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Partition Entropy as a Measure of Regularity of Music Scales

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Abstract: The entropy of the partition generated by an n-tone music scale is proposed to quantify its regularity. The normalized entropy relative to a regular partition, and its complementary, here referred to as the bias, allow to analyze various conditions of similarity between an arbitrary scale and a regular scale. Interesting particular cases are scales having limited bias because their tones are distributed along interval fractions of a regular partition. The most typical case in music concerns partitions associated with well-formed scales generated by a single tone h. These scales are maximal even sets that combine two elementary intervals. Then, the normalized entropy depends on the number of tones as well as the relative size of both elementary intervals. When well-formed scales are refined, several nested families stand out with increasing regularity. It is proven that a scale of minimal bias, i.e. with less bias than those with fewer tones, is always a best rational approximation of $log_2 h$.

Keywords: metric entropy; convex sets; convex functions; continued fractions; best rational approximation; maximal even sets; well-formed scales; pythagorean tuning

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1. Introduction

An n-tone music scale determines a partition of the octave in n intervals. Regarding their regularity, scales can be of equal divisions of the octave, i.e., with n tones of equal temperament (n-TET scale), where discretization precision increases as the number of tones grows, or of different divisions, where precision increases in two ways, on the one hand, as the size of the intervals decreases and, on the other hand, as its regularity increases. In particular, well-formed scales of one generator, hereinafter referred to as cyclic scales, if they are non-degenerate (of equal temperament) are formed from two elementary intervals [1,2], one longer than the interval of an n-TET scale and the other of shorter width. These scales also fulfill the condition of maximum evenness [3,4], that is, they present the most even distribution, which does not depend on the relative size of the intervals. In many cases, as increasing the number of tones of a cyclic scale, the regularity of the intervals decreases, still satisfying the condition of maximal even set. Therefore, for cyclic scales it is not obvious how to quantify precision and regularity of the partition as increasing the number of tones. The present work has the purpose of analyzing it, as well as studying the bias from regular temperament in more general cases.

The regularity of the intervals of an n-tone scale E_n can be quantified in several ways. A fairly common way is from their standard deviation. In the octave, the intervals A_i, $i \in I = \{1, \ldots, n\}$, between consecutive tones, considered as a discrete random variable of equal probability such that $\sum_{i\in I} A_i = 1$, have expected value $t = \frac{1}{n}$, which is the size of the elementary interval of the n-TET scale. A measure of dispersion of these values about the mean is the standard deviation σ, defined from its square, the variance, as $\sigma^2 = \frac{1}{n}\sum_{i\in I}(A_i - t)^2$. In this way it is possible to compare different n-tone scales in terms of the average dispersion of their intervals relatively to the n-TET scale. However, this procedure does not present interesting properties to deal, for instance, with successive refinements of cyclic scales.

Another way to measure the regularity of E_n may be based on the quadratic sum of its intervals. We consider it in a more general way. Let us assume that $(\mathcal{M}, \mu)$ is a Lebesgue-measurable space

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such that $\mu(\mathcal{M}) = 1$, and $\alpha = \{A_i\}_{i\in I}$ is a finite measurable partition of $\mathcal{M}$, that is[1], $\mu(\cup_{i\in I} A_i) = 1$ and $\mu(A_i \cap A_j) = 0$ if $i \neq j$. Then, the sum of squares for the partition α relative to the measure μ is $S(\alpha) = \sum_{i\in I} \mu(A_i)^2$. The function $S(\alpha)$ is a convex function of $\mu(A_i)$, which constrained to $\sum_{i\in I} \mu(A_i) = 1$ has an absolute minimum for $\mu(A_i) = \frac{1}{n}, \forall i$. In this case, $S(\alpha) = \frac{1}{n}$ is a decreasing function of n. When the partition is refined, that is, if $\beta = \{B_j\}_{j\in J}$ is another partition of the octave, then its sum $\alpha \vee \beta = \{A_i \cap B_j\}_{i\in I, j\in J}$ is a refinement of them, in order to make successive refinements it would be desirable to be able to express $S(\alpha \vee \beta) = S(\alpha) + F(\alpha, \beta)$, with a function F that is a linear combination of squares of $\mu(A_i \cap B_j)$, i.e., also a sum of squares of the refinement. But this is not so, because in F it will necessarily appear cross products.

Fortunately, the notions of regularity and fineness of a partition have in several fields a well-known way of being quantified, which is the partition entropy. The concept of entropy was introduced by Clasius in thermodynamics (in 1850), Boltzmann applied it to statistical mechanics (1877), Planck related it to probability theory (1906), Shannon applied it to information theory (1948), Jaynes used it as a measure of uncertainty (1957), and Kolmogorov extended it to deterministic systems (1958).

Certainly, entropy has been used in music almost from the beginning of information theory [5–7] by considering the musical language as a source that produces a sequence of symbols representing musical tones and by associating them with certain probabilities according to their frequency of appearance. In particular, a wide range of works have used entropy to identify music styles, e.g., [8–11] although results may vary depending on the preanalytical assumptions made in order to treat the data (scale degrees, pitch class, octave equivalence, weighting by duration, key-signature dependency, modal bias, etc.). With the same purpose, cross-entropy, e.g., [12] has been used to quantify stylistic similarity between two sequences [13]. An up-to-date review on entropy and other physical parameters applied to music is provided by Gündüz [14].

Nevertheless, it seems that entropy as a measure of regularity of music scales has been neglected. Therefore, in the current paper it is proposed to use the normalized entropy as a measure of their regularity. Its application is illustrated in two cases. On one hand to study the similarity between an arbitrary scale and an n-TET scale. On the other hand, to study the regularity of cyclic scales generated by a tone h and the relationship between their bias and the rational approximation of $\log_2 h$ they provide.

2. Partition Entropy

Following the notation and terminology of Arnold and Avez [15], the entropy of the partition[2] α is defined from a concave function $z(t)$ as follows,

$$H(\alpha) = \sum_{i\in I} z(\mu(A_i))\,; \quad z(t) = \begin{cases} -t\log_2 t, & 0 < t \leq 1 \\ 0, & t = 0 \end{cases} \tag{1}$$

It is fulfilled[3],

$$H(\alpha) = n\,\frac{1}{n}\sum_i z(\mu(A_i)) \leq n\,z\left(\frac{1}{n}\sum_i \mu(A_i)\right) = n\,z\left(\frac{1}{n}\right) = \log_2 n$$

Therefore, for a partition of n elements, the value $\log_2 n$ is the maximum value that the entropy can reach, and it is reached when the elements of the partition are of equal measure.

Thus, $\sum_i z(\mu(A_i)) = \sum_i \mu(A_i)\log_2 \frac{1}{\mu(A_i)}$ is the weighted average of the values $\log_2 \frac{1}{\mu(A_i)}$ (what in probabilities would be the *information* associated with the value of the random variable A_i, i.e., the

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1 Equalities that involve measures are understood to be true except in a null set, that is, strictly we should write $\mu(\mathcal{M} \setminus \cup_{i\in I} A_i) = 0$, but we will not do it to simplify the notation.

2 It is also called metric entropy, and it consists in changing the metric of the theory of probabilities, i.e., the probability, by a generic metric, where the concepts relative to the partitions remain.

3 If the function of a random variable $f(x)$ is concave, the expected value satisfies $\mathrm{E}(f(x)) \leq f(\mathrm{E}(x))$ (Jensen's inequality).

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expected value of the information content or self-information of the variable A_i). The smaller $\mu(A_i)$, the greater is the above logarithm, so this average gives an idea of how much refined the partition is.

Shannon (1948) showed that the entropy defined in this way is the *only function* that, except for a multiplicative constant, satisfies the following postulates:

1. $H(\alpha)$ is a continuous function of $\mu(A_i)$.
2. If $\mu(A_1) = \cdots = \mu(A_n) = \frac{1}{n}$, then $H(\alpha)$ is an increasing function of n.
3. Let's consider partitions $\alpha = \{A_i\}_{i\in I}$, $\beta = \{B_j\}_{j\in J}$ and a refinement of both $\alpha \vee \beta = \{A_i \cap B_j\}_{i\in I, j\in J}$. The conditional entropy of α relative to β is defined as

$$H(\alpha/\beta) = \sum_j \mu(B_j) \sum_i z(\mu(A_i/B_j)) \equiv \sum_j \mu(B_j) \sum_i z(\frac{\mu(A_i \cap B_j)}{\mu(B_j)}) \tag{2}$$

where $\mu(A_i/B_j)$ is the conditional measure of A_i relative to B_j. Then, the following specific property of entropy (sub-additivity)[4] holds,

$$H(\alpha \vee \beta) = H(\alpha) + H(\beta/\alpha) \leq H(\alpha) + H(\beta) \tag{3}$$

4. If φ is an automorfism of $(\mathcal{M}, \mu)$ and $\alpha = \{A_i\}_{i\in I}$, $\varphi\alpha = \{\varphi A_i\}_{i\in I}$ are both partitions, then $\varphi(\alpha \vee \beta) = \varphi\alpha \vee \varphi\beta$ and $H(\alpha/\beta) = H(\varphi\alpha/\varphi\beta)$.

Therefore, the partition entropy seems an appropriate parameter to measure how much a scale of n arbitrary tones differs from an n-TET scale. It is also the parameter that provides us with an estimate of how refined a scale is, because the entropy of a scale of n tones will always be greater than that of a subscale of $n-1$ tones, and less than or equal to that of an n-TET scale. This will be useful when dealing with cyclic scales. In addition, as the intervals of a partition are subdivided, the entropy is additive with regard to the intervals being refined, which saves calculations.

3. Cyclic Scales

Computation of entropy for cyclic scales requires a brief review of their properties, which hereinafter are summarized by following Cubarsi [16,17].

For any frequency ratio $\nu \in \Omega \equiv (0, \infty)$, the values $2^k\nu$, $k \in \mathbb{Z}$, define one equivalence class. A ratio of 2 between values corresponds to the frequency range of one octave. The set Ω is a commutative group for multiplication. The set of all the octaves of the fundamental frequency ratio ($\nu_0 = 1$) is a monogenous subgroup of Ω of infinite cardinal, $\Omega^2 = \{2^k, k \in \mathbb{Z}\}$. The frequency classes are the elements of the quotient group $\Omega_0 = \Omega/\Omega^2$, also commutative for multiplication. For each equivalence class, we choose a representative in $[1, 2)$, the reference octave, which we identify with Ω_0. A finite set of these representatives will be referred to as *scale tones*.

An n-tone cyclic scale E_n^h is a scale of one generator, a real positive value h, which satisfy the symmetry condition, consisting of displaying several degrees of rotational symmetry that are equivalent to the closure condition [18–20], although such an equivalence does not hold for scales with more than two generators [21,22]. The partition of the octave induced by the scale notes has exactly two sizes of scale steps, and each number of generic intervals occurs in two different sizes, which is known as Myhill's property [1,2]. For $h = 3$ we meet the particular case of the 12-tone Pythagorean scale, generated by fifths, as well as those listed in Table 1, and for h other than a rational power of 2 (which would lead to degenerate cases of equal temperament scales) we get generalized Pythagorean scales.

The scale tones are $\nu_k = \frac{h^k}{2^{[\![k]\!]}}$; $k = 0, \ldots, n-1$; with $[\![k]\!] = \lfloor k \log_2 h \rfloor$ (floor function). When the scale tones are ordered clockwise direction from lowest to highest pitch in $[1, 2)$ (say, in *cyclic order*) we find two extreme tones, the minimum tone ν_m and the maximum tone ν_M, which determine the two elementary factors $U = \nu_m = \frac{h^m}{2^{[\![m]\!]}}$ (up the fundamental) and $D = \frac{2}{\nu_M} = \frac{2^{[\![M]\!]+1}}{h^M}$ (down the

4 In general, for three measurable partitions α, β, γ, $H(\alpha \vee \beta/\gamma) = H(\alpha/\gamma) + H(\beta/\alpha \vee \gamma) \leq H(\alpha/\gamma) + H(\beta/\gamma)$.

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fundamental) associated with the generic widths of the step interval, so that $U^M D^m = 2$. The indices satisfy $n = m + M$, all of them coprime.

Table 1. Properties of cyclic scales: GRA=Good rational approximation (accurate scale); RID=Regular interval distributed; BRA=Best rational approximation (optimal scale); MB=Minimal bias; I | Xn=maximum distance $1/xn$ to a note of the n-TET scale.

n						N	m	M	δ
2						3	1	1	0
3	GRA	RID		I | 2n		5	2	1	1
5	GRA	RID	BRA	I | 3n	MB	8	2	3	1
7	GRA	RID				11	2	5	0
12	GRA	RID	BRA	I | 4n	MB	19	7	5	0
17		RID				27	12	5	1
29	GRA					46	12	17	1
41	GRA	RID	BRA			65	12	29	1
53	GRA	RID	BRA	I | 6n	MB	84	12	41	0
94						149	53	41	1
147						233	53	94	1
200	GRA					317	53	147	1
253	GRA					401	53	200	1
306	GRA	RID	BRA	I | 2n		485	53	253	1
359	GRA	RID				569	53	306	0
665	GRA	RID	BRA	I | 8n	MB	1054	359	306	0
971						1539	665	306	1
1636						2593	665	971	1
2301						3647	665	1636	1
2966						4701	665	2301	1
3631						5755	665	2966	1
4296						6809	665	3631	1
4961						7863	665	4296	1
5626						8917	665	4961	1
6291						9971	665	5626	1
6956						11025	665	6291	1
7621						12079	665	6956	1
8286	GRA					13133	665	7621	1
8951	GRA					14187	665	8286	1
9616	GRA					15241	665	8951	1
10281	GRA					16295	665	9616	1
10946	GRA					17349	665	10281	1
11611	GRA					18403	665	10946	1
12276	GRA					19457	665	11611	1
12941	GRA					20511	665	12276	1
13606	GRA					21565	665	12941	1
14271	GRA					22619	665	13606	1
14936	GRA					23673	665	14271	1
15601	GRA	RID	BRA	I | 2n		24727	665	14936	1
16266		RID				25781	665	15601	0
31867	GRA	RID	BRA	I | 2n		50508	16266	15601	0
47468		RID				75235	31867	15601	1
79335	GRA	RID	BRA	I | 2n		125743	31867	47468	1
111202	GRA	RID	BRA			176251	31867	79335	0
190537	GRA	RID	BRA	I | 8n	MB	301994	111202	79335	1
301739						478245	111202	190537	0
492276						780239	301739	190537	0
682813						1082233	492276	190537	0
873350						1384227	682813	190537	0
1063887						1686221	873350	190537	0

The tone $\nu_n = \frac{h^n}{2^{[\![n]\!]}}$, which does not belong to the scale E_n^h, provides the closure condition (either $\nu_n \to 1^+$ or $\nu_n \to 2^-$) determining the n-order comma $\kappa_n = \min(\nu_n, \frac{2}{\nu_n})$ (in the frequency space), i.e., the error in closing the scale near the fundamental, *with no other scale notes between them*. The comma itself does not provide information about whether ν_n closes above or below the fundamental. By using the index $N = [\![m]\!] + [\![M]\!] + 1 = \lfloor n \log_2 h + \frac{1}{2} \rfloor$, two parameters provide this information. On the one hand, the *scale closure*, $\gamma_n = \frac{h^n}{2^N}$, which is a value close to 1 satisfying $\frac{U}{D} = \gamma_n$. On the other hand, the *scale digit* $\delta = N - [\![n]\!]$, taking values 0 or 1. Then, $\delta = 0 \iff \gamma_n > 1$ $(\nu_n \to 1^+, \gamma_n = \kappa_n)$ or $\delta = 1 \iff \gamma_n < 1$ $(\nu_n \to 2^-, \gamma_n = \kappa_n^{-1})$. The value $|\log_2 \gamma_n| = \log_2 \kappa_n$ multiplied by 1200 measures the distance from ν_n to 1 in cents (¢).

Alternatively, each tone ν_k in the frequency space is associated with a *note* or pitch class $\log_2 \nu_k$ in the octave $S_0 = \mathbb{R}/\mathbb{Z}$, so that the above quantities have the corresponding one in S_0. Thus, the elementary intervals $u = \log_2 U, d = \log_2 D$ generate the partition of the octave in n intervals, $Mu + md = 1$, satisfying $u - d = \phi_n$, with interval closure $\phi_n = \log_2 \gamma_n$ and interval comma $|\phi_n| = \log_2 \kappa_n$. Sometimes is more useful to work in the multiplicative space of tones and sometimes in the additive space of notes.

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The fractions $\frac{N}{n}$ are associated with convergent or semi-convergent continued fraction expansions of $\log_2 h$ [23,24]. Among cyclic scales, two categories may be pointed out. On the one hand, *optimal scales*, associated with the best closure $\gamma_n \approx 1$, corresponding to the *best rational approximations*[5] $\frac{N}{n}$ of $\log_2 h$, i.e., the convergents of its canonical continued fraction expansions from both sides. On the other hand, what we shall name *accurate scales*, which are associated with the best estimations of the generator tone $\frac{2^{\frac{N}{n}}}{h} \approx 1$, corresponding to the *good rational approximations* $\frac{N}{n}$ of $\log_2 h$. Apart from *accurate scales*, which include *optimal scales*, there are cyclic scales not associated with good or best rational approximations, still corresponding to semiconvergents. For all these scales, the values U, D and γ_n have bounds according to Appendix A. In particular, $|\log_2 \gamma_n|$ quantifies the error of the rational approximation of $\log_2 h$. These bounds determine the interval between a note of the cyclic scale E_n^h and the one with the same ordinal in the n-TET scale, which is analyzed in Appendix B.

Finally, the family of cyclic scales follow a chain, $E_n^h \subset E_{n^+}^h$, so that starting from the indices (m, M) of E_n^h, the same values for the next scale $E_{n^+}^h$ are (i) $m^+ = m + M,\ M^+ = M \iff \delta = 0$, and (ii) $m^+ = m,\ M^+ = m + M \iff \delta = 1$ (see Table 1).

4. Entropy of a Cyclic Scale

For a cyclic scale E_n^h, with the measure defined for tone intervals as $\mu([x_0, x_1]) = \log_2 \frac{x_1}{x_0}$ for $0 < x_0 \leq x_1$, consider the partition of the octave $\alpha \equiv (u^M, d^m)$ composed of M intervals of width u and m intervals of width d, regardless of the order in which the intervals follow each other. The partition entropy is

$$H(u^M, d^m) = Mz(u) + mz(d) \tag{4}$$

We will explicitly write it in terms of the indices of the minimum and maximum tones m and M. In order to do so, we define the values

$$|x\rangle = \{\log_2 h^x\}, \quad \langle x| = 1 - \{\log_2 h^x\}; \qquad x \in R$$

where $\{\log_2 x\}$ is the mantissa of $\log_2 x$, that is, $\log_2 h^x - \lfloor \log_2 h^x \rfloor$, so that the elemental intervals u, d can be expressed in terms of the indices of the minimum and maximum tones as

$$u = |m\rangle, \quad d = \langle M| \tag{5}$$

In this way, Eq. 4 becomes

$$H(|m\rangle^M, \langle M|^m) = Mz(|m\rangle) + mz(\langle M|) \tag{6}$$

Let's see how to express the entropy of the cyclic scale $E_{n^+}^h$, the one following E_n^h in the chain of cyclic scales. The scale refinement process is as follows. The partition (u^M, d^m) breaks, so that the major interval splits into two, one of the same size as the minor one plus a remainder. This residual is only smaller than the size of the smaller interval when the scale is optimal. This process is iterated.

As explained in §3, we must distinguish two cases, depending on whether γ_n is greater or less than 1, or equivalently, if δ equals 0 or 1:

[5] The terms "good" and "best" rational approximation [25] are equivalent to best approximation "of the first kind" and "of the second kind", respectively [26]. For the current case, the conditions mean:
One-sided best approximation of $\log_2 h^+$ satisfies, if $\gamma_n < 1$, $([\![k]\!] + 1) - k \log_2 h > N - n \log_2 h > 0;\ 0 < k \leq n$.
One-sided best approximation of $\log_2 h^-$ satisfies, if $\gamma_n > 1$, $0 < n \log_2 h - N < k \log_2 h - [\![k]\!];\ 0 < k \leq n$.
Best rational approximation satisfies $0 < |n \log_2 h - N| < |k \log_2 h - ([\![k]\!] + 1)|$ and $0 < |n \log_2 h - N| < |k \log_2 h - [\![k]\!]|;\ 0 < k \leq n$.
One-sided good approximation of $\log_2 h^+$ satisfies, if $\gamma_n < 1$, $\frac{[\![k]\!]+1}{k} - \log_2 h > \frac{N}{n} - \log_2 h > 0;\ 0 < k \leq n$.
One-sided good approximation of $\log_2 h^-$ satisfies, if $\gamma_n > 1$, $0 < \log_2 h - \frac{N}{n} < \log_2 h - \frac{[\![k]\!]}{k};\ 0 < k \leq n$.
Good rational approximation satisfies $0 < |\log_2 h - \frac{N}{n}| < |\log_2 h - \frac{[\![k]\!]+1}{k}|$ and $0 < |\log_2 h - \frac{N}{n}| < |\log_2 h - \frac{[\![k]\!]}{k}|;\ 0 < k \leq n$.

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(i) $\delta = 0, \gamma_n > 1, U > D, u > d$. In this case, $N = [\![n]\!], \quad m^+ = n, \quad M^+ = M$.
The refinement is done in the M intervals of size u and the m intervals of size d are maintained,

$$\begin{array}{l} u' = u - d = \phi_n \iff u' \equiv |m^+\rangle = |m\rangle - \langle M| \\ d' = d \iff d' \equiv \langle M^+| = \langle M| \end{array}$$

We express[6] the entropy according to Eq. 3, being the new partition $\alpha \vee \beta = (u'^{M^+}, d'^{m^+})$,

$$\begin{array}{c} H(u'^{M^+}, d'^{m^+}) = H(u^M, d^m) + M\, u\, H(\frac{d}{u}, 1 - \frac{d}{u}) = \\ = H(u^M, d^m) + M\,[z(d) - z(u) + z(\kappa_n)] \end{array}$$

In terms of the indices of the extreme tones, by noting $H_n = H(E_n^h)$, it can be written as

$$H_{n^+} = H_n + \Delta_M; \quad \Delta_M = M\,[z(\langle M|) - z(|m\rangle) + z(\kappa_n)] \tag{7}$$

(ii) $\delta = 1, \gamma_n < 1, U < D, u < d$. In this case, $N = [\![n]\!] + 1, \quad m^+ = m, \quad M^+ = n$.
The refinement is done in the m intervals of size d and the M intervals of size u are maintained,

$$\begin{array}{l} d' = d - u = -\phi_n \iff d' \equiv \langle M^+| = \langle M| - |m\rangle \\ u' = u \iff u' \equiv |m^+\rangle = |m\rangle \end{array}$$

The entropy of the new partition $\alpha \vee \beta = (u'^{M^+}, d'^{m^+})$ is

$$\begin{array}{c} H(u'^{M^+}, d'^{m^+}) = H(u^M, d^m) + m\, d\, H(\frac{u}{d}, 1 - \frac{u}{d}) = \\ = H(u^M, d^m) + m\,[z(u) - z(d) + z(\kappa_n)]) \end{array}$$

In terms of the indices of the extreme tones it is equivalent to

$$H_{n^+} = H_n + \Delta_m; \quad \Delta_m = m\,[z(|m\rangle) - z(\langle M|) + z(\kappa_n)] \tag{8}$$

5. Partition Modulation

According to Eq. A2, we write the elementary intervals of a cyclic scale as

$$u = \frac{1 + m\phi_n}{n}, \quad d = \frac{1 - M\phi_n}{n} \tag{9}$$

We have referred to one degenerate case of cyclic scale, the limiting case when the two intervals are one, $u \to d$. In this case $u = d = \frac{1}{n}$ and $E_n^h = E_n^\top$, the n-TET scale. This is equivalent to $\phi_n \to 0$ in the expressions of Eq. 9. However, we should consider two more degenerate cases. The larger is one interval, the smaller is the other, always filling a full octave, $Mu + md = 1$. Therefore, if $d \to 0$, then the cyclic scale becomes an equal temperament scale of M tones and, if $u \to 0$, the scale becomes an equal temperament scale of m tones. In other words,

(i) According to Appendix A, for $\phi_n > 0$, by Eq. A5, $\phi_n < \frac{1}{M}$. If $\phi_n \to \frac{1}{M}$ then $d \to 0$. In this case, the intervals of the cyclic scale satisfy $0 < d < \phi_n < u$. Since $u - d = \phi_n$, then, when $u \to \phi_n$ the scale in no more non-degenerate and becomes an M-TET scale, with entropy

$$H_n = H(u^M, d^m) \xrightarrow[d \to 0]{} H(u^M) = H_M = \log_2(M)$$

Obviously, an optimal cyclic scale is far from this situation, because it satisfies $0 < \phi_n < d < u$.

[6] Since $b[z(\frac{a}{b}) + z(1 - \frac{a}{b})] = z(a) - z(b) + z(b - a)$ and $u - d = \kappa_n$, it holds $u\, H(\frac{d}{u}, 1 - \frac{d}{u}) = z(d) - z(u) + z(\kappa_n)$.

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(ii) Also, according to Appendix A, for $\phi_n < 0$, owing to Eq. A8, $\phi_n > -\frac{1}{m}$. If $\phi_n \to -\frac{1}{m}$ then, $u \to 0$. In this case, the intervals of the cyclic scale satisfy $0 < u < |\phi_n| < d$. Then $d \to |\phi_n|$, so that the scale becomes a degenerate m-TET scale, with entropy

$$H_n = H(u^M, d^m) \xrightarrow[u\to 0]{} H(d^m) = H_m = \log_2(m)$$

An optimal cyclic scale case is also far from this situation, since it satisfies $0 < |\phi_n| < u < d$.

Lemma 1. *The scales formed from the two elementary intervals $u' = \frac{1+m\xi}{n}$ and $d' = \frac{1-M\xi}{n}$ with values $-\frac{1}{m} < \xi < \frac{1}{M}$ generate an infinite and continuous family of n-tone cyclic scales $C_n^h(\xi)$, which are neighbors of E_n^h, such that $C_n^h(\phi_n) = E_n^h$ and $C_n^h(0) = E_n^\top$. In addition, if $-\frac{1}{n+m} < \xi < \frac{1}{n+M}$, the scales $C_n^h(\xi)$ are optimal.*

These results are immediate consequence of the bounds obtained in Appendix A.

5.1. Modulating Temperament Scales

The most usual case of cyclic scale E_n^h is the one associated with a generator corresponding to a harmonic $h \in \mathbb{Z}^+$ of the fundamental tone. Then, the tonal class of the generator is $g = \frac{h}{\lfloor \log_2 h \rfloor} \in (1,2)$, which can be written as $g = 2^{\frac{\mu+\phi_n}{n}}$, where $\mu = N - [\![1]\!] n$ is the ordinal of the scale tone that better approximates g [17], known as chromatic length of the pitch class of the generator. The value $\mu = \lfloor n \log_2 g + \frac{1}{2} \rfloor$ also determines the indices m and M of the extreme tones, corresponding to the ordinals 1 and $n-1$ of the scale notes. Since, as $\mu\, m - n\, [\![m]\!] = 1$, m is the positive integer, $0 < m < n$, so that $\mu\, m = 1 \bmod n$.

For fixed n (and also m and M), the family of cyclic scales $C_n^h(\xi)$ has generators that, in general, are not the tonal class of any harmonic, that is, they are real scales. In this case we write them as $g' = 2^{\frac{\mu+\xi}{n}}$, so that, for $\xi \in (-\frac{1}{m}, \frac{1}{M})$, we get a partition modulation, which is a continuum of irregular temperaments [27,28] close to E_n^h and $E_n^\top$, also corresponding to cyclic scales. Hence, the n-tone cyclic scales $C_n^h(\xi)$ is a family of *modulating temperament scales* around the generator $2^{\frac{\mu}{n}}$, namely $\mathcal{T}_n^\mu = \{C_n^h(\xi);\ \xi \in (-\frac{1}{m}, \frac{1}{M})\}$.

For example, for $n = 12$, and $g' = \sqrt[4]{5}$, the value $\xi = \frac{12}{4}\log_2 5 - \lfloor \frac{12}{4}\log_2 5 + \frac{1}{2} \rfloor = -0.03422$ gives rise to the *quarter-comma meantone temperament*, which fits well the class of the fifth harmonic, in exchange for decreasing in 5¢ the accuracy of the third one. Since the value of $|\xi|$ is lower than $\frac{1}{n+m} = \frac{1}{19} = 0.05263$, it results in an optimal scale.

5.2. Entropy in Terms of the Closure

Let us consider an n-tone modulating the temperament scale $C_n^h(\xi) \in \mathcal{T}_n^\mu$. According to equations 4 and 9, we write the entropy of $C_n^h(\xi)$ as

$$H_n = \mathcal{H}(m, M, \xi) = Mz(\tfrac{1+m\xi}{n}) + mz(\tfrac{1-M\xi}{n})$$

Using the relationship $z(\frac{a}{b}) = \frac{1}{b}z(a) + az(\frac{1}{b})$, it is immediate to see that

$$\mathcal{H}(m, M, \xi) = \log_2 n + \tfrac{M}{n}z(1+m\xi) + \tfrac{m}{n}z(1-M\xi)$$

that is,

$$\mathcal{H}(m, M, \xi) = \log_2 n - \tfrac{M}{n}(1+m\xi)\log_2(1+m\xi) - \tfrac{m}{n}(1-M\xi)\log_2(1-M\xi) \tag{10}$$

By deriving Eq. 10 respect of ξ, we get,

$$\frac{\partial \mathcal{H}}{\partial \xi} = -\frac{Mm}{n}\left[\log_2(1+m\xi) - \log_2(1-M\xi)\right] \tag{11}$$

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If $0 < \xi < \frac{1}{M}$, the two terms of the previous equation are positive and the derivative is negative, therefore, in this interval the entropy is a decreasing function. If $-\frac{1}{m} < \xi < 0$, both terms in the above equation are negative and the derivative is positive, so in this interval the entropy is an increasing function. At $\xi = 0$ there is a local maximum, since $\log_2(1 + m\xi) - \log_2(1 - M\xi) = 0$ if and only if $\xi = 0$. Thus, $\mathcal{H}$ in terms of ξ is a concave function, since $\frac{\partial^2 \mathcal{H}}{\partial \xi^2} = -\frac{Mm}{n \ln 2}(\frac{m}{1+m\xi} + \frac{M}{1-M\xi}) < 0$. Therefore,

(a) If $0 < \xi < \frac{1}{M}$, then $\quad \mathcal{H}(m, M, 0) = \log_2 n > \mathcal{H}(m, M, \xi) > \log_2 M = \mathcal{H}(m, M, \frac{1}{M})$.
(b) If $-\frac{1}{m} < \xi < 0$, then $\quad \mathcal{H}(m, M, -\frac{1}{m}) = \log_2 m < \mathcal{H}(m, M, \xi) < \log_2 n = \mathcal{H}(m, M, 0)$.

In addition, it is straightforward to see the following properties,

(c) If $m < M$, $\mathcal{H}(m, M, \xi) < \mathcal{H}(m, M, -\xi)$ for $\xi > 0$ and $\mathcal{H}(m, M, \xi) > \mathcal{H}(m, M, -\xi)$ for $\xi < 0$.
(d) If $m > M$, $\mathcal{H}(m, M, \xi) > \mathcal{H}(m, M, -\xi)$ for $\xi > 0$ and $\mathcal{H}(m, M, \xi) < \mathcal{H}(m, M, -\xi)$ for $\xi < 0$.
(e) $\mathcal{H}(m, M, \xi) > \mathcal{H}(M, m, -\xi)$ for $-\frac{1}{m} < \xi < \frac{1}{M}$

Properties (c) and (d) refer to the slight asymmetry of the entropy far from $\xi = 0$, which, as we shall see in the last section, is negligible around $\xi = 0$. Property (e) means that a cyclic scale generated by h and its inverse scale, generated counter-clockwise with swapped indices of the extreme tones, although they have different notes that differ in one comma, have the same entropy.

Therefore, the entropy of any n-tone modulating temperament scale in the family $\mathcal{T}_n^\mu$ only depends on the closure.

6. Normalized Entropy

Since the entropy increases as the partition is refined, to be able to compare qualities of different scales, whether they are cyclic or not, we will consider the *normalized entropy*, i.e., relatively to the maximum value it can reach, that of the equally tempered scale. For an n-tone scale E_n it is defined as[7]

$$\eta(E_n) = \frac{1}{\log_2(n)} H(E_n); \quad 0 \leq \eta \leq 1$$

so that η tends to 1 as E_n approaches to an n-TET scale. For a cyclic scale E_n^h, we will write $\eta_n = \eta(E_n^h)$.

Let us see in which case the normalized entropy increases when a scale is being refined. Let us assume two scales, not necessarily cyclic, $E_n \subset E_{n^+}$ with $n < n^+$ (i.e., E_{n^+} is a refinement of E_n).

Lemma 2. *The normalized entropy increases if and only if the relative increment of entropy in refining the partition is greater than the relative increment of entropy of the corresponding equal temperament scales.*

Proof. By the sub-additivity property, $H(E_{n^+}) = H(E_n) + \Delta$. It will be $\eta(E_n) < \eta(E_{n^+})$ if and only if $\frac{H(E_n)}{\log_2 n} < \frac{H(E_{n^+})}{\log_2 n^+}$, so $\frac{H(E_{n^+})}{\log_2 n} < \frac{H(E_{n^+})+\Delta}{\log_2 n^+}$. Hence

$$\frac{\log_2 n^+ - \log_2 n}{\log_2 n} < \frac{\Delta}{H(E_n)} \tag{12}$$

□

The deviation of an n-tone scale E_n, not necessarily cyclic, relatively to the regular n-TET scale will be measured from the complementary of the normalized entropy, which we will call *bias*,

$$\theta(E_n) = 1 - \eta(E_n); \quad 0 \leq \theta \leq 1 \tag{13}$$

[7] This ratio has also been called efficiency and relative entropy [29].

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7. Bias of a Cyclic Scale

For a non-degenerate n-tone cyclic scale $C_n^h(\xi) \in \mathcal{T}_n^\mu$ the bias $\theta(C_n^h(\xi)) = \theta_n(\xi)$ depends on the divergence of its elementary intervals u and d with regard to the elementary interval of the n-TET scale. Thus, bearing in mind Eq. 10,

$$\theta_n(\xi) = \frac{1}{n \ln n} \left[M(1+m\xi)\ln(1+m\xi) + m(1-M\xi)\ln(1-M\xi) \right] \quad (14)$$

The graph of the function θ_n is the mirror image up-to-down of H_n, scaled by the factor $\frac{1}{\log_2 n}$, hence it is convex (Figure 1); the value $\theta_n(0) = 0$ is its minimum in the interval $\xi \in (-\frac{1}{m}, \frac{1}{M})$, and at the extremes (corresponding to degenerate scales) it takes the values $\theta_n(-\frac{1}{m}) = 1 - \frac{\ln m}{\ln n}$ and $\theta_n(\frac{1}{M}) = 1 - \frac{\ln M}{\ln n}$. Therefore, in the interval $(-\frac{1}{m}, 0)$, $\theta_n(\xi)$ is a decreasing function and in $(0, \frac{1}{M})$ it is an increasing function of ξ.

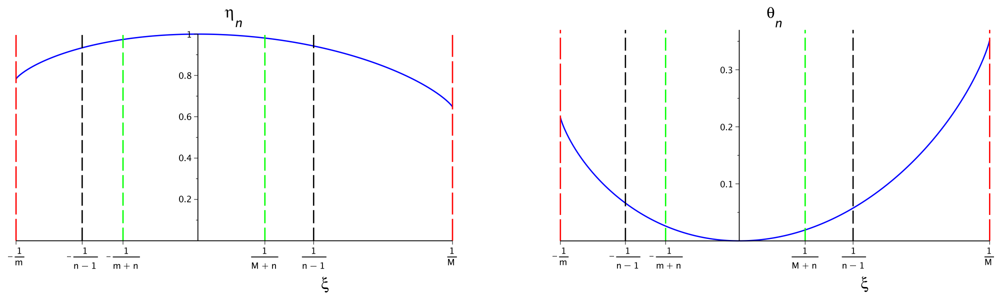

Figure 1. Graphs of normalized entropy and bias.

Remark that this behavior of θ_n in terms of ξ holds when n is fixed. In other words, $\theta_n = \theta(\xi; m, M)$. Then, although for another n'-tone cyclic scale, $n' \neq n$, the bias $\theta_{n'}$ has a similar behavior with respect to ξ', we cannot assure that if $\xi < \xi'$ then $\theta(\xi; m, M) < \theta(\xi'; m', M')$, since this also depends on the values of m' and M', such that $m' + M' = n'$.

Figure 2 shows the trends of the bias θ_n and the interval comma $|\phi_n|$ for Pythagorean scales (generated by $h = 3$) in terms of n. In general, optimal scales have low bias, but not always a scale is refined the bias decreases. For example, the first optimal scales are those of $n = 5; 12; 41; 53; 306; 665 \ldots$ and the bias decreases for $n = 5; 12; 53; 665$ but the scale of $n = 41$ has greater bias than that of $n = 5; 12$; and the one of $n = 306$ has greater bias than those of $n = 12; 53$. Also notice that there are intervals where $|\phi_n|$ decreases but θ_n increases.

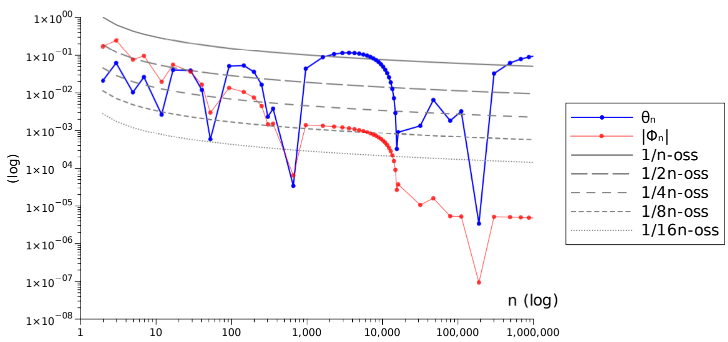

Figure 2. Behavior of $|\phi_n| = \log_2 \kappa_n$ (red) and $\theta_n = 1 - \eta_n$ (blue) in terms of the number of tones n (bilogarithmic scales). In gray, maximum bias for several similarity levels to n-TET scales.

We will say that a cyclic scale E_n^h is of *minimal bias* (MB) if for any cyclic scale $E_{n'}^h$ with $n' < n$, then $\theta_n < \theta_{n'}$. Obviously, this is tantamount to say that $\eta_n > \eta_{n'}$, hence, the scale has greater normalized entropy than the previous ones. Not all optimal cyclic scales are MB.

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Before a deeper analysis, in an approximate way we may estimate how close an MB scale and an equal temperament scale of the same number of tones are. We may use the criterion for which a cyclic scale (not necessarily optimal) unambiguously approximates an n-TET scale if every note of the former is at a distance equal or less than half an elementary interval from the latter. According to Eq. A12 with $\lambda = 2$, this leads to the condition $\cent(\kappa_n) \leq \frac{600}{n-1}$ (say condition I|2n).

This condition can be satisfied by optimal and non-optimal cyclic scales. For example, for $n = 3$ condition I|2n is hold although the scale is not optimal. On the contrary, $n = 41; 306; 111\,202$ are optimal scales but the condition I|2n is not met.

Let's see what happens with the MB condition. In this case, up to $n = 2 \cdot 10^5$, all scales that satisfy MB also satisfy I|2n, but there are scales that are I|2n and not MB scales, for example for $n = 3; 306; 15\,601; 31\,867; 79\,335$. Hence, the condition that a note of the cyclic scale is closer than half an interval of a note of a tempered scale is weaker than the MB condition. On the contrary, if we further restrict the above condition, let's say, that the respective notes are at most at a third interval, i.e., $\cent(\kappa_n) \leq \frac{400}{n-1}$ (say condition I|3n), then, up to $n = 2 \cdot 10^5$ all I|3n scales are MB scales. In Table 1 these and other properties are displayed.

8. Scales with Limited Bias

8.1. Scale Distributed within Regular Intervals

We study several families of scales for which it is possible to estimate a priori the lower limit of the entropy.

We say that two n-tone scales E_n and E'_n sharing the fundamental tone *alternate* if their notes in cyclic order, $s_j = \log_2 \sigma_j \in E_n$, $s'_j = \log_2 \sigma'_j \in E'_n$, $0 < j < n$, fulfill one of the following conditions:

$$\begin{array}{lllll} \text{(a)} & s_{j-1} < s'_j \leq s_j; & 0 < j < n & (E_n \text{ alternate by the right of } E'_n\,) \\ \text{(b)} & s'_{j-1} < s_j \leq s'_j; & 0 < j < n & (E_n \text{ alternate by the left of } E'_n\,) \end{array} \tag{15}$$

If E'_n is the equal temperament scale $E_n^{\top}$, then we write $\sigma'_j = \vartheta'_j$ and say that E_n is *regular interval distributed* (RID). In this case, the alternance can also be defined from the following relations involving intervals between tones[8],

$$\text{(a)} \quad 0 \leq I(\sigma_j, \vartheta'_j) < \frac{1}{n} \qquad \text{(b)} \quad 0 \leq I(\vartheta'_j, \sigma_j) < \frac{1}{n}; \qquad 0 < j < n \tag{16}$$

Note that these conditions are generally more restrictive than the condition

$$d(\sigma_j, \vartheta'_j) < \frac{1}{n} \iff -\frac{1}{n} < I(\sigma_j, \vartheta'_j) < \frac{1}{n} \tag{17}$$

Obviously, Eq. 16 implies the condition of Eq. 17, although, as seen in Appendix B, for cyclic scales they are equivalent.

For example, suppose case (a). If $0 \leq I(\sigma_j, \vartheta'_j) < \frac{1}{n}$, then the interval between two consecutive notes of E_n satisfies $I(\sigma_j, \sigma_{j+1}) = I(\sigma_j, \vartheta'_{j+1}) + I(\vartheta'_{j+1}, \sigma_{j+1}) = \frac{1}{n} - I(\sigma_j, \vartheta'_j) + I(\vartheta'_{j+1}, \sigma_{j+1}) < \frac{2}{n}$, so that $d(\sigma_j, \sigma_{j+1}) < \frac{2}{n}$. Instead, if $d(\sigma_j, \vartheta'_j) < \frac{1}{n}$, then $d(\sigma_j, \sigma_{j+1}) \leq d(\sigma_j, \vartheta'_j) + d(\vartheta'_j, \vartheta'_{j+1}) + d(\sigma_{j+1}, \vartheta'_{j+1}) < \frac{3}{n}$. In a similar way would be reasoned case (b).

Lemma 3. *The interval between two consecutive notes of an n-tone RID-scale is lower than* $\frac{2}{n}$.

---

[8] The interval between two tones $\nu_1 < \nu_2$ in $(0, \infty)$ is $I(\nu_2, \nu_1) = \log_2 \nu_2 - \log_2 \nu_1$.

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8.2. Scale r-Similar to n-TET

A criterion for measuring the proximity between scales is *similarity* [30]. Two n-tone scales T and T' are similar at level $r > 0$ (r-similar) if for each tone $\tau_i \in T$, $\exists \tau'_j \in S$, such that $d(\tau_i, \tau'_j) \leq r$ and $d(\tau_i, \tau'_k) > r, \forall k \neq j$.

For example, a cyclic scale E_n^h and the n-TET scale have level of similarity $\frac{1}{2n}$ if and only if any pair of notes with ordinal j, i.e., $\vartheta_j \in E_n^h$, $\vartheta'_j \in E_n^\top$, satisfy $d(\vartheta_j, \vartheta'_j) \leq \frac{1}{2n}$. This condition is equivalent to the condition of Eq. A12 with $\lambda = 2$, $\cent(\kappa_n) \leq \frac{600}{n-1}$.

However, for the current purpose of evaluating and comparing entropy partitions, we will slightly modify such a concept by assuming that the fundamental is shared by both scales. Then, we say a scale E_n is *similar* to the n-TET scale at level r, with $0 < r \leq \frac{1}{2n}$, if their tones satisfy

$$d(\sigma_j, \vartheta'_j) \leq r \,; \quad 0 < j < n \tag{18}$$

8.3. Scale One-Side r-Similar to n-TET

We introduce a concept that will mix the one of similarity with the one of distribution within regular intervals, which will be appropriate to study the bounds of the entropy for cyclic scales.

Let ϑ'_j be the tones of the n-TET scale. We say that a scale E_n is *one-side r-similar* at level r (r-OSS) to the n-TET scale, with $0 < r \leq \frac{1}{n}$, if

$$\text{(a)} \quad 0 \leq I(\sigma_j, \vartheta'_j) < r \qquad \text{(b)} \quad 0 \leq I(\vartheta'_j, \sigma_j) < r; \qquad 0 < j < n \tag{19}$$

It is OSS by the right (a) or by the left (b), respectively. Therefore, when $r = \frac{1}{n}$ it matches the definition of a RID-scale.

9. Entropy of a Scale $\frac{1}{\lambda n}$-OSS to n-TET

Consider an n-tone scale E_n, not necessarily cyclic, which is $\frac{1}{\lambda n}$-OSS to n-TET for $\lambda \geq 1$. We already know that the maximum entropy is $\log_2 n$. Let's see the minimum entropy it can reach. We assume case (a) of Eq. 19 and consider the n notes ($n \geq 2$) of E_n in cyclic order in $[0,1) \cup \{1\}$, since the entropy does not change if we add a null set. We also extend the range of possible variation of the intervals between notes at the extremes to include the limiting, degenerate cases. The points determining the division of the octave are

$$s_0 = 0; \quad s_i = \frac{i}{n} + t_i, \quad 0 \leq t_i \leq \frac{1}{\lambda n}, \quad 1 \leq i \leq n-1; \quad s_n = 1$$

By writing $T = \frac{1}{n}$, the respective intervals are, by assuming $t_0 = t_n = 0$,

$$T_i = s_i - s_{i-1} = T + t_i - t_{i-1}, \; 1 \leq i \leq n$$

Then, scale E_n generates the partition $\alpha = \{T_i\}, i \in \{1, \ldots, n\}$, with entropy

$$H(\alpha) = \sum_{i=1}^{n} z(T_i) = -\sum_{i=1}^{n} (T + t_i - t_{i-1}) \log_2 (T + t_i - t_{i-1})$$

The entropy $H(\alpha)$ is a concave and differentiable function $f(t_1, \ldots, t_{n-1})$, $f{:}V \to \mathrm{R}$, with $V = [0, \frac{1}{\lambda n}]^{n-1}$, a hypercube, which is a convex and compact space. Then, f has a local and global maximum at $(t_1, \ldots, t_{n-1}) = (0, \ldots, 0)$, corresponding to an n-TET scale with $T_i = T$ and $H = \log_2 n$, and the global minimum is reached at one or some of the vertices of V. These vertices are determined by the possible values $t_j \in \{0, \frac{1}{\lambda n}\}$, $1 \leq j \leq n-1$, and the resulting intervals T_i can only take the following values: interval T_1, values $\frac{1}{n}$ and $\frac{\lambda+1}{\lambda n}$; interval T_n, values $\frac{1}{n}$ and $\frac{\lambda-1}{\lambda n}$; and the intermediate intervals, values $\frac{\lambda-1}{\lambda n}$, $\frac{1}{n}$ and $\frac{\lambda+1}{\lambda n}$.

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The octave is covered by a number of different intervals satisfying

$$A + B + C = n \tag{20}$$

of which A in number have width $\frac{\lambda-1}{\lambda n}$, B width $\frac{1}{n}$, and C width $\frac{\lambda+1}{\lambda n}$, so that[9] $A\,\frac{\lambda-1}{\lambda n} + B\,\frac{\lambda}{\lambda n} + C\,\frac{\lambda+1}{\lambda n} = 1$. Therefore,

$$A\,(\lambda-1) + B\,\lambda + C\,(\lambda+1) = \lambda n \tag{21}$$

The entropy at one of these vertices can be written in terms of the respective number of intervals as

$$\begin{aligned} f &= A\,z(\tfrac{\lambda-1}{\lambda n}) + B\,z(\tfrac{\lambda}{\lambda n}) + C\,z(\tfrac{\lambda+1}{\lambda n}) = \\ &= \log_2 n + \tfrac{1}{n}\,g(A,C); \quad g(A,C) = A\,z(\tfrac{\lambda-1}{\lambda}) + C\,z(\tfrac{\lambda+1}{\lambda}) \end{aligned} \tag{22}$$

where z is the concave function defined in Eq. 1, also extended for values $t > 1$, where $z(t) < 0$. Among possible configurations, we look for the minimum value that f can take, which will correspond to the minimum value of g. Notice that this quantity is added to the entropy of an n-TET scale, being it consistent with the fact that the B intervals of width $\frac{1}{n}$ are not involved in the expression. However, the intervals in number C, of greater width than $\frac{1}{n}$, contribute to decreasing the entropy, while those in number A, of width lower than $\frac{1}{n}$, contribute to increasing it, since the corresponding function z evaluated in values less than 1 is positive.

In the current case, by combining equations 20 and 21, we get $A = C$ and $B + 2C = n$. Therefore,

$$g(C) = C\,Z(\lambda,1); \qquad Z(\lambda,a) = z(\tfrac{\lambda-a}{\lambda}) + z(\tfrac{\lambda+a}{\lambda}), \quad \lambda \geq a \tag{23}$$

If n is even, the maximum value for C, by assuming $B = 0$, is $C = \frac{n}{2}$. Hence, the number of intervals of null-size is also $A = \frac{n}{2}$. Then[10], $g(\frac{n}{2}) = \frac{n}{2}Z(\lambda,1)$, corresponding to a degenerate scale of $\frac{n}{2}$ non-null intervals. Therefore, if $n = \dot{2}$ the entropy satisfies $\log_2 n + \frac{1}{2}Z(\lambda,1) \leq H(\alpha)$.

If n is odd, the maximum value for C and A is $\frac{n-1}{2}$, with $B = 1$. Then, $g = \frac{n-1}{2}Z(\lambda,1)$, corresponding to a degenerate scale of $\frac{n+1}{2}$ non-null intervals. Therefore, if $n \neq \dot{2}$ the entropy satisfies $\log_2 n + \frac{1}{2}Z(\lambda,1)(1-\frac{1}{n}) \leq H(\alpha)$.

Case (b) is similar. It would be reasoned by considering that $f(t_1,\ldots,t_{n-1}) = f(-t_{n-1},\ldots,-t_1)$, i.e., it is equivalent to case (a), but following the intervals from right to left starting at 1. Therefore,

Theorem 1. *The entropy and bias of a scale $\frac{1}{\lambda n}$-OSS to n-TET satisfy*

$$\begin{aligned} &n = \dot{2}, \quad \log_2 n + \tfrac{1}{2}Z(\lambda,1) \leq H(\alpha) \leq \log_2 n; \qquad 0 \leq \theta(\alpha) \leq -\tfrac{1}{2\log_2 n}Z(\lambda,1) \\ &n \neq \dot{2}, \quad \log_2 n + \tfrac{1}{2}Z(\lambda,1)(1-\tfrac{1}{n}) \leq H(\alpha) \leq \log_2 n; \quad 0 \leq \theta(\alpha) \leq -\tfrac{1}{2\log_2 n}Z(\lambda,1)(1-\tfrac{1}{n}) \end{aligned} \tag{24}$$

We explicitly write two cases. For $\lambda = 1$, since $Z(1,1) = -2$, the entropy of a RID-scale satisfies, if $n = \dot{2}$, $\log_2 \frac{n}{2} \leq H(\alpha) \leq \log_2 n$ and, if $n \neq \dot{2}$, $\log_2 \frac{n}{2} + \frac{1}{n} \leq H(\alpha) \leq \log_2 n$.

For $\lambda = 2$, since $Z(2,1) = -0.189$, the entropy of a scale $\frac{1}{2n}$-OSS to n-TET satisfies, if $n = \dot{2}$, $\log_2 n - 0.189 \leq H(\alpha) \leq \log_2 n$ and, if $n \neq \dot{2}$, $\log_2 n - 0.189(1-\frac{1}{n}) \leq H(\alpha) \leq \log_2 n$.

Levels for $\lambda = 1, 2, 4, 8, 16$ are displayed in Fig. 2.

---

[9] Some of these intervals may have width zero, giving rise to a degenerate scale with less than n non-null intervals.

[10] The function $Z(\lambda,a) = z(\frac{\lambda-a}{\lambda}) + z(\frac{\lambda+a}{\lambda})$ with $a > 0$ is defined for $\lambda \geq a$, where satisfies $Z(\lambda,a) < 0$, always increasing from $Z(a,a) = -2$ until $Z(\lambda,a) \to 0$ when $\lambda \to \infty$.

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10. Entropy of a Scale $\frac{1}{\lambda n}$-Similar to n-TET

Let us calculate the minimum entropy that can reach an n-tone scale E_n, not necessarily cyclic, having a similarity level $\frac{1}{\lambda n}$ with the n-TET scale for $\lambda \geq 2$. According to the previous notation and considerations, the points that determine the division of the octave are

$$s_0 = 0; \quad s_i = \frac{i}{n} + t_i, \quad -\frac{1}{\lambda n} \leq t_i \leq \frac{1}{\lambda n}, \quad 1 \leq i \leq n-1; \quad s_n = 1$$

The respective intervals are, $T_i = s_i - s_{i-1} = \frac{1}{n} + t_i - t_{i-1}$, $1 \leq i \leq n$, by assuming $t_0 = t_n = 0$. As before, we write the entropy of the partition $\alpha = \{T_i\}$, $i \in \{1,\ldots,n\}$, as $H(\alpha) = \sum_{i=1}^{n} z(T_i) = f(t_1,\ldots,t_{n-1})$, where $f{:}V \to \mathrm{R}$, with $V = [-\frac{1}{\lambda n}, \frac{1}{\lambda n}]^{n-1}$, a convex and compact space. The local and global maximum of f takes place at $(t_1,\ldots,t_{n-1}) = (0,\ldots,0)$, corresponding to the scale of n-TET, with $T_i = T$ and $H = \log_2 n$. The global minimum is reached at one or some of the vertices of V. These vertices are determined by the possible values $t_j \in \{-\frac{1}{\lambda n}, \frac{1}{\lambda n}\}$, $1 \leq j \leq n-1$, and the resulting intervals T_i can only take the following values: the extreme intervals T_1 and T_n values $\frac{\lambda-1}{\lambda n}$ and $\frac{\lambda+1}{\lambda n}$, while the intermediate intervals can have values $\frac{\lambda-2}{\lambda n}$, $\frac{1}{n}$, and $\frac{\lambda+2}{\lambda n}$. The octave is covered by a number of different intervals satisfying

$$A + B + C + D + E = n \tag{25}$$

of which A in number have width $\frac{\lambda-2}{\lambda n}$, B width $\frac{\lambda-1}{\lambda n}$, C width $\frac{1}{n}$, D width $\frac{\lambda+1}{\lambda n}$ and E width $\frac{\lambda+2}{\lambda n}$, so that $A\,\frac{\lambda-2}{\lambda n} + B\,\frac{\lambda-1}{\lambda n} + C\,\frac{\lambda}{\lambda n} + D\,\frac{\lambda+1}{\lambda n} + E\,\frac{\lambda+2}{\lambda n} = 1$. Therefore,

$$A\,(\lambda-2) + B\,(\lambda-1) + C\,\lambda + D\,(\lambda+1) + E\,(\lambda+2) = \lambda n \tag{26}$$

The entropy at one of these vertices can be written in terms of the respective number of intervals as

$$\begin{aligned} f &= A\,z(\tfrac{\lambda-2}{\lambda n}) + B\,z(\tfrac{\lambda-1}{\lambda n}) + C\,z(\tfrac{\lambda}{\lambda n}) + D\,z(\tfrac{\lambda+1}{\lambda n}) + E\,z(\tfrac{\lambda+2}{\lambda n}) = \\ &= \log_2 n + \tfrac{1}{n}\,g(A,B,D,E), \quad g(A,B,D,E) = A\,z(\tfrac{\lambda-2}{\lambda}) + B\,z(\tfrac{\lambda-1}{\lambda}) + D\,z(\tfrac{\lambda+1}{\lambda}) + E\,z(\tfrac{\lambda+2}{\lambda}) \end{aligned} \tag{27}$$

Once again, the intervals of width $\frac{1}{n}$ are not involved in this expression. The intervals in number D and E, of greater width than $\frac{1}{n}$, contribute to decreasing the entropy, while those who are there in number A and B, of width lower than $\frac{1}{n}$, contribute to increasing it.

The minimum value of g is obtained by the highest possible values of E and D (in that order) and minimum of A and B.

In the current case, by combining equations 25 and 26, we get $A = E + \frac{1}{2}D - \frac{1}{2}B$ and

$$B + 2C + 3D + 4E = 2n \tag{28}$$

If n is even, at most there can be $E = \frac{n}{2}$ intermediate intervals of width $\frac{\lambda+2}{\lambda n}$. By substitution of this value into Eq. 28, we get $B + 2C + 3D = 0$, so that $B{=}C{=}D{=}0$ and $A = \frac{n}{2}$. Hence, the minimum g is $g(\frac{n}{2},0,0,\frac{n}{2}) = \frac{n}{2}Z(\lambda,2)$, corresponding to a degenerate scale of $\frac{n}{2}$ non-null intervals. Therefore, if $n = \dot{2}$ the entropy satisfies $\log_2 n + \frac{1}{2}Z(\lambda,2) \leq H(\alpha)$.

If n is odd, we should examine two alternatives. On the one hand, if we assume there are $E = \frac{n-1}{2}$ intermediate intervals of width $\frac{\lambda+2}{\lambda n}$, by substitution into Eq. 28 we get $B + 2C + 3D - 2 = 0$. Possible interval values are $(B,C,D) = (0,1,0); (2,0,0)$. In this case, the minimum value of g is provided by the first triad, i.e., $B{=}D{=}0$, $C = 1$, $A = \frac{n-1}{2}$, which yields $g(\frac{n-1}{2},0,0,\frac{n-1}{2}) = \frac{n-1}{2}Z(\lambda,2)$, corresponding to a degenerate scale of $\frac{n+1}{2}$ non-null intervals. In this case,

$$\log_2 n + \tfrac{1}{2}Z(\lambda,2)(1 - \tfrac{1}{n}) \leq H(\alpha) \tag{29}$$

On the other hand, as second alternative, if there are $E = \frac{n-3}{2}$ intermediate intervals of width $\frac{\lambda+2}{\lambda n}$, it is hold $B + 2C + 3D - 6 = 0$. Possible interval values are $(B,C,D) = (0,0,2); (1,1,1); (3,0,1); (0,3,0); (4,1,0)$.

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The minimum of g is provided by the first triad, so that $B{=}C{=}0$, $D=2$, $A=\frac{n-1}{2}$, and $g(\frac{n-1}{2},0,2,\frac{n-3}{2})=\frac{n-1}{2}z(\frac{\lambda-2}{\lambda})+2z(\frac{\lambda+1}{\lambda})+\frac{n-3}{2}z(\frac{\lambda+2}{\lambda})$ corresponding to a degenerate scale of $\frac{n+1}{2}$ non-null intervals. Hence,

$$\log_2 n+\frac{1}{2}Z(\lambda,2)\left[1-\frac{1}{n}\frac{z(\frac{\lambda-2}{\lambda})-4z(\frac{\lambda+1}{\lambda})+3z(\frac{\lambda+2}{\lambda})}{Z(\lambda,2)}\right]\leq H(\alpha) \tag{30}$$

The factor multiplying $\frac{1}{n}$ is a positive value between $3(2-\log_2 3)=1.245$ and $\frac{3}{2}$ for $\lambda\geq 2$. Since the value $Z(\lambda,2)$ is negative, the above factor increases it. Hence, the entropy of Eq. 29 is lower than the entropy of Eq. 30. Therefore,

Theorem 2. *The entropy and bias of a scale $\frac{1}{\lambda n}$-similar to n-TET satisfy*

$$\begin{aligned} n&=\dot{2}, && \log_2 n+\tfrac{1}{2}Z(\lambda,2)\leq H(\alpha)\leq\log_2 n; && 0\leq\theta(\alpha)\leq-\tfrac{1}{2\log_2 n}Z(\lambda,2)\\ n&\neq\dot{2}, && \log_2 n+\tfrac{1}{2}Z(\lambda,2)(1-\tfrac{1}{n})\leq H(\alpha)\leq\log_2 n; && 0\leq\theta(\alpha)\leq-\tfrac{1}{2\log_2 n}Z(\lambda,2)(1-\tfrac{1}{n})\end{aligned} \tag{31}$$

We explicitly evaluate the case for $\lambda=2$. Since $Z(2,2)=-2$, if $n=\dot{2}$, then $\log_2\frac{n}{2}\leq H(\alpha)\leq\log_2 n$ and, if $n\neq\dot{2}$, then $\log_2\frac{n}{2}+\frac{1}{n}\leq H(\alpha)\leq\log_2 n$, which are the same bounds as for a RID-scale.

11. Cyclic RID-Scales

We may reformulate Theorem A1 according to the above definitions.

Theorem 3. *Optimal cyclic scales are RID-scales.*

Nevertheless, there are many non-optimal cyclic scales that are also RID-scales. Cyclic RID-scales correspond to partial convergents of continued fractions where the comma is not as low as for optimal scales, i.e., their closure is not a *best approximation*, although they belong to a family of relatively good partial convergents. However, although in most cases a convergent that gives a *good approximation* (i.e., an accurate scale) generates a RID-scale, there are exceptions, such as for $n=200$[11]. Therefore, there exist accurate scales that are not RID, and RID-scales that are not accurate.

11.1. Comma and Elementary Intervals of a RID-Scale

It is immediate to identify a RID-scale from its interval comma $|\phi_n|=|\log_2\gamma_n|=|n\log_2 h-N|$. As seen in Appendix B, the distance between the tones $\vartheta'_j\in E_n^{\top}$ and $\vartheta_j=\nu_k\in E_n^h$ for $0\leq j<n$ is $\frac{k}{n}|\phi_n|$. So, $\frac{k}{n}|\phi_n|<\frac{1}{n}$ for all $0\leq k<n$ if and only if $\frac{n-1}{n}|\phi_n|<\frac{1}{n}$, that is, the interval comma is limited as $|\phi_n|<\frac{1}{n-1}$.

Lemma 4. *A cyclic scale is RID if and only if $|\phi_n|<\frac{1}{n-1}$.*

This condition, which is satisfied by all optimal scales, assures us that the value of $|\phi_n|$ is small enough to be far from the previously seen degenerate cases of less than n notes, since for every RID-scale it is fulfilled $|\phi_n|<\frac{1}{n-1}\leq\min(\frac{1}{m},\frac{1}{M})$.

From another point of view, while considering RID-scales we are excluding "bad approximations" of $\log_2 h$ that satisfy, if $\phi_n>0$, $\frac{1}{n-1}\leq n\log_2 h-N<\frac{1}{M}$ and, if $\phi_n<0$, $-\frac{1}{m}<N-n\log_2 h\leq-\frac{1}{n-1}$. This fact also has implications on the bounds of the two elementary intervals of cyclic RID-scales, We distinguish the following cases.

---

[11] Between the notes $j=83$ of the respective scales E_{200}^3 and $E_{200}^{\top}$ there is a distance of nearly 1.5 elementary intervals of the equal temperament scale and between two consecutive notes of the cyclic scale there may be a distance equivalent to 2.1 regular intervals, what means that within some regular intervals there are two notes of the cyclic scale.

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(i) If $0 < \phi_n < \frac{1}{n-1}$, according to Eq. 9, $u < \frac{1}{n}(1 + \frac{m}{n-1}) < \frac{1}{n}(1 + \frac{n-1}{n-1}) = \frac{2}{n}$. Hence[12],

$$\frac{1}{n} < u < \min(\tfrac{1}{M}, \tfrac{2}{n}) \tag{32}$$

(ii) If $-\frac{1}{n-1} < \phi_n < 0$, according to Eq. 9, $d < \frac{1}{n}(1 + \frac{M}{n-1}) < \frac{1}{n}(1 + \frac{n-1}{n-1}) = \frac{2}{n}$. Hence[13],

$$\frac{1}{n} < d < \min(\tfrac{1}{m}, \tfrac{2}{n}) \tag{33}$$

In both cases, the size of the greatest elementary interval of a cyclic RID-scale is lower than $\frac{2}{n}$, as stated in Lemma 3.

11.2. Bias of RID-Scales

For large values of $|\phi_n|$, that is, $|\phi_n| > \frac{1}{m}$ or $|\phi_n| > \frac{1}{M}$, Eq. 14 can have a quite arbitrary and non-symmetrical behavior, but for values $|\phi_n| < \frac{1}{n-1}$, and in particular for optimal scales with $|\phi_n| < \frac{1}{n+M}$ or $|\phi_n| < \frac{1}{n+m}$, depending on the value δ, the bias θ_n behaves as proportional to ϕ_n^2.

Being $|m\phi_n| < 1$ and $|M\phi_n| < 1$, we can approximate the logarithms in the following expressions as

$$(1 + m\phi_n)\ln(1 + m\phi_n) = m\phi_n + \tfrac{1}{2}m^2\phi_n^2 - \tfrac{1}{6}m^3\phi_n^3 + O_4(m\phi_n)$$
$$(1 - M\phi_n)\ln(1 - M\phi_n) = -M\phi_n + \tfrac{1}{2}M^2\phi_n^2 + \tfrac{1}{6}M^3\phi_n^3 + O_4(M\phi_n)$$

and by substitution into Eq. 14 we get

$$\theta_n = \frac{1}{\ln n}\left[\frac{Mm^2+mM^2}{2n}\phi_n^2 - \frac{Mm^3-mM^3}{6n}\phi_n^3 + \frac{Mm^4+mM^4}{12n}\,O_4(\phi_n)\right] =$$
$$= \frac{mM}{2\ln n}\left[\phi_n^2 - \frac{m-M}{3}\phi_n^3 + \frac{m^2-mM+M^2}{6}\,O_4(\phi_n)\right]$$

Notice that $m|\phi_n|^3 = \frac{1}{m^2}|m\phi_n|^3 = O_3(|\phi_n|) < O_3(|m\phi_n|)$ with $|m\phi_n| < 1$ and $M|\phi_n|^3 = \frac{1}{M^2}|M\phi_n|^3 = O_3(|\phi_n|) < O_3(|M\phi_n|)$ with $|M\phi_n| < 1$. The same happens with the higher order terms. Therefore, for enough small $|\phi_n|$ we can use the approximation derived from the following result,

Lemma 5. *The bias of a cyclic RID-scale satisfies*

$$\theta_n = \frac{mM}{2\ln(m+M)}\,\phi_n^2 + O_3(|\phi_n|) \tag{34}$$

11.3. Cyclic Scales of Minimal Bias

As explained in §4, while refining a cyclic scale, in each iteration one of the indices of the extreme tones remains fixed, while the other increases by a value equal to the one that remains fixed. Let's see that in each refinement, the following function appearing in Eq. 34 always increases,

$$\psi(m, M) = \frac{mM}{\ln(m+M)} \tag{35}$$

Lemma 6. *For $n = m + M \geq 2$, $\psi(m^+, M) > \psi(m, M)$ with $m^+ = m + M$, and $\psi(m, M^+) > \psi(m, M)$ with $M^+ = m + M$.*

---

[12] We must distinguish two cases: (a) $M > m$, therefore $\frac{1}{n} < \frac{1}{M} < \frac{2}{n}$ and $\frac{2}{n} < \frac{1}{m}$; since $Mu + md = 1$, we have $\frac{1}{n} < u < \frac{1}{M}$ and $0 < d < \frac{1}{n}$, which does not add any new limitation. (b) $M < m$, therefore $\frac{1}{n} < u < \frac{2}{n}$ and $\frac{1}{n}(1 - \frac{M}{m}) < d < \frac{1}{n}$. Therefore, the extremes still correspond to n-tone scales by avoiding degenerate cases.

[13] We must distinguish two cases: (a) $M > m$, therefore $\frac{1}{n} < \frac{1}{M} < \frac{2}{n}$ and $\frac{2}{n} < \frac{1}{m}$; hence, $\frac{1}{n} < d < \frac{2}{n}$ and $\frac{1}{n}(1 - \frac{m}{M}) < u < \frac{1}{n}$. Once again, the extremes correspond to n-tone scales, not degenerating toward scales of fewer tones. (b) $M < m$, therefore $\frac{2}{n} < \frac{1}{M}$ and $\frac{1}{n} < \frac{1}{m} < \frac{2}{n}$; hence, $\frac{1}{n} < d < \frac{1}{m}$ and $0 < u < \frac{1}{n}$, which does not add any new limitation.

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Proof. Indeed, it suffices to check that the function $\psi(m,M) = \frac{mM}{\ln(m+M)}$ satisfies $\frac{\partial\psi}{\partial m} > 0$, $\frac{\partial\psi}{\partial M} > 0$.

$$\frac{\partial}{\partial m}\frac{mM}{\ln(m+M)} = \frac{M\ln(m+M) - \frac{mM}{m+M}}{\ln^2(m+M)} = \frac{M^2\ln(m+M) + mM(\ln(m+M)-1)}{\ln^2(m+M)} > 0 \iff m+M \geq 2$$

$$\frac{\partial}{\partial M}\frac{mM}{\ln(m+M)} = \frac{m\ln(m+M) - \frac{mM}{m+M}}{\ln^2(m+M)} = \frac{m^2\ln(m+M) + mM(\ln(m+M)-1)}{\ln^2(m+M)} > 0 \iff m+M \geq 2$$

Note that, for $m + M > 2$, the above inequalities hold and, if $m + M = 2$, then $M = m = 1$, so that the above results are also valid. □

Corollary 1. *Let us write $\psi_n = \psi(m,M)$. If $n > n'$ then $\psi_n > \psi_{n'}$.*

With this notation, the bias of a cyclic RID-scale can be estimated from

$$\theta_n = \frac{1}{2}\psi_n\phi_n^2 \tag{36}$$

Theorem 4. *Every cyclic RID-scale of minimal bias is optimal.*

Proof. We prove it by denying the consequent. Assume two cyclic RID-scales E_n^h i $E_{n'}^h$ such that $n > n'$ with interval commas satisfying $|\phi_n| > |\phi_{n'}|$, i.e., E_n^h is not optimal. Then, applying the previous corollary to Eq. 36, we have

$$\psi_n\phi_n^2 > \psi_{n'}\phi_{n'}^2 \Rightarrow \theta_n > \theta_{n'}$$

Therefore, if a cyclic RID-scale is not optimal, it cannot be MB. □

The first cyclic MB-scales for $h = 3$, i.e., Pythagorean scales, are for $n = 5; 12; 53; 665\ldots$ as shown in Fig. 2, as well as in Table 1.

12. Conclusions

In the current paper it is proposed to measure the regularity of the intervals of a music scale from its partition entropy. Among other properties, the fact of being a continuous increasing function of n for an n-TET scale, which is always the maximum value that the entropy of any n-tone scale can reach, together with the sub-additivity property, which guarantees that as refining the partition, the entropy always increases, make this parameter very suitable for our purpose. In order to compare scales with different number of tones, the entropy relative to the corresponding regular scale is used, which is the normalized entropy, so that their complementary to 1 quantifies the bias relative to the n-TET scale.

The main application of these concepts is to cyclic scales, whose properties are reviewed and further investigated in the Appendices. Two situations have been analyzed. First, cyclic scales with a fixed number of tones, which is a family of modulating temperament scales around one generator. In this case, the bias only depends on the closure, i.e., the relative size of both elementary intervals. Second, as cyclic scales are refined, the bias also depends on how many intervals of each size there are.

In order to study such a dependency, it has been necessary to restrict the scales in two ways. On the one hand, to those having a lower limit of the entropy determined by the condition that their notes are distributed along each of the intervals of a regular scale (RID-scales). Such a study has been made in a general way, by determining the maximum bias of several scales, not necessarily cyclic, having different levels of similarity with a n-TET scale, either from one side or from both sides. Figure 2 displays the similarity levels for cyclic scales. On the other hand, to scales with the comma not exceeding that of a RID-scale, since in this case the dependency bias-closure is well defined.

We prove that any cyclic scale of minimal bias (MB-scale), i.e., whose bias is lower than that of the cyclic scales of fewer tones, is necessarily optimal, i.e., corresponds to a best rational aproximation of $\log_2 h$. Therefore, in relation to the closure, scales can be ordered in nested families, from worst to

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best, as cyclic, accurate, and optimal scales, whilst in relation to their regularity, they can be ordered in nested families as cyclic, RID, optimal, and MB-scales.

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Abbreviations

The following abbreviations are used in this manuscript:

BRA	Best rational approximation (optimal scale)
GRA	Good rational approximation (accurate scale)
I | Xn	maximum distance $1/x_n$ to n-TET
MB	Minimal bias
n-TET	n-tones of equal temperament
OSS	One-side similar
RID	Regular interval distributed

Appendix A. Bounds for the Elementary Intervals and Closure of Cyclic Scales

The notes of a cyclic scale E_n^h divide the circle of the octave into n intervals of width $\log_2 U$ or $\log_2 D$, beginning and ending in the pitch class of the fundamental, so that

$$M \log_2 U + m \log_2 D = 1 \tag{A1}$$

According to the definitions of U, D, and γ_n, these intervals can be referred to the elementary interval of the equal temperament scale as

$$\log_2 U = \tfrac{1}{n} + \tfrac{m}{n} \log_2 \gamma_n; \quad \log_2 D = \tfrac{1}{n} - \tfrac{M}{n} \log_2 \gamma_n \tag{A2}$$

We may determine their bounds by distinguishing between the cases γ_n greater or less than 1.

(i) If $\gamma_n > 1$, then $U > D$. Depending on the scale, we have:

(a) The closure of an *optimal cyclic scale* satisfies $1 < \gamma_n < \frac{2}{\nu_M} < \nu_m$, hence $1 < \gamma_n < D < U$. By taking logarithms and taking into account Eq. A2 we get

$$0 < \log_2 \gamma_n < \log_2 D = \tfrac{1}{n} - \tfrac{M}{n} \log_2 \gamma_n \quad \Longrightarrow \quad 0 < \log_2 \gamma_n < \tfrac{1}{n+M} \tag{A3}$$

The elementary intervals satisfy

$$\tfrac{1}{n+M} < \log_2 D < \tfrac{1}{n}; \quad \tfrac{1}{n} < \log_2 U < \tfrac{2}{n+M}$$

(b) The closure of a *non-optimal cyclic scale* satisfies $1 < \frac{2}{\nu_M} < \gamma_n < \nu_m$, hence $1 < D < \gamma_n < U$. Then, by taking logarithms,

$$\begin{aligned} \log_2 D = \tfrac{1}{n} - \tfrac{M}{n} \log_2 \gamma_n < \log_2 \gamma_n \quad &\Longrightarrow \quad \tfrac{1}{n+M} < \log_2 \gamma_n \\ \log_2 \gamma_n < \log_2 U = \tfrac{1}{n} + \tfrac{m}{n} \log_2 \gamma_n \quad &\Longrightarrow \quad \log_2 \gamma_n < \tfrac{1}{M} \end{aligned} \tag{A4}$$

Therefore,

$$0 < \log_2 D < \tfrac{1}{n+M}; \quad \tfrac{2}{n+M} < \log_2 U < \tfrac{1}{M}$$

(c) For an *accurate cyclic scale*, to the above condition $0 < \log_2 D < \log_2 \gamma_n < \log_2 U$ we must add that of a *good rational approximation*, that can be written as

$$0 < \tfrac{1}{n} \log_2 \gamma_n < \tfrac{1}{M} \log_2 D < \tfrac{1}{m} \log_2 U$$

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Then, $\frac{1}{n}\log_2\gamma_n < \frac{1-M\log_2\gamma_n}{Mn}$, hence $\log_2\gamma_n < \frac{1}{2M}$, and, since Eq. A4 is still valid, we get the following bounds,

$$\frac{1}{n+M} < \log_2\gamma_n < \frac{1}{2M}; \quad \frac{1}{2n} < \log_2 D < \frac{1}{n+M}; \quad \frac{2}{n+M} < \log_2 U < \frac{1}{2n} + \frac{1}{2M}$$

Thus, in case (i), either for optimal or non-optimal scales, it is satisfied,

$$\gamma_n > 1 \Longrightarrow \quad 0 < \log_2\gamma_n < \frac{1}{M}; \quad 0 < \log_2 D < \frac{1}{n}; \quad \frac{1}{n} < \log_2 U < \frac{1}{M} \tag{A5}$$

(ii) If $\gamma_n < 1$, then $D > U$. Depending on the scale, we have:

(a) The closure of an *optimal cyclic scale* satisfies $\frac{\nu_M}{2} < \frac{1}{\nu_m} < \gamma_n < 1$, hence $\frac{1}{D} < \frac{1}{U} < \gamma_n < 1$. Then, by taking logarithms,

$$-\log_2 U = -\frac{1}{n} - \frac{m}{n}\log_2\gamma_n < \log_2\gamma_n < 0 \quad \Longrightarrow \quad -\frac{1}{n+m} < \log_2\gamma_n < 0 \tag{A6}$$

The elementary intervals satisfy

$$\frac{1}{n+m} < \log_2 U < \frac{1}{n}; \quad \frac{1}{n} < \log_2 D < \frac{2}{n+m}$$

(b) The closure of a *non-optimal cyclic scale* satisfies $\frac{\nu_M}{2} < \gamma_n < \frac{1}{\nu_m} < 1$, hence $\frac{1}{D} < \gamma_n < \frac{1}{U} < 1$. Then, by taking logarithms,

$$\begin{aligned} -\log_2 D = -\frac{1}{n} + \frac{M}{n}\log_2\gamma_n < \log_2\gamma_n \quad &\Longrightarrow \quad -\frac{1}{m} < \log_2\gamma_n < 0 \\ \log_2\gamma_n < -\log_2 U = -\frac{1}{n} - \frac{m}{n}\log_2\gamma_n \quad &\Longrightarrow \quad \log_2\gamma_n < -\frac{1}{n+m} \end{aligned} \tag{A7}$$

Then, the elementary intervals satisfy

$$0 < \log_2 U < \frac{1}{n+m}; \quad \frac{2}{n+m} < \log_2 D < \frac{1}{m}$$

(c) For an *accurate cyclic scale*, to the above condition $-\log_2 D < \log_2\gamma_n < -\log_2 U < 0$ we must add that of a *good rational approximation*, that can be written as

$$-\frac{1}{M}\log_2 D < -\frac{1}{m}\log_2 U < \frac{1}{n}\log_2\gamma_n < 0$$

Then, $\frac{1}{n}\log_2\gamma_n > -\frac{1+m\log_2\gamma_n}{mn}$, hence $-\frac{1}{2m} < \log_2\gamma_n$, and, since Eq. A7 is still valid, we get the following bounds,

$$-\frac{1}{2m} < \log_2\gamma_n < -\frac{1}{n+m}; \quad \frac{2}{n+m} < \log_2 D < \frac{1}{2n} + \frac{1}{2m}; \quad \frac{1}{2n} < \log_2 U < \frac{1}{n+m}$$

Thus, in case (ii), either for optimal or non-optimal scales, it is satisfied,

$$\gamma_n < 1 \Longrightarrow \quad -\frac{1}{m} < \log_2\gamma_n < 0; \quad 0 < \log_2 U < \frac{1}{n}; \quad \frac{1}{n} < \log_2 D < \frac{1}{m} \tag{A8}$$

Table A1 summarizes the above results over the octave S_0. Notice that, for optimal scales, equations A3 and A6 give the following bounds, similarly to the theory of continued fractions,

$$\text{(i)} \quad \left|\log_2 h - \frac{N}{n}\right| < \frac{1}{n(n+M)}; \quad \text{(ii)} \quad \left|\log_2 h - \frac{N}{n}\right| < \frac{1}{n(n+m)}$$

Appendix B. Deviation of a Cyclic Scale from n-TET

We calculate the interval between the notes with ordinal j of the respective scales E_n^h and $E_n^{\top}$, corresponding to the tones $\vartheta_j \in E_n^h$ and $\vartheta'_j = 2^{\frac{j}{n}} \in E_n^{\top}$. According to equations A2 and A1, the

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cyclic scale	$\phi_n > 0$	$\phi_n < 0$
optimal	$0 < \phi_n < \frac{1}{n+M}$	$-\frac{1}{n+m} < \phi_n < 0$
	$\frac{1}{n} < u < \frac{2}{n+M}$	$\frac{1}{n+m} < u < \frac{1}{n}$
	$\frac{1}{n+M} < d < \frac{1}{n}$	$\frac{1}{n} < d < \frac{2}{n+m}$
non-optimal	$\frac{1}{n+M} < \phi_n < \frac{1}{M}$	$-\frac{1}{m} < \phi_n < -\frac{1}{n+m}$
	$\frac{2}{n+M} < u < \frac{1}{M}$	$0 < u < \frac{1}{n+m}$
	$0 < d < \frac{1}{n+M}$	$\frac{2}{n+m} < d < \frac{1}{m}$
accurate	$\frac{1}{n+M} < \phi_n < \frac{1}{2M}$	$-\frac{1}{2m} < \phi_n < -\frac{1}{n+m}$
	$\frac{2}{n+M} < u < \frac{1}{2n} + \frac{1}{2M}$	$\frac{1}{2n} < u < \frac{1}{n+m}$
	$\frac{1}{2n} < d < \frac{1}{n+M}$	$\frac{2}{n+m} < d < \frac{1}{2n} + \frac{1}{2m}$
in general	$0 < \phi_n < \frac{1}{M}$	$-\frac{1}{m} < \phi_n < 0$
	$\frac{1}{n} < u < \frac{1}{M}$	$1 < u < \frac{1}{n}$
	$0 < d < \frac{1}{n}$	$\frac{1}{n} < d < \frac{1}{m}$

Table A1. Bounds for $\phi_n = \log_2 \gamma_n$, $u = \log_2 U$ and $d = \log_2 D$, depending on the type of cyclic scale.

deviations of each note of E_n^h with regard to $E_n^\top$ compensate each other and, in the end, they close exactly the octave. If the tone ϑ_j corresponds to the iteration ν_k, then $j = Nk - n[\![k]\!]$ [17, Eq. 28]. Therefore, $[\![k]\!] = \frac{Nk-j}{n}$. Bearing in mind that $\log_2 \gamma_n = n \log_2 h - N$, the scale note $\log_2 \vartheta_j = \log_2 \nu_k$ is

$$\log_2 \vartheta_j = k \log_2 h - [\![k]\!] = \frac{k}{n} \log_2 \gamma_n + \frac{j}{n} \tag{A9}$$

Lemma A1. *The interval that separates the two notes is*

$$I(\vartheta_j, \vartheta'_j) = \log_2 \vartheta_j - \log_2 \vartheta'_j = \frac{k}{n} \log_2 \gamma_n; \quad j = 0, \ldots, n-1 \tag{A10}$$

Therefore, all of these intervals have the same sign, which is positive if $\gamma_n > 1$ and negative if $\gamma_n < 1$. By taking absolute value, the interval between these close notes becomes a distance,

$$d(\vartheta_j, \vartheta'_j) = \frac{k}{n} |\log_2 \gamma_n| = \frac{k}{n} \kappa_n; \quad j = 0, \ldots, n-1 \tag{A11}$$

As expected, these distances increase with the iterations, so that the maximum is reached by last iterate, $k = n-1$, i.e., $d_{\max} = \max_j d(\vartheta_j, \vartheta'_j) = \frac{n-1}{n} \kappa_n$. Then, a condition such as $d_{\max} \leq \frac{1}{\lambda n}$, for $\lambda > 0$, becomes $\kappa_n \leq \frac{1}{\lambda (n-1)}$, which, in cents is

$$¢(\kappa_n) \leq \frac{1200}{\lambda (n-1)} \tag{A12}$$

For an optimal cyclic scale, according to equations A3 and A6, $-\frac{1}{n+m} < \log_2 \gamma_n < \frac{1}{n+M}$, so that $|\log_2 \gamma_n| < \frac{1}{n}$ and $k|\log_2 \gamma_n| < 1$ in Eq. A11. Therefore, we conclude $d(\vartheta_j, \vartheta'_j) < \frac{1}{n}$.

Theorem A1. *The notes ϑ_i, $i \neq 0$, of an* optimal *cyclic scale E_n^h and the ones of the corresponding n-TET scale $E_n^\top$ alternate, i.e., for $0 < i < n$ either* (i) $\vartheta'_i < \vartheta_i < \vartheta'_{i+1}$ *or* (ii) $\vartheta'_{i-1} < \vartheta_i < \vartheta'_i$.

Such a situation is also possible for non-optimal scales, although there are exceptions where the tones do not alternate. For instance, in a non-optimal scale with $\gamma_n < 1$, Eq. A7 implies $m|\log_2 \gamma_n| < 1$. But, in a cyclic scale where $M > m$, the condition $k|\log_2 \gamma_n| < 1, \forall k$, stated in case (ii) is not fulfilled. An example of such a situation is the 29-tone scale, with $m = 12$ and $M = 17$, where $|\log_2 \gamma_n|^{-1} = 27.7$.

The 28-th tone $\nu_{28} = \vartheta_{12}$ (454.74 cents) is below ϑ'_{12} (496.55 cents) more than a factor $2^{\frac{1}{29}}$ (41.38 cents), and between ϑ_{12} and $\vartheta_{13} = \nu_{11}$ there are two consecutive tones ϑ'_{11} and ϑ'_{12} of the 29-TET scale.

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