Article

Not peer-reviewed version

Basis-Dependent Quantum Nonlocality

Kaifeng Hu , Zhihua Guo , Huaixin Cao * , Ling Lu

Posted Date: 23 April 2024

doi: 10.20944/preprints202404.1505.v1

Keywords: basis-dependence; density operator; density matrix; classical correlation; separability; unsteerability; Bell locality

Article

Basis-Dependent Quantum Nonlocality

Kaifeng Hu [1], Zhihua Guo [2], Huaixin Cao [2],* and Ling Lu [3]

1 China People's Police University, Guangzhou 510663, China
2 School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710119, China
3 T&R office of Mathematics, Army Academy of Border and Coastal Defence, Xi'an 710108, China
* Correspondence: guozhihua@snnu.edu.cn; caohx@snnu.edu.cn

Abstract: Quantum nonlocality represents correlation properties between subsystems of a composite quantum system, usually including the four types: Bell nonlocality, steerability, entanglement, and quantum correlation (quantum discord). Given a basis $e_{AB} = \{|e_{ij}\rangle\}_{i\in[d_A],j\in[d_B]}$ for the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of a bipartite system AB, a density operator ρ^{AB} (quantum state) of AB can be represented as a $d_A d_B \times d_A d_B$ matrix $\hat{\rho}_{e_{AB}} = [\langle e_{ij}|\rho^{AB}|e_{ij}\rangle]$, called the density matrix of a density operator ρ^{AB}. A natural question is what is the relationship between the quantum nonlocality of a density operator ρ^{AB} and its corresponding density matrix $\hat{\rho}_{e_{AB}}$? In this work, we discuss the relationships between quantum locality and basis, and prove that one type of quantum locality of density operators and that of their density matrices under a basis are the same if and only if the chosen basis is the tensor product of the bases of subsystems. Consequently, different bases define different quantum nonlocality density operators.

Keywords: basis-dependence; density operator; density matrix; classical correlation; separability; unsteerability; Bell locality

---

1. Introduction

Quantum nonlocality, also called quantum correlation, represents correlation properties between subsystems of a composite quantum system, including Bell nonlocality, steerability, entanglement, and quantum correlation.

Bell nonlocality of bipartite states is demonstrated by some local quantum measurements whose statistics of the measurement outcomes cannot be explained by a local hidden variable (LHV) model [1,2]. Such a nonclassical feature of quantum mechanics can be used in device-independent quantum information processing [2]. For more discussions on Bell nonlocality, please refer to Clauser and Shimony [3], Home and Selleri [4], Khalfin and Tsirelson [5], Tsirelson [6], Zeilinger [7], Werner and Wolf [8], Genovese [9], and Buhrman *et al.* [10], and [11–14].

Einstein-Podolsky-Rosen (EPR) steering, as a form of quantum correlation, was first observed by Schrödinger [15] in the context of the well-known EPR paradox [16–19]. EPR steering arises in scenarios wherein some local quantum measurements on one part of a bipartite system are used to steer the other part. This scenario demonstrates EPR steering if the obtained ensembles cannot be explained by a local hidden state (LHS) model [20]. Following a close analogy of criteria for other forms of quantum nonlocality, Cavalcanti *et al.* [21] developed a general theory of experimental EPR steering criteria and derived several criteria applicable to discrete and continuous-variable observables. Saunders *et al.* [22] contributed experimental EPR steering by using Bell local states. Bennet *et al.* [23] derived arbitrarily loss-tolerant tests, thereby enabling us to perform a detection loophole free demonstration of EPR steering with parties separated by a coiled 1-km-long optical fiber. Händchen *et al.* [24] presented an experimental realization of two entangled Gaussian modes of light that shows a steering effect in one direction but not in the other. The generated one-way steering provides new insight into quantum physics and may open a new field of applications in quantum information. EPR steering, as an intermediate form of quantum correlation between entanglement and Bell nonlocality, allows for entanglement certification when the measurements performed by one of the parties are not characterized (or are untrusted); it has many applications, such as quantum key distribution [25–27],

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

quantum benchmark for qubit teleportation[28], subchannel discrimination in a quantum evolution [29], and so on. Therefore, it has been widely researched, see [30–39].

Just like quantum steering, quantum entanglement was also recognized by Einstein, Podolsky, Rosen [16], and Schrödinger [15] in 1935. It is a property of a composite quantum system involving nonclassical correlations between subsystems and then having potential for many quantum processes, including canonical ones: quantum cryptography, quantum teleportation, and dense coding.

A bipartite entanglement state reveals correlations between the two subsystems. However, some separable states may reveal some correlations. Such states are just the states that have nonzero quantum discord. Quantum discord induced by H. Ollivier and W.H. Zurek in [40] is a measure of the quantumness of correlations but not a quantum property of states. Considered measurement-induced disturbance, Luo [41] classified correlations between subsystems into classical correlations and quantum correlations by introducing classical correlated (CC) states and quantum correlated states. It was found that a CC state is a separable state [40–43] and so any entangled state must be a QC state. Thus, quantum correlations have been applied in some quantum computing tasks without entanglement [44], the study of quantum key distribution [45], three-spin XXZ chain with three-spin interaction [46] and so on.

It is well-known that the four types of quantum nonlocality have the following relations:

$$\text{Bell nonlocality} \Rightarrow \text{steerability} \Rightarrow \text{entanglement} \Rightarrow \text{quantum correlation},$$

equivalently,

$$\text{Bell locality} \Leftarrow \text{unsteerability} \Leftarrow \text{separability} \Leftarrow \text{classical correlation}.$$

From the mathematical definitions of quantum locality (Bell locality, separability and classical correlation), we see that these properties depend only on the algebraic structures of the density operators ρ^{AB} (quantum states) and should be independent of the choice of basis for the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of a system AB. However, in applications and experiments, density operators are usually written as matrices under a chosen basis e_{AB} for $\mathcal{H}_A \otimes \mathcal{H}_B$. As we known, different bases lead to different matrix representations of operators. Thus, different choice of bases may induce different quantum locality of density matrices, which we called basis-dependent quantum nonlocality.

In the sequence sections, we discuss the relationships between these quantum locality and basis, and find that quantum locality of density operators and their density matrices under a basis are the same only when the basis is a tensor product of the bases of subsystems.

2. Basis Dependent Separability

According to quantum mechanics, a quantum system S is described by a d_S-dimensional complex Hilbert space $\mathcal{H}_S$ (called the *state space* of S) with a right-linear inner product $\langle \cdot|\cdot \rangle$ and the states of the system are denoted by positive operators of trace 1 on $\mathcal{H}_S$. The set of all states of S is denoted by $D(\mathcal{H}_S)$. Thus,

$$D(\mathcal{H}_S) = \{\rho \in B(\mathcal{H}_S) : \rho \geq 0, \mathrm{tr}\rho = 1\},$$

where $B(\mathcal{H}_S)$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}_S$. The elements of $D(\mathcal{H}_S)$ are called the *mixed states* of S. A unit vector $|\psi\rangle$ in $\mathcal{H}_S$ is said to be a *pure state* of S and the set of all pure states of S is denoted by $PS(\mathcal{H}_S)$. We also use $[d]$ to denote the set $\{1, 2, \ldots, d\}$.

By the postulates of quantum mechanics, the state space of the composite system AB of A and B is given by the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ of the state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of A and B, respectively.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

2.1. Concepts and Notations

Recall that a state $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be *separable* if it can be written as the form of

$$\rho = \sum_{i=1}^{k} c_i \rho_i^A \otimes \rho_i^B \tag{1}$$

for some states $\rho_i^A \in D(\mathcal{H}_A), \rho_i^B \in D(\mathcal{H}_B)$ and $c_i \geq 0 (i \in [k])$ with $\sum_{i=1}^k c_i = 1$. Otherwise, it is said to be *entangled*. Moreover, a pure state $|\psi\rangle \in PS(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be *separable* if it can be written as the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ for some $|\psi_A\rangle \in PS(\mathcal{H}_A)$ and $|\psi_B\rangle \in PS(\mathcal{H}_B)$. Otherwise, it is called to be *entangled*.

2.2. Separability Depending on Basis

Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ be an orthonormal basis (ONB) for $\mathcal{H}_A \otimes \mathcal{H}_B$ and $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathbb{C}^{d_A d_B} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ be the coordinate mapping:

$$U|\psi^{AB}\rangle = (c_{11}, c_{12}, \ldots, c_{1d_B}, c_{21}, c_{22}, \ldots, c_{2d_B}, \ldots, c_{d_A1}, c_{d_A2}, \ldots, c_{d_Ad_B})^{\mathrm{T}} \tag{2}$$

for all $|\psi^{AB}\rangle = \sum_{i,j} c_{ij}|e_{ij}\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$. We call the vector $U|\psi^{AB}\rangle$ the coordinate state of a state $|\psi^{AB}\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_B$.

We want to discuss the relationship between the separability of the coordinate state $U|\psi^{AB}\rangle$ and that of the original (abstract) state $|\psi^{AB}\rangle$. For convenience, we write $U|\psi^{AB}\rangle = |\psi^{AB}\rangle_{e_{AB}}$. Let $\{|i_A\rangle\}_{i=1}^{d_A}$ and $\{|j_B\rangle\}_{j=1}^{d_B}$ be the canonical $\{0,1\}$-bases for $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, respectively. Then we get the canonical $\{0,1\}$-basis (with $m = d_A, n = d_B$)

$$\{|i_A j_B\rangle\}_{i,j} = \{|1_A 1_B\rangle, |1_A 2_B\rangle, \ldots, |1_A n_B\rangle, \ldots, |m_A 1_B\rangle, |m_A 2_B\rangle, \ldots, |m_A n_B\rangle\}$$

for $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. Thus, the definition (2) of U becomes

$$U|\psi^{AB}\rangle = \sum_{i,j} c_{ij}|i_A j_B\rangle \tag{3}$$

for all $|\psi^{AB}\rangle = \sum_{i,j} c_{ij}|e_{ij}\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$. For example,

$$U|e_{ij}\rangle = |i_A\rangle \otimes |j_B\rangle \equiv |i_A\rangle|j_B\rangle \equiv |i_A j_B\rangle, \ \ \forall i \in [d_A], j \in [d_B], \tag{4}$$

leading to

$$\langle i_A j_B|U\rho^{AB}U^\dagger|k_A \ell_B\rangle = \langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle = \langle i_A j_B|\rho^{AB}_{e_{AB}}|k_A \ell_B\rangle \tag{5}$$

for all $i, k \in [d_A], j, \ell \in [d_B]$, where $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is the $d_A d_B \times d_A d_B$ matrix with $(ij, k\ell)$-entry $\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle$, called the density matrix of ρ^{AB} under basis e_{AB} and written as $\rho^{AB}_{e_{AB}} = U\rho^{AB}U^\dagger$ due to the relation (5).

Equation (4) shows that the coordinate mapping U transforms the basis states $|e_{ij}\rangle$ as separable states $|i_A\rangle \otimes |j_B\rangle$. Generally, $U|\psi^{AB}\rangle$ is not necessarily separable even if $|\psi^{AB}\rangle$ is separable. For example, we let $d_A = d_B = 2$ and $e_A = \{|e_i^A\rangle\}_{i=1}^2$ and $e_B = \{|e_i^B\rangle\}_{i=1}^2$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and define an ONB $e_{AB} = \{|e_{11}\rangle, |e_{12}\rangle, |e_{21}\rangle, |e_{22}\rangle\}$ for $\mathcal{H}_A \otimes \mathcal{H}_B$ by

$$\begin{cases} |e_{11}\rangle = \frac{\sqrt{2}}{2}(|e_1^A e_1^B\rangle + |e_2^A e_2^B\rangle); \\ |e_{12}\rangle = \frac{\sqrt{2}}{2}(|e_1^A e_2^B\rangle + |e_2^A e_1^B\rangle); \\ |e_{21}\rangle = \frac{\sqrt{2}}{2}(|e_1^A e_2^B\rangle - |e_2^A e_1^B\rangle); \\ |e_{22}\rangle = \frac{\sqrt{2}}{2}(|e_1^A e_1^B\rangle - |e_2^A e_2^B\rangle). \end{cases}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

Clearly, $|\psi^{AB}\rangle := |e_1^A e_1^B\rangle = \frac{\sqrt{2}}{2}(|e_{11}\rangle + |e_{22}\rangle)$ is a separable state of $\mathcal{H}_A \otimes \mathcal{H}_B$, but

$$U|\psi^{AB}\rangle = |\psi^{AB}\rangle_{e_{AB}} = \left(\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right)^{\mathrm{T}} \in \mathbb{C}^2 \otimes \mathbb{C}^2,$$

which can not be written as a tensor product of two qubits and then not separable. If, however, we choose a "separable basis":

$$e_A \otimes e_B = \{|e_i^A\rangle \otimes |e_j^B\rangle\}_{i,j=1}^2 = \{|e_1^A\rangle \otimes |e_1^B\rangle, |e_1^A\rangle \otimes |e_2^B\rangle, |e_2^A\rangle \otimes |e_1^B\rangle, |e_2^A\rangle \otimes |e_2^B\rangle\},$$

then the coordinate state of $|\psi^{AB}\rangle = |e_1^A e_1^B\rangle = \frac{\sqrt{2}}{2}(|e_{11}\rangle + |e_{22}\rangle)$ reads

$$|\psi^{AB}\rangle_{e_A \otimes e_B} = (1, 0, 0, 0)^{\mathrm{T}} \in \mathbb{C}^2 \otimes \mathbb{C}^2,$$

which is separable.

This observation leads to the following conclusion.

Theorem 1. *Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is a separable state of $\mathcal{H}_A \otimes \mathcal{H}_B$ if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is a separable state of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

(b) A density operator ρ^{AB} is a separable state of $\mathcal{H}_A \otimes \mathcal{H}_B$ if and only if its density matrix $\rho_{e_{AB}}^{AB}$ is a separable state of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

Proof. (a) For every separable state $|\psi^{AB}\rangle = |\psi^A\rangle|\psi^B\rangle$, it holds that

$$c_{ij} := \langle e_{ij}|\psi^{AB}\rangle = \langle e_i^A|\psi^A\rangle \cdot \langle e_j^B|\psi^B\rangle$$

for all i, j. Writing $a_i = \langle e_i^A|\psi^A\rangle$ and $b_j = \langle e_j^B|\psi^B\rangle$ implies that

$$|\psi^{AB}\rangle_{e_{AB}} = (a_1, a_2, \ldots, a_{d_A})^{\mathrm{T}} \otimes (b_1, b_2, \ldots, b_{d_B})^{\mathrm{T}}, \tag{6}$$

and so $|\psi^{AB}\rangle_{e_{AB}}$ is a separable state in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

Conversely, let $|\psi^{AB}\rangle_{e_{AB}}$ be a separable state. Then Equation (6) holds for some states $|a\rangle = (a_1, a_2, \ldots, a_{d_A})^{\mathrm{T}}$ and $|b\rangle = (b_1, b_2, \ldots, b_{d_B})^{\mathrm{T}}$ of $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, respectively. Put $|\psi^A\rangle = \sum_{i=1}^{d_A} a_i|e_i^A\rangle$ and $|\psi^B\rangle = \sum_{j=1}^{d_B} b_j|e_j^B\rangle$, then $U(|\psi^A\rangle|\psi^B\rangle) = |a\rangle|b\rangle = U|\psi^{AB}\rangle$ and so $|\psi^{AB}\rangle = |\psi^A\rangle|\psi^B\rangle$ since U is injective. Hence, $|\psi^{AB}\rangle$ is separable. This shows that $U|\psi^{AB}\rangle = |\psi^{AB}\rangle_{e_{AB}}$ is separable if and only if $\psi^{AB}\rangle$ is separable.

(b) First, we let $\rho^{AB} = \sum_t c_t \rho_t^A \otimes \rho_t^B$ be a separable state of AB. Then

$$[\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle] = \sum_t c_t[\langle e_i^A|\rho_t^A|e_k^A\rangle \cdot \langle e_j^B|\rho_t^B|e_\ell^B\rangle] = \sum_t c_t[\langle e_i^A|\rho_t^A|e_k^A\rangle] \otimes [\langle e_j^B|\rho_t^B|e_\ell^B\rangle],$$

which is the tensor product of two density matrices. Thus, $\rho_{e_{AB}}^{AB}$ is a separable density matrix. □

Conversely, we let $\rho_{e_{AB}}^{AB}$ be a separable density matrix. Then $\rho_{e_{AB}}^{AB} = \sum_{n=1}^m t_n[a_{ik}^{(n)}] \otimes [b_{j\ell}^{(n)}]$ for some density matrices $[a_{ik}^{(n)}]$ and $[b_{j\ell}^{(n)}]$ where $t_n \geq 0$ and $\sum_n t_n = 1$. Define density operators ρ_n^A on $\mathcal{H}_A$ and ρ_n^B on $\mathcal{H}_B$ by

$$\langle e_i^A|\rho_n^A|e_k^A\rangle = a_{ik}^{(n)}, \langle e_j^B|\rho_n^B|e_\ell^B\rangle = b_{j\ell}^{(n)},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

for all i, j, k, ℓ, then we compute that

$$
\begin{aligned}
\left[\langle e_{ij}|\sum_{n=1}^{m} t_n(\rho_n^A \otimes \rho_n^B)|e_{k\ell}\rangle\right] &= \sum_{n=1}^{m} t_n[\langle e_i^A|\rho_n^A|e_k^A\rangle \cdot \langle e_j^B|\rho_n^B|e_\ell^B\rangle] \\
&= \sum_{n=1}^{m} t_n[\langle e_i^A|\rho_n^A|e_k^A\rangle] \otimes [\langle e_j^B|\rho_n^B|e_\ell^B\rangle] \\
&= \sum_{n=1}^{m} t_n[a_{ik}^{(n)}]_{i,k} \otimes [b_{j\ell}^{(n)}] \\
&= \rho_{e_{AB}}^{AB} \\
&= [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle],
\end{aligned}
$$

and so $\rho^{AB} = \sum_{n=1}^{m} t_n(\rho_n^A \otimes \rho_n^B)$, which is separable. The proof is completed.

Similarly, one can check the following.

Theorem 2. *Let $d_A = d_B = d$, $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is separable if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is separable.

(b) A density operator ρ^{AB} is separable if and only if its density matrix $\rho_{e_{AB}}^{AB}$ is separable.

The following theorem shows that the condition $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ (resp. $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$) is necessary for the conditions (a) and (b) in Theorem 1 (resp. Theorem 2) to be satisfied.

Theorem 3. *Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ be an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ and and U be fined by Equation (2). Then the following statements are equivalent (TFSAE).*

(a) The coordinate mapping U preserves separability of pure states in both directions, i.e., $|\psi^{AB}\rangle$ is separable if and only if $U|\psi^{AB}\rangle$ is separable.

(b) The coordinate mapping U preserves separability of density operators in both directions, i.e., a density operator ρ^{AB} is separable if and only if its density matrix $\rho_{e_{AB}}^{AB}$ is separable.

(c) There exist unitary operators $W_X : \mathcal{H}_X \to \mathbb{C}^{d_X} (X = A, B)$ such that when $d_A \neq d_B$, $U = W_A \otimes W_B$; when $d_A = d_B = d$, either $U = W_A \otimes W_B$, or $U = S(W_A \otimes W_B)$, where $S : \mathbb{C}^d \otimes \mathbb{C}^d$ is the swap operator: $S(|x\rangle \otimes |y\rangle) = |y\rangle \otimes |x\rangle$.

(d) There exist orthonormal bases $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that when $d_A \neq d_B$, $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ for all i, j; When $d_A = d_B = d$, either $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ for all i, j, or $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ for all i, j.

Proof. $(a) \Leftrightarrow (b)$: Suppose that (a) holds. Let $\rho^{AB} = \sum_{n=1}^{m} c_n \rho_n^A \otimes \rho_n^B$ be a convex combination of product states. It suffices to prove that $U(\rho_n^A \otimes \rho_n^B)U^\dagger$ is separable for each n. Fixed n and write

$$
\rho_n^A = \sum_s \lambda_s |\psi_s^A\rangle\langle\psi_s^A|, \rho_n^B = \sum_t \mu_t |\psi_t^B\rangle\langle\psi_t^B|,
$$

then

$$
U(\rho_n^A \otimes \rho_n^B)U^\dagger = \sum_{s,t} \lambda_s \mu_t U|\psi_s^A \psi_t^B\rangle\langle\psi_s^A \psi_t^B|U^\dagger,
$$

which is separable using (a). Hence, $U\rho^{AB}U^\dagger = \sum_{n=1}^{m} c_n U(\rho_n^A \otimes \rho_n^B)U^\dagger$ is separable. It follows from Theorem 1(b) that the density matrix $[\langle i_A j_B|U\rho^{AB}U^\dagger|k_A \ell_B\rangle]$ is separable, i.e., $\rho_{e_{AB}}^{AB} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is separable. Conversely, let $[\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ be separable. Then the density matrix $[\langle i_A j_B|U\rho^{AB}U^\dagger|k_A \ell_B\rangle]$ is separable and therefore the density operator $U\rho^{AB}U^\dagger$ is separable (Theorem 1(b)). Thus, we can write $U\rho^{AB}U^\dagger = \sum_n c_n |\alpha_n^{AB}\rangle\langle\alpha_n^{AB}|$ for some separable pure states $|\alpha_n^{AB}\rangle$ of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, where $c_n \geq 0$ with $\sum_n c_n = 1$. Since $\rho^{AB} = \sum_n c_n U^\dagger|\alpha_n^{AB}\rangle\langle\alpha_n^{AB}|U$ and $U^\dagger|\alpha_n^{AB}\rangle$ is separable (using (a)) for all n, we see that ρ^{AB} is separable. Now, (b) follows. Clearly, (b) implies (a). □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

$(a) \Rightarrow (c)$: Take ONBs $e_A = \{|x_i\rangle\}_{i=1}^{d_A}$ and $e_B = \{|y_j\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and define unitary operators $U_A : \mathcal{H}_A \to \mathbb{C}^{d_A}$ and $U_B : \mathcal{H}_B \to \mathbb{C}^{d_B}$ by

$$U_A\left(\sum_{i=1}^{d_A} c_i|x_i\rangle\right) = (c_1, c_2, \ldots, c_{d_A})^{\mathrm{T}},$$

$$U_B\left(\sum_{j=1}^{d_B} d_j|y_j\rangle\right) = (d_1, d_2, \ldots, d_{d_A})^{\mathrm{T}},$$

and put $U_{AB} = U_A \otimes U_B$, $V = UU_{AB}^{-1}$. Then we obtain a commutative diagram Figure 1:

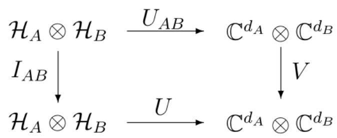

Figure 1. The unitary operator V on $\mathbb{C}^{d_A}\mathbb{C}^{d_B}$.

Let (a) be valid. Since both U and U_{AB} preserve separability of pure states in both directions, so does V, i.e., $V|\psi\rangle$ is separable if and only if $|\psi\rangle$ is separable. It follows from [47] that there are unitary operators V_A on $\mathbb{C}^{d_A}$ and V_B on $\mathbb{C}^{d_B}$ such that when $d_A \neq d_B$, $V = V_A \otimes V_B$; when $d_A = d_B = d$, either $V = V_A \otimes V_B$, or $V = S(V_A \otimes V_B)$. Thus, condition (c) follows by letting $W_X = V_X U_X$ $(X = A, B)$. See Figure 3 for the last case.

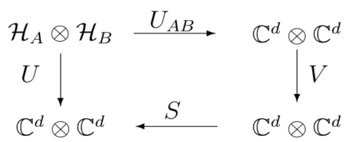

Figure 2. Decomposition of U.

$(c) \Rightarrow (d)$: Suppose that condition (c) is satisfied. From the definition of U, we see that $U|e_{ij}\rangle = |i_A\rangle \otimes |j_B\rangle$ for all $i \in [d_A]$ and $j \in [d_B]$ where $\{|i_A\rangle\}_{i=1}^{d_A}$ and $\{|j_B\rangle\}_{j=1}^{d_B}$ are the canonical $\{0,1\}$-bases for $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, respectively. Put $|e_i^A\rangle = W_A^{-1}|i_A\rangle$ and $|e_j^B\rangle = W_B^{-1}|j_B\rangle$ for all i, j.

When $d_A \neq d_B$, we have

$$|e_i^A\rangle \otimes |e_j^B\rangle = W_A^{-1}|i_A\rangle \otimes W_B^{-1}|j_B\rangle = U^{-1}(|i_A\rangle \otimes |j_B\rangle) = |e_{ij}\rangle$$

for all i, j;

When $d_A = d_B = d$, if $U = W_A \otimes W_B$, then we have

$$|e_i^A\rangle \otimes |e_j^B\rangle = W_A^{-1}|i_A\rangle \otimes W_B^{-1}|j_B\rangle = U^{-1}(|i_A\rangle \otimes |j_B\rangle) = |e_{ij}\rangle$$

for all i, j; if $U = S(W_A \otimes W_B)$, then we have

$$\begin{aligned} |e_j^A\rangle \otimes |e_i^B\rangle &= W_A^{-1}|j_A\rangle \otimes W_B^{-1}|i_B\rangle = U^{-1}S(|j_A\rangle \otimes |i_B\rangle) = U^{-1}(|i_B\rangle \otimes |j_A\rangle) \\ &= U^{-1}(|i_A\rangle \otimes |j_B\rangle) = |e_{ij}\rangle \end{aligned}$$

for all i, j, here the fact that $|i_B\rangle = |i_A\rangle$ and $|j_A\rangle = |j_B\rangle$ were used. Now, condition (d) follows.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

$(d) \Rightarrow (a)$: Suppose that condition (d) is satisfied. Let $|\psi^A\rangle$ and $|\psi^B\rangle$ be any states of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then

$$|\psi^A\rangle = \sum_{i=1}^{d_A} c_i|e_i^A\rangle, |\psi^B\rangle = \sum_{j=1}^{d_B} d_j|e_j^B\rangle.$$

When $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ for all i, j, we have

$$U(|\psi^A\rangle \otimes |\psi^B\rangle) = \sum_{i,j} c_i d_j U(|e_i^A\rangle \otimes |e_j^B\rangle) = \sum_{i,j} c_i d_j U|e_{ij}\rangle = \sum_{i,j} c_i d_j |i_A\rangle \otimes |j_B\rangle = |u\rangle \otimes |v\rangle,$$

where $|u\rangle = \sum_i c_i|i_A\rangle$ and $|v\rangle = \sum_j d_j|j_B\rangle$.

When $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ for all i, j, we have $d_A = d_B = d$ and write

$$|\psi^A\rangle = \sum_{j=1}^{d} d_j|e_j^A\rangle, |\psi^B\rangle = \sum_{i=1}^{d} c_i|e_i^B\rangle.$$

Thus,

$$U(|\psi^A\rangle \otimes |\psi^B\rangle) = \sum_{i,j} c_i d_j U(|e_j^A\rangle \otimes |e_i^B\rangle) = \sum_{i,j} c_i d_j U|e_{ji}\rangle = \sum_{i,j} c_i d_j |j_A\rangle \otimes |i_B\rangle = |u\rangle \otimes |v\rangle,$$

where $|u\rangle = \sum_i c_i|i_A\rangle$ and $|v\rangle = \sum_j d_j|j_B\rangle$. This shows that U maps any separable pure state as a separable pure state.

Conversely, we assume that $U|\psi^{AB}\rangle = |u\rangle \otimes |v\rangle$ for some states $|u\rangle$ and $|v\rangle$ of $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, respectively. Let us show that $|\psi^{AB}\rangle$ is separable. To do this, we let

$$|\psi^{AB}\rangle = \sum_{i,j} t_{ij}|e_{ij}\rangle, |u\rangle = \sum_i c_i|i_A\rangle, |v\rangle = \sum_j d_j|j_B\rangle.$$

Then

$$\sum_{i,j} c_i d_j |i_A\rangle|j_B\rangle = |u\rangle \otimes |v\rangle = U|\psi^{AB}\rangle = \sum_{i,j} t_{ij}|i_A\rangle|j_B\rangle$$

and so $t_{ij} = c_i d_j$ for all i, j. Hence, $|\psi^{AB}\rangle = \sum_{i,j} c_i d_j|e_{ij}\rangle$. Since either $|e_{ij}\rangle = |d_i^A\rangle \otimes |e_j^B\rangle$ for all i, j, or $d_A = d_B = d$ and $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ for all i, j, we have either

$$|\psi^{AB}\rangle = \left(\sum_{i=1}^{d} c_i|e_i^A\rangle\right) \otimes \left(\sum_{j=1}^{d} d_j|e_j^B\rangle\right),$$

or

$$|\psi^{AB}\rangle = \left(\sum_{j=1}^{d} d_j|e_j^A\rangle\right) \otimes \left(\sum_{i=1}^{d} c_i|e_i^B\rangle\right).$$

This shows that $|\psi^{AB}\rangle$ is separable. Thus, U preserves separability of pure states in both directions and so condition (a) is satisfied. The proof is completed.

3. Basis Dependent Bell Locality

In this section, we will discuss relationship between Bell locality and basis. To describe and discuss Bell locality, the following mathematical definitions were given by [14] abstracted from the literatures [2,20].

3.1. Concepts and Notations

To describe Bell locality of a bipartite state ρ^{AB}, we use x and y to denote the labels of POVMs of Alice and Bob and use a and b to denote their measurement outcomes, respectively. Thus, their POVM choices are denoted by $M^x = \{M_{a|x}\}_{a=1}^{o_A}$ and $N^y = \{N_{b|y}\}_{b=1}^{o_B}$, respectively, where $x \in [m_A], y \in [m_B]$.

These POVMs form *measurement assemblages* (MA) of A and B: $\mathcal{M}_A = \{M^x\}_{x=1}^{m_A}$ and $\mathcal{N}_B = \{N^y\}_{y=1}^{m_B}$, respectively.

The following concepts were given in [14].

(1) A state ρ^{AB} is said to be *Bell local* for a given measurement assemblage $(\mathcal{M}_A, \mathcal{N}_B)$ if there exists a probability distribution (PD) $\{\pi_\lambda\}_{\lambda=1}^d$ such that, for each (λ, x) and (λ, y), there exist PDs $\{P_A(a|x,\lambda)\}_{a=1}^{o_A}$ and $\{P_B(b|y,\lambda)\}_{b=1}^{o_B}$, respectively, for which it holds that

$$\mathrm{tr}[(M_{a|x} \otimes N_{b|y})\rho^{AB}] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda) P_B(b|y,\lambda) \tag{7}$$

for all a, b, x, y.

Equation (7) is said to be a *local hidden variable model (LHVM)* of ρ^{AB} w.r.t. MA $(\mathcal{M}_A, \mathcal{N}_B)$ and λ is said to be an LHV with PD $\{\pi_\lambda\}_{\lambda=1}^d$.

(2) A state ρ^{AB} is said to be *Bell nonlocal* for $(\mathcal{M}_A, \mathcal{N}_B)$ if it is not Bell local for $(\mathcal{M}_A, \mathcal{N}_B)$.

(3) A state ρ^{AB} is said to be *Bell local* if for every $(\mathcal{M}_A, \mathcal{N}_B)$, there exists a PD $\{\pi_\lambda\}_{\lambda=1}^d$ such that Equation (7) holds.

(4) A state ρ^{AB} is said to be *Bell nonlocal* if it is not Bell local, i.e., ρ^{AB} is not Bell local for some $(\mathcal{M}_A, \mathcal{N}_B)$.

Let $\mathcal{BL}(\mathcal{M}_A, \mathcal{N}_B)$ denote the set of all states that are Bell local for $\mathcal{M}_A \otimes \mathcal{N}_B$, $\mathcal{BNL}(\mathcal{M}_A, \mathcal{N}_B)$ denote the set of all states that are Bell nonlocal for $(\mathcal{M}_A, \mathcal{N}_B)$, $\mathcal{BL}(AB)$ the set of all Bell local states of AB; $\mathcal{BNL}(AB)$ denote the set of all states that are Bell nonlocal. Thus, we see from the definition that

$$\mathcal{BL}(AB) = \bigcap_{\mathcal{M}_A, \mathcal{N}_B} \mathcal{BL}(\mathcal{M}_A, \mathcal{N}_B); \mathcal{BNL}(AB) = \bigcup_{\mathcal{M}_A, \mathcal{N}_B} \mathcal{BNL}(\mathcal{M}_A, \mathcal{N}_B).$$

3.2. Bell Locality Depending on Basis

Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_i^A\rangle\}_{i=1}^{d_A}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Define unitary operators:

$$U_A : \mathcal{H}_A \to \mathbb{C}^{d_A}, \ U_A \left(\sum_{i=1}^{d_A} c_i |e_i^A\rangle \right) = (c_1, c_2, \ldots, c_{d_A})^{\mathrm{T}}, \tag{8}$$

$$U_B : \mathcal{H}_B \to \mathbb{C}^{d_B}, \ U_B \left(\sum_{j=1}^{d_B} d_j |e_j^B\rangle \right) = (d_1, d_2, \ldots, d_{d_A})^{\mathrm{T}}, \tag{9}$$

and for every linear operator $T : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$, define a $d_A d_B \times d_A d_B$ matrix

$$\widetilde{T} = [t_{ij,k\ell}] \text{ with } (ij, k\ell)\text{-entry } t_{ij,k\ell} = \langle e_i^A e_j^B | T | e_k^A e_\ell^B \rangle.$$

Since

$$\langle i_A j_B | (U_A \otimes U_B) T (U_A^\dagger \otimes U_B^\dagger) | k_A \ell_B \rangle = \langle e_i^A e_j^B | T | e_k^A e_\ell^B \rangle = t_{ij,k\ell} = \langle i_A j_B | \widetilde{T} | k_A \ell_B \rangle$$

for all $i, j, k\ell$, we have

$$\mathrm{tr}(\widetilde{T}) = \mathrm{tr}((U_A \otimes U_B) T (U_A^\dagger \otimes U_B^\dagger)). \tag{10}$$

Let $\mathcal{M}_A = \{M^x\}_{x=1}^{m_A}$ and $\mathcal{N}_B = \{N^y\}_{y=1}^{m_B}$ be measurement assemblages (MAs) of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, where $M^x = \{M_{a|x}\}_{a=1}^{o_A}$ and $N^y = \{N_{b|y}\}_{b=1}^{o_B}$ are POVMs of systems A and B, respectively. Then $\widetilde{M^x} := \{\widetilde{M_{a|x}}\}_{a=1}^{o_A}$ and $\widetilde{N^y} = \{\widetilde{N_{b|y}}\}_{b=1}^{o_B}$ are POVMs of systems $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, respectively, where

$$\widetilde{M_{a|x}} = U_A M_{a|x} U_A^\dagger, \widetilde{N_{b|y}} = U_B N_{a|x} U_B^\dagger.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

Hence, $\widetilde{\mathcal{M}_A} = \{\widetilde{M^x}\}_{x=1}^{m_A}$ and $\widetilde{\mathcal{N}_B} = \{\widetilde{N^y}\}_{y=1}^{m_B}$ be MAs of $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, respectively. Using Equation (10) yields that for all x, y, a, b,

$$\mathrm{tr}[(\widetilde{M_{a|x}} \otimes \widetilde{N_{b|y}})\rho^{AB}_{e_{AB}}] = \mathrm{tr}[(M_{a|x} \otimes N_{b|y})\rho^{AB}].$$

Thus, $\rho^{AB} \in \mathcal{BL}(\mathcal{M}_A, \mathcal{N}_B)$ if and only if $\rho^{AB}_{e_{AB}} \in \mathcal{BL}(\widetilde{\mathcal{M}_A}, \widetilde{\mathcal{N}_B})$. Since Bell locality and separability of pure states are the same [48] for pure states, we see from Theorem 1(a) that a pure state $|\psi^{AB}\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ is Bell local if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is Bell local.

As a conclusion, we obtain the following.

Theorem 4. *Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is a Bell local state of $\mathcal{H}_A \otimes \mathcal{H}_B$ if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is a Bell local state of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

(b) A density operator ρ^{AB} is a Bell local state of $\mathcal{H}_A \otimes \mathcal{H}_B$ if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is a Bell local state of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} = \mathbb{C}^{d_A d_B}$ with $(ij, k\ell)$-entry $\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle$.

Similarly, one can check the following.

Theorem 5. *Let $d_A = d_B = d$, $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is a Bell local state of $\mathcal{H}_A \otimes \mathcal{H}_B$ if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is a Bell local state of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

(b) A density operator ρ^{AB} is Bell local if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is Bell local with $(ij, k\ell)$-entry $\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle$.

Using Theorem 3, Theorems 4 and 5, once can check the following.

Theorem 6. *Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ be an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ and and U be fined by Equation (2). Then TFSAE.*

(a) The coordinate mapping U preserves Bell locality of pure states in both directions, i.e., $|\psi^{AB}\rangle$ is Bell local if and only if $U|\psi^{AB}\rangle$ is Bell local.

(b) A density operator ρ^{AB} is Bell local if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is Bell local with $(ij, k\ell)$-entry $\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle$.

(c) There exist unitary operators $W_X : \mathcal{H}_X \to \mathbb{C}^{d_X} (X = A, B)$ such that when $d_A \neq d_B$, $U = W_A \otimes W_B$; when $d_A = d_B = d$, either $U = W_A \otimes W_B$, or $U = S(W_A \otimes W_B)$, where $S : \mathbb{C}^d \otimes \mathbb{C}^d$ is the swap operator: $S(|x\rangle \otimes |y\rangle) = |y\rangle \otimes |x\rangle$.

(d) There exist orthonormal bases $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that when $d_A \neq d_B$, $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ for all i, j; When $d_A = d_B = d$, either $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ for all i, j, or $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ for all i, j.

Proof. Use the implications:

$$(a) \Leftrightarrow (d)(\text{Theorem 3}), (\mathrm{d}) \Leftrightarrow (\mathrm{c})(\text{Theorem 3}), (\mathrm{d}) \Rightarrow (\mathrm{b})(\text{Theorems 4 and 5}), (\mathrm{b}) \Rightarrow (\mathrm{a}).$$

□

4. Basis Dependent Unsteerability

In this section, we will discuss relationship between unsteerability and basis.

4.1. Concepts and Notations

Recall that [14] a state ρ^{AB} of $\mathcal{H}_A \otimes \mathcal{H}_B$ is said to be unsteerable from A to B with an MA $\mathcal{M}_A = \{M^x\}_{x=1}^{m_A}$ where $M^x = \{M_{a|x}\}_{a=1}^{o_A}$ is a POVM of system A if there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{n}$ and a set $\{\sigma_\lambda\}_{\lambda=1}^{n}$ of system B such that

$$\mathrm{tr}_A[(M_{a|x} \otimes I_B)\rho^{AB}] = \sum_{\lambda=1}^{n} \pi_\lambda P_A(a|x,\lambda)\sigma_\lambda, \ \ \forall x, a, \tag{11}$$

where $\{P_A(a|x,\lambda)\}_{a=1}^{o_A}$ is a PD for all x, λ. ρ^{AB} is said to be unsteerable from A to B if it is unsteerable from A to B with any MA $\mathcal{M}_A$. ρ^{AB} is said to be steerable from A to B if it is not unsteerable for some MA $\mathcal{M}_A$. A pure state $|\psi^{AB}\rangle$ is said to be unsteerable from A to B if so is its density operator $|\psi^{AB}\rangle\langle\psi^{AB}|$. Similarly, one define unsteerability from B to A. It is easy to see that a state that is unsteerable either from A to B or from B to A is Bell local. Thus, a pure state is unsteerable either from A to B or from B to A is separable.

4.2. Unsteerability Depending on Basis

Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_i^A\rangle\}_{i=1}^{d_A}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then a density operator ρ^{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ has the corresponding density matrix $\rho_{e_{AB}}^{AB}$ of ρ^{AB} under the basis $e_{AB} = e_A \otimes e_B$ is

$$\rho_{e_{AB}}^{AB} = (U_A \otimes U_B)\rho^{AB}(U_A^\dagger \otimes U_B^\dagger), \tag{12}$$

where U_A and U_B are unitary operators given by Equations (8) and (9). In this case, for any MA $\mathcal{M}_A$ of system A, we have

$$\mathrm{tr}_A[(\widetilde{M_{a|x}} \otimes E_{d_B})\rho_{e_{AB}}^{AB}] = \mathrm{tr}_A[(M_{a|x} \otimes I_B)\rho^{AB}],$$

where E_{d_B} is the $d_B \times d_B$ unit matrix. Thus, ρ^{AB} is unsteerable from A to B (resp. with MA $\mathcal{M}_A = \{M^x\}_{x=1}^{m_A}$) if and only if its density matrix $\rho_{e_{AB}}^{AB}$ is unsteerable from A to B (resp. with MA $\widetilde{\mathcal{M}_A} := \{\widetilde{M^x}\}_{x=1}^{m_A}$).

Similarly, when $d_A = d_B = d$ and $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ for all $i, j \in [d]$, we have

$$\rho_{e_{AB}}^{AB} = S(U_A \otimes U_B)\rho^{AB}(U_A^\dagger \otimes U_B^\dagger)S^\dagger, \tag{13}$$

where S is the swap operator on $\mathbb{C}^d \otimes \mathbb{C}^d$. Thus,

$$\mathrm{tr}_B[(E_{d_A} \otimes \widetilde{M_{a|x}})\rho_{e_{AB}}^{AB}] = \mathrm{tr}_A[(M_{a|x} \otimes I_B)\rho^{AB}].$$

This shows that ρ^{AB} is unsteerable from A to B (resp. with MA $\mathcal{M}_A = \{M^x\}_{x=1}^{m_A}$) if and only if its density matrix $\rho_{e_{AB}}^{AB}$ is unsteerable from B to A (resp. with MA $\widetilde{\mathcal{M}_A} := \{\widetilde{M^x}\}_{x=1}^{m_A}$).

From these observations, we obtain the following.

Theorem 7. *Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is unsteerable from A to B if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is unsteerable from A to B.

(b) A density operator ρ^{AB} is unsteerable from A to B if and only if its density matrix $\rho_{e_{AB}}^{AB} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is unsteerable from A to B.

Theorem 8. *Let $d_A = d_B = d$, $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is unsteerable from A to B if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is unsteerable from B to A.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 April 2024 doi:10.20944/preprints202404.1505.v1

(b) A density operator ρ^{AB} is unsteerable from A to B if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is unsteerable from B to A.

Using Theorem 1 and the fact that a pure state $|\psi^{AB}\rangle$ is unsteerable from A to B if and only if it is separable [48], combining Theorem 7, one can check the following.

Theorem 9. *Let $d_A \neq d_B$ and $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ be an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ and and U be defined by Equation (2). Then TFSAE.*

(a) $|\psi^{AB}\rangle$ is unsteerable from A to B if and only if $U|\psi^{AB}\rangle$ is unsteerable from A to B.

(b) A density operator ρ^{AB} is unsteerable from A to B if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is unsteerable from A to B.

(c) There exist unitary operators $W_X : \mathcal{H}_X \to \mathbb{C}^{d_X} (X = A, B)$ such that when $d_A \neq d_B$, $U = W_A \otimes W_B$.

(d) There exist orthonormal bases $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_i^A\rangle\}_{i=1}^{d_A}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ for all i, j.

Using Theorem 1 and the fact that a pure state $|\psi^{AB}\rangle$ is unsteerable from A to B if and only if it is separable [48], combining Theorem 8, one can check the following.

Theorem 10. *Let $d_A = d_B = d$ and $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ be an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ and U be defined by Equation (2). Then the following (a) and (b) are equivalent.*

(a) $|\psi^{AB}\rangle$ is unsteerable from A to B if and only if $U|\psi^{AB}\rangle$ is unsteerable from A to B.

(b) A density operator ρ^{AB} is unsteerable from A to B if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is unsteerable from A to B.

If (a) or (b) holds, then

(c) There exist unitary operators $W_X : \mathcal{H}_X \to \mathbb{C}^{d_X} (X = A, B)$ such that $U = W_A \otimes W_B$, or $U = S(W_A \otimes W_B)$.

(d) There exist orthonormal bases $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ for all i, j, or $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ for all i, j.

5. Basis Dependent Classical Correlation

In this section, we will discuss relationship between classical correlation and basis.

5.1. Concepts and Notations

Recall that [41–43] a state ρ^{AB} of $\mathcal{H}_A \otimes \mathcal{H}_B$ is said to classically correlated (CC) if there exists a rank-1 projective measurement $\Pi = \{\Pi_s^A \otimes \Pi_t^B : s \in [m], t \in [n]\}$ such that

$$\Pi(\rho^{AB}) := \sum_{s=1}^{m}\sum_{t=1}^{n}(\Pi_s^A \otimes \Pi_t^B)\rho^{AB}(\Pi_s^A \otimes \Pi_t^B) = \rho^{AB}; \tag{14}$$

otherwise, ρ^{AB} is said to be *quantum correlated* (QC). A pure state $|\psi\rangle$ is said to be CC (resp. QC) if its density operator $|\psi\rangle\langle\psi|$ is CC (resp. QC).

Luo in [41] proved that a state ρ^{AB} of $\mathcal{H}_A \otimes \mathcal{H}_B$ is CC if and only if it can be represented as

$$\rho^{AB} = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|, \tag{15}$$

where $\{p_{ij}\}$ is a PD, $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ are some ONBs for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. This implies that every CC state is separable while a separable state is not necessarily CC. But a bipartite pure state is CC if and only if it is separable ([48] [Theorem 5.1]).

5.2. Classical Correlation Depending on Basis

Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then a density operator ρ^{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ has the corresponding density matrix of ρ^{AB} under the basis $e_{AB} = e_A \otimes e_B$:

$$\rho^{AB}_{e_{AB}} = [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle] = [\langle e_i^A e_j^B|\rho^{AB}|e_k^A e_\ell^B\rangle],$$

which is equal to the matrix representation of the operator

$$(U_A \otimes U_B)\rho^{AB}(U_A^\dagger \otimes U_B^\dagger) : \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \to \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$$

under the canonical $\{0,1\}$-basis (with $m = d_A, n = d_B$)

$$\{|i_A j_B\rangle\}_{i,j} = \{|1_A 1_B\rangle, |1_A 2_B\rangle, \ldots, |1_A n_B\rangle, \ldots, |m_A 1_B\rangle, |m_A 2_B\rangle, \ldots, |m_A n_B\rangle\}$$

for $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} = \mathbb{C}^{d_A d_B}$. With this basis, a linear operator X on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is usually identified with matrix representation $[\langle i_A j_B|X|k_A \ell_B\rangle]$. Thus,

$$\rho^{AB}_{e_{AB}} = (U_A \otimes U_B)\rho^{AB}(U_A^\dagger \otimes U_B^\dagger), \tag{16}$$

where U_A and U_B are unitary operators given by Equations (8) and (9).

Similarly, when $d_A = d_B = d$ and $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ for all $i, j \in [d]$, we have

$$\rho^{AB}_{e_{AB}} = S(U_A \otimes U_B)\rho^{AB}(U_A^\dagger \otimes U_B^\dagger)S^\dagger, \tag{17}$$

where S is the swap operator on $\mathbb{C}^d \otimes \mathbb{C}^d$.

Using the characterization Equation (15) and Formula (16), we obtain the following.

Theorem 11. *Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is CC if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is CC.

(b) A density operator ρ^{AB} is CC if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is CC with $(ij, k\ell)$-entry $\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle$.

Similarly, using the characterization Equation (15) and Formula (17), we obtain the following.

Theorem 12. *Let $d_A = d_B = d$, $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ where $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle (i \in [d_A], j \in [d_B])$ for some ONBs $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ and $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then*

(a) A pure state $|\psi^{AB}\rangle$ is CC if and only if its coordinate state $|\psi^{AB}\rangle_{e_{AB}}$ is CC.

(b) A density operator ρ^{AB} is CC if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is CC with $(ij, k\ell)$-entry $\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle$.

Using Theorems 11 and 12, once can check the following.

Theorem 13. *Let $e_{AB} = \{|e_{ij}\rangle : i \in [d_A], j \in [d_B]\}$ be an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ and U be fined by Equation (2). Then TFSAE.*

(a) $|\psi^{AB}\rangle$ is CC if and only if $U|\psi^{AB}\rangle$ is CC.

(b) A density operator ρ^{AB} is CC if and only if its density matrix $\rho^{AB}_{e_{AB}} := [\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle]$ is CC with $(ij, k\ell)$-entry $\langle e_{ij}|\rho^{AB}|e_{k\ell}\rangle$.

(c) There exist unitary operators $W_X : \mathcal{H}_X \to \mathbb{C}^{d_X} (X = A, B)$ such that when $d_A \neq d_B$, $U = W_A \otimes W_B$; when $d_A = d_B = d$, either $U = W_A \otimes W_B$, or $U = S(W_A \otimes W_B)$, where $S : \mathbb{C}^d \otimes \mathbb{C}^d$ is the swap operator: $S(|x\rangle \otimes |y\rangle) = |y\rangle \otimes |x\rangle$.

(d) There exist orthonormal bases $e_A = \{|e_i^A\rangle\}_{i=1}^{d_A}$ *and* $e_B = \{|e_j^B\rangle\}_{j=1}^{d_B}$ *for* $\mathcal{H}_A$ *and* $\mathcal{H}_B$*, respectively, such that when* $d_A \neq d_B$, $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ *for all* i, j*; When* $d_A = d_B = d$*, either* $|e_{ij}\rangle = |e_i^A\rangle \otimes |e_j^B\rangle$ *for all* i, j*, or* $|e_{ij}\rangle = |e_j^A\rangle \otimes |e_i^B\rangle$ *for all* i, j.

6. Conclusions

Usually, a density operator ρ^{AB} (quantum state) of a bipartite system AB is represented as a matrix $\hat{\rho}_{e_{AB}} = [\rho_{ij}]$ under a basis $e_{AB} = \{e_{ij}\}_{i\in[d_A],j\in[d_B]}$ for the Hilbert space $H_A \otimes H_B$ of the system AB, called the density matrix of the density operator ρ^{AB}. In this work, we have discussed the relationships between quantum locality and basis, and observed that all density operators and their density matrices under a basis e_{AB} have the same quantum locality if and only if the basis e_{AB} is a product basis of two bases of subsystems. Consequently, different choices of bases may induce different quantum locality; equivalently, different bases define different quantum nonlocality, which we called basis-dependent quantum nonlocality. Also, entangled basis can generate quantum nonlocality of a density matrix from a separable density operator.

Acknowledgments: This work was supported by the National Natural Science Foundation of China under Grant Nos. 12271325, 11871318, and the Special Plan for Young Top-notch Talent of Shaanxi Province under Grant No. 1503070117.

Author Contributions: The work of this paper was accomplished by Kaifeng Hu, Zhihua Guo, Huaixin Cao and Ling Lu. Moreover, all authors have read the paper carefully and approved the research contents that were written in the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: Not applicable.

References

1. Bell, J.S.: On the Einstein Podolsky Rosen paradox. Physics **1**, 195-200 (1964)
2. Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V. and Wehner, S.: Bell nonlocality. Rev. Mod. Phys. **86**, 419-478 (2014)
3. Clauser, J.F., Shimony, A.: Experimental tests and implications. Rep. Prog. Phys. **41**, 1881-1927 (1978)
4. Home, D., Selleri, F.: Bell's theorem and the EPR paradox. Rivista Del Nuovo Cimento. **14**, 1-96 (1991)
5. Khalfin, L., Tsirelson, B.: Quantum/classical correspondence in the light of Bell's inequalities. Found. Phys. **22**, 879-948 (1992)
6. Tsirelson, B.S.: Some results and problems on quantum Bell-type inequalities. Hadronic J. Suppl. **8**, 329-345 (1993)
7. Zeilinger, A.: Experiment and the foundations of quantum physics. Rev. Mod. Phys. **71**, S288-S297 (1999)
8. Werner, R.F., Wolf, M.M.: All-multipartite Bell-correlation inequalities for two dichotomic observables per site. Phys. Rev. A **64**, 032112 (2001)
9. Genovese, M.: Research on hidden variable theories: a review of recent progresses. Phys. Rep. **413**, 319-396 (2005)
10. Buhrman, H., Cleve, R., Massar, S., de Wolf, R.: Nonlocality and communication complexity. Rev. Mod. Phys. **82**, 665-698 (2010)
11. Zhao, L.J., Guo, Y.M., Li-Jost, X., Fei, S.M.: Quantum nonlocality can be distributed via separable states. Sci. China-Phys. Mech. Astron. **61**, 070321 (2018)
12. Long, G.L., Qin, W., Yang, Z., Li, J.L.: Realistic interpretation of quantum mechanics and encounter- delayed-choice experiment. Sci. China-Phys. Mech. Astron. **61**, 030311 (2018)
13. Yang, Y., Cao, H.X., Chen, L., Huang, Y.F.: Λ_k-Nonlocality of multipartite states and the related nonlocality inequalities. Int. J. Theor. Phys. **57**, 1498-1515 (2018)
14. Cao, H.X., Guo, Z.H.: Characterizing Bell nonlocality and EPR steering. Sci. China-Phys. Mech. Astron. **62**, 030311 (2019)

15. Schrödinger, E.: Discussion of probability relations between separated systems. Math. Proc. Camb. Phil. Soc. **31**, 555-563 (1935)
16. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev., **47**, 777 (1935)
17. Reid, M.D.: Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification. Phys. Rev. A **40**, 913-923 (1989)
18. Werner, R.F.: Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Phys. Rev. A **40**, 4277-4281 (1989)
19. Qu, Z.Y., Pereira, S.F., Kimble, H.J., Peng, K.C.: Realization of the Einstein-Podolsky-Rosen paradox for continuous variables. Phys. Rev. Lett. **68**, 3663 (1992)
20. Wiseman, H.M., Jones, S.J., Doherty, A.C.: Steering, entanglement, nonlocality, and the Einstein- Podolsky-Rosen paradox. Phys. Rev. Lett. **98**, 140402 (2007)
21. Cavalcanti, E.G., Jones, S. J., Wiseman, H. M., Reid, M.D.: Experimental criteria for steering and the EinsteinPodolsky-Rosen paradox, Phys. Rev. A **80**, 032112 (2009)
22. Saunders, D.J., Jones, S.J., Wiseman, H.M., Pryde, G.J.: Experimental EPR-steering using Bell-local states. Nat. Phys. **6**, 845-849 (2010)
23. Bennet, A.J., Evans, D.A., Saunders, D.J., Branciard, C., Cavalcanti, E.G., Wiseman, H.M., Pryde, G.J.: Arbitrarily loss-tolerant Einstein-Podolsky-Rosen steering allowing a demonstration over 1 km of optical fiber with no detection loophole. Phys. Rev. X **2**, 031003 (2012)
24. Händchen, V., Eberle, T.,Steinlechner, S. *et al.*: Observation of one-way Einstein-Podolsky-Rosen steering. Nat. Photonics **6**, 596-599 (2012)
25. Branciard, C., Cavalcanti, E.G., Walborn, S.P., *et al.*: One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering. Phys. Rev. A **85**, 010301(R) (2012)
26. Wittmann, B., Ramelow, S., Steinlechner, F., *et al.*: New J. Phys. **14**, 053030 (2012)
27. Steinlechner, S., Bauchrowitz, J., Eberle, T., *et al.*: Strong Einstein-Podolsky-Rosen steering with unconditional entangled states. Phys. Rev. A **87**, 022104 (2013)
28. Reid, M.D.: Signifying quantum benchmarks for qubit teleportation and secure quantum communication using Einstein-Podolsky-Rosen steering inequalities. Phys. Rev. A **88**, 062338 (2013)
29. Piani, M., Watrous, J.: Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering. Phys. Rev. Lett. **114**, 060404 (2015)
30. Zukowski, M., Dutta, A., Yin, Z.: Geometric Bell-like inequalities for steering. Phys. Rev. A **91**, 032107 (2015)
31. Quintino, M.T., Vèrtesi, T., Cavalcanti, D., *et al.*: Inequivalence of entanglement, steering, and Bell nonlocality for general measurements. Phys. Rev. A **92**, 032107 (2015)
32. Zhu, H., Hayashi, M., Chen, L.: Universal steering inequalities. Phys. Rev. Lett. **116**, 070403 (2016)
33. Sun, K., Ye, X.J., Xu, J.S., *et al.*: Experimental quantification of asymmetric Einstein-Podolsky-Rosen steering. Phys. Rev. Lett. **116**, 160404 (2016)
34. Quan, Q., Zhu, H.J., Liu, S.Y., *et al.*: Steering Bell-diagonal states. Sci. Rep. **6**, 22025 (2016)
35. Chen, J.L., Ren, C.L., Chen, C.B.: Bell's nonlocality can be detected by the violation of Einstein-Podolsky-Rosen steering inequality. Sci. Rep. **6**, 39063 (2016)
36. Cavalcanti, D., Skrzypczyk, P.: Quantum steering: a review with focus on semidefinite programming. Rep. Prog. Phys. **80**, 024001 (2017)
37. Yu, S., Chen, Q., Zhang, C., *et al.*: All entangled pure states violate a single Bell's inequality. Phys. Rev. Lett. **109**, 120402 (2012)
38. Sun, K., Ye, X.J., Xu, J.S., *et al.*: Experimental quantification of asymmetric Einstein-Podolsky-Rosen steering. Phys. Rev. Lett. **116**, 160404 (2016)
39. Skrzypczyk, P., Navascues, M., Cavalcanti, D.: Quantifying einstein-podolsky-rosen steering. Phys. Rev. Lett. **112**, 180404 (2014)
40. Ollivier, H., Zurek, W.H.: Quantum discord: A measure of the quantumness of correlations. Phys. Rev. Lett. **88**, 017901 (2001)
41. Luo, S.: Using measurement-induced disturbance to characterize correlations as classical or quantum. Phys. Rev. A **77**, 022301 (2008)
42. Guo, Z.H., Cao, H.X., Chen, Z.L.: Distinguishing classical correlations from quantum correlations. J. Phys. A: Math. Theor. **45**, 145301 (2012)

 doi:10.20944/preprints202404.1505.v1

43. Guo, Z.H., Cao, H.X., Qu, S.X.: Structures of three types of local quantum chan- nels based on quantum correlations. Found. Phys. **45**, 355-369 (2015)
44. Knill, E., Laflamme, R.: Power of one bit of quantum information. Phys. Rev. Lett. **81**, 5672 (1998)
45. Su, X.L.: Applying Gaussian quantum discord to quantum key distribution. Chin. Sci. Bull. **59**, 1083-1090 (2014)
46. Yue, H.D., Zhang, Y., Gong, J.: Quantum discord in three-spin XXZ chain with three-spin interaction. Sci. China-Phys. Mech. Astron. **55**, 1641-1645 (2012)
47. Zhang, Y., Cao, H.X., Guo, Z.H.: Strongly separable operators on finite dimensional tensor product spaces. Acta Math. Sinica, **54**, 959-972 (2011)
48. Cao, H.X., Zhang, C.Y., , Guo, Z.H.: Some measurement-based characterizations of separability of bipartite states. Int. J. Theor. Phys. **60**, 2558-2572 (2021)