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Jeśmanowicz Conjecture and Gaussian Integer Ring

Proof of Jeśmanowicz Conjecture Based on Gaussian Integer Ring

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Abstract: Let a,b,c be positive integers such that $a^2 + b^2 = c^2, 2|b, \gcd(a, b) = 1$. In 1956, Jeśmanowicz conjectured that for any positive integer w, the only solution of $(aw)^x + (bw)^y = (cw)^z$ in positive integers is (x, y, z) = (2, 2, 2). In this paper, based on Gaussian integer ring, we show that Jeśmanowicz' conjecture is true for any positive integer $w \geq 1$.

Keywords: Jeśmanowicz conjecture; Diophantine equation; Gaussian integer ring; $4k + 1$ type prime number

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1. Notation

N^+: Set of positive integers.
$Z[i] = \{z = x + yi : x, y \in N, i = \sqrt{-1}\}$:Gaussian integer ring.
$x + yi$: Pythagorean Gaussian integer, where $x, y \in N^+, x > y, 2|xy, \gcd(x, y) = 1$.
All circles are centered at the origin of the complex plane.

2. Introduction

Let $a, b, c \in N^+$ satisfy $a^2 + b^2 = c^2$, where $2|b, \gcd(a, b) = 1$. Such a triplet (a, b, c) is called a Pythagorean triple. In 1956, Jeśmanowicz conjectured that for any positive integer w, the only solution of

$$(aw)^x + (bw)^y = (cw)^z, \qquad x, y, z \in N^+ \tag{1}$$

in positive integers is $(x, y, z) = (2, 2, 2)$. This is a unsolved problem on Pythagorean numbers. In the same year, Sierpiński[2] proved that the equation $3^x + 4^y = 5^z$ has $x = y = z = 2$ as its only solution in positive integers. Jeśmanowicz[1] showed that when $(a, b, c) \in \{(5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)\}$, the equation (1) has only the solution $(x, y, z) = (2, 2, 2)$. Ma and Chen[5] proved that if $4 \nmid mn$, then the equation

$$(m^2 - n^2)^x + (2mn)^y = (m^2 + n^2)^z, y \geq 2$$

only has the positive integer solution $(x, y, z) = (2, 2, 2)$. M.Tang[4] showed that Jeśmanowicz' conjecture is true for Pythagorean triples $(a, b, c) = (2^{2^k} - 1, 2^{2^{k-1}+1}, 2^{2^k} + 1)$. It should be emphasized that this is the first paper to prove Jeśmanowicz' conjecture holds on an infinite subset of Pythagorean triples (a, b, c).

Research shows that c in Pythagorean triple (a, b, c), each prime factor of which is a prime number modulo 4 with remainder 1, and both c and its prime factors can be factorized on the Gaussian integer ring. Based on these characteristics, we obtain the following result.

Theorem 1. *If* $\xi = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \cdots p_m^{e_m} (m \geq 1, e_k \geq 1)$ *and the prime factor* $p_k \equiv 1 (\mod 4)$, *then*

(1) *On the circumference with a radius equal to* $\sqrt{\xi}$, *there are* 2^{m-1} *Pythagorean Gaussian integers* $\alpha_k + i\beta_k (1 \leq k \leq 2^{m-1})$.

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()2) On this circumference with a radius equal to $(\sqrt{\xi})^n (n \geq 2)$, there are always 2^{m-1} Pythagorean Gaussian integers $(\alpha_k + i\beta_k)^n$.

(3) On the circumference with a radius equal to $(\sqrt{\xi})^n (n \geq 2)$, the real part and imaginary part of the kth Pythagorean Gaussian integer are polynomials about α_k and β_k.

Lemma 1. *If a positive integer ξ contains at least one prime factor of type 4k+3, then there is no Pythagorean Gaussian integer on the circumference with a radius of $(\sqrt{\xi})^n (n \geq 1)$.*

Theorem 2. *For each Pythagorean triplet (a, b, c), the Diophantine equation*

$$a^x + b^y = c^z \tag{2}$$

only has the solution $(x, y, z) = (2, 2, 2)$.

Theorem 3. *For each Pythagorean triplet (a, b, c) and any $w \in \mathbb{N}^+$, the Diophantine equation*

$$(aw)^x + (bw)^y = (cw)^z \tag{3}$$

only has the solution $(x, y, z) = (2, 2, 2)$.

3. The Basic Properties of Prime Number

Assuming the prime number $p \equiv 1(\mod 4)$. On the integer ring, it can be uniquely represented as $p = \alpha^2 + \beta^2$, where $\alpha, \beta \in N^+, \gcd(\alpha, \beta) = 1$, $\alpha > \beta$. However, on $Z[i]$, p is not a prime number and has four different factorizations.

$$p = (\alpha + \beta i)(\alpha - \beta i) \tag{4}$$
$$p = (-\alpha + \beta i)(-\alpha - \beta i) \tag{5}$$
$$p = (\beta + \alpha i)(\beta - \alpha i) \tag{6}$$
$$p = (-\beta + \alpha i)(-\beta - \alpha i) \tag{7}$$

The formula (4) is called the main decomposition formula, and the other three formulas are called

Figure 1. The four conjugate pairs of p

auxiliary decomposition formulas. These complex numbers in the four decomposition formulas are Gaussian integers on the circumference with the radius equal to $\sqrt{p}$, as shown in Figure 1. Among these 8 complex numbers, only $\alpha + \beta i$ is a Pythagorean Gaussian integer, and the other 7 complex numbers are called the images of $\alpha + \beta i$. Define a transformation function $\Theta(z)$ that maps the images to Pythagorean Gaussian integers.

$$\alpha + \beta i = \begin{cases} \Theta(\pm\alpha \pm \beta i) \\ \Theta(\pm\beta \pm \alpha i) \end{cases}$$

4. Lemmas

Let p be a prime number satisfying $p \equiv 1(\mod 4)$, and $p = \alpha^2 + \beta^2$, where $\alpha > \beta$ and $\gcd(\alpha, \beta) = 1$. For any positive integer n, there is only one Pythagorean Gaussian integer on the circumference of a circle with radius $(\sqrt{p})^n$.

Proof. Proof. On $Z[i]$, the primary decomposition of p is

$$(\sqrt{p})^2 = (\alpha + i\beta)(\alpha - i\beta).$$

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n power on both sides of the equation, we have

$$[(\sqrt{p})^2]^n = [(\alpha + i\beta)(\alpha - i\beta)]^n$$
$$[(\sqrt{p})^n]^2 = (\alpha + i\beta)^n(\alpha - i\beta)^n \tag{8}$$

Due to Both $(\alpha + i\beta)^n$ and $(\alpha - i\beta)^n$ have no integer factors, there are only Pythagorean Gaussian integer $\Theta((\alpha + i\beta)^n)$ on the circumference of radius $(\sqrt{p})^n(n \geq 1)$.

This completes the proof of Lemma 4 □

□

According to Lemma 4, we obtain an ordered set

$$\Lambda(\sqrt{p}, \alpha + i\beta) = \{\alpha + i\beta, \Theta((\alpha + i\beta)^2), \cdots, \Theta((\alpha + i\beta)^n), \cdots\}. \tag{9}$$

It is not difficult to see that the Gaussian integer $\Theta((\alpha + i\beta)^{2k})$ in the set $\Lambda(\sqrt{p}, \alpha + i\beta)$, its real part, imaginary part, and radius $(\sqrt{p})^{2k}$ of the circle, form a Pythagorean triple.

For example:

$$\Lambda(\sqrt{5}, 2 + i) = \{2 + i, \Theta((2 + i)^2), \cdots, \Theta((2 + i)^n), \cdots\}$$

$\Theta((2 + i)^2)$ in the set is a Gaussian integer on a circumference with radius equal to $(\sqrt{5})^2$, and its real and imaginary parts are

$$\Re[\Theta((2 + i)^2)] = 4, \Im[\Theta((2 + i)^2)] = 3$$

It is evident that $(3, 4, 5)$ is a Pythagorean triple.

If $\xi = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \cdots p_m^{e_m} (k \geq 1, e_k \geq 1)$ and the prime factor $p_k \equiv 1(\mod 4)$, then there are 2^{m-1} Pythagorean Gaussian integers on the circumference with a radius equal to $\sqrt{\xi}$.

Proof. Proof. Let

$$p_k = (\alpha_k + i\beta_k)(\alpha_k - i\beta_k)$$

According to Lemma 4, ξ can be decomposed on $Z[i]$ as:

$$\begin{aligned}
\xi &= p_1^{e_1} p_2^{e_2} \cdots p_i^{e_i} \cdots p_m^{e_m} \\
&= \prod_{k=1}^{m} [(\alpha_k + i\beta_k)(\alpha_k - i\beta_k)]^{e_k} \\
&= \prod_{k=1}^{m} [(\alpha_k + i\beta_k)^{e_k}(\alpha_k - i\beta_k)^{e_k}]
\end{aligned}$$

Let $\chi(\xi)$ and $\overline{\chi(\xi)}$ be the conjugate pair of ξ. We have

$$(\sqrt{\xi})^2 = \chi(\xi)\overline{\chi(\xi)},$$

where

$$\chi(\xi) = \prod_{k=1}^{m} g(k), \quad g(k) \in \{(\alpha_k + i\beta_k)^{e_k}, (\alpha_k - i\beta_k)^{e_k}\}.$$

Since there are two choices for $g(k)$, $\chi(\xi)$ has 2^{m-1} distinct values, denoted as

$$\chi_1(\xi), \chi_2(\xi), \cdots, \chi_{2^{m-1}}(\xi).$$

Representing these numbers as an ordered set, we have

$$Circle(\sqrt{\xi}) = \{u_k + iv_k : 1 \leq k \leq 2^{m-1}, u_k = \Re[\Theta(\chi_k(\xi))], v_k = \Im[\Theta(\chi_k(\xi))]\}. \tag{10}$$

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It follows that there are 2^{m-1} Pythagorean Gaussian integers on the circumference with a radius equal to $\sqrt{\xi}$.

This completes the proof of Lemma 4 □

□

For example.

$$\xi = 5 \cdot 13 \cdot 17 = (1+2i)(1-2i)(3+2i)(3-2i)(1+4i)(1-4i).$$

From the above,we get

$$\chi_1(\xi) = \Theta((1+2i)(3+2i)(1+4i)) = 33+4i$$
$$\chi_2(\xi) = \Theta((1+2i)(3+2i)(1-4i)) = 31+12i$$
$$\chi_3(\xi) = \Theta((1+2i)(3-2i)(1+4i)) = 23+24i$$
$$\chi_4(\xi) = \Theta((1+2i)(3-2i)(1-4i)) = 32+9i$$

namely

$$Circle(\sqrt{5 \cdot 13 \cdot 17}) = \{33+4i, 32+9i, 31+12i, 23+24i\}$$

If $\xi = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \cdots p_m^{e_m} (k \geq 1, e_k \geq 1)$ and $p_k \equiv 1 (\mod 4)$, then the number of Pythagorean Gaussian integers on the circumference with a radius equal to $(\sqrt{\xi})^n$ depends only on the number of prime factors of ξ and is independent of the exponent n.

Proof. Proof. Let

$$p_k = (\alpha_k + i\beta_k)(\alpha_k - i\beta_k)$$

Based on the conditions, we have

$$\begin{aligned} \xi^n &= \left(p_1^{e_1} p_2^{e_2} \cdots p_i^{e_i} \cdots p_m^{e_m}\right)^n \\ &= \prod_{k=1}^{m} [(\alpha_k + i\beta_k)(\alpha_k - i\beta_k)]^{ne_k} \end{aligned}$$

$$[(\sqrt{\xi})^n]^2 = \prod_{k=1}^{m} [(\alpha_k + i\beta_k)^{ne_k} (\alpha_k - i\beta_k)^{ne_k}]$$

Set

$$[(\sqrt{\xi})^n]^2 = \omega(\xi^n)\overline{\omega(\xi^n)}$$

where

$$\omega(\xi^n) = \prod_{k=1}^{m} h(k), \quad h(k) \in \{(\alpha_k + i\beta_k)^{ne_k}, (\alpha_k - i\beta_k)^{ne_k}\}.$$

Based on the values of $h(k)$, we obtain 2^{m-1} distinct values for $\omega(\xi^n)$. Using Lemma 4 and the expression for $\omega_k(\xi^n)$, we have

$$\omega_k(\xi^n) = \chi_k^n(\xi). \tag{11}$$

By (11) and the ordered set (10), We obtain an ordered set of 2^{m-1} Pythagorean Gaussian integers on the circumference with a radius equal to $(\sqrt{\xi})^n$

$$Circle(\sqrt{\xi}^n) = \{\omega_k(\xi^n) = (u_k + iv_k)^n : 1 \leq k \leq 2^{m-1}, u_k = \Re[\Theta(\chi_k(\xi))], v_k = \Im[\Theta(\chi_k(\xi))]\}. \tag{12}$$

This completes the proof of Lemma 4 □ □

Let $u_k + iv_k \in Circle(\sqrt{\xi})$. Based on Lemma 4, we obtain the ordered set

$$\Lambda(\sqrt{\xi}, u_k + iv_k) = \{u_k + iv_k, (u_k + iv_k)^2, \cdots, (u_k + iv_k)^n, \cdots\}. \quad (13)$$

For example.

$$\xi = 5 \cdot 13 \cdot 17 = (1+2i)(1-2i)(3+2i)(3-2i)(1+4i)(1-4i)$$

$$Circle(\sqrt{5 \cdot 13 \cdot 17}) = \{33+4i, 31+12i, 23+24i, 32+9i\}$$

$$Circle(\sqrt{5 \cdot 13 \cdot 17}^n) = \{\Theta(33+4i)^n, \Theta(31+12i)^n, \Theta(23+24i)^n, \Theta(32+9i)^n\}$$

Taking $31 + 12i \in Circle(\sqrt{5 \cdot 13 \cdot 17})$, the ordered set generated by it is

$$\Lambda(\sqrt{5 \cdot 13 \cdot 17}, 31+12i) = \{31+12i, (31+12i)^2, \cdots, (31+12i)^n, \cdots\}.$$

5. Proof of theorem 1

Proof. Proof. Lemma 4 and 4 are the proofs of parts (1) and (2) of Theorem 1, respectively. Here we focus on proving part (3) of Theorem 1. Consider the mth Gaussian integer $(u_k + iv_k)^m$ in the ordered set $\Lambda(\sqrt{\xi}, u_k + iv_k)$, where its real part and imaginary part satisfy the following relationship with the radius $(\sqrt{\xi})^m$ of the circle

$$[(\sqrt{\xi})^m]^2 = (\Re(u_k + iv_k)^m)^2 + (\Im(u_k + iv_k)^m)^2.$$

We discuss the properties of the real and imaginary parts of Gaussian integer $(u_k + iv_k)^m$ when $m = 2n$ and $m = 2n + 1$, respectively.

(1). The circle with a radius equal to $(\sqrt{\xi})^{2n} (n > 1)$

Consider the $2n$th term in the set (13)

$$(u_n + iv_n)^{2n} = (\alpha_n + i\beta_n)^n$$

where $\alpha_n = u_n^2 - v_n^2, \beta_n = 2u_n v_n$. Correspondingly,

$$[(\sqrt{\xi})^{2n}]^2 = (\Re[(\alpha_n + i\beta_n)^n])^2 + (\Im[(\alpha_n + i\beta_n)^n])^2$$

Without loss of generality, set $\alpha = \alpha_n, \beta = \beta_n, u = u_n, v = v_n$. Expand $(\alpha + i\beta)^n$ as a binomial

$$(\alpha + i\beta)^n = \alpha^n + \binom{n}{1}\alpha^{n-1}(i\beta) + \binom{n}{2}\alpha^{n-2}(i\beta)^2 + \cdots + \binom{n}{n-1}\alpha(i\beta)^{n-1} + (i\beta)^n \quad (14)$$

- When $n \equiv 0(\mod 2)$, by (14),we have

$$\Re[(\alpha + i\beta)^n] = \alpha^n - \binom{n}{2}\alpha^{n-2}\beta^2 + \binom{n}{4}\alpha^{n-4}\beta^4 + \cdots \mp \binom{n}{n-2}\alpha^2\beta^{n-2} \pm \beta^n \quad (15)$$

$$\Im[(\alpha + i\beta)^n] = \binom{n}{1}\alpha^{n-1}\beta - \binom{n}{3}\alpha^{n-3}\beta^3 + \cdots \mp \binom{n}{n-3}\alpha^3\beta^{n-3} \pm \binom{n}{1}\alpha\beta^{n-1} \quad (16)$$

According to (15) and (16), we obtain

$$\Re[(\alpha + i\beta)^n] \neq 0(\mod \alpha),$$

$$\Re[(\alpha + i\beta)^n] \neq 0(\mod \beta),$$

$$\Im[(\alpha + i\beta)^n] \equiv 0(\mod \alpha\beta).$$

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Therefore, both formulas (15) and (16) are polynomials in terms of α and β, rather than values in the form of α^x and β^y.

- When $n \equiv 1(\mod 2)$, from (14) we get

$$\Re[(\alpha+i\beta)^n] = \alpha^n - \binom{n}{2}\alpha^{n-2}\beta^2 + \cdots \mp \binom{n}{n-3}\alpha^3\beta^{n-3} \pm \binom{n}{1}\alpha\beta^{n-1} \tag{17}$$

$$\Im[(\alpha+i\beta)^n] = \binom{n}{1}\alpha^{n-1}\beta - \binom{n}{3}\alpha^{n-3}\beta^3 + \cdots \mp \binom{n}{n-2}\alpha^2\beta^{n-2} \pm \beta^n \tag{18}$$

Suppose that β is even. From $\alpha \neq \beta(\mod 2)$ and equation (18), we have

$$\Im[(\alpha+i\beta)^n] \equiv n\alpha^{n-1}\beta \neq 0(\mod \beta^2),$$

$$\Im[(\alpha+i\beta)^n] \equiv \beta^n \neq 0(\mod \alpha).$$

Therefore, the equation (18) is a polynomial about α and β.

(2). This circle with a radius of $(\sqrt{\xi})^{2n+1}(n \geq 1)$

Considering (13), the $2n+1$ term in the set

$$(u_n+iv_n)^{2n+1} = (u_n+iv_n)(\alpha_n+i\beta_n)^n$$

where $\alpha_n = u_n^2 - v_n^2, \beta_n = 2u_nv_n$. Correspondingly,

$$(\sqrt{\xi}^{2n+1})^2 = \left(\Re[(\alpha_n+i\beta_n)^n(u_n+iv_n)]\right)^2 + \left(\Im[(\alpha_n+i\beta_n)^n(u_n+iv_n)]\right)^2$$

Set $\alpha = \alpha_n, \beta = \beta_n, u = u_n, v = v_n$, we have

$$\Re[(\alpha+i\beta)^n(u+vi)] = \Re[(\alpha+i\beta)^n]\cdot u - \Im[(\alpha+i\beta)^n]\cdot v \tag{19}$$

$$\Im[(\alpha+i\beta)^n(u+vi)] = \Re[(\alpha+i\beta)^n]\cdot v + \Im[(\alpha+i\beta)^n]\cdot u \tag{20}$$

- When $n \equiv 1(\mod 2)$, substituting equations (17) and (18) into equations (19) and (20), we obtain

$$\begin{aligned}\Re[(\alpha+i\beta)^n(u+vi)] = u&\left[\alpha^n - \binom{n}{2}\alpha^{n-2}\beta^2 + \cdots \mp \binom{n}{n-3}\alpha^3\beta^{n-3} \pm \binom{n}{1}\alpha\beta^{n-1}\right]\\ -v&\left[\binom{n}{1}\alpha^{n-1}\beta - \binom{n}{3}\alpha^{n-3}\beta^3 + \cdots \mp \binom{n}{n-2}\alpha^2\beta^{n-2} \pm \beta^n\right]\end{aligned}$$

$$\begin{aligned}\Im[(\alpha+i\beta)^n(u+vi)] = v&\left[\alpha^n - \binom{n}{2}\alpha^{n-2}\beta^2 + \cdots \mp \binom{n}{n-3}\alpha^3\beta^{n-3} \pm \binom{n}{1}\alpha\beta^{n-1}\right]\\ +u&\left[\binom{n}{1}\alpha^{n-1}\beta - \binom{n}{3}\alpha^{n-3}\beta^3 + \cdots \mp \binom{n}{n-2}\alpha^2\beta^{n-2} \pm \beta^n\right]\end{aligned}$$

From the above two equations, we obtain

$$\Re[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \alpha),$$

$$\Re[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \beta),$$

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$$\Im[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \alpha),$$
$$\Im[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \beta).$$

Therefore, equations (19) and (20) are polynomials in α and β, not values of the type α^x and β^y.

- When $n \equiv 0(\mod 2)$, substituting equations (15) and (16) into equations (19) and (20), we obtain

$$\Re[(\alpha+i\beta)^n(u+vi)] = u\left[\alpha^n - \binom{n}{2}\alpha^{n-2}\beta^2 + \binom{n}{4}\alpha^{n-4}\beta^4 + \cdots \mp \binom{n}{n-2}\alpha^2\beta^{n-2} \pm \beta^n\right] - v\left[\binom{n}{1}\alpha^{n-1}\beta - \binom{n}{3}\alpha^{n-3}\beta^3 + \cdots \mp \binom{n}{n-3}\alpha^3\beta^{n-3} \pm \binom{n}{1}\alpha\beta^{n-1}\right]$$

$$\Im[(\alpha+i\beta)^n(u+vi)] = v\left[\alpha^n - \binom{n}{2}\alpha^{n-2}\beta^2 + \binom{n}{4}\alpha^{n-4}\beta^4 + \cdots \mp \binom{n}{n-2}\alpha^2\beta^{n-2} \pm \beta^n\right] + u\left[\binom{n}{1}\alpha^{n-1}\beta - \binom{n}{3}\alpha^{n-3}\beta^3 + \cdots \mp \binom{n}{n-3}\alpha^3\beta^{n-3} \pm \binom{n}{1}\alpha\beta^{n-1}\right]$$

From the above two equations, we get

$$\Re[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \alpha),$$
$$\Re[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \beta),$$
$$\Im[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \alpha),$$
$$\Im[(\alpha+i\beta)^n(u+vi)] \neq 0(\mod \beta).$$

Therefore, both equations (19) and (20) are polynomials in α and β.
□

6. Proof of Theorem 2

Proof. Proof. As is well known, all prime numbers in the form of $4k+3$ cannot be expressed as the sum of squares of two coprime positive integers. It is not difficult to deduce that any positive integer containing at least one $4k+3$ type prime factor also has this property. So, c in the Pythagorean triplet (a,b,c) must not contain a $4k+3$ type prime factor.

Assume $c = q_1^{d_1} q_2^{d_2} \cdots q_k^{d_k} \cdots q_s^{d_s} (s \geq 1)$ and $q_k \equiv 1(\mod 4)$. According to Lemma 4, there are 2^{s-1} Pythagorean Gaussian integers on the circumference of a circle with radius $\sqrt{c}$. Expressing these integers as an ordered set, we obtain

$$Circle(\sqrt{c}) = \{\tau_k = u_k + v_k i : 1 \leq k \leq 2^{s-1}, u_k, v_k \in \mathbb{N}^+\} \quad (21)$$

According to Lemma 4, we obtain the Gaussian integer set of the Pythagorean type on the circumference with radius equal to $(\sqrt{c})^n$

$$Circle((\sqrt{c})^n) = \{\omega_k(c^n) = (u_k + iv_k)^n : 1 \leq k \leq 2^{m-1}, c^n = \Re(\omega_k(c^n))^2 + \Im(\omega_k(c^n))^2\}. \quad (22)$$

Let $n=2$, from (22) we have

$$Circle(c) = \{a_k + b_k i \quad : a_k = u_k^2 - v_k^2, b_k = 2u_k v_k, \quad 1 \leq k \leq 2^{s-1}\}. \quad (23)$$

Obviously, $a+bi \in Circle(c)$. Suppose that $a+bi$ is the mth number in it, then

$$\Lambda(\sqrt{c}, u_m + iv_m) = \{u_m + iv_m, a+bi, \cdots, \Theta((u_m+iv_m)^n), \cdots\}. \quad (24)$$

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It is clear that the set (24) is equivalent to the set (13). According to Theorem 1, for any $n > 2$, the real and imaginary parts of each Gaussian integer of the Pythagorean type in this set are polynomials in a and b, that is, $a^x + b^y = c^z$ only when x=y=z=2.

This completes the proof of Theorem 2 □

7. Proof of Theorem 3

Proof. Proof. The geometric meaning of $a^2 + b^2 = c^2$ and $(aw)^2 + (bw)^2 = (cw)^2$ is two concentric circles with a ratio of their radii equal to w, as shown in Figure 2. Let's assume that $c = q_1^{d_1} q_2^{d_2} \cdots q_s^{d_s}$. According to Lemma 4, we obtain a set of Pythagorean Gaussian integers on the circumference with a radius equal to $\sqrt{cw}$

$$Circle(\sqrt{cw}) = \{\tau_k = u_k w + v_k wi : 1 \leq k \leq 2^{s-1}, u_k, v_k \in \mathbb{N}^+\} \tag{25}$$

According to Lemma 4, we obtain the Gaussian integer set of the Pythagorean type on the circumference with radius equal to $(\sqrt{cw})^n$

$$Circle((\sqrt{cw})^n) = \{(u_k w + i v_k w)^n : 1 \leq k \leq 2^{m-1}\}. \tag{26}$$

Set $n = 2$, by (26), we get

$$Circle(cw) = \{a_k w + b_k wi \quad : a_k = u_k^2 - v_k^2, b_k = 2u_k v_k, \quad 1 \leq k \leq 2^{s-1}\}. \tag{27}$$

Obviously, $aw + bwi \in Circle(cw)$. Let's assume that $aw + bwi$ is the mth number in this set, then

$$\Lambda(\sqrt{cw}, u_m w + i v_m w) = \{u_m w + i v_m w, aw + bwi, \cdots, \Theta((u_m w + i v_m w)^n), \cdots\}. \tag{28}$$

When we enlarge a circle with radius equal to$\sqrt{c}^n$ to a circle with radius equal to $\sqrt{cw}^n$, the Gaussian integer $(a + bi)^n$ on the original circumference is correspondingly enlarged to $(aw + bwi)^n$, as shown in Figures3 and 4. According to Theorem1 and Theorem 2, the real and imaginary parts of each Gaussian integer in the set (28) are polynomials in aw and bw, that is,$(aw)^x + (bw)^y = (cw)^z$ has only one solution for $(x, y, z) = (2, 2, 2)$.

Figure 2. $\odot(c)$ is magnified by w times

Figure 3. $\odot(c^n)$ is magnified by w^n times

Figure 4. $\odot(c^n\sqrt{c})$ is magnified by $w^n\sqrt{w}$ times

This completes the proof of Theorem 3 □

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