c^4)d - 90ce + \frac{50d}{(1+c)^2}e^2,$$

$$\frac{\partial\Psi}{\partial d} = 50 - 50c^2 + 50c^5 - 150d^2 - 90de - \frac{50}{1+c}e^2,$$

and

$$\frac{\partial\Psi}{\partial e} = 45 - 45c^2 - 45d^2 - \frac{100d}{1+c}e.$$

By putting $\frac{\partial\Psi}{\partial c} = \frac{\partial\Psi}{\partial d} = \frac{\partial\Psi}{\partial e} = 0$, and simplifying, we obtain

$$\begin{cases}(250c^6 + 500c^5 + 250c^4 - 100c^3 - 200c^2 - 100c)d - 900c(1+c)^2e + 50de^2 = 0,\\ 50c^6 + 50c^5 - 50c^3 - 50c^2 + 50c + 50 - 150(1+c)d^2 - 90(1+c)de - 50e^2 = 0,\\ -45c^3 - 45c^2 + 45c + 45 - 45(1+c)d^2 - 100de = 0.\end{cases}$$

By a numberical caculation, we get

$$\begin{cases} c_1 = -1,\\ d_1 = d,\\ e_1 = 0,\end{cases} \quad \begin{cases} c_2 = 0.8441,\\ d_2 = 0.4127,\\ e_2 = 0.2354,\end{cases} \quad \begin{cases} c_3 = 0.8441,\\ d_3 = -0.4127,\\ e_3 = -0.2354.\end{cases}$$

Thus, there's no critical point which satisfies $0 \le c \le 1$ and $0 \le d \le 1 - c^2$.

(1)For $c = 0$,

$$\Psi(0,d,e) = 50d - 50d^3 + (45-45d^2)e - 50de^2 = \Lambda_1(d,e).$$

Consider

$$\begin{cases}\frac{\partial\Lambda_1}{\partial d} = 50 - 150d^2 - 90de - 50e^2 = 0,\\ \frac{\partial\Lambda_1}{\partial e} = 45 - 45d^2 - 100de = 0.\end{cases}$$

A numberrical caculation that there is no critical point in $(0,1) \times (0, 1-d^2)$.

(2)For $d = 0$,

$$\Psi(c,0,e) = (45-45c^2)e \le 45(1-c^2)^2 \le 45.$$

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(3)For $e=0$,

$$\Psi(c,d,0)=(50-50c^2+50c^5)d-50d^3=\Lambda_2(c,d)\leq\Lambda_2(0.7368,0.2828)=8.403278.$$

(4)For $e=1-c^2-\frac{d^2}{1+c}$,

$$\begin{aligned}\Psi(c,d,1-c^2-\frac{d^2}{1+c})&=45-90c^2+45c^4+(50c-50c^3+50c^5)d+(-90+45c+45c^2)d^2\\&+\frac{50-150c}{1+c}d^3+\frac{45}{1+c}d^4-\frac{50}{(1+c)^3}d^5=\Lambda_3(c,d).\end{aligned}$$

Differentiating Λ_3 partially with respect to c and d, we yield

$$\begin{aligned}\frac{\partial\Lambda_3}{\partial c}&=-180c+180c^3+(50-150c^2+250c^4)d+(45+90c)d^2-\frac{100d^3}{(1+c)^2}\\&-\frac{45d^4}{(1+c)^2}+\frac{150d^5}{(1+c)^4}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\Lambda_3}{\partial d}&=50c-50c^3+50c^5+(-180+90c+90c^2)d+\frac{150-450c}{1+c}d^2+\frac{180d^3}{1+c}\\&-\frac{250d^4}{(1+c)^3}.\end{aligned}$$

Setting $\frac{\partial\Lambda_3}{\partial c}=\frac{\partial\Lambda_3}{\partial d}=0$ and simplifying, we receive

$$\begin{cases}180c^7+720c^6+900c^5-900c^3-720c^2-180c+(250c^8+1000c^7+1350c^6+400c^5\\-600c^4-400c^3+150c^2+200c+50)d+(90c^5+405c^4+720c^3+630c^2+270c+45)d^2\\-100(1+c)^2d^3-45(1+c)^2d^4+150d^5=0,\\50c^8+150c^7+100c^6-100c^5-100c^4+100c^3+150c^2+50c+(90c^5+360c^4+360c^3-180c^2\\-450c-180)d+(-450c^3-750c^2-150c+150)d^2+180(1+c)^2d^3-250d^4=0\end{cases}$$

Applying Newton's methods, we recieve

$$\begin{cases}c_1=-1\\d_1=0,\end{cases}\quad\begin{cases}c_2=0,\\d_2=0,\end{cases}\quad\begin{cases}c_3=0.8336,\\d_3=0.4141,\end{cases}\quad\begin{cases}c_4=-0.9194,\\d_4=-0.0957.\end{cases}$$

Thus, there is no critical point satisfing $0\leq c\leq 1$ and $0\leq d\leq 1-c^2$.

(5)For $d=c=0$,

$$\Psi(0,0,e)=45e\leq 45.$$

(6)For $e=d=0$,

$$\Psi(c,0,0)=0.$$

(7)For $c=e=0$,

$$\Psi(0,d,0)=50d-50d^3=\Lambda_4(d)\leq\Lambda_4(\frac{\sqrt{3}}{3})=\frac{100\sqrt{3}}{9}=19.2444.$$

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(8)For $d = 1 - c^2$ and $e = 0$,

$$\Psi(c, 1 - c^2, 0) = 50c^2 - 100c^4 + 50c^5 + 50c^6 - 50c^7 = \Lambda_5(c) \leq \Lambda_5(0.7145) = 10.6734.$$

(9)For $e = 1 - c^2$ and $d = 0$,

$$\Psi(c, 0, 1 - c^2) = 45(1 - c^2)^2 \leq 45.$$

(10)For $e = 1 - d^2$ and $c = 0$,

$$\Psi(0, d, 1 - c^2) = 45 - 90d^2 + 50d^3 + 45d^4 - 50d^5 = \Lambda_6(d) \leq \Lambda_6(0) = 45.$$

Hence, we get

$$|b_3 b_6 - b_4 b_5| \leq \frac{45}{1296} = \frac{5}{144}.$$

The equality holds for $c_1 = c_2 = 0$ and $c_3 = c_4 = 1$. □

Theorem 9. *If $g \in \mathcal{B}T_{4l}$, then*

$$|H_{3,3}(g)| \leq 0.016103.$$

Proof. Let $g \in \mathcal{B}T_{4l}$. From Lemma 3, Theorem 1, Theorem 2, Theorem 3 and 4, we receive

$$\begin{aligned} |H_{3,3}(g)| &\leq |b_5||b_4 b_6 - b_5^2| + |b_6||b_3 b_6 - b_4 b_5| + |b_7||b_3 b_5 - b_4^2| \\ &\leq \frac{1}{6} \times 0.035888 + \frac{5}{36} \times \frac{5}{144} + \frac{5}{42} \times 0.044516 \\ &= 0.016103. \end{aligned}$$

□

3. The bounds of the logarithmic coefficients for $g \in \mathcal{B}T_{4l}$

Theorem 10. *If $g \in \mathcal{B}T_{4l}$, then*

$$|\delta_1| \leq \frac{5}{24}, \quad |\delta_2| \leq \frac{5}{36}, \quad |\delta_3| \leq \frac{5}{48}, \quad |\delta_4| \leq \frac{13827}{165888} = 0.083351, |\delta_5| \leq \frac{173400}{2488320} = 0.069686.$$

The first three bounds are the best possible.

Proof. Let $g \in \mathcal{B}T_{4l}$. From(6),(8) and (9), we have

$$\delta_1 = \frac{5c_1}{24}, \tag{15}$$

$$\delta_2 = \frac{5}{36}(c_2 - \frac{5}{8}c_1^2), \tag{16}$$

$$\delta_3 = \frac{5}{48}(c_3 - \frac{5}{9}c_1 c_2 + \frac{25}{216}c_1^3), \tag{17}$$

$$\delta_4 = \frac{1}{165888}(13824c_4 - 7200c_1 c_3 + 4000c_1^2 c_2 - 3200c_2^2 - 625c_1^4). \tag{18}$$

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$$\delta_5 = \frac{1}{2488320}(172800c_5 - 86400c_1c_4 - 72000c_2c_3 + 45000c_1^2c_3 + 40000c_1c_2^2 - 5000c_1^3c_2 + 37685c_1^5). \quad (19)$$

Applying Lemma 4 to (15), we have

$$|\delta_1| \le \frac{5}{24}$$

Applying Lemma 5 to (16), we get

$$|\delta_2| \le \frac{5}{36}.$$

Utilizing the triangle inequality and Lemma 6 with $\gamma = -\frac{5}{9}$ and $\zeta = \frac{25}{216}$, we yield

$$|\delta_3| \le \frac{5}{48}.$$

Rearranging (18), we obtain

$$\begin{aligned} |\delta_4| &= \frac{1}{165888}|6912(c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{1}{8}c_1^4) + 6912c_4 - 288c_1c_3 + 256c_2^2 - 1184c_1^2c_2 + 239c_1^4| \\ &\le \frac{1}{165888}|6912(c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{1}{8}c_1^4)| + \frac{1}{165888}|6912c_4 - 288c_1c_3 + 256c_2^2 - 1184c_1^2c_2 + 239c_1^4| \\ &= \frac{1}{165888}D_1 + \frac{1}{165888}D_2, \end{aligned}$$

where $D_1 = |6912(c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{1}{8}c_1^4)|$

$$D_2 = |6912c_4 - 288c_1c_3 + 256c_2^2 - 1184c_1^2c_2 + 239c_1^4|. \quad (20)$$

Using Lemma 7 with $\gamma = -\frac{1}{2}$, we get $D_1 \le 6912$. Rearranging (20), we get

$$D_2 = |6912c_4 - 288c_1(c_3 + 2c_1^2c_2 + c_1^4) + 256c_2^2 - 608c_1^2c_2 + 527c_1^4|.$$

Using Lemma 1, Lemma 6 and the triangle inequality, we receive

$$\begin{aligned} D_2 &\le 6912(1 - |c_1|^2 - |c_2|^2) + 288|c_1| + 256|c_2|^2 + 608|c_1|^2|c_2| + 527|c_1|^4 \\ &= 6912 + 288|c_1| - 6912|c_1|^2 + 527|c_1|^4 + 608|c_1|^2|c_2| - 6656|c_2|^2 = \Upsilon_1(c,d) \end{aligned}$$

where $|c_1| = c, |c_2| = d$.

Consider

$$\begin{cases} \frac{\partial \Upsilon_1}{\partial c} = 288 - 13824c + 2108c^3 + 1216cd = 0, \\ \frac{\partial \Upsilon_1}{\partial d} = 608c^2 - 13312d = 0 \end{cases}$$

Applying Newton's methods, we have

$$\begin{cases} c_1 = 2.5173 \\ d_1 = 0.2894, \end{cases} \quad \begin{cases} c_2 = -2.5173, \\ d_2 = 0.2942, \end{cases} \quad \begin{cases} c_3 = 0.0208, \\ d_3 = 0. \end{cases}$$

Thus, in $(0,1) \times (0, 1-c^2)$, there is no critical point.

(1)For $c = 0$,

$$\Upsilon_1(0,d) = 6912 - 6912d^2 \le 6912.$$

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(2)For $d=0$,

$$\Upsilon_1(c,0)=6912+288c-6912c^2+527c^4=\gamma_1(c)\leq\gamma_1(0.0208)=6915.$$

(3)For $d=1-c^2$,

$$\Upsilon_1(c,1-c^2)=256+288c+7008c^2-6737c^4=\gamma_2(d)\leq\gamma_2(0.7313)=2287.6.$$

Therefore, we have

$$|\gamma_4|\leq\frac{13827}{165888}=0.083351.$$

Rearranging (19), we obtain

$$\begin{aligned}|\delta_5| &= \frac{1}{2488320}|86400(c_5-c_1c_4-c_2c_3+\frac{3}{4}c_1c_2^2+\frac{3}{4}c_1^2c_3-\frac{1}{2}c_1^3c_2+\frac{1}{16}c_1^5)\\ &+86400c_5+14400c_2c_3-19800c_1^2c_3-24800c_1c_2^2+38200c_1^3c_2+32285c_1^5|\\ &\leq\frac{1}{2488320}|86400(c_5-c_1c_4-c_2c_3+\frac{3}{4}c_1c_2^2+\frac{3}{4}c_1^2c_3-\frac{1}{2}c_1^3c_2+\frac{1}{16}c_1^5)|\\ &+\frac{1}{2488320}|86400c_5+14400c_2c_3-19800c_1^2c_3-24800c_1c_2^2+38200c_1^3c_2+32285c_1^5|\\ &=\frac{1}{2488320}D_3+\frac{1}{2488320}D_4,\end{aligned}$$

where $D_3=|86400(c_5-c_1c_4-c_2c_3+\frac{3}{4}c_1c_2^2+\frac{3}{4}c_1^2c_3-\frac{1}{2}c_1^3c_2+\frac{1}{16}c_1^5)|$ and

$$D_4 = |86400c_5+14400c_2c_3-19800c_1^2c_3-24800c_1c_2^2+38200c_1^3c_2+32285c_1^5|. \tag{21}$$

Using Lemma 8 with $\gamma=-\frac{1}{2}$, we get $D_3\leq 86400$. Rearranging (21), we get

$$D_4 = |86400c_5+14400c_2(c_3-2c_1c_2+c_1^3)-19800c_1^2(c_3-c_1c_2-c_1^3)+4000c_1c_2^2+4000c_1^3c_2+12485c_1^5|.$$

Utilizing the triangle inequality, Lemma 1, Lemma 6 and 5, we obtain

$$\begin{aligned}D_4 &\leq 86400|c_5|+14400|c_2||c_3-2c_1c_2+c_1^3|+19800|c_1|^2|c_3-c_1c_2-c_1^3|+4000|c_1||c_2|^2+4000|c_1|^3|c_2|+12485|c_1|^5\\ &\leq 86400(1-|c_1|^2-|c_2|^2-\frac{|c_3|^2}{1+|c_1|})+14400|c_2|+19800|c_1|^2+4000|c_1||c_2|^2+4000|c_1|^3|c_2|+12485|c_1|^5\\ &=86400-666000|c_1|^2+12485|c_1|^5+(14400+4000|c_1|^3)|c_2|+(-86400+4000|c_1|)|c_2|^2-86400\frac{|c_3|^2}{1+|c_1|}\\ &\leq 86400-666000|c_1|^2+12485|c_1|^5+(14400+4000|c_1|^3)|c_2|+(-86400+4000|c_1|)|c_2|^2=\Upsilon_2(c,d)\end{aligned}$$

where $|c_1|=c$, $|c_2|=d$. Consider

$$\begin{cases}\frac{\partial\Upsilon_2}{\partial c}=-133200c+62425c^4+12000c^2d+4000d^2=0,\\ \frac{\partial\Upsilon_2}{\partial d}=14400+4000c^3+(-172800+8000c)d=0.\end{cases}$$

We have a critical point $(0.000209, 0.083334)$. Thus, we get $\Upsilon_2(c,d)\leq\Upsilon_2(0.000209,0.083334)=86999.968$.

(1)For $c=0$,

$$\Upsilon_2(0,d)=86400+14400d-86400d^2=\gamma_3(d)\leq\gamma_3(\frac{1}{12})=87000.$$

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(2)For $d = 0$,

$$\mathrm{Y}_2(c,0) = 86400 - 66600c^2 + 12485c^5 \leq 864000.$$

(3)For $d = 1 - c^2$,

$$\mathrm{Y}_2(c,1-c^2) = 14400 + 4000c + 91800c^2 - 4000c^3 - 86400c^4 + 12485c^5 = \gamma_4(c) \leq \gamma_4(0.7771) = 43098..$$

Therefore, we receive

$$|\delta_5| \leq \frac{173400}{2488320} = 0.069686.$$

The proof of Theorem 10 is completed. □

4. Conclusion

In this paper, we considered a subclass of bounded turning functions linked with a four-leaf-type domain. Utilizing the estimates of the coefficients of the Schwartz function, we obtained the coefficients of $|b_6|, |b_7|, |b_8|$, and the third-order determinants of $H_{3,2}, H_{3,3}$ of the class $\mathcal{BT}_{4l}$ for the first time. Also, one can easily use this new methodology to obtain the bounds the coefficients of $|b_6|, |b_7|, |b_8|$, and the third-order Hankel determinant of $H_{3,2}, H_{3,3}$ for other subclasses of univalent functions.

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