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Article

Cohomology of Modified Rota-Baxter Pre-Lie Algebras and Its Applications

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Abstract: Semenov-Tian-Shansky has introduced the modified classical Yang-Baxter equation, which is called the modified r-matrix. Relevant studies have been extensive in recent times. This paper is devoted to study cohomology theory of modified Rota-Baxter pre-Lie algebras and its applications. First we introduce the concept and representations of modified Rota-Baxter pre-Lie algebras. We then develop the cohomology of modified Rota-Baxter pre-Lie algebras with coefficients in a suitable representation. As applications, we consider the infinitesimal deformations, abelian extensions and skeletal modified Rota-Baxter pre-Lie 2-algebra in terms of lower degree cohomology groups.

Keywords: Pre-Lie algebras; modified Rota-Baxter operator; cohomology; deformation; abelian extension; pre-Lie 2-algebras

MSC: 17A01; 17B30; 17B10; 17B38; 17B56

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1. Introduction

Cayley [1] first introduced pre-Lie algebras (also called left-symmetric algebra) in the context of rooted tree algebras. Independently, Gerstenhaber [2] also introduced pre-Lie algebras in the deformation theory of rings and algebras. Pre-Lie algebras arose from the study of affine manifolds, affine structures on Lie groups and convex homogeneous cones [3], and then appeared in in geometry and physics, such as integrable systems, classical and quantum Yang-Baxter equations [4,5], quantum field theory, Poisson brackets, operands, complex and symplectic structures on Lie groups and Lie algebras [6]. See also in [7–15] for some interesting related about pre-Lie algebras.

Rota-Baxter operators on associative algebras were first introduced by Baxter [16] in his study of probability fluctuation theory, it was further developed by Rota [17]. Rota-Baxter operator has been widely used in many fields of mathematics and physics, including combinatorics, number theory, operads and quantum field theory [18]. The cohomology and deformation theory of Rota-Baxter operators of weight zero have been studied on various algebraic structures, see [19–23]. Recently, Wang and Zhou [24], Das [25] studied Rota-Baxter associative algebras of any weight by different methods respectively. Inspired by Wang and Zhou's work, Das [26] considered the cohomology and deformations of weighted Rota-Baxter Lie algebras. The authors [27,28] developed the cohomology, extensions and deformations of Rota-Baxter 3-Lie algebras with any weight. In [29], Chen, Lou and Sun studied the cohomology and extensions of Rota-Baxter Lie triple systems. In [30], Guo and his collaborators explored the cohomology, deformations and extensions of Rota-Baxter pre-Lie algebras of arbitrary weights.

The term modified Rota-Baxter operator stemmed from the notion of the modified classical Yang-Baxter equation, which was also introduced in the work of Semenov-Tian-Shansky [31] as a modification of the operator form of the classical Yang-Baxter equation. Due to the importance of Rota-Baxter algebras and modified Rota-Baxter algebras, Zheng, Guo and Qiu [32] studied properties of extended Rota-Baxter operators. Recently, Jiang and Sheng have been established cohomology and deformation theory of modified r-matrices in [33]. Inspired by [33], modified Rota-Baxter algebraic structures have been widely studied in [34–36].

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However, there was very few study about the modified Rota-Baxter pre-Lie algebras. The purpose of the paper is to study the cohomology of a modified Rota-Baxter pre-Lie algebra and its applications. In precisely, we introduce the concept of a modified Rota-Baxter pre-Lie algebra, which includes a pre-Lie algebra and a modified Rota-Baxter operator. And then, we propose a representation of a modified Rota-Baxter pre-Lie algebra. We define a cochain map Υ, and then the cohomology of modified Rota-Baxter pre-Lie algebras with coefficients in a representation is constructed. Finally, as applications of our propose cohomology theory, we consider the infinitesimal deformations and abelian extensions of a modified Rota-Baxter pre-Lie algebra in terms of second cohomology groups. In addition, we prove that any skeletal modified Rota-Baxter pre-Lie 2-algebra can be classified by the third cohomology group.

The paper is organized as follows. In Section 2, we introduce the concept of modified Rota-Baxter pre-Lie algebras, and give its representations. In Section 3, we establish the cohomology theory of modified Rota-Baxter pre-Lie algebras with coefficients in a representation, and apply it to the study of infinitesimal deformation. In Section 4, we discuss an abelian extension of the modified Rota-Baxter pre-Lie algebras in terms of our second cohomology groups. Finally, in Section 5, we classify skeletal modified Rota-Baxter pre-Lie 2-algebra using the third cohomology group.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All the vector spaces and (multi)linear maps are taken over $\mathbb{K}$.

2. Representations of Modified Rota-Baxter Pre-Lie Algebras

In this section, we introduce the concept of modified Rota-Baxter pre-Lie algebras motivated by the modified r-matrices in [33] and give some examples. Next we propose the representation of modified Rota-Baxter pre-Lie algebras. Finally, we establish a new modified Rota-Baxter pre-Lie algebra and give its representation.

First, let's recall some definitions and results about pre-Lie algebra and its representations from [2].

Definition 2.1. [2] A pre-Lie algebra is a pair $(\mathcal{P}, \bullet)$ consisting of a vector space $\mathcal{P}$ and a binary operation $\bullet : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ such that for $a, b, c \in \mathcal{P}$, the associator

$$(a, b, c) = (a \bullet b) \bullet c - a \bullet (b \bullet c),$$

is symmetric in a, b, i.e.

$$(a, b, c) = (b, a, c), \text{ or equivalently, } (a \bullet b) \bullet c - a \bullet (b \bullet c) = (b \bullet a) \bullet c - b \bullet (a \bullet c). \tag{2.1}$$

Given a pre-Lie algebra $(\mathcal{P}, \bullet)$, the commutator $[a, b]^c = a \bullet b - b \bullet a$, defines a Lie algebra structure on $\mathcal{P}$, which is called the sub-adjacent Lie algebra of $(\mathcal{P}, \bullet)$ and we denote it by $\mathcal{P}^c$.

Definition 2.2. (i) Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra. A modified Rota-Baxter operator on $\mathcal{P}$ is a linear map $M : \mathcal{P} \to \mathcal{P}$ subject to

$$Ma \bullet Mb = M(Ma \bullet b + a \bullet Mb) - a \bullet b, \ \forall\, a, b \in \mathcal{P}. \tag{2.2}$$

Furthermore, the triple $(\mathcal{P}, \bullet, M)$ is called modified Rota-Baxter pre-Lie algebra, simply denoted by $(\mathcal{P}, M)$.

(ii) A homomorphism between two modified Rota-Baxter pre-Lie algebras $(\mathcal{P}_1, M_1)$ and $(\mathcal{P}_2, M_2)$ is a pre-Lie algebra homomorphism $F : \mathcal{P}_1 \to \mathcal{P}_2$ such that $F \circ M_1 = M_2 \circ F$. Furthermore, F is called an isomorphism from $(\mathcal{P}_1, M_1)$ to $(\mathcal{P}_2, M_2)$ if F is nondegenerate.

Example 2.3. An identity map $\mathrm{id}_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ is a modified Rota-Baxter operator.

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Example 2.4. Let $(\mathcal{P}, \bullet)$ be a 2-dimensional pre-Lie algebra and $\{\epsilon_1, \epsilon_2\}$ be a basis, whose nonzero products are given as follows:

$$\epsilon_1 \bullet \epsilon_2 = \epsilon_1, \quad \epsilon_2 \bullet \epsilon_2 = \epsilon_2.$$

Then, for $k \in \mathbb{K}$, the operator

$$M = \begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}$$

is a modified Rota-Baxter operator on $\mathcal{P}$.

Example 2.5. Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra. If a linear map $M : \mathcal{P} \to \mathcal{P}$ is a modified Rota-Baxter operator, then $-M$ is also a modified Rota-Baxter operator.

Definition 2.6. [13] Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra. A Rota-Baxter operator of weight -1 on $\mathcal{P}$ is a linear map $R : \mathcal{P} \to \mathcal{P}$ subject to

$$Ra \bullet Rb = R(Ra \bullet b + a \bullet Rb - a \bullet b), \ \forall \ a, b \in \mathcal{P}.$$

And then, the triple $(\mathcal{P}, \bullet, R)$ is called Rota-Baxter pre-Lie algebra of weight -1.

Proposition 2.7. *Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra. If a linear map $R : \mathcal{P} \to \mathcal{P}$ is a Rota-Baxter operator of weight -1, then the map $2R - \mathrm{id}_{\mathcal{P}}$ is a modified Rota-Baxter operator on $\mathcal{P}$.*

Proof. For any $a, b \in \mathcal{P}$, we have

$$\begin{aligned}
&(2R - \mathrm{id}_{\mathcal{P}})a \bullet (2R - \mathrm{id}_{\mathcal{P}})b \\
&= (2Ra - a) \bullet (2Rb - b) \\
&= 4Ra \bullet Rb - 2Ra \bullet b - 2a \bullet Rb + a \bullet b \\
&= 4R(Ra \bullet b + a \bullet Rb - a \bullet b) - 2Ra \bullet b - 2a \bullet Rb + a \bullet b \\
&= (2R - \mathrm{id}_{\mathcal{P}})\big((2R - \mathrm{id}_{\mathcal{P}})a \bullet b + a \bullet (2R - \mathrm{id}_{\mathcal{P}})b\big) - a \bullet b.
\end{aligned}$$

The proposition follows. □

Recall from [13] that a Nijenhuis operator on a pre-Lie algebra $(\mathcal{P}, \bullet)$ is a linear map $N : \mathcal{P} \to \mathcal{P}$ satisfies

$$Na \bullet Nb = N(Na \bullet b + a \bullet Nb - N(a \bullet b)),$$

for all $a, b \in \mathcal{P}$. The relationship between the modified Rota-Baxter operator and Nijenhuis operator is as follows, which proves to be obvious.

Proposition 2.8. *Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra and $N : \mathcal{P} \to \mathcal{P}$ be a linear map. If $N^2 = \mathrm{id}$, then N is a Nijenhuis operator if and only if N is a modified Rota-Baxter operator.*

Definition 2.9. [8] Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra and V a vector space. A representation of $\mathcal{P}$ on V consists of a pair $(\bullet_l, \bullet_r)$, where $\bullet_l : \mathcal{P} \times V \to V$ and $\bullet_r : V \times \mathcal{P} \to V$ are two linear maps satisfying

$$\begin{aligned}
a \bullet_l (b \bullet_l u) - (a \bullet b) \bullet_l u &= b \bullet_l (a \bullet_l u) - (b \bullet a) \bullet_l u, \\
a \bullet_l (u \bullet_r b) - (a \bullet_l u) \bullet_r b &= u \bullet_r (a \bullet b) - (u \bullet_r a) \bullet_r b, \forall a, b \in \mathcal{P}, u \in V.
\end{aligned}$$

Definition 2.10. A representation of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$ is a quadruple $(V; \bullet_l, \bullet_r, M_V)$ such that the following conditions are satisfied:

(i) $(V; \bullet_l, \bullet_r)$ is a representation of the pre-Lie algebra $(\mathcal{P}, \bullet)$;

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(ii) $M_V : V \to V$ is a linear map satisfying the following equations

$$Ma \bullet_l M_V u = M_V(Ma \bullet_l u + a \bullet_l M_V u) - a \bullet_l u, \tag{2.3}$$
$$M_V u \bullet_r Ma = M_V(M_V u \bullet_r a + u \bullet_r Ma) - u \bullet_r a, \tag{2.4}$$

for $a \in \mathcal{P}$ and $u \in V$.

Example 2.11. $(\mathcal{P}; \bullet_l = \bullet_r = \bullet, M)$ is an adjoint representation of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$.

Next we construct the semidirect product of the modified Rota-Baxter pre-Lie algebra.

Proposition 2.12. *If $(V; \bullet_l, \bullet_r, M_V)$ is a representation of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$, then $\mathcal{P} \oplus V$ is a modified Rota-Baxter pre-Lie algebra with the following maps:*

$$(a + u) \bullet_\ltimes (b + v) := a \bullet b + a \bullet_l v + u \bullet_r b,$$
$$M \oplus M_V(a + u) = Ma + M_V u,$$

for $a \in \mathcal{P}$ and $u \in V$. In the case, the modified Rota-Baxter pre-Lie algebra $\mathcal{P} \oplus V$ is called a semidirect product of $\mathcal{P}$ and V, denoted by $\mathcal{P} \ltimes V = (\mathcal{P} \oplus V, \bullet_\ltimes, M \oplus M_V)$.

Proof. Firstly, it is easy to verify that $(\mathcal{P} \oplus V, \bullet_\ltimes)$ is a pre-Lie algebra. In addition, for any $a, b \in \mathcal{P}$ and $u, v \in V$, by Equations (2.2)- (2.4) we have

$$\begin{aligned}
&M \oplus M_V(a + u) \bullet_\ltimes M \oplus M_V(b + v) \\
&= (Ma + M_V u) \bullet_\ltimes (Mb + M_V v) \\
&= Ma \bullet Mb + Ma \bullet_l M_V v + M_V u \bullet_r Mb \\
&= M(Ma \bullet b + a \bullet Mb) - a \bullet b + M_V(Ma \bullet_l u + a \bullet_l M_V u) - a \bullet_l u \\
&\quad + M_V(M_V u \bullet_r b + u \bullet_r Mb) - u \bullet_r b \\
&= M \oplus M_V\big((a + u) \bullet_\ltimes M \oplus M_V(b + v) + M \oplus M_V(a + u) \bullet_\ltimes (b + v)\big) - (a + u) \bullet_\ltimes (b + v),
\end{aligned}$$

which means that $(\mathcal{P} \oplus V, \bullet_\ltimes, M \oplus M_V)$ is a modified Rota-Baxter pre-Lie algebra. □

Proposition 2.13. *Let $(\mathcal{P}, \bullet, M)$ be a modified Rota-Baxter pre-Lie algebra, Define new operation as follows:*

$$a \bullet_M b = Ma \bullet b + a \bullet Mb, \forall a, b \in \mathcal{P}. \tag{2.5}$$

Then, (i) $(\mathcal{P}, \bullet_M)$ is a pre-Lie algebra. We denote this pre-Lie algebra by $\mathcal{P}_M$.
(ii) $(\mathcal{P}_M, M)$ is a modified Rota-Baxter pre-Lie algebra.

Proof. (i) For any $a, b, c \in \mathcal{P}$, by Equations (2.1) and (2.2), we have

$$\begin{aligned}
&(a \bullet_M b) \bullet_M c - a \bullet_M (b \bullet_M c) \\
&= M(Ma \bullet b + a \bullet Mb) \bullet c + (Ma \bullet b + a \bullet Mb) \bullet Mc - Ma \bullet (Mb \bullet c + b \bullet Mc) \\
&\quad - a \bullet M(Mb \bullet c + b \bullet Mc) \\
&= M(Mb \bullet a + b \bullet Ma) \bullet c + (Mb \bullet a + b \bullet Ma) \bullet Mc - Mb \bullet (Ma \bullet c + a \bullet Mc) \\
&\quad - b \bullet M(Ma \bullet c + a \bullet Mc) \\
&= (b \bullet_M a) \bullet_M c - b \bullet_M (a \bullet_M c)
\end{aligned}$$

Thus, $(\mathcal{P}, \bullet_M)$ is a pre-Lie algebra.

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(ii) For any $a, b \in \mathcal{P}$, by Eq. (2.2), we have

$$\begin{aligned} Ma \bullet_M Mb =& M^2a \bullet Mb + Ma \bullet M^2b \\ =& M(M^2a \bullet b + Ma \bullet Mb) - Ma \bullet b + M(Ma \bullet Mb + a \bullet M^2b) - a \bullet Mb \\ =& M(Ma \bullet_M b + Ma \bullet_M b) - a \bullet_M b. \end{aligned}$$

Hence, $(\mathcal{P}_M, M)$ is a modified Rota-Baxter pre-Lie algebra. □

Proposition 2.14. *Let $(V; \bullet_l, \bullet_r, M_V)$ be a representation of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$, Define two bilinear maps $\bullet_l^M : \mathcal{P} \times V \to V$ and $\bullet_r^M : V \times \mathcal{P} \to V$ by*

$$a \bullet_l^M u := Ma \bullet_l u - M_V(a \bullet_l u), \tag{2.6}$$

$$u \bullet_r^M a := u \bullet_r Ma - M_V(u \bullet_r a), \forall a \in \mathcal{P}, u \in V. \tag{2.7}$$

Then $(V; \bullet_l^M, \bullet_r^M)$ is a representation of a pre-Lie algebra $\mathcal{P}_M$. Moreover, $(V; \bullet_l^M, \bullet_r^M, M_V)$ is a representation of a modified Rota-Baxter pre-Lie algebra $(\mathcal{P}_M, M)$.

Proof. First, by direct verification, $(V; \bullet_l^M, \bullet_r^M)$ is a representation of the pre-Lie algebra $\mathcal{P}_M$. Further, for any $a \in \mathcal{P}$ and $u \in V$, by Eq. (2.3), we have

$$\begin{aligned} & Ma \bullet_l^M M_V u \\ &= M^2a \bullet_l M_V u - M_V(Ma \bullet_l M_V u) \\ &= M_V(M^2a \bullet_l u + Ma \bullet_l M_V u) - Ma \bullet_l u - M_V^2(Ma \bullet_l u + a \bullet_l M_V u) + M_V(a \bullet_l u) \\ &= M_V\big(M^2a \bullet_l u + Ma \bullet_l M_V u - M_V(Ma \bullet_l u + a \bullet_l M_V u)\big) - \big(Ma \bullet_l u - M_V(a \bullet_l u)\big) \\ &= M_V(Ma \bullet_l^M u + a \bullet_l^M M_V u) - a \bullet_l^M u. \end{aligned}$$

Similarly, by Eq. (2.4), there is also $M_V u \bullet_r^M Ma = M_V(M_V u \bullet_r^M a + u \bullet_r^M Ma) - u \bullet_r^M a$. Hence, $(V; \bullet_l^M, \bullet_r^M, M_V)$ is a representation of $(\mathcal{P}_M, M)$. □

Example 2.15. $(\mathcal{P}; \bullet_l^M = \bullet_r^M = \bullet^M, M)$ is an adjoint representation of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}_M, M)$, where

$$a \bullet^M b := Ma \bullet b - M(a \bullet b),$$

for any $a, b \in \mathcal{P}$.

3. Cohomology of Modified Rota-Baxter Pre-Lie Algebras

In this section, we develop the cohomology of a modified Rota-Baxter pre-Lie algebra with coefficients in its representation.

Let us recall the cohomology theory of pre-Lie algebras in [14]. Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra and $(V; \bullet_l, \bullet_r)$ be a representation of it. Denote the $n-$cochains of $\mathcal{P}$ with coefficients in representation V by

$$C^n_{\mathrm{PLie}}(\mathcal{P}, V) := \mathrm{Hom}(\mathcal{P}^{\otimes n}, V).$$

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The coboundary operator $\delta : C^n_{\mathrm{PLie}}(\mathcal{P}, V) \to C^{n+1}_{\mathrm{PLie}}(\mathcal{P}, V)$, for $a_1, \cdots, a_{n+1} \in \mathcal{P}$ and $g \in C^n_{\mathrm{PLie}}(\mathcal{P}, V)$, as

$$\begin{aligned} &\delta g(a_1, \ldots, a_{n+1}) \\ =& \sum_{i=1}^{n}(-1)^{i+1}a_i \bullet_l g(a_1, \ldots, \widehat{a}_i, \cdots, a_{n+1}) + \sum_{i=1}^{n}(-1)^{i+1}g(a_1, \ldots, \widehat{a}_i, \ldots, a_n, a_i) \bullet_r a_{n+1} \\ &- \sum_{i=1}^{n}(-1)^{i+1}g(a_1, \ldots, \widehat{a}_i, \ldots, a_n, a_i \bullet a_{n+1}) + \sum_{1\leq i<j\leq n}(-1)^{i+j}g([a_i, a_j]^c, a_1, \ldots, \widehat{a}_i, \ldots, \widehat{a}_j, \ldots, a_{n+1}). \end{aligned} \tag{3.1}$$

Then, it has been proved in [14] that $\delta^2 = 0$. Let us denote by $H^*_{\mathrm{PLie}}(\mathcal{P}, V)$, the cohomology group associated to the cochain complex $(C^*_{\mathrm{PLie}}(\mathcal{P}, V), \delta)$.

We first study the cohomology of the modified Rota-Baxter operator.

Let $(\mathcal{P}, \bullet, M)$ be a modified Rota-Baxter pre-Lie algebra and $(V; \bullet_l, \bullet_r, M_V)$ be a representation of it, Recall that Proposition 2.13 and Proposition 2.14 give a new pre-Lie algebra $\mathcal{P}_M$ and a new representation $V_M = (V; \bullet^M_l, \bullet^M_r)$ over $\mathcal{P}_M$. Consider the cochain complex of $\mathcal{P}_M$ with coefficients in V_M:

$$(C^*_{\mathrm{PLie}}(\mathcal{P}_M, V_M), \delta_M) = (\oplus_{n=1}^{\infty} C^n_{\mathrm{PLie}}(\mathcal{P}_M, V_M), \delta_M).$$

More precisely, $C^n_{\mathrm{PLie}}(\mathcal{P}_M, V_M) := \mathrm{Hom}(\mathcal{P}_M^{\otimes n}, V_M)$ and its coboundary map $\delta_M : C^n_{\mathrm{PLie}}(\mathcal{P}_M, V_M) \to C^n_{\mathrm{PLie}}(\mathcal{P}_M, V_M)$, for $a_1, \cdots, a_{n+1} \in \mathcal{P}_R$ and $f \in C^n_{\mathrm{PLie}}(\mathcal{P}_M, V_M)$, is given as follows:

$$\begin{aligned} &\delta_M f(a_1, \ldots, a_{n+1}) \\ =& \sum_{i=1}^{n}(-1)^{i+1}\Big(Ma_i \bullet_l f(a_1, \ldots, \widehat{a}_i, \cdots, a_{n+1}) - M_V\big(a_i \bullet_l f(a_1, \ldots, \widehat{a}_i, \cdots, a_{n+1})\big)\Big) \\ &+ \sum_{i=1}^{n}(-1)^{i+1}\Big(f(a_1, \ldots, \widehat{a}_i, \ldots, a_n, a_i) \bullet_r Ma_{n+1} - M_V\big(f(a_1, \ldots, \widehat{a}_i, \ldots, a_n, a_i) \bullet_r a_{n+1}\big)\Big) \\ &- \sum_{i=1}^{n}(-1)^{i+1}f(a_1, \ldots, \widehat{a}_i, \ldots, a_n, Ma_i \bullet a_{n+1} + a_i \bullet Ma_{n+1}) \\ &+ \sum_{1\leq i<j\leq n}(-1)^{i+j}f(Ma_i \bullet a_j + a_i \bullet Ma_j - Ma_j \bullet a_i - a_j \bullet Ma_i, a_1, \ldots, \widehat{a}_i, \ldots, \widehat{a}_j, \ldots, a_{n+1}). \end{aligned} \tag{3.2}$$

Definition 3.1. Let $(\mathcal{P}, \bullet, M)$ be a modified Rota-Baxter pre-Lie algebra and $(V; \bullet_l, \bullet_r, M_V)$ be a representation of it. Then the cochain complex $(C^*_{\mathrm{PLie}}(\mathcal{P}_M, V_M), \delta_M)$ is called the cochain complex of modified Rota-Baxter operator M with coefficients in V_M, denoted by $(\mathcal{C}^*_{\mathrm{MRBO}}(\mathcal{P}, V), \delta_M)$. The cohomology of $(\mathcal{C}^*_{\mathrm{MRBO}}(\mathcal{P}, V), \delta_M)$, denoted by $\mathcal{H}^*_{\mathrm{MRBO}}(\mathcal{P}, V)$, is called the cohomology of modified Rota-Baxter operator M with coefficients in V_M.

In particular, when $(\mathcal{P}; \bullet^M_l = \bullet^M_r = \bullet^M, M)$ is the adjoint representation of $(\mathcal{P}_M, M)$, we denote $(\mathcal{C}^*_{\mathrm{MRBO}}(\mathcal{P}, \mathcal{P}), \delta_M)$ by $(\mathcal{C}^*_{\mathrm{MRBO}}(\mathcal{P}), \delta_M)$ and call it the cochain complex of modified Rota-Baxter operator M, and denote $\mathcal{H}^*_{\mathrm{MRBO}}(\mathcal{P}, \mathcal{P})$ by $\mathcal{H}^*_{\mathrm{MRBO}}(\mathcal{P})$ and call it the cohomology of modified Rota-Baxter operator M.

Next, we will combine the cohomology of pre-Lie algebras and the cohomology of modified Rota-Baxter operators to construct a cohomology theory for modified Rota-Baxter pre-Lie algebras.

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Let's construct the following cochain map. For any $n \geq 1$, we define a linear map $\Upsilon : C^n_{\mathrm{PLie}}(\mathcal{P}, V) \to \mathcal{C}^n_{\mathrm{MRBO}}(\mathcal{P}, V)$ by

$$
\begin{aligned}
&(\Upsilon f)(a_1, \ldots, a_n) \\
&= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} \Big(\sum_{1 \leqslant j_1 < \cdots < j_{2i-2} \leqslant n} f(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-2}}, \ldots, a_n) \\
&\quad - \sum_{1 \leqslant j_1 < \cdots < j_{2i-3} \leqslant n} M_V f(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-3}}, \ldots, a_n) \Big), \text{if } n \text{ is an even,}
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
&(\Upsilon f)(a_1, \ldots, a_n) \\
&= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} \Big(\sum_{1 \leqslant j_1 < \cdots < j_{2i-1} \leqslant n} f(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-1}}, \ldots, a_n) \\
&\quad - \sum_{1 \leqslant j_1 < \cdots < j_{2i-2} \leqslant n} M_V f(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-2}}, \ldots, a_n) \Big), \text{if } n \text{ is an odd,}
\end{aligned} \tag{3.4}
$$

among them, when the subscript of j_{2i-3} is negative, f is a zero map. For example, when $n = 1$, by Eq. (3.4), the map $\Upsilon : C^1_{\mathrm{PLie}}(\mathcal{P}, V) \to \mathcal{C}^1_{\mathrm{MRBO}}(\mathcal{P}, V)$ is as follows:

$$
(\Upsilon f)(a_1) = f(Ma_1) - M_V f(a_1). \tag{3.5}
$$

Lemma 3.2. *The map Υ is a cochain map, i.e., $\Upsilon \circ \delta = \delta_M \circ \Upsilon$. In other words, the following diagram is commutative:*

$$
\begin{array}{ccccccccc}
C^1_{\mathrm{PLie}}(\mathcal{P}, V) & \xrightarrow{\delta} & C^2_{\mathrm{PLie}}(\mathcal{P}, V) & \cdots\cdots & C^n_{\mathrm{PLie}}(\mathcal{P}, V) & \xrightarrow{\delta} & C^{n+1}_{\mathrm{PLie}}(\mathcal{P}, V) & \cdots\cdots & \\
\downarrow{\scriptstyle \Upsilon} & & \downarrow{\scriptstyle \Upsilon} & & \downarrow{\scriptstyle \Upsilon} & & \downarrow{\scriptstyle \Upsilon} & & \\
\mathcal{C}^1_{\mathrm{MRBO}}(\mathcal{P}, V) & \xrightarrow{\delta_M} & \mathcal{C}^2_{\mathrm{MRBO}}(\mathcal{P}, V) & \cdots\cdots & \mathcal{C}^n_{\mathrm{MRBO}}(\mathcal{P}, V) & \xrightarrow{\delta_M} & \mathcal{C}^{n+1}_{\mathrm{MRBO}}(\mathcal{P}, V) & \cdots\cdots & .
\end{array}
$$

Proof. It can be proved by using similar arguments to Appendix A in [28]. Because of space limitations, here we only prove the case of $n = 1$. For any $f \in C^1_{\mathrm{PLie}}(\mathcal{P}, V)$ and $a, b \in \mathcal{P}$, by Equations(2.2)-(2.7), (3.1)-(3.3) and (3.5), we have

$$
\begin{aligned}
&\Upsilon(\delta f)(a, b) \\
&= (\delta f)(Ma, Mb) - M_V\big((\delta f)(Ma, b) + (\delta f)(a, Mb)\big) + (\delta f)(a, b) \\
&= Ma \bullet_l f(Mb) + f(Ma) \bullet_r Mb - f(Ma \bullet Mb) - M_V\big(Ma \bullet_l f(b) + f(Ma) \bullet_r b - f(Ma \bullet b) \\
&\quad + a \bullet_l f(Mb) + f(a) \bullet_r Mb - f(a \bullet Mb)\big) + a \bullet_l f(b) + f(a) \bullet_r b - f(a \bullet b)
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
&\delta_M(\Upsilon f)(a, b) \\
&= Ma \bullet_l \big(f(Mb) - M_V f(b)\big) - M_V\big(a \bullet_l \big(f(Mb) - M_V f(b)\big)\big) + \big(f(Ma) - M_V f(a)\big) \bullet_r Mb \\
&\quad - M_V\big(\big(f(Ma) - M_V f(a)\big) \bullet_r b\big) - f(Ma \bullet Mb + a \bullet b) + M_V f(Ma \bullet b + a \bullet Mb)
\end{aligned} \tag{3.7}
$$

Further comparing Equations (3.6) and (3.7), we have (3.6)=(3.7). Therefore, $\Upsilon \circ \delta = \delta_M \circ \Upsilon$. $\square$

Definition 3.3. Let $(\mathcal{P}, \bullet, M)$ be a modified Rota-Baxter pre-Lie algebra and $(V; \bullet_l, \bullet_r, M_V)$ be a representation of it. We define the cochain complex $(\mathcal{C}^*_{\mathrm{MRBPLie}}(\mathcal{P}, V), \partial)$ of modified Rota-Baxter pre-

Lie algebra $(\mathcal{P}, \bullet, M)$ with coefficients in $(V; \bullet_l, \bullet_r, M_V)$ to the negative shift of the mapping cone of Υ, that is, let

$$\mathcal{C}^1_{\text{MRBPLie}}(\mathcal{P}, V) = C^1_{\text{PLie}}(\mathcal{P}, V) \text{ and } \mathcal{C}^n_{\text{MRBPLie}}(\mathfrak{g}, V) := C^n_{\text{PLie}}(\mathcal{P}, V) \oplus \mathcal{C}^{n-1}_{\text{MRBO}}(\mathcal{P}, V), \forall n \geq 2,$$

and the coboundary map $\partial : \mathcal{C}^1_{\text{MRBPLie}}(\mathcal{P}, V) \to \mathcal{C}^2_{\text{MRBPLie}}(\mathcal{P}, V)$ is given by

$$\partial(f) = (\delta f, -\Upsilon f), \forall f \in \mathcal{C}^1_{\text{MRBPLie}}(\mathcal{P}, V);$$

for $n \geq 2$, the coboundary map $\partial : \mathcal{C}^n_{\text{MRBPLie}}(\mathcal{P}, V) \to \mathcal{C}^{n+1}_{\text{MRBPLie}}(\mathcal{P}, V)$ is given by

$$\partial(f, g) = (\delta f, -\delta_M g - \Upsilon f), \forall (f, g) \in \mathcal{C}^n_{\text{MRBPLie}}(\mathcal{P}, V).$$

The cohomology of $(\mathcal{C}^*_{\text{MRBPLie}}(\mathcal{P}, V), \partial)$, denoted by $\mathcal{H}^*_{\text{MRBPLie}}(\mathcal{P}, V)$, is called the cohomology of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$ with coefficients in $(V; \bullet_l, \bullet_r, M_V)$. In particular, when $(V; \bullet_l, \bullet_r, M_V) = (\mathcal{P}; \bullet_l = \bullet_r = \bullet, M)$, we just denote $(\mathcal{C}^*_{\text{MRBPLie}}(\mathcal{P}, \mathcal{P}), \partial)$, $\mathcal{H}^*_{\text{MRBPLie}}(\mathcal{P}, \mathcal{P})$ by $(\mathcal{C}^*_{\text{MRBPLie}}(\mathcal{P}), \partial)$, $\mathcal{H}^*_{\text{MRBPLie}}(\mathcal{P})$ respectively, and call them the cochain complex, the cohomology of modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$ respectively.

It is obvious that there is a short exact sequence of cochain complexes:

$$0 \to \mathcal{C}^{*-1}_{\text{MRBO}}(\mathcal{P}, V) \longrightarrow \mathcal{C}^*_{\text{MRBPLie}}(\mathcal{P}, V) \longrightarrow C^*_{\text{PLie}}(\mathcal{P}, V) \to 0.$$

It induces a long exact sequence of cohomology groups:

$$\cdots \to \mathcal{H}^n_{\text{MRBPLie}}(\mathcal{P}, V) \to H^n_{\text{PLie}}(\mathcal{P}, V) \to \mathcal{H}^n_{\text{MRBO}}(\mathcal{P}, V) \to \mathcal{H}^{n+1}_{\text{MRBPLie}}(\mathcal{P}, V) \to H^{n+1}_{\text{PLie}}(\mathcal{P}, V) \to \cdots.$$

At the end of this section, we use the established cohomology theory to characterize infinitesimal deformations of modified Rota-Baxter pre-Lie algebras.

Definition 3.4. A infinitesimal deformation of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$ is a pair $(\bullet_t, M_t)$ of the forms

$$\bullet_t = \bullet + \bullet_1 t, \quad M_t = M + M_1 t,$$

such that the following conditions are satisfied:

(i) $(\bullet_1, M_1) \in \mathcal{C}^2_{\text{MRBPLie}}(\mathcal{P})$,

(ii) and $(\mathcal{P}[[t]], \bullet_t, M_t)$ is a modified Rota-Baxter pre-Lie algebra over $\mathbb{K}[[t]]$.

Proposition 3.5. *Let $(\mathcal{P}[[t]], \bullet_t, M_t)$ be a infinitesimal deformation of modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$. Then $(\bullet_1, M_1)$ is a 2-cocycle in the cochain complex $(\mathcal{C}^*_{\text{MRBPLie}}(\mathcal{P}), \partial)$.*

Proof. Suppose $(\mathcal{P}[[t]], \bullet_t, M_t)$ is a modified Rota-Baxter pre-Lie algebra. Then for any $a, b, c \in \mathcal{P}$, we have

$$\begin{aligned}(a \bullet_t b) \bullet_t c - a \bullet_t (b \bullet_t c) =& (b \bullet_t a) \bullet_t c - b \bullet_t (a \bullet_t c),\\ M_t a \bullet_t M_t b =& M_t(M_t a \bullet_t b + a \bullet_t M_t b) - a \bullet_t b.\end{aligned}$$

Comparing coefficients of t^1 on both sides of the above equations, we have

$$\begin{aligned}&(a \bullet_1 b) \bullet c + (a \bullet b) \bullet_1 c - a \bullet (b \bullet_1 c) - a \bullet_1 (b \bullet c)\\ &= (b \bullet_1 a) \bullet c + (b \bullet a) \bullet_1 c - b \bullet_1 (a \bullet c) - b \bullet (a \bullet_1 c),\\ &M_1 a \bullet Mb + Ma \bullet M_1 b + Ma \bullet_1 Mb\\ &= M(M_1 a \bullet b + Ma \bullet_1 b + a \bullet M_1 b + a \bullet_1 Mb) + M_1(Ma \bullet b + a \bullet Mb) - a \bullet_1 b.\end{aligned}$$

Therefore, $\partial(\bullet_1, M_1) = (\delta \bullet_1, -\delta_M M_1 - \Upsilon \bullet_1) = 0$, that is, $(\bullet_1, M_1)$ is a 2-cocycle. □

Definition 3.6. The 2-cocycle $(\bullet_1, M_1)$ is called the infinitesimal of the infinitesimal deformation $(\mathcal{P}[[t]], \bullet_t, M_t)$ of modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$.

Definition 3.7. Let $(\mathcal{P}[[t]], \bullet_t, M_t)$ and $(\mathcal{P}[[t]], \bullet'_t, M'_t)$ be two infinitesimal deformations of modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$. An isomorphism from $(\mathcal{P}[[t]], \bullet'_t, M'_t)$ to $(\mathcal{P}[[t]], \bullet_t, M_t)$ is a linear map $\varphi_t = \mathrm{id} + t\varphi_1$, where $\varphi_1 : \mathcal{P} \to \mathcal{P}$ is linear map, such that:

$$\varphi_t \circ \bullet'_t = \bullet_t \circ (\varphi_t \otimes \varphi_t), \tag{3.8}$$
$$\varphi_t \circ M'_t = M_t \circ \varphi_t. \tag{3.9}$$

In this case, we say that the two infinitesimal deformations $(\mathcal{P}[[t]], \bullet_t, M_t)$ and $(\mathcal{P}[[t]], \bullet'_t, M'_t)$ are equivalent.

Proposition 3.8. *The infinitesimals of two equivalent infinitesimal deformations of $(\mathcal{P}, \bullet, M)$ are in the same cohomology class in $\mathcal{H}^2_{\mathrm{MRBPLie}}(\mathcal{P})$.*

Proof. Let $\varphi_t : (\mathcal{P}[[t]], \bullet'_t, M'_t) \to (\mathcal{P}[[t]], \bullet_t, M_t)$ be an isomorphism. By expanding Equations (3.8) and (3.9) and comparing the coefficients of t^1 on both sides, we have

$$\bullet'_1 - \bullet_1 = \varphi_1 \bullet \mathrm{id} + \mathrm{id} \bullet \varphi_1 - \varphi_1 \circ \bullet = \delta\varphi_1,$$
$$M'_1 - M_1 = M \circ \varphi_1 - \varphi_1 \circ M = -\Upsilon\varphi_1,$$

that is, we have

$$(\bullet'_1, M'_1) - (\bullet_1, M_1) = (\delta\varphi_1, -\Upsilon\varphi_1) = \partial(\varphi_1) \in \mathcal{B}^2_{\mathrm{MRBPLie}}(\mathcal{P}).$$

Therefore, $(\bullet'_1, M'_1)$ and $(\bullet_1, M_1)$ are cohomologous and belongs to the same cohomology class in $\mathcal{H}^2_{\mathrm{MRBPLie}}(\mathcal{P})$. □

4. Abelian Extensions of Modified Rota-Baxter Pre-Lie Algebras

In this section, we prove that any abelian extension of a modified Rota-Baxter pre-Lie algebra has a representation and a 2-cocycle. It is further proved that they are classified by the second cohomology, as one would expect of a good cohomology theory.

Definition 4.1. Let $(\mathcal{P}, \bullet, M)$ be a modified Rota-Baxter pre-Lie algebra and $(V, \bullet_V, M_V)$ an abelian modified Rota-Baxter pre-Lie algebra with the trivial product $\bullet_V$. An abelian extension $(\hat{\mathcal{P}}, \hat{\bullet}, \hat{M})$ of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ is a short exact sequence of morphisms of modified Rota-Baxter pre-Lie algebras

$$0 \longrightarrow (V, \bullet_V, M_V) \xrightarrow{\mathbf{i}} (\hat{\mathcal{P}}, \hat{\bullet}, \hat{M}) \xrightarrow{\mathbf{p}} (\mathcal{P}, \bullet, M) \longrightarrow 0$$

such that $\hat{M}u = M_V u$ and $u \hat{\bullet} v = 0$, for $u, v \in V$, i.e., V is an abelian ideal of $\hat{\mathcal{P}}$.

Definition 4.2. A section of an abelian extension $(\hat{\mathcal{P}}, \hat{\bullet}, \hat{M})$ of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ is a linear map $\mathbf{s} : \mathcal{P} \to \hat{\mathcal{P}}$ such that $\mathbf{p} \circ \mathbf{s} = \mathrm{id}_{\mathcal{P}}$.

Definition 4.3. Let $(\hat{\mathcal{P}}_1, \hat{\bullet}_1, \hat{M}_1)$ and $(\hat{\mathcal{P}}_2, \hat{\bullet}_2, \hat{M}_2)$ be two abelian extensions of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$. They are said to be equivalent if there is an isomorphism of modified Rota-Baxter pre-Lie algebras $F : (\hat{\mathcal{P}}_1, \hat{\bullet}_1, \hat{M}_1) \to (\hat{\mathcal{P}}_2, \hat{\bullet}_2, \hat{M}_2)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V, \bullet_V, M_V) & \xrightarrow{\mathbf{i}_1} & (\hat{\mathcal{P}}_1, \hat{\bullet}_1, \hat{M}_1) & \xrightarrow{\mathbf{p}_1} & (\mathcal{P}, \bullet, M) & \longrightarrow & 0 \\ & & \| & & \downarrow F & & \| & & \\ 0 & \longrightarrow & (V, \bullet_V, M_V) & \xrightarrow{\mathbf{i}_2} & (\hat{\mathcal{P}}_2, \hat{\bullet}_2, \hat{M}_2) & \xrightarrow{\mathbf{p}_2} & (\mathcal{P}, \bullet, M) & \longrightarrow & 0. \end{array} \tag{4.1}$$

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Now for an abelian extension $(\hat{\mathcal{P}}, \hat{\bullet}, \hat{M})$ of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ with a section $\mathbf{s} : \mathcal{P} \to \hat{\mathcal{P}}$, we define two bilinear maps $\bullet_l : \mathcal{P} \times V \to V$, $\bullet_r : V \times \mathcal{P} \to V$ by

$$a \bullet_l u = \mathbf{s}(a) \hat{\bullet} u, u \bullet_r a = u \hat{\bullet} \mathbf{s}(a), \quad \forall a \in \mathcal{P}, u \in V.$$

Proposition 4.4. *With the above notations, $(V; \bullet_l, \bullet_r, M_V)$ is a representation of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$ and does not depend on the choice of $\mathbf{s}$.*

Proof. First, for any other section $\mathbf{s}' : \mathcal{P} \to \hat{\mathcal{P}}$, for any $a \in \mathcal{P}$, we have

$$\mathbf{p}(\mathbf{s}(a) - \mathbf{s}'(a)) = \mathbf{p}(\mathbf{s}(a)) - \mathbf{p}(\mathbf{s}'(a)) = a - a = 0.$$

Thus, there exists an element $u \in V$, such that $\mathbf{s}'(a) = \mathbf{s}(a) + u$. Note that V is an abelian ideal of $\hat{\mathcal{P}}$, this yields that

$$\mathbf{s}'(x) \hat{\bullet} u = (\mathbf{s}(x) + v) \hat{\bullet} u = \mathbf{s}(x) \hat{\bullet} u, \;\; u \hat{\bullet} \mathbf{s}'(x) = u \hat{\bullet} (\mathbf{s}(x) + v) = u \hat{\bullet} \mathbf{s}(x).$$

This means that $\bullet_l, \bullet_r$ does not depend on the choice of $\mathbf{s}$.

Next, for any $a, b \in \mathcal{P}$ and $u \in V$, by V is an abelian ideal of $\hat{\mathcal{P}}$ and $\mathbf{s}(a) \hat{\bullet} \mathbf{s}(b) - \mathbf{s}(a \bullet b) \in V$, we have

$$\begin{aligned} a \bullet_l (b \bullet_l u) - (a \bullet b) \bullet_l u &= \mathbf{s}(a) \hat{\bullet} (\mathbf{s}(b) \hat{\bullet} u) - \mathbf{s}(a \bullet b) \hat{\bullet} u \\ &= \mathbf{s}(a) \hat{\bullet} (\mathbf{s}(b) \hat{\bullet} u) - (\mathbf{s}(a) \hat{\bullet} \mathbf{s}(b)) \hat{\bullet} u \\ &= \mathbf{s}(b) \hat{\bullet} (\mathbf{s}(a) \hat{\bullet} u) - (\mathbf{s}(b) \hat{\bullet} \mathbf{s}(a)) \hat{\bullet} u \\ &= b \bullet_l (a \bullet_l u) - (b \bullet a) \bullet_l u. \end{aligned}$$

By the same token, there is also $a \bullet_l (u \bullet_r b) - (a \bullet_l u) \bullet_r b = u \bullet_r (a \bullet b) - (u \bullet_r a) \bullet_r b$. This shows that $(V; \bullet_l, \bullet_r)$ is a representation of the pre-Lie algebra $(\mathcal{P}, \bullet)$

On the other hand, by $\hat{M}\mathbf{s}(a) - \mathbf{s}(Ma) \in V$, we have

$$\begin{aligned} Ma \bullet_l M_V u =& \mathbf{s}(Ma) \hat{\bullet} M_V u = \hat{M}\mathbf{s}(a) \hat{\bullet} M_V u = \hat{M}\mathbf{s}(a) \hat{\bullet} \hat{M} u \\ =& \hat{M}(\hat{M}\mathbf{s}(a) \hat{\bullet} u + \mathbf{s}(a) \hat{\bullet} \hat{M} u) - \mathbf{s}(a) \hat{\bullet} u \\ =& M_V(\mathbf{s}(Ma) \hat{\bullet} u + \mathbf{s}(a) \hat{\bullet} M_V u) - \mathbf{s}(a) \hat{\bullet} u \\ =& M_V(Ma \bullet_l u + a \bullet_l M_V u) - a \bullet_l u. \end{aligned}$$

In the same way, there is also $M_V u \bullet_r Ma = M_V(M_V u \bullet_r a + u \bullet_r Ma) - u \bullet_r a$. Hence, $(V; \bullet_l, \bullet_r, M_V)$ is a representation of $(\mathcal{P}, \bullet, M)$. $\square$

Let $(\hat{\mathcal{P}}, \hat{\bullet}, \hat{M})$ be an abelian extension of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ and $\mathbf{s} : \mathcal{P} \to \hat{\mathcal{P}}$ be a section of it. Define the following maps $\omega : \mathcal{P} \times \mathcal{P} \to V$ and $\chi : \mathcal{P} \to V$ respectively by

$$\begin{aligned} \omega(a, b) &= \mathbf{s}(a) \hat{\bullet} \mathbf{s}(b) - s(a \bullet b), \\ \chi(a) &= \hat{M}\mathbf{s}(a) - \mathbf{s}(Ma), \quad \forall a, b \in \mathcal{P}. \end{aligned}$$

We transfer the modified Rota-Baxter pre-Lie algebra structure on $\hat{\mathcal{P}}$ to $\mathcal{P} \oplus V$ by endowing $\mathcal{P} \oplus V$ with a multiplication $\bullet_\omega$, and a modified Rota-Baxter operator M_χ defined by

$$(a + u) \bullet_\omega (b + v) = a \bullet b + a \bullet_l v + u \bullet_r b + \omega(a, b), \tag{4.2}$$

$$M_\chi(a + u) = Ma + \chi(a) + M_V u, \, \forall a, b \in \mathcal{P}, \, u, v \in V. \tag{4.3}$$

Proposition 4.5. *The triple $(\mathcal{P}\oplus V,\bullet_\omega,M_\chi)$ is a modified Rota-Baxter pre-Lie algebra if and only if (ω,χ) is a 2-cocycle of the modified Rota-Baxter pre-Lie algebra $(\mathcal{P},\bullet,M)$ with the coefficient in $(V,\bullet_V,M_V)$. In this case,*

$$0 \longrightarrow (V,\bullet_V,M_V) \xrightarrow{\ \mathbf{i}\ } (\mathcal{P}\oplus V,\bullet_\omega,M_\chi) \xrightarrow{\ \mathbf{p}\ } (\mathcal{P},\bullet,M) \longrightarrow 0$$

is an abelian extension.

Proof. The triple $(\mathcal{P}\oplus V,\bullet_\omega,M_\chi)$ is a modified Rota-Baxter pre-Lie algebra if and only if for any $a,b,c\in\mathcal{P}$ and $u,v,w\in V$, the following equations hold:

$$\begin{aligned}
&((a+u)\bullet_\omega(b+v))\bullet_\omega(c+w)-(a+u)\bullet_\omega((b+v)\bullet_\omega(c+w))\\
&=((b+v)\bullet_\omega(a+u))\bullet_\omega(c+w)-(b+v)\bullet_\omega((a+u)\bullet_\omega(c+w)), \qquad (4.4)\\
&M_\chi(a+u)\bullet_\omega M_\chi(b+v)\\
&=M_\chi(M_\chi(a+u)\bullet_\omega(b+v)+(a+u)\bullet_\omega M_\chi(b+v))-(a+u)\bullet_\omega(b+v). \qquad (4.5)
\end{aligned}$$

Further, Equations (4.4) and (4.5) are equivalent to the following equations:

$$\begin{aligned}
&\omega(a,b)\bullet_r c+\omega(a\bullet b,c)-a\bullet_l\omega(b,c)-\omega(a,b\bullet c)\\
&=\omega(b,a)\bullet_r c+\omega(b\bullet a,c)-b\bullet_l\omega(a,c)-\omega(b,a\bullet c), \qquad (4.6)\\
&Ma\bullet_l\chi(b)+\chi(a)\bullet_r Mb+\omega(Ma,Mb)\\
&=\chi(Ma\bullet b+a\bullet Mb)+M_V\big(\chi(a)\bullet_r b+a\bullet_l\chi(b)+\omega(Ma,b)+\omega(a,Mb)\big)-\omega(a,b). \qquad (4.7)
\end{aligned}$$

Using Equations (4.6) and (4.7), we have $\delta\omega=0$ and $-\delta_M\chi-\Upsilon\omega=0$, respectively. Therefore, $\partial(\omega,\chi)=(\delta\omega,-\delta_M\chi-\Upsilon\omega)=0$, that is, (ω,χ) is a 2-cocycle.

Conversely, if (ω,χ) is a 2-cocycle of $(\mathcal{P},\bullet,M)$ with the coefficient in $(V,\bullet_V,M_V)$, then we have $\partial(\omega,\chi)=(\delta\omega,-\delta_M\chi-\Upsilon\omega)=0$, in which Equations (4.4) and (4.5) hold. Hence $(\mathcal{P}\oplus V,\bullet_\omega,M_\chi)$ is a modified Rota-Baxter pre-Lie algebra. □

Proposition 4.6. *Let $(\hat{\mathcal{P}},\hat\bullet,\hat M)$ be an abelian extension of $(\mathcal{P},\bullet,M)$ by $(V,\bullet_V,M_V)$ and $\mathbf{s}$ be a section of it. If the pair (ω,χ) is a 2-cocycle of $(\mathcal{P},\bullet,M)$ with the coefficient in $(V,\bullet_V,M_V)$ constructed using the section $\mathbf{s}$, then its cohomology class does not depend on the choice of $\mathbf{s}$.*

Proof. Let $\mathbf{s}_1,\mathbf{s}_2:\mathcal{P}\to\hat{\mathcal{P}}$ be two distinct sections, by Proposition 4.5, we have two corresponding 2-cocycles (ω_1,χ_1) and (ω_2,χ_2) respectively. Define a linear map $\gamma:\mathcal{P}\to V$ by $\gamma(a)=\mathbf{s}_1(a)-\mathbf{s}_2(a)$. Then

$$\begin{aligned}
\omega_1(a,b)&=\mathbf{s}_1(a)\hat\bullet_1\mathbf{s}_1(b)-\mathbf{s}_1(a\bullet b)\\
&=(\mathbf{s}_2(a)+\gamma(a))\hat\bullet_1(\mathbf{s}_2(b)+\gamma(b))-(\mathbf{s}_2(a\bullet b)+\gamma(a\bullet b))\\
&=\mathbf{s}_2(a)\hat\bullet_2\mathbf{s}_2(b)-\mathbf{s}_2(a\bullet b)+\mathbf{s}_2(a)\hat\bullet_2\gamma(b)+\gamma(a)\hat\bullet_2\mathbf{s}_2(b)+\gamma(a)\hat\bullet_2\gamma(b)-\gamma(a\bullet b)\\
&=\mathbf{s}_2(a)\hat\bullet_2\mathbf{s}_2(b)-\mathbf{s}_2(a\bullet b)+a\bullet_l\gamma(b)+\gamma(a)\bullet_r b-\gamma(a\bullet b)\\
&=\omega_2(a,b)+\delta\gamma(a\bullet b),\\
\chi_1(a)&=\hat M\mathbf{s}_1(a)-\mathbf{s}_1(Ma)\\
&=\hat M(\mathbf{s}_2(a)+\gamma(a))-(\mathbf{s}_2(Ma)+\gamma(Ma))\\
&=\hat M\mathbf{s}_2(a)-\mathbf{s}_2(Ma)+\hat M\gamma(a)-\gamma(Ma)\\
&=\chi_2(a)+M_V\gamma(a)-\gamma(Ma)\\
&=\chi_2(a)-\Upsilon\gamma(a).
\end{aligned}$$

Hence, $(\omega_1,\chi_1)-(\omega_2,\chi_2)=(\delta\gamma,-\Upsilon\gamma)=\partial(\gamma)\in\mathcal{B}^2_{\mathrm{MRBPLie}}(\mathcal{P},V)$, that is (ω_1,χ_1) and (ω_2,χ_2) form the same cohomological class in $\mathcal{H}^2_{\mathrm{MRBPLie}}(\mathcal{P},V)$. □

Next we are ready to classify abelian extensions of a modified Rota-Baxter pre-Lie algebra.

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Theorem 4.7. *Abelian extensions of a modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ are classified by the second cohomology group $\mathcal{H}^2_{\mathrm{MRBPLie}}(\mathcal{P}, V)$.*

Proof. Assume that $(\hat{\mathcal{P}}_1, \hat{\bullet}_1, \hat{M}_1)$ and $(\hat{\mathcal{P}}_2, \hat{\bullet}_2, \hat{M}_2)$ are equivalent abelian extensions of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ with the associated isomorphism $F : (\hat{\mathcal{P}}_1, \hat{\bullet}_1, \hat{M}_1) \to (\hat{\mathcal{P}}_2, \hat{\bullet}_2, \hat{M}_2)$ such that the diagram in (4.1) is commutative. Let $\mathbf{s}_1$ be a section of $(\hat{\mathcal{P}}_1, \hat{\bullet}_1, \hat{M}_1)$. As $\mathbf{p}_2 \circ F = \mathbf{p}_1$, we have

$$\mathbf{p}_2 \circ (F \circ \mathbf{s}_1) = \mathbf{p}_1 \circ \mathbf{s}_1 = \mathrm{id}_{\mathcal{P}}.$$

That is, $F \circ \mathbf{s}_1$ is a section of $(\hat{\mathcal{P}}_2, \hat{\bullet}_2, \hat{M}_2)$. Denote $\mathbf{s}_2 := F \circ \mathbf{s}_1$. Since F is an isomorphism of modified Rota-Baxter pre-Lie algebras such that $F|_V = \mathrm{id}_V$, we have

$$\begin{aligned}
\omega_2(a,b) &= \mathbf{s}_2(a)\hat{\bullet}_2\mathbf{s}_2(b) - \mathbf{s}_2(a \bullet b)\\
&= F \circ \mathbf{s}_1(a)\hat{\bullet}_2 F \circ \mathbf{s}_1(b) - F \circ \mathbf{s}_1(a \bullet b)\\
&= F\big(\mathbf{s}_1(a)\hat{\bullet}_1\mathbf{s}_1(b) - \mathbf{s}_1(a \bullet b)\big)\\
&= F(\omega_1(a,b))\\
&= \omega_1(a,b)
\end{aligned}$$

and

$$\begin{aligned}
\chi_2(a) &= \hat{M}\mathbf{s}_2(a) - \mathbf{s}_2(Ma)\\
&= \hat{M}(F \circ \mathbf{s}_1(a)) - F \circ \mathbf{s}_1(Ma)\\
&= \hat{M}(\mathbf{s}_1(a)) - \mathbf{s}_1(M(a))\\
&= \chi_1(a).
\end{aligned}$$

So, two isomorphic abelian extensions give rise to the same element in $\mathcal{H}^2_{\mathrm{MRBPLie}}(\mathcal{P}, V)$.

Conversely, given two 2-cocycles (ω_1, χ_1) and (ω_2, χ_2) , we can construct two abelian extensions $(\mathcal{P} \oplus V, \bullet_{\omega_1}, M_{\chi_1})$ and $(\mathcal{P} \oplus V, \bullet_{\omega_2}, M_{\chi_2})$ via Proposition 4.5. If they represent the same cohomology class in $\mathcal{H}^2_{\mathrm{MRBPLie}}(\mathcal{P}, V)$, then there is a linear map $\iota : \mathcal{P} \to V$ such that

$$(\omega_1, \chi_1) - (\omega_2, \chi_2) = \partial(\iota).$$

Define a linear map $F_\iota : \mathcal{P} \oplus V \to \mathcal{P} \oplus V$ by $F_\iota(a + u) := a + \iota(a) + u,\ a \in F_\iota, u \in V$. Then it is easy to verify that F_ι is an isomorphism of these two abelian extensions $(\mathcal{P} \oplus V, \bullet_{\omega_1}, M_{\chi_1})$ and $(\mathcal{P} \oplus V, \bullet_{\omega_2}, M_{\chi_2})$. □

5. Skeletal Modified Rota-Baxter Pre-Lie 2-Algebras

In this section, we introduce the notion of modified Rota-Baxter pre-Lie 2-algebras and show that skeletal modified Rota-Baxter pre-Lie 2-algebras are classified by 3-cocycles of modified Rota-Baxter pre-Lie algebras.

We first recall the definition of pre-Lie 2-algebras from [15], which is a categorization of a pre-Lie algebra.

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A pre-Lie 2-algebra is a quintuple $(\mathcal{P}_0, \mathcal{P}_1, h, l_2, l_3)$, where $h : \mathcal{P}_1 \to \mathcal{P}_0$ is a linear map, $l_2 : \mathcal{P}_i \times \mathcal{P}_j \to \mathcal{P}_{i+j}$ are bilinear maps and $l_3 : \mathcal{P}_0 \times \mathcal{P}_0 \times \mathcal{P}_0 \to \mathcal{P}_1$ is a trilinear map, such that for any $a, b, c, x \in \mathcal{P}_0$ and $u, v \in \mathcal{P}_1$, the following equations are satisfied:

$$hl_2(a,u) = l_2(a,h(u)), \tag{5.1}$$
$$hl_2(u,a) = l_2(h(u),a), \tag{5.2}$$
$$l_2(h(u),v) = l_2(u,h(v)), \tag{5.3}$$
$$hl_3(a,b,c) = l_2(a,l_2(b,c)) - l_2(l_2(a,b),c) - l_2(b,l_2(a,c)) + l_2(l_2(b,a),c), \tag{5.4}$$
$$l_3(a,b,h(u)) = l_2(a,l_2(b,u)) - l_2(l_2(a,b),u) - l_2(b,l_2(a,u)) + l_2(l_2(b,a),u), \tag{5.5}$$
$$l_3(h(u),b,c) = l_2(u,l_2(b,c)) - l_2(l_2(u,b),c) - l_2(b,l_2(u,c)) + l_2(l_2(b,u),c), \tag{5.6}$$
$$\begin{aligned} &l_2(x,l_3(a,b,c)) - l_2(a,l_3(x,b,c)) + l_2(b,l_3(x,a,c)) + l_2(l_3(a,b,x),c) - l_2(l_3(x,b,a),c) \\ &+ l_2(l_3(x,a,b),c) - l_3(a,b,l_2(x,c)) + l_3(x,b,l_2(a,c)) - l_3(x,a,l_2(b,c)) - l_3(l_2(x,a) - l_2(a,x),b,c) \\ &+ l_3(l_2(x,b) - l_2(b,x),a,c) - l_3(l_2(a,b) - l_2(b,a),x,c) = 0. \end{aligned} \tag{5.7}$$

Motivated by [23] and [30], we propose the definition of a modified Rota-Baxter pre-Lie 2-algebra.

Definition 5.1. A modified Rota-Baxter pre-Lie 2-algebra consists of a pre-Lie 2-algebra $\mathfrak{P} = (\mathcal{P}_0, \mathcal{P}_1, h, l_2, l_3)$ and a modified Rota-Baxter 2-operator $\mathfrak{M} = (M_0, M_1, M_2)$ on $\mathfrak{P}$, where $M_0 : \mathcal{P}_0 \to \mathcal{P}_0$, $M_1 : \mathcal{P}_1 \to \mathcal{P}_1$ and $M_2 : \mathcal{P}_0 \times \mathcal{P}_0 \to \mathcal{P}_1$, for any $a, b, c \in \mathcal{P}_0, u \in \mathcal{P}_1$, satisfying the following equations:

$$M_0 \circ h = h \circ M_1, \tag{5.8}$$
$$hM_2(a,b) + l_2(M_0a, M_0b) = M_0\big(l_2(M_0(a),b) + l_2(a,M_0(b))\big) - l_2(a,b), \tag{5.9}$$
$$M_2(h(u),b) + l_2(M_1u, M_0b) = M_1\big(l_2(M_1(u),b) + l_2(u,M_0(b))\big) - l_2(u,b), \tag{5.10}$$
$$M_2(a,h(u)) + l_2(M_0a, M_1u) = M_1\big(l_2(M_0(a),u) + l_2(a,M_1(u))\big) - l_2(a,u), \tag{5.11}$$
$$\begin{aligned} &M_1l_2(a,M_2(b,c)) - l_2(M_0a,M_2(b,c)) + l_2(M_0b,M_2(a,c)) - M_1l_2(b,M_2(a,c)) \\ &- l_2(M_2(b,a),M_0c) + M_1l_2(M_2(b,a),c) + l_2(M_2(a,b),M_0c) - M_1l_2(M_2(a,b),c) \\ &+ M_2(b,l_2(M_0a,c) + l_2(a,M_0c)) - M_2(a,l_2(M_0b,c) + l_2(b,M_0c)) \\ &+ M_2(l_2(M_0a,b) + l_2(a,M_0b) - l_2(M_0b,a) - l_2(b,M_0a),c) - l_3(M_0a,M_0b,M_0c) \\ &+ M_1(l_3(a,M_0b,M_0c) + l_3(M_0a,b,M_0c) + l_3(M_0a,M_0b,c)) \\ &- l_3(M_0a,b,c) - l_3(a,M_0b,c) - l_3(a,b,M_0c) + M_1l_3(a,b,c) = 0. \end{aligned} \tag{5.12}$$

We denote a modified Rota-Baxter pre-Lie 2-algebra by $(\mathfrak{P}, \mathfrak{M})$.

A modified Rota-Baxter pre-Lie 2-algebra is said to be skeletal (resp. strict) if $h = 0$ (resp. $l_3 = 0, M_2 = 0$).

First we have the following trivial example of strict modified Rota-Baxter pre-Lie 2-algebra.

Example 5.2. For any modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$, $(\mathcal{P}_0 = \mathcal{P}_1 = \mathcal{P}, h = 0, l_2 = \bullet, M_0 = M_1 = M)$ is a strict modified Rota-Baxter pre-Lie 2-algebra.

Proposition 5.3. *Let $(\mathfrak{P}, \mathfrak{M})$ be a modified Rota-Baxter pre-Lie 2-algebra.*

(i) If $(\mathfrak{P}, \mathfrak{M})$ is skeletal or strict, then $(\mathcal{P}_0, \bullet_0, M_0)$ is a modified Rota-Baxter pre-Lie algebra, where $a \bullet_0 b = l_2(a,b)$ for any $a, b \in \mathcal{P}_0$.

(ii) If $(\mathfrak{P}, \mathfrak{M})$ is strict, then $(\mathcal{P}_1, \bullet_1, M_1)$ is a modified Rota-Baxter pre-Lie algebra, where $u \bullet_1 v = l_2(h(u),v) = l_2(u,h(v))$ for any $u, v \in \mathcal{P}_1$.

(iii) If $(\mathfrak{P}, \mathfrak{M})$ is skeletal or strict, then $(\mathcal{P}_1; \bullet_l, \bullet_r, M_1)$ is a representation of $(\mathcal{P}_0, \bullet_0, M_0)$ where $a \bullet_l u = l_2(a,u)$ and $u \bullet_r a = l_2(u,a)$ for $a \in \mathcal{P}_0, u \in \mathcal{P}_1$.

Proof. The (i),(ii) and (iii) can be obtained by direct verification. □

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Theorem 5.4. *There is a one-to-one correspondence between skeletal modified Rota-Baxter pre-Lie 2-algebras and 3-cocycles of modified Rota-Baxter pre-Lie algebras.*

Proof. Let $(\mathfrak{P}, \mathfrak{M})$ be a skeletal modified Rota-Baxter pre-Lie 2-algebra. By Proposition 5.3, we can consider the cohomology of modified Rota-Baxter pre-Lie algebra $(\mathcal{P}_0, \bullet_0, M_0)$ with coefficients in the representation $(\mathcal{P}_1; \bullet_l, \bullet_r, M_1)$. For any $a, b, c, x \in \mathcal{P}_0$, combining Equations (3.1) and (5.7), we have

$$\begin{aligned}
&\delta l_3(x,a,b,c)\\
=&x \bullet_l l_3(a,b,c) - a \bullet_l l_3(x,b,c) + b \bullet_l l_3(x,a,c) + l_3(a,b,x) \bullet_r c - l_3(x,b,a) \bullet_r c + l_3(x,a,b) \bullet_r c\\
&- l_3(a,b,x \bullet_0 c) + l_3(x,b,a \bullet_0 c) - l_3(x,a,b \bullet_0 c) - l_3(x \bullet_0 a - a \bullet_0 x, b, c) + l_3(x \bullet_0 b - b \bullet_0 x, a, c)\\
&- l_3(a \bullet_0 b - b \bullet_0 a, x, c)\\
=&l_2(x, l_3(a,b,c)) - l_2(a, l_3(x,b,c)) + l_2(b, l_3(x,a,c)) + l_2(l_3(a,b,x), c) - l_2(l_3(x,b,a), c) + l_2(l_3(x,a,b), c)\\
&- l_3(a,b,l_2(x,c)) + l_3(x,b,l_2(a,c)) - l_3(x,a,l_2(b,c)) - l_3(l_2(x,a) - l_2(a,x), b, c)\\
&+ l_3(l_2(x,b) - l_2(b,x), a, c) - l_3(l_2(a,b) - l_2(b,a), x, c)\\
=&0.
\end{aligned}$$

By Equations (3.2) and (5.12), there holds that

$$\begin{aligned}
&(-\delta_M M_2 - \Upsilon l_3)(a,b,c) = -\delta_M M_2(a,b,c) - \Upsilon l_3(a,b,c)\\
=&- M_0 a \bullet_l M_2(b,c) + M_1(a \bullet_l M_2(b,c)) + M_0 b \bullet_l M_2(a,c) - M_1(b \bullet_l M_2(a,c))\\
&- M_2(b,a) \bullet_r M_0 c + M_1(M_2(b,a) \bullet_r c) + M_2(a,b) \bullet_r M_0 c - M_1(M_2(a,b) \bullet_r c)\\
&+ M_2(b, M_0 a \bullet_0 c + a \bullet_0 M_0 c) - M_2(a, M_0 b \bullet_0 c + b \bullet_0 M_0 c)\\
&+ M_2(M_0 a \bullet_0 b + a \bullet_0 M_0 b - M_0 b \bullet_0 a - b \bullet_0 M_0 a, c) - l_3(M_0 a, M_0 b, M_0 c)\\
&+ M_1(l_3(a, M_0 b, M_0 c) + l_3(M_0 a, b, M_0 c) + l_3(M_0 a, M_0 b, c))\\
&- l_3(M_0 a, b, c) - l_3(a, M_0 b, c) - l_3(a, b, M_0 c) + M_1 l_3(a,b,c)
\end{aligned}$$

$$\begin{aligned}
=&- l_2(M_0 a, M_2(b,c)) + M_1 l_2(a, M_2(b,c)) + l_2(M_0 b, M_2(a,c)) - M_1 l_2(b, M_2(a,c))\\
&- l_2(M_2(b,a), M_0 c) + M_1 l_2(M_2(b,a), c) + l_2(M_2(a,b), M_0 c) - M_1 l_2(M_2(a,b), c)\\
&+ M_2(b, l_2(M_0 a, c) + l_2(a, M_0 c)) - M_2(a, l_2(M_0 b, c) + l_2(b, M_0 c))\\
&+ M_2(l_2(M_0 a, b) + l_2(a, M_0 b) - l_2(M_0 b, a) - l_2(b, M_0 a), c) - l_3(M_0 a, M_0 b, M_0 c)\\
&+ M_1(l_3(a, M_0 b, M_0 c) + l_3(M_0 a, b, M_0 c) + l_3(M_0 a, M_0 b, c))\\
&- l_3(M_0 a, b, c) - l_3(a, M_0 b, c) - l_3(a, b, M_0 c) + M_1 l_3(a,b,c)\\
=&0.
\end{aligned}$$

Thus, $\partial(l_3, M_2) = (\delta l_3, -\delta_M M_2 - \Upsilon l_3) = 0$, that is $(l_3, M_2) \in \mathcal{C}^3_{\mathrm{MRBPLie}}(\mathcal{P}_0, \mathcal{P}_1)$ is a 3-cocycle of modified Rota-Baxter pre-Lie algebra $(\mathcal{P}_0, \bullet_0, M_0)$ with coefficients in the representation $(\mathcal{P}_1; \bullet_l, \bullet_r, M_1)$.

Conversely, suppose that $(l_3, M_2) \in \mathcal{C}^3_{\mathrm{MRBPLie}}(\mathcal{P}, V)$ is a 3-cocycle of modified Rota-Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$ with coefficients in the representation $(V; \bullet_l, \bullet_r, M_V)$. Then $(\mathfrak{P}, \mathfrak{M})$ is a skeletal modified Rota-Baxter pre-Lie 2-algebra, where $\mathfrak{P} = (\mathcal{P}_0 = \mathcal{P}, \mathcal{P}_1 = V, h = 0, l_2, l_3)$ and $\mathfrak{M} = (M_0 = M, M_1 = M_V, M_2)$ with $l_2(a,b) = a \bullet b, l_2(a,u) = a \bullet_l u, l_2(u,a) = u \bullet_r a$ for any $a, b \in \mathcal{P}_0, u \in \mathcal{P}_1$. □

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