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Posted Date: 18 April 2024

doi: 10.20944/preprints202404.1204.v1

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A Novel Version Three Parameter Nadarajah Haghighi Model: Entropy Measures, Inference and Applications

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Abstract: The fitting and modeling of skewed, complex and asymmetric datasets is an exciting research topic in many fields of applied sciences, notably lifetime, medical, and financial sciences. The Heavy tailed Nadarajah Haghighi model is introduced by compounding the heavy tailed family and Nadarajah Haghighi distribution in this paper. The so obtained model has three parameters accounting for the scale and shape of the distribution. The proposed distribution's fundamental characteristics such as probability density, cumulative distribution, hazard rate, and survival functions are provided, and several key statistical properties are established, and several entropy information measures are proposed. The estimation of model parameter is performed via the MLE procedure. Further, different simulation experiments are conducted to demonstrate the performance of proposed estimator's using some measure, like the average estimate, the average bias, and associated mean square error. Finally, we are applied our proposed model to analyze three different real datasets. In our illustration, we are compared the practicality of the recommended model with several well-known competing models.

Keywords: asymmetric dataset; heavy tailed; MLE procedure; nadarajah Haghighi; simulation experiments; survival function

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1. Introduction

In the last few decades, the classical distributions have typically been found to lack fit in many areas of studies, such as economic, actuarial science, engineering, environmental science, lifetime, medical science, and a number of others. For this, the authors have elaborated seriously to establish several of distinctive models adapted for specific datasets. This flexibility may be achieved by adding new factors to the classic distribution, including adding one or more parameters, or made several transformation to the baseline distribution, increasing its ability to handle complicated scenarios. In this context, numerous class of distributions have been obtained by applying different procedures to reinforce the adaptability of models, thereby creating a more versatile and potent distribution for dataset modeling, for example, see the studies of Madhavi and Kundu [18], Marshall and Olkin [19], Meraou and Raqab [23], Meraou el al. [24], Amal S. Hassan et al. [6], Cordeiro et al. [13], Bourguignon et al [10], Meraou et al. [22], Zagrofos and Balakrishana [30], Eugene et al. [14], Alzaatreh et al. [5], Afify et al. [2], and Meraou et al. [21].

Recently, a novel class of distributions was provided by Ahmed et al. [3] called as the heavy-tailed (HT) class of distributions. This new extension is an appropriate model for modeling the HT, symmetric, complex, and skewed datasets, and it is a powerful method for generating new models to analyze

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datasets underlying patterns better. The HT class of distributions is capable of portraying several real-realistic application studies in many areas, notably hydrology, biology, agriculture, production, survival, and finance. The cdf and pdf of the HT family of distributions can be expressed, respectively, as follows:

$$F(x;\lambda,\xi) = \frac{\lambda G(x,\xi)}{\lambda - 1 + G(x,\xi)}, \quad x \in \mathbb{R},\ \lambda > 1, \tag{1}$$

and

$$f(x;\lambda,\xi) = \frac{\lambda(\lambda-1)g(x;\xi)}{\left[\lambda - 1 + G(x,\xi)\right]^2}, \tag{2}$$

where $G(x,\xi)$ and $g(x,\xi)$ represent respectively the cdf and pdf of the baseline distribution with vector parameter ξ. The survival function (sf) and associated hazard rate function (hrf) of the proposed HT class of distribution are expressed, respectively, as

$$S(x;\lambda,\xi) = \frac{(\lambda-1)(1-G(x,\xi))}{\lambda - 1 + G(x,\xi)}, \quad x \in \mathbb{R},\ \lambda > 1, \tag{3}$$

and

$$h(x;\lambda,\xi) = \frac{\lambda(\lambda-1)g(x;\xi)}{S(x;\lambda,\xi)\left[\lambda - 1 + G(x,\xi)\right]^2}, \tag{4}$$

It is worth mentioning that the Nadarajah Haghighi (NH) distribution is widely used in statistical modeling. It is innovative extension of the exponential distribution to serve as an alternative model to the gamma and Exponentiated-exponential distribution. The NH distribution is introduced firstly by Nadarajah and Haghighi [25], which can be applied to fit several kind of datasets notably the skewed, complex and heavy tailed, as well as it can be used in different fields of application, particularly in reliability and engineering. It is well documented that the NH model can be extensively investigated in many study domains. Noteworthy studies involving this model include Almetwally and Meraou [4], Korkmaz et al. [16], Tahir et al. [28], Lone et al. [17], and Anum Shafiq et al. [8]. The cdf and pdf of ME distribution are formulated respectively as

$$G(y;\alpha,\beta) = 1 - e^{1-(1+\beta y)^{\alpha}},\ y,\ \alpha,\beta > 0, \tag{5}$$

and

$$g(y;\alpha,\beta) = \alpha\beta(1+\beta y)^{\alpha-1}\, e^{1-(1+\beta y)^{\alpha}}. \tag{6}$$

This research aims to enhance the NH distribution by adding one additional parameter, resulting in a generalized distribution termed the Heavy Tailed Nadarajah Haghighi model (HTNH). These supplementary parameter has great potential in modeling the tail behaviour of the specified probability function. The pdf of the proposed model can be increasing and skewed, and a sub model can be obtained as a special case. The estimation of the model's parameters is performed via the maximum likelihood estimator (MLE) technique. Further, we introduced the confidence interval (CI) for model parameters using the asymptotic distribution of MLE procedure, and various entropy information measures have been introduced.

The following factors have been involved in the design of this paper. In Section 2, we defined our HTNH model and established its distributional properties. Many mathematical properties of our proposed model are provided in Section 3. The final expressions of six proposed entropy information measures for the HTNH model are formulated in Section 4. Section 5 demonstrates the estimation of model parameters via MLE technique as well as the CIs of unknown parameters for our HTNH model are constructed also in this Section. In Section 6, a Monte Carlo (MC) simulation analysis are illustrated

 doi:10.20944/preprints202404.1204.v1

to show the consistency and unbiased of recommended estimator, and three real datasets are utilized for selecting the best fitting models in Section 7. In the last section, closing remarks are devoted.

2. The HTNH Model and Its Distributional Properties

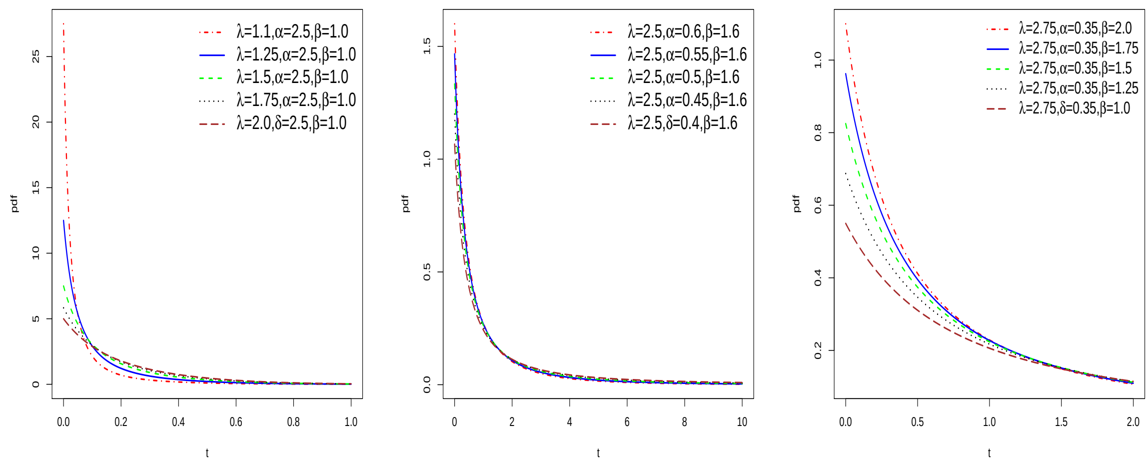
In this part of the work, weconsider certain distributional properties of the proposed HTNH model, including cdf, pdf, and corresponding reliability functions like survival function (sf), hazard rate function (hrf), cumulative hazard rate function (chrf), and reverse hazard function(rhrf). Let T be a random variable following the HTNH model with parameters λ, α, and β ($T \sim \text{HTNH}(\lambda, \alpha, \beta)$). According to the Eqs. (5) and (1), the cdf and pdf of T are obtained, and they are expressed, respectively, as follows:

$$\Delta(t;\lambda,\alpha,\beta) = \frac{\lambda\left[1 - e^{1-(1+\beta t)^{\alpha}}\right]}{\lambda - e^{1-(1+\beta t)^{\alpha}}},\ t > 0,\ \text{and},\ \lambda > 1,\ \alpha,\ \beta > 0, \tag{7}$$

and

$$\delta(t;\lambda,\alpha,\beta) = \frac{\alpha\beta\lambda(\lambda-1)(1+\beta t)^{\alpha-1}\, e^{1-(1+\beta t)^{\alpha}}}{\left[\lambda - e^{1-(1+\beta t)^{\alpha}}\right]^{2}}. \tag{8}$$

Under Eq. (7), the HT exponential (HTE) can be obtained as special case when α=1. Figure (1) depicts a few of the most likely contours of the pdf using several parameter values of HTNH distribution. In tha all situation, the pdf of the suggested HTNH model is decreasing and positively skewed.



(a) (b) (c)

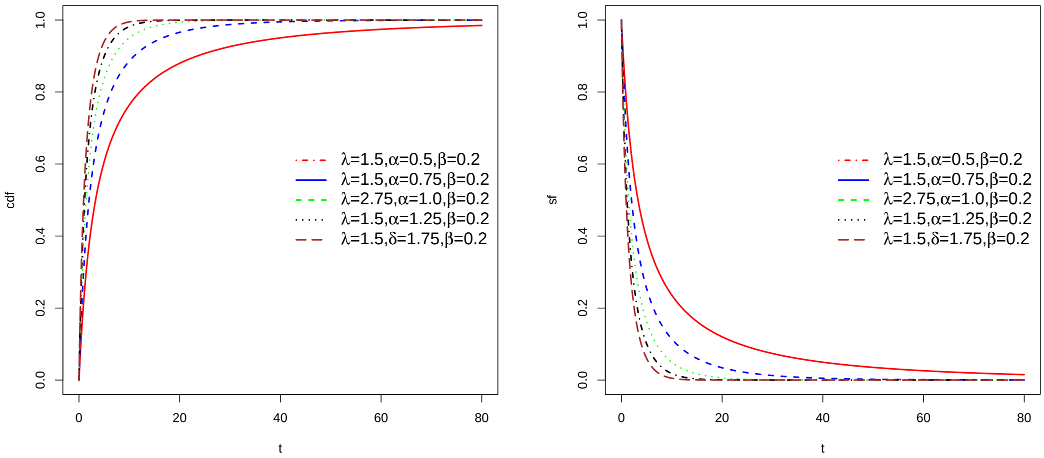
Figure 1. pdf plots of the HTNH model using different parameter values of (λ, α, β).

Next, the sf and hrf of T are structured, respectively, by

$$S(t;\lambda,\alpha,\beta) = \frac{e^{1-(1+\beta t)^{\alpha}}(\lambda-1)}{\lambda - e^{1-(1+\beta t)^{\alpha}}},$$

and

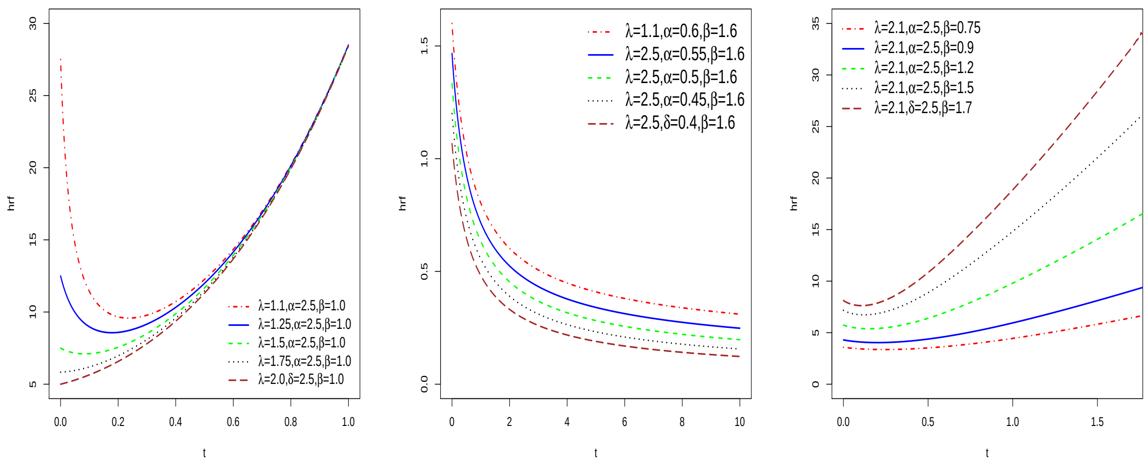
$$h(t;\lambda,\alpha,\beta) = \frac{\alpha\beta\lambda(1+\beta t)^{\alpha-1}}{\lambda - e^{1-(1+\beta t)^{\alpha}}}.$$



(a) cdf (b) sf

Figure 2. cdf and sf plots of the HTNH model using different parameter values of (λ, α, β).

The cdf and sf plots of the proposed HTNH model are drawn in Figure (2), and Figure (3) demonstrates the graphs of the hrfs curves of the proposed HTNH model. From the Figue (3), The hrf of the HTNH model has a variety of shapes. It is increasing, decreasing and J-shaped functions. Consequently,it can be deduced that our HTNH model is important in modeling various kinds of datasets.



(a) (b) (c)

Figure 3. hrf plots of the HTNH model using different parameter values of (λ, α, β).

The chrf and rhrf of the HTNH model are written, respectively, as

$$H(t; \lambda, \alpha, \beta) = -\log\left\{\frac{e^{1-(1+\beta t)^{\alpha}}(\lambda - 1)}{\lambda - e^{1-(1+\beta t)^{\alpha}}}\right\},$$

and

$$R(t; \lambda, \alpha, \beta) = \frac{\alpha\beta(\lambda - 1)(1+\beta t)^{\alpha}\, e^{1-(1+\beta t)^{\alpha}})}{(\lambda - e^{1-(1+\beta t)^{\alpha}})(1 - e^{1-(1+\beta t)^{\alpha}})}.$$

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3. Statistical Properties of HTNH Model

In this part of the study, we established several mathematical properties properties of the proposed HTNH model. In this subsection, let us consider $T \sim HTNH(\lambda, \alpha, \beta)$.

3.1. Quantile Function

The quantile function of T is defined as

$$\mathcal{QF}_\omega =, \frac{1}{\beta}\left\{\left(1-\log\left[\frac{\lambda(\omega-1)}{\omega-\lambda}\right]\right)^{\frac{1}{\alpha}}-1\right\}\ 0<\omega<1, \tag{9}$$

Bu using the above equation, a random sample from our suggested HTNH model may be generated with ω follows a uniform random number (0,1), which will be useful for further development.
The coefficients for skewness ($\mathcal{SK}$) and the kurtosis ($\mathcal{KR}$) of Y can be formulated, respectively, as

$$\mathcal{SK} = \frac{\mathcal{QF}_{1/4}+\mathcal{QF}_{3/4}-2\mathcal{QF}_{1/2}}{\mathcal{QF}_{3/4}-\mathcal{QF}_{1/4}},$$

and

$$\mathcal{KR} = \frac{\mathcal{QF}_{7/8}-\mathcal{QF}_{5/8}+\mathcal{QF}_{3/8}-\mathcal{QF}_{1/8}}{\mathcal{QF}_{6/8}-\mathcal{QF}_{2/8}}.$$

3.2. Useful Expansion

For simplicity, let us consider the following series

$$(1+\rho)^{-2} = \sum_{m=1}^{\infty}(-1)^{m-1}m\rho^{m-1}.$$

By replacing the above equation in Eq. (8), the pdf of HTNH model is reformulated by

$$\delta(t;\lambda,\alpha,\beta) = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}m\lambda\alpha\beta}{(\lambda-1)^m}\, e^{1-(1+\eta t)^\alpha}\,(1+\beta t)^{\alpha-1}\left(1-e^{1-(1+\beta t)^\alpha}\right)^{m-1}.$$

The associated cdf is rewritten as

$$\Delta(t;\lambda,\alpha,\beta) = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}m\lambda\alpha\beta}{(\lambda-1)^m}\left(1-e^{1-(1+\beta t)^\alpha}\right)^{m}.$$

3.3. Moment and Related measures

The corresponding r^{th}-moment of T can be expressed by

$$u'_r = \lambda\alpha\beta\sum_{m=1}^{\infty}m(\lambda-1)^{-m}\eta_{r,\alpha,\beta}(t), \tag{10}$$

where $\eta_{r,\alpha,\beta}(t) = \int_0^\infty t^r\, e^{1-(1+\beta t)^\alpha}\,(1+\beta t)^{\alpha-1}\left(1-e^{1-(1+\beta t)^\alpha}\right)^{m-1} dt$.
Using Eq. (10), the first and second moments of T can be resulted, and they are written as

$$u'_1 = \lambda\alpha\beta\sum_{m=1}^{\infty}m(\lambda-1)^{-m}\eta_{1,\alpha,\beta}(t),$$

and

$$u'_2 = \lambda\alpha\beta\sum_{m=1}^{\infty}m(\lambda-1)^{-m}\eta_{2,\alpha,\beta}(t).$$

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Furthermore The variance and index of dispersion for T are

$$Var(T) = u_2' - u_1'^2,$$

and

$$ID = \frac{var(Y)}{u_1'}.$$

The moment generating function (mgf) and characteristic function (cf) of T are presented, respectively, as follows :

$$M(y) = \lambda\alpha\beta \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{m\, y^k}{(\lambda-1)^m k!} \eta_{r,\alpha,\beta}(t),$$

and

$$\Phi(y) = \lambda\alpha\beta \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{m\, (iy)^k}{(\lambda-1)^m k!} \eta_{r,\alpha,\beta}(t),$$

Tables (1) and (2) represent several suggested statistical properties of the HTNH model, likely u_1', variance, ID, $\mathcal{K}$, and $\mathcal{R}$, which also displayed in Figures (4) and (5). All these numerical values and plots confirm that the suggested HTNH model is an efficient distribution for fitting several types of datasets.

Table 1. Numerous statistical measures for the HTNH model at $\alpha = 2$.

	β	u_1'	$Var(T)$	ID	$\mathcal{SK}$	$\mathcal{KR}$
λ=2	0.2	1.3647	1.8501	1.3557	1.7282	3.6433
	0.4	0.6823	0.4625	0.6778	1.7282	3.6433
	0.6	0.4549	0.2056	0.4519	1.7282	3.6433
	0.8	0.3412	0.1156	0.3389	1.7282	3.6433
λ=3	0.2	1.5724	2.1078	1.3405	1.5172	2.7109
	0.4	0.7862	0.5270	0.6703	1.5172	2.7109
	0.6	0.5241	0.2342	0.4468	1.5172	2.7109
	0.8	0.3931	0.1317	0.3351	1.5172	2.7109
λ=4	0.2	1.6627	2.2149	1.3321	1.4361	2.3869
	0.4	0.8314	0.5537	0.6661	1.4361	2.3869
	0.6	0.5542	0.2461	0.4440	1.4361	2.3869
	0.8	0.4157	0.1384	0.3330	1.4361	2.3869

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Table 2. Numerous statistical measures for the HTNH model at $\alpha = 4$.

	β	u_1'	$Var(T)$	ID	$\mathcal{SK}$	$\mathcal{KR}$
λ=2	0.2	0.6127	0.3156	0.5151	1.4198	2.1529
	0.4	0.3063	0.0789	0.2576	1.4198	2.1529
	0.6	0.2042	0.0351	0.1717	1.4198	2.1529
	0.8	0.1532	0.0197	0.1288	1.4198	2.1529
λ=3	0.2	0.7014	0.3564	0.5082	1.2194	1.3991
	0.4	0.3507	0.0891	0.2541	1.2194	1.3991
	0.6	0.2338	0.0396	0.1694	1.2194	1.3991
	0.8	0.1753	0.0223	0.1271	1.2194	1.3991
λ=4	0.2	0.7395	0.3718	0.5028	1.1427	1.1605
	0.4	0.3697	0.0929	0.2514	1.1427	1.1605
	0.6	0.2465	0.0413	0.1676	1.1427	1.1605
	0.8	0.1849	0.0232	0.1257	1.1427	1.1605

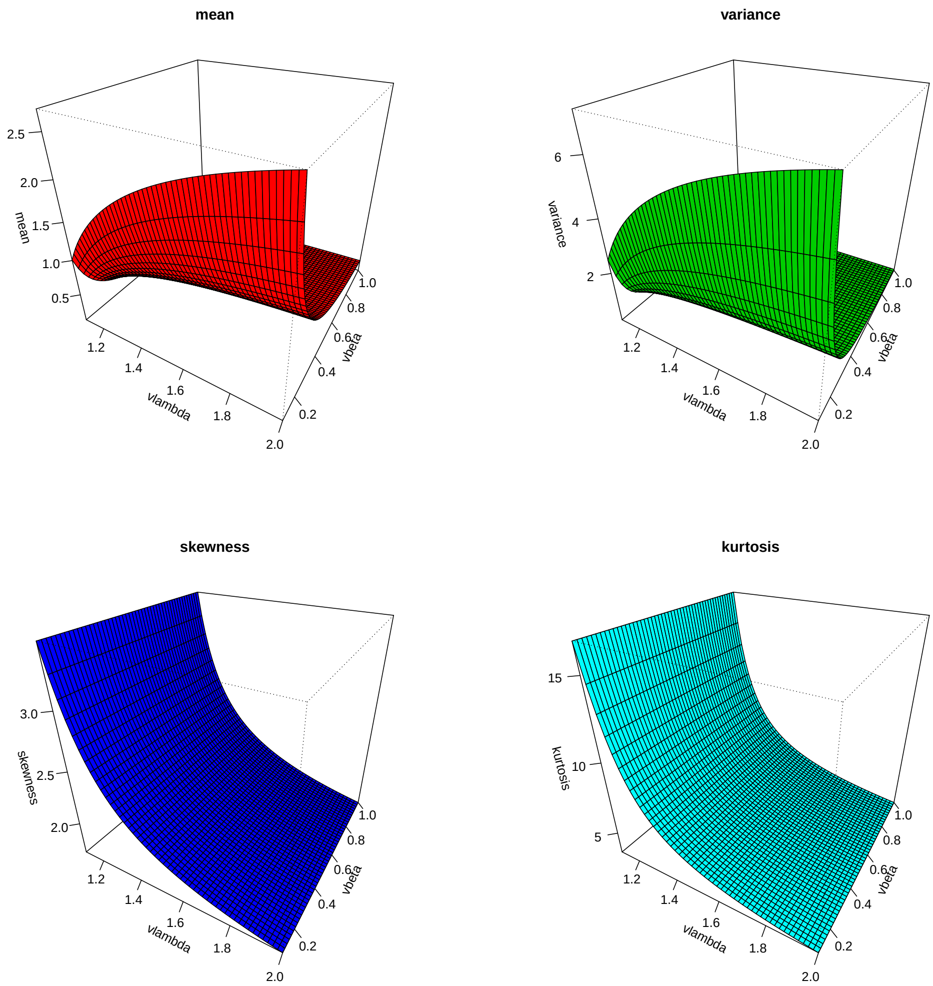

Figure 4. 3D curves for the proposed statistical properties of HTNH model at $\alpha = 2$.

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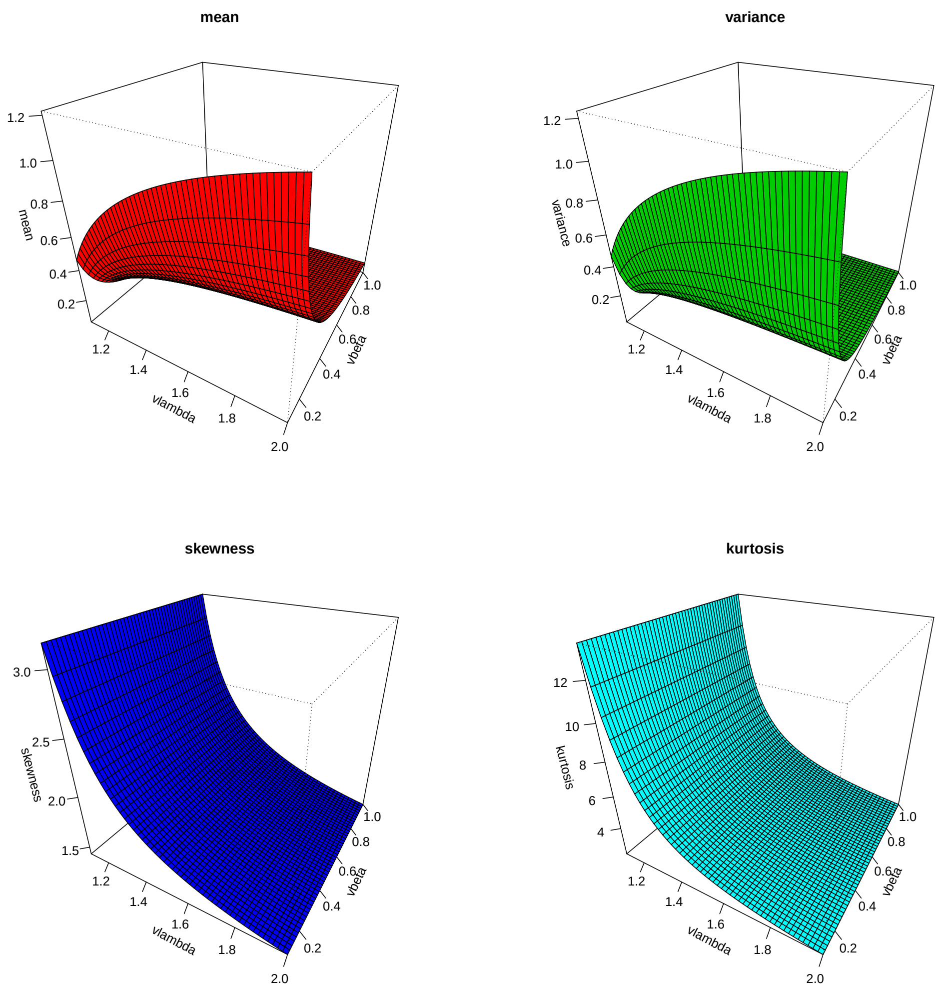

Figure 5. 3D curves for the proposed statistical properties of HTNH model at $\alpha = 4$.

3.4. *Residual and Reverse Residual Life*

Let T be a random variable follows the HTNH model. The residual reliability function of Y, abbreviated as $D_l(t)$, can be defined as

$$\begin{aligned} D_l(t) &= \frac{S(t+l;\lambda,\alpha,\beta)}{S(l;\lambda,\alpha,\beta)} \\ &= \frac{(1-G(t+l;\alpha,\beta))(\lambda-1+G(l;\alpha,\beta))}{[1-G(l;\alpha,\beta)][\lambda-1+G(t+l;\alpha,\beta)]} \\ &= \frac{e^{1-(1+\beta(t+l))^{\alpha}}(\lambda-e^{1-(1+\beta l)^{\alpha}})}{e^{1-(1+\beta l)^{\alpha}}(\lambda-e^{1-(1+\beta(t+l))^{\alpha}})}. \end{aligned}$$

Similarly, the reverse residual reliability function of Y, abbreviated as $\overline{D}_l(t)$, is

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$$
\begin{aligned}
\overline{D}_l(t) &= \frac{S(t-l;\lambda,\alpha,\beta)}{S(l;\lambda,\alpha,\beta)} \\
&= \frac{(1-G(t-l;\alpha,\beta))(\lambda-1+G(l;\alpha,\beta))}{[1-G(l;\alpha,\beta)][\lambda-1+G(t-l;\alpha,\beta)]} \\
&= \frac{e^{1-(1+\beta(t-l))^{\alpha}}\left(\lambda-e^{1-(1+\beta l)^{\alpha}}\right)}{e^{1-(1+\beta l)^{\alpha}}\left(\lambda-e^{1-(1+\beta(t-l))^{\alpha}}\right)}.
\end{aligned}
$$

3.5. *pdf and cdf of order Statistics of HTNH Model*

Assume that $T_1, T_2, \ldots, T_n$ is random sample taken from the HTNH distribution, and $T_{(1:n)}, T_{(2:n)}, \ldots, T_{(n:n)}$ represents its order statistics. Further, the j^{th} pdf of $T_{(j:n)}$ is

$$
\begin{aligned}
\psi_{(j:n)}(t;\lambda,\alpha,\beta) &= \frac{n!}{(j-1)!(n-j)!}\delta(t;\lambda,\alpha,\beta)[\Delta(t;\lambda,\alpha,\beta)]^{j-1}[1-\Delta(t;\lambda,\alpha,\beta)]^{n-j} \\
&= \frac{\alpha\beta\lambda(\lambda-1)n!\,(1+\beta t)^{\alpha-1}e^{1-(1+\beta t)^{\alpha}}}{(j-1)!(n-j)!}\left(\lambda-e^{1-(1+\beta t)^{\alpha}}\right)^{j-3}\left(\lambda\left[1-e^{1-(1+\beta t)^{\alpha}}\right]\right)^{j-1} \\
&\times \left(1-\frac{\lambda\left[1-e^{1-(1+\beta t)^{\alpha}}\right]}{\lambda-e^{1-(1+\beta t)^{\alpha}}}\right)^{n-j}.
\end{aligned}
$$

The corresponding jth cdf of $T_{(j:n)}$ can be provided as

$$
\begin{aligned}
\Psi_{(j:n)}(t;\lambda,\alpha,\beta) &= \sum_{i=j}^{n}\Delta^{i}(t;\lambda,\alpha,\beta)[1-\Delta(t;\lambda,\alpha,\beta)]^{n-i} \\
&= \sum_{i=j}^{n}\sum_{w=0}^{n-i}\binom{n}{i}\binom{n-i}{w}(-1)^{i}\,\Delta^{w+i}(t;\lambda,\alpha,\beta) \\
&= \sum_{i=j}^{n}\sum_{w=0}^{n-i}\binom{n}{i}\binom{n-i}{w}(-1)^{i}\left(\frac{\lambda\left[1-e^{1-(1+\beta t)^{\alpha}}\right]}{\lambda-e^{1-(1+\beta t)^{\alpha}}}\right)^{w+i}.
\end{aligned}
$$

Hence, the pdf of the minimum and maximum order statistics, defined respectively by $T_{(n:n)} = \max\{t_1, t_2, \ldots, t_n\}$ and $T_{(1:n)} = \min\{t_1, t_2, \ldots, t_n\}$ are written as

$$
\psi_{(1:n)}(t;\lambda,\alpha,\beta) = n\alpha\beta\lambda(\lambda-1)\,(1+\beta t)^{\alpha-1}e^{1-(1+\beta t)^{\alpha}}\left(\lambda-e^{1-(1+\beta t)^{\alpha}}\right)^{-2}\left(1-\frac{\lambda\left[1-e^{1-(1+\beta t)^{\alpha}}\right]}{\lambda-e^{1-(1+\beta t)^{\alpha}}}\right)^{n-1},
$$

and

$$
\psi_{(n:n)}(t;\lambda,\alpha,\beta) = n\alpha\beta\lambda(\lambda-1)\,(1+\beta t)^{\alpha-1}e^{1-(1+\beta t)^{\alpha}}\left(\lambda-e^{1-(1+\beta t)^{\alpha}}\right)^{n-3}\left(\lambda\left[1-e^{1-(1+\beta t)^{\alpha}}\right]\right)^{n-1}.
$$

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4. Certain Entropy Measures

In this subsection, numerous information entropy's notably Rényi, Shannon, Havrda and Charvat, Tsallis, Arimoto, and Mathai–Haubold are considered. First, we start by defined the Rényi [26] ($\Xi_1(\tau)$) entropy of our HTNH model. It is formulated by

$$\begin{aligned}
\Xi_1(\tau) &= \frac{1}{1-\tau}\log\left(\int_0^\infty \delta^\tau(t;\lambda,\alpha,\beta)\,dt\right) \quad \tau \neq 1, \tau > 0. \\
&= \frac{1}{1-\tau}\log\left(\left\{\int_0^\infty \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m\lambda\alpha\beta}{(\lambda-1)^m}\, e^{1-(1+\eta t)^\alpha}\,(1+\beta t)^{\alpha-1}\left(1-e^{1-(1+\beta t)^\alpha}\right)^{m-1} dt\right\}^\tau\right) \\
&= \frac{(\lambda\alpha\beta)^\tau}{1-\tau}\log\left(\left\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}m}{(\lambda-1)^m}\,\Phi_{m,\alpha,\beta}(t)\right\}^\tau\right),
\end{aligned}$$

with, $\Phi_{m,\alpha,\beta}(t) = \int_0^\infty e^{1-(1+\beta t)^\alpha}\,(1+\beta t)^{\alpha-1}\left(1-e^{1-(1+\beta t)^\alpha}\right)^{m-1} dt.$

The Shannon entropy is another uncertainty information measure, and it is defined by

$$\begin{aligned}
\Xi_2 &= \mathrm{E}(-\log\delta(t;\lambda,\alpha,\beta)) \\
&= \mathrm{E}\left(-\log\left[\frac{\alpha\beta\lambda(\lambda-1)(1+\beta t)^{\alpha-1}\,e^{1-(1+\beta t)^\alpha}}{\left[\lambda-e^{1-(1+\beta t)^\alpha}\right]^2}\right]\right) \\
&= -\log(\alpha\beta\lambda(\lambda-1)) - (\alpha-1)\mathrm{E}(\log[1+\beta t]) - \mathrm{E}(1-(1+\beta t)^\alpha) + 2\mathrm{E}\left(\log\left[\lambda-e^{1-(1+\beta t)^\alpha}\right]\right)
\end{aligned}$$

Next, we consider a novel Havrda and Charvat entropy [15] ($\Xi_3(\tau)$) of the proposed HTNH distribution in this work. It is presented as

$$\begin{aligned}
\Xi_3(\tau) &= \frac{1}{2^{1-\tau}-1}\left(\int_0^\infty \delta(t;\lambda,\alpha,\beta)\,dt - 1\right) \quad \tau \neq 1, \tau > 0. \\
&= \frac{\alpha\beta\lambda}{2^{1-\tau}-1}\left(\sum_{m=1}^{\infty}\frac{(-1)^{m-1}m}{(\lambda-1)^m}\,\Phi_{m,\alpha,\beta}(t) - 1\right).
\end{aligned}$$

The Tsallis entropy [29] $\Xi_4(\tau)$ of HTNH model is

$$\begin{aligned}
\Xi_4(\tau) &= \frac{1}{1-\tau}\left(1-\int_0^\infty \delta^\tau(t;\lambda,\alpha,\beta)\,dt\right) \quad \tau \neq 1, \tau > 0. \\
&= \frac{(\lambda\alpha\beta)^\tau}{\tau-1}\left(1-\left\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}m}{(\lambda-1)^m}\,\Phi_{m,\alpha,\beta}(t)\right\}^\tau\right).
\end{aligned}$$

Further, we expressed the Arimoto entropy [9] ($\Xi_5(\tau)$) of the HTNH model. It can be examined by

$$\begin{aligned}
\Xi_5(\tau) &= \frac{\tau}{1-\tau}\left(\int_0^\infty \{\delta^\tau(t;\lambda,\alpha,\beta)\,dt\}^{\frac{1}{\tau}} - 1\right) \quad \tau \neq 1, \tau > 0. \\
&= \frac{\lambda\alpha\beta\tau}{\tau-1}\left(\left\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}m}{(\lambda-1)^m}\,\Phi_{m,\alpha,\beta}(t)\right\}^{\frac{1}{\tau}} - 1\right).
\end{aligned}$$

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At the end, a new flexible entropy measure called the Mathai–Haubold entropy [20] ($\Xi_6(\tau)$) is established. It can be defined as follows:

$$\begin{aligned}\Xi_6(\tau) &= \frac{1}{\tau-1}\left(\int_0^\infty \delta^{2-\tau}(t;\lambda,\alpha,\beta)\,dt - 1\right) \quad \tau\neq 1, \tau>0.\\ &= \frac{(\lambda\alpha\beta)^{2-\tau}}{\tau-1}\left(\left\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}m}{(\lambda-1)^m}\,\Phi_{m,\alpha,\beta}(t)\right\}^{2-\tau} - 1\right).\end{aligned}$$

Tables (3) and (4) summarized certain information entropy measures of the HTNH model as discused in this Section using several selected parameters of λ, α, and β. Further, Figures (6) and (7) depict the 3D curves of these information entropy's.

Table 3. Different numerical records of proposed entropy measures at $\tau = 0.5$ and $\lambda = 1.5$.

	α	$\Xi_1(\tau)$	Ξ_2	$\Xi_3(\tau)$	$\Xi_4(\tau)$	$\Xi_5(\tau)$	$\Xi_6(\tau)$
β=1	0.2	3.0866	2.2944	8.8843	7.3600	20.9023	-18.2487
	0.4	2.7092	1.7463	6.9416	5.7506	14.0179	-13.2577
	0.6	1.9997	1.1071	4.1473	3.4358	6.3869	-6.9614
	0.8	1.3829	0.6686	2.4060	1.9932	2.9864	-3.6424
β=2	0.2	2.9204	1.9489	7.9834	6.6136	17.5486	-15.8757
	0.4	2.2667	1.1079	5.0845	4.2121	8.6477	-8.9483
	0.6	1.3250	0.4178	2.2685	1.8793	2.7622	-3.4027
	0.8	0.6882	-0.0192	0.9916	0.8215	0.9902	-1.3512
β=3	0.2	2.8067	1.7044	7.4088	6.1376	15.5552	-14.4146
	0.4	1.9620	0.7159	4.0248	3.3342	6.1135	-6.7115
	0.6	0.9200	0.0168	1.4100	1.1681	1.5092	-1.9874
	0.8	0.2812	-0.4180	0.3645	0.3019	0.3247	-0.4696

Table 4. Different numerical records of proposed entropy measures at $\tau = 1.25$ and $\lambda = 3$.

	α	$\Xi_1(\tau)$	Ξ_2	$\Xi_3(\tau)$	$\Xi_4(\tau)$	$\Xi_5(\tau)$	$\Xi_6(\tau)$
β=1	0.2	4.0285	2.5078	3.9894	2.5389	2.7661	-2.1206
	0.4	2.1733	2.3272	2.6347	1.6768	1.7626	-1.3387
	0.6	1.4374	1.6300	1.8973	1.2075	1.2493	-0.9450
	0.8	0.9841	1.1330	1.3707	0.8724	0.8933	-0.6740
β=2	0.2	3.1994	2.3623	3.4607	2.2024	2.3632	-1.8045
	0.4	1.4736	1.7315	1.9368	1.2326	1.2763	-0.9656
	0.6	0.7493	0.9386	1.0736	0.6833	0.6958	-0.5243
	0.8	0.2973	0.4418	0.4503	0.2865	0.2887	-0.2169
β=3	0.2	2.7427	2.2202	3.1191	1.9850	2.1111	-1.6082
	0.4	1.0705	1.3473	1.4758	0.9392	0.9637	-0.7274
	0.6	0.3488	0.5347	0.5249	0.3340	0.3369	-0.2532
	0.8	-0.1017	0.0388	-0.1619	-0.1030	-0.1028	0.0770

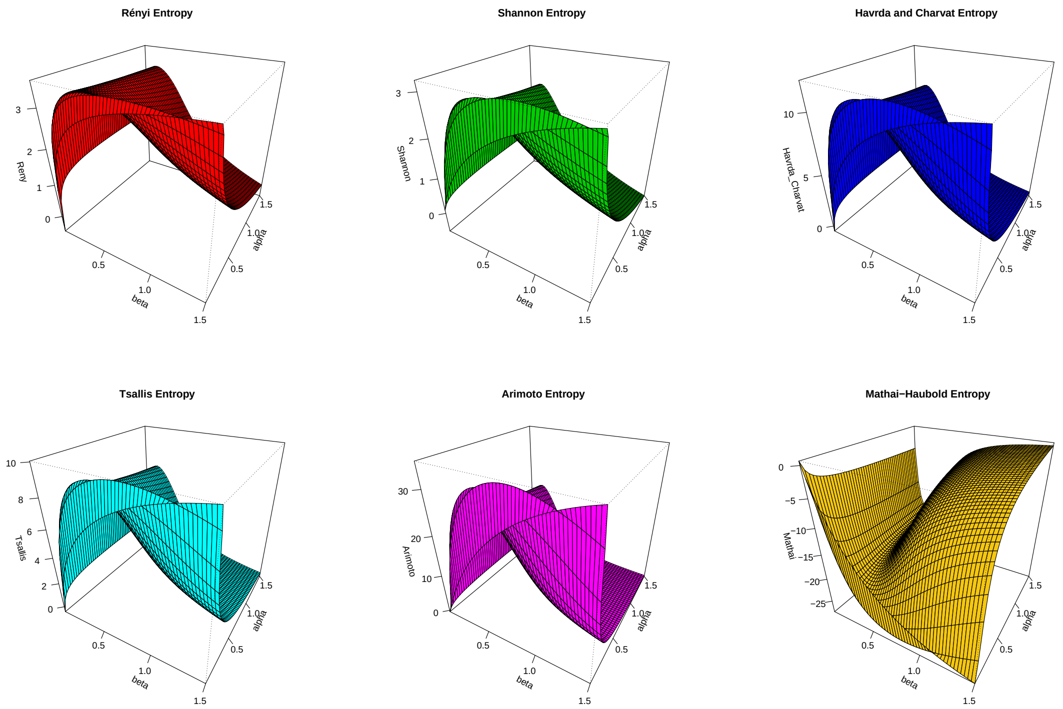

Figure 6. 3D curves of proposed entropy measures at $\lambda = 1.5$ and $\tau = 0.5$.

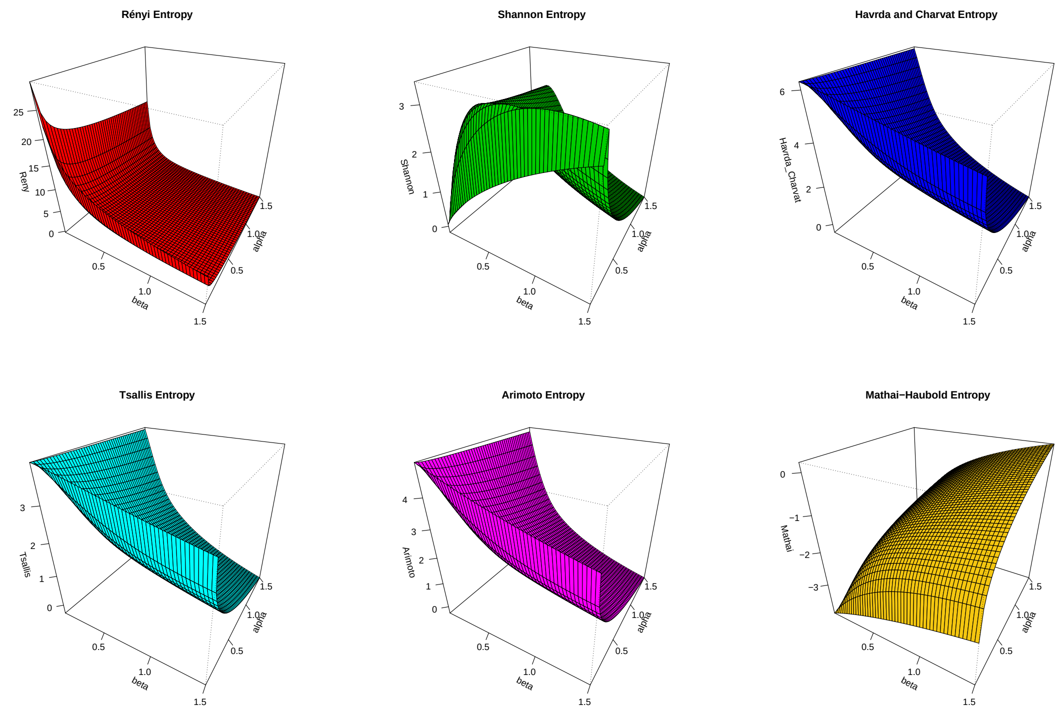

Figure 7. 3D curves of proposed entropy measures at $\lambda = 3$ and $\tau = 1.25$.

5. Statistical Inference

In this section, we introduce different estimator's techniques for determining the estimate parameters of our proposed model.

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5.1. Maximum Likelihood Estimator

Assume that $(t_1, t_2 \ldots, t_n)$ is a random sample drawn from our HTNH model of size n. With $\boldsymbol{\Sigma} = (\lambda, \alpha, \beta)$ denote the parameter vector,tThe log-likelihood function of our model can be written as follows:

$$\begin{aligned}\mathcal{LL}(\boldsymbol{\Sigma}) &= n\log\alpha + n\log\beta + n\log\lambda + n\log(\lambda-1) + (\alpha-1)\sum_{i=1}^{n}\log(1+\beta t_i) + \sum_{i=1}^{n}(1-(1+\beta t_i)^{\alpha}) \\ &- 2\sum_{i=1}^{n}\log(\lambda - e^{1-(1+\beta t_i)^{\alpha}}). \end{aligned} \tag{11}$$

The partial derivatives of Eq. (11), with respect parameters are

$$\frac{\partial \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \lambda} = \frac{n}{\lambda} + \frac{n}{\lambda-1} - 2\sum_{i=1}^{n}\left(\lambda - e^{1-(1+\beta t_i)^{\alpha}}\right)^{-1}, \tag{12}$$

$$\frac{\partial \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1+\beta t_i) - \sum_{i=1}^{n}(1+\beta t_i)^{\alpha}\log(1+\beta t_i)^{\alpha} - 2\sum_{i=1}^{n}\frac{(1+\beta t_i)^{\alpha}\, e^{1-(1+\beta t_i)^{\alpha}}}{\lambda - e^{1-(1+\beta t_i)^{\alpha}}}, \tag{13}$$

and

$$\frac{\partial \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \beta} = \frac{n}{\beta} + (\alpha-1)\sum_{i=1}^{n}t_i(1+\beta t_i)^{-1} - \alpha\sum_{i=1}^{n}t_i(1+\beta t_i)^{\alpha-1} - 2\alpha\sum_{i=1}^{n}\frac{t_i(1+\beta t_i)^{\alpha-1}\, e^{1-(1+\beta t_i)^{\alpha}}}{\lambda - e^{1-(1+\beta t_i)^{\alpha}}}. \tag{14}$$

The Eqs (12), (13) and (14) describe non-linear equations that are tricky to write concisely, rendering it difficult to solve them directly for the vector parameter $\boldsymbol{\Sigma} = (\lambda, \alpha, \beta)$. Iterative approaches, notably fixed point, secant and Newton-Raphson methods are employed. The aforementioned methods enable it to be quicker to determine the MLE of $\mathcal{LL}(\boldsymbol{\Sigma})$.

5.2. Approximate Confidence Interval

Now, for constructing the confidence intervals (CIs) of the parameters. We use the asymptotic distribution of MLE of $\boldsymbol{\Sigma}$. Precisely,

$$(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) \xrightarrow{D} N_3(\mathbf{0}, \boldsymbol{I}^{-1}(\boldsymbol{\Sigma})),$$

where $\hat{\boldsymbol{\Sigma}}$ is the MLE of $\boldsymbol{\Sigma}$ and $\boldsymbol{I}^{-1}(\boldsymbol{\Sigma})$ is the inverse of the observed information matrix of $\boldsymbol{\Sigma}$, which has a size of 3 by 3, and it is presented as

$$\boldsymbol{I}^{-1}(\boldsymbol{\Sigma}) = \begin{pmatrix} \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \lambda^2} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \lambda \partial \alpha} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \lambda \partial \beta} \\ \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \beta \partial \lambda} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \beta^2} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \beta \partial \alpha} \\ \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \alpha \partial \lambda} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \alpha \partial \beta} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\Sigma})}{\partial \alpha^2} \end{pmatrix}$$

Finally, with $\Sigma_1 = \lambda$, $\Sigma_2 = \alpha$, and $\Sigma_3 = \beta$. The lower confidence limit (LCL) and upper confidence limit (UCL) of $(1-\gamma)\%$ CI of Σ_k are

$$\text{LCB} = \hat{\Sigma}_k - z_{\gamma/2}\sqrt{I^{-1}(\hat{\Sigma}_k)}, \ \ k = 1,2,3,$$

and

$$\text{UCB} = \hat{\Sigma}_k + z_{\gamma/2}\sqrt{I^{-1}(\hat{\Sigma}_k)}, \ \ k = 1,2,3,$$

where $z_{\gamma/2}$ is upper $\gamma/2$ quantile of the standard normal distribution, $N(0,1)$.

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6. Simulation Experiments

We performed a Monte Carlo (MC) simulation experiments in this subsection to demonstrate the potential of the MLE method of tour suggested HTNH model using several sample sizes $n = \{300, 500, 700, 1000\}$, and different parameter case values of (λ, α, β) including Case1=$(1.1, 0.25, 0.25)$, Case2=$(1.2, 0.25, 0.5)$, Case3=$(1.4, 0.2, 0.75)$, and Case4-$(1.6, 0.2, 1.2)$. By Utilizing the Eq (9), random numbers were generated applying the following expression:

1. Generate ω from Uniforme(0,1).
2. Generate t as
$$t = \frac{1}{\beta}\left\{\left(1 - \log\left[\frac{\lambda(\omega-1)}{\omega-\lambda}\right]\right)^{\frac{1}{\alpha}} - 1\right\}.$$

Consequently, with M =1000 times repetition of the process, we calculated certain measures including the average estimate (AE), the average biases (AB), and average mean square errors (MSEs). These findings are summarized in Tables 5-8. A noteworthy trend emerges from the values of Tables 5-8 as the sample size grows: the AEs tend to the actual values of parameters, and the ABs and MSEs exhibit a reduction for all parameter sets. This observation underscores the consistent and unbiased of the MLE technique in accurately estimating parameters within the HTNH model. Further, for the CIs of the HTNH parameters, it appears from Tables (5)-(8) that the ALs constructed under MLE decline as n tends to be grow.

Table 5. The AEs, ABs, and MSEs of the HTNH model using Case1.

Simple size	Est.	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
300	AE	1.1693	0.2578	0.3750
	AB	0.0693	0.0078	0.1250
	MSE	0.0406	0.0015	0.0961
	LCL	1.0160	0.1850	0.0348
	UCL	1.49699	0.34430	0.8798
	AL	0.48096	0.1592	0.8450
500	AE	1.1185	0.2540	0.2754
	AB	0.0185	0.0040	0.0254
	MSE	0.0091	0.0009	0.0303
	LCL	1.0329	0.2034	0.0749
	UCL	1.3272	0.3122	0.7323
	AL	0.2942	0.1087	0.6573
700	AE	1.1066	0.2490	0.2671
	AB	0.0066	0.0009	0.0171
	MSE	0.0032	0.0004	0.0193
	LCL	1.0255	0.2096	0.0609
	UCL	1.2568	0.2918	0.5987
	AL	0.2312	0.0821	0.5378
1000	AE	1.1140	0.2521	0.2768
	AB	0.0140	0.0021	0.0268
	MSE	0.0028	0.0004	0.0118
	LCL	1.0385	0.2189	0.0778
	UCL	1.2872	0.2932	0.5797
	AL	0.2486	0.0743	0.5018

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Table 6. The AEs, ABs, and MSEs of the HTNH model using Case2.

Simple size	Est.	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
300	AE	1.3907	0.2564	0.7021
	AB	0.1907	0.0064	0.2021
	MSE	0.4052	0.0008	0.3405
	LCL	1.0382	0.2130	0.0958
	UCL	2.7006	0.3201	2.4076
	AL	1.6623	0.1071	2.3118
500	AE	1.2729	0.2526	0.5977
	AB	0.0729	0.0026	0.0977
	MSE	0.0502	0.0004	0.1159
	LCL	1.0664	0.2183	0.1743
	UCL	1.9156	0.2925	1.4563
	AL	0.8491	0.0742	1.2819
700	AE	1.2412	0.2500	0.5632
	AB	0.0412	0.0009	0.0632
	MSE	0.0357	0.0003	0.0990
	LCL	1.0809	0.2173	0.2124
	UCL	1.7205	0.2906	1.2128
	AL	0.6396	0.0732	1.0003
1000	AE	1.2234	0.2485	0.5482
	AB	0.0234	0.0014	0.0482
	MSE	0.0112	0.0001	0.0499
	LCL	1.0876	0.2229	0.2304
	UCL	1.5196	0.2730	1.1347
	AL	0.4319	0.0501	0.9043

Table 7. The AEs, ABs, and MSEs of the HTNH model using model using Case3.

Simple size	Est.	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
300	AE	1.5444	0.1998	0.8704
	AB	0.1444	0.0001	0.1204
	MSE	0.2738	0.0004	0.2116
	LCL	1.1726	0.1767	0.3458
	UCL	2.9200	0.2257	2.0828
	AL	1.7474	0.0490	1.7360
500	AE	1.48077	0.2042	0.7789
	AB	0.0807	0.0042	0.0289
	MSE	0.1231	0.0003	0.1334
	LCL	1.1843	0.18292	0.3564
	UCL	2.1959	0.2296	1.5652
	AL	1.0115	0.0466	1.2088
700	AE	1.4606	0.2006	0.8202
	AB	0.0606	0.0006	0.0702
	MSE	0.0493	0.0002	0.0981
	LCL	1.2300	0.1848	0.4369
	UCL	1.9319	0.2190	1.4832
	AL	0.7019	0.0342	1.0462
1000	AE	1.4503	0.2002	0.8179
	AB	0.0503	0.0002	0.0679
	MSE	0.0332	0.0001	0.0744
	LCL	1.2239	0.1861	0.4696
	UCL	1.8541	0.2163	1.4488
	AL	0.6308	0.0302	0.9792

 doi:10.20944/preprints202404.1204.v1

Table 8. The AEs, ABs, and MSEs of the HTNH model using Case4.

Simple size	Est.	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
300	AE	1.7282	0.2018	1.1295
	AB	0.1282	0.0018	0.1295
	MSE	0.2834	0.0006	0.2045
	LCL	1.2100	0.1708	0.3726
	UCL	3.2956	0.2301	2.0673
	AL	2.0856	0.0593	1.6947
500	AE	1.7273	0.2003	1.1062
	AB	0.1273	0.0003	0.1062
	MSE	0.2051	0.0004	0.1678
	LCL	1.2528	0.1829	0.5263
	UCL	2.7325	0.2278	2.0077
	AL	1.4797	0.0449	1.4814
700	AE	1.6982	0.1999	1.0477
	AB	0.0982	0.0001	0.0477
	MSE	0.2316	0.0002	0.1696
	LCL	1.2841	0.1843	0.4820
	UCL	2.7068	0.2185	1.9236
	AL	1.4227	0.0342	1.4615
1000	AE	1.6350	0.1991	1.0195
	AB	0.0350	0.0008	0.0195
	MSE	0.0829	0.0001	0.0921
	LCL	1.3273	0.18282	0.6125
	UCL	2.4934	0.2126	1.7508
	AL	1.1661	0.0297	1.1383

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7. Dataset Analysis

This segment focuses on appraising the effectiveness of the proposed HTNH distribution using three real datasets.

7.1. First dataset

This considered dataset defined the remission times of bladder cancer patients, and it is previously studied by Abouelmagd et al. [1] and Cordeiro et al. [12]. The values of dataset are reported as follows:

Table 9. The remission times of bladder cancer patients.

17.36	17.14	17.12	16.62	15.96	14.83	14.77	14.76	14.24	13.80	13.29	13.11	12.63
12.07	12.03	12.02	11.98	11.79	11.64	11.25	10.75	10.66	10.34	10.06	9.74	9.47
9.22	9.02	8.66	8.65	8.53	8.37	8.26	7.93	7.87	7.66	7.63	7.62	7.59
7.39	7.32	7.28	7.26	7.09	6.97	6.94	6.93	6.76	6.54	6.25	5.85	5.71
5.62	5.49	5.41	5.41	5.34	5.32	5.32	5.17	5.09	5.06	4.98	4.87	4.51
4.50	3.02	4.40	4.34	4.33	4.26	4.23	4.18	3.88	3.82	3.70	3.64	3.57
3.52	3.48	3.36	3.36	3.31	3.25	2.87	2.83	2.75	2.69	2.69	2.64	2.62
2.54	2.46	2.26	2.23	2.09	2.07	2.02	2.02	1.76	1.46	1.40	1.35	1.26
1.19	1.05	0.90	0.81	0.51	0.50	0.40	0.20	0.08				

7.2. Second dataset

This considered dataset defined the Kevlar 373/epoxywas subjected to a continuous 90% stress level until all fatigue fractures failed, and its available in Andrews and Herzberg [7]. The values of proposed dataset were given as

Table 10. The values of second dataset.

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566	0.6748	0.6751
0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.9120	0.9836	1.0483	1.0596
1.0773	1.1733	1.2570	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595	1.4880	1.5728	1.5733
1.7083	1.7263	1.7460	1.7630	1.7746	1.8475	1.8375	1.8503	1.8808	1.8878	1.8881	1.9316
1.9558	2.0048	2.0408	2.0903	2.1093	2.1330	2.2100	2.2460	2.2878	2.3203	2.3470	2.3513
2.4951	2.5260	2.9911	3.0256	3.2678	3.4045	3.4846	3.7433	3.7455	3.9143	4.8073	5.4005
5.4435	5.5295										

7.3. Third Dataset

This considered dataset defined the infant mortality rate per 1000 live births for a few chosen nations in 2021, as reported by a https://data.worldbank.org/indicator/SP.DYN.IMRT.IN, and it is studied by Chinedu rt al. [11]. The values of this dataset were written as

Table 11. The values of third dataset.

56	10	22	3	69	6	7	11	4
4	19	13	7	27	12	3	4	11
84	27	25	6	35	14	11	12	6

Table (12) represents the description statistics of suggested three datasets, and the kernel density, TTT, and box plots for the observed datasets are drawn, respectively, in Figure (8).

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Table 12. Summary statistic measures for the three considered datasets.

Dataset	Q_1	Median	Mean	Q_3	ID	$\mathcal{SK}$	$\mathcal{KR}$
1	2.870	5.340	6.408	8.660	3.012	0.738	-0.312
2	0.891	1.717	1.801	2.237	0.851	1.196	1.3517
3	6.000	11.000	18.810	23.500	22.355	1.846	2.610

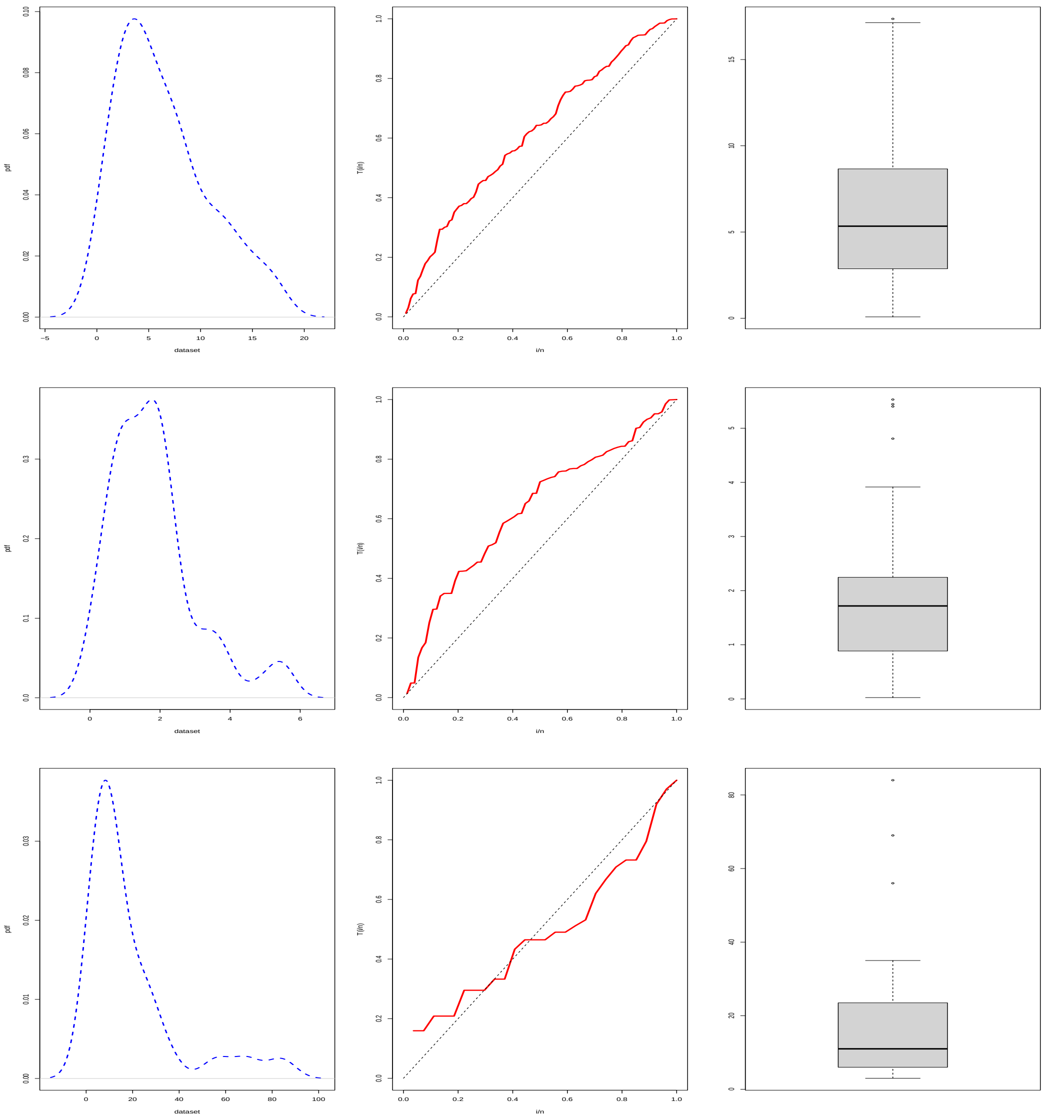

Figure 8. Kernel density, TTT and box plots of the proposed three datasets.

Furthermore, to enable comparisons, the HTNH model was fitted against several alternative models. These models include Burr power inverse Nadarajah Haghighi (PINH), Nadarajah Haghighi(NH), Heavy tailed exponential (HTE), Sin Nadarajah Haghighi (SNH), and alpha power transformed exponential (APTE) models.
Table (13) displayed the results of the estimated parameter values with corresponding log likelihood function ($\mathcal{LL}$). Now, in the evaluation process, specific metrics are employed to ascertain the most robust model among the contenders. This involves utilizing well-established criteria, notably Akaike Information Criterion ($\mathcal{A}$), Bayesian Information Criterion ($\mathcal{B}$), and Kolmogorov-Smirnov ($\mathcal{KS}$) statis-

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tics with its associated $\mathcal{P}$-values. Among the competing models, the one exhibiting the lowest values for these indicators along with the highest p-value is regarded as the most suitable choice, As a result, according to the values of Table (13) we can be concluded that the HTNH model is more appropriate for modelling the three considered datasets. Figures (9), (10), and (11) discuss the fitted density and cdf curves of fitting proposed distributions. All these Figures demonstrate that our HTNH model fits the considered three datasets well.

Table 13. Parameter estimations with various statistic comparison measures for the two considered datasets.

Dataset	Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\mathcal{KS}$	$\mathcal{P}$-value	$\mathcal{LL}$	$\mathcal{A}$	$\mathcal{B}$
I	HTNH	32.327	17.255	0.0056	0.0743	0.5596	-312.774	631.549	639.732
	PINH	0.2289	218.607	2.1747	0.1273	0.0512	-328.966	663.932	672.114
	NH		2.7633	0.0418	0.0983	0.2239	-318.406	640.825	646.280
	HTE		53.497	0.1540	0.1472	0.0149	-323.134	650.268	650.377
	SNH		1.1934	0.0702	0.1212	0.0723	-318.628	641.272	646.727
	APTE		0.1955	2.4602	0.1100	0.1297	-319.784	642.918	648.372
II	HTNH	37.658	9.2309	0.0379	0.1081	0.3289	-111.694	229.388	236.300
	PINH	0.1176	235.45	4.0355	0.1544	0.0524	-116.978	239.956	246.868
	NH		2.3901	0.1771	0.1381	0.1077	-114.485	232.970	237.578
	HTE		50.236	0.5470	0.1844	0.0112	-117.712	239.424	244.032
	SNH		1.5596	0.1815	0.1493	0.0661	-115.357	234.714	239.322
	APTE		0.7621	3.6160	0.1302	0.1488	-114.203	232.406	237.014
III	HTNH	13.359	0.9402	0.0562	0.1561	0.5257	-106.209	218.418	222.306
	PINH	51.976	0.0899	0.9757	0.1605	0.4897	-106.214	218.428	222.315
	NH		0.8113	0.0750	0.1640	0.4620	-111.238	226.476	229.067
	HTE		1.3786	0.0223	0.2013	0.2235	-113.421	230.842	233.433
	SNH		0.6403	0.0577	0.1598	0.4853	-108.696	221.392	223.983
	APTE		0.0297	0.1434	0.1784	0.3565	-112.401	228.802	231.393

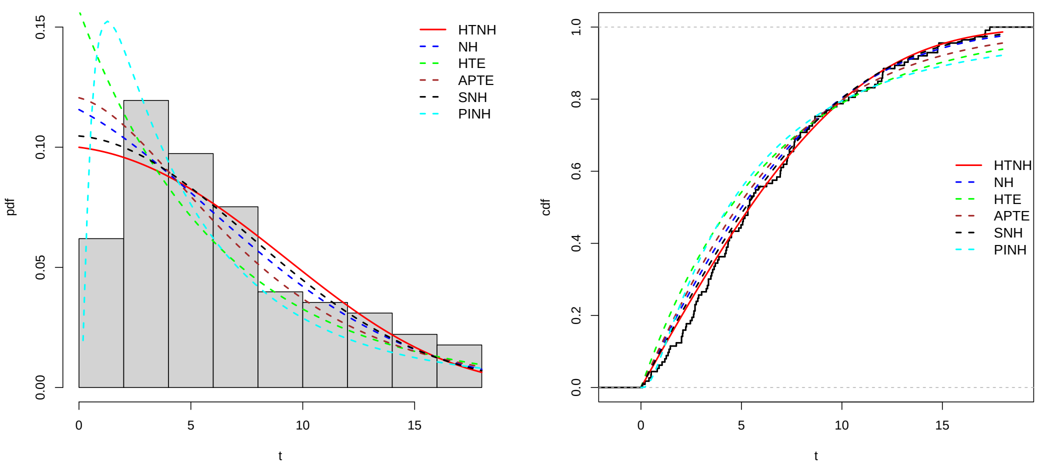

Figure 9. Estimation plots of pdf and cdf of the fitting distributions using first dataset.

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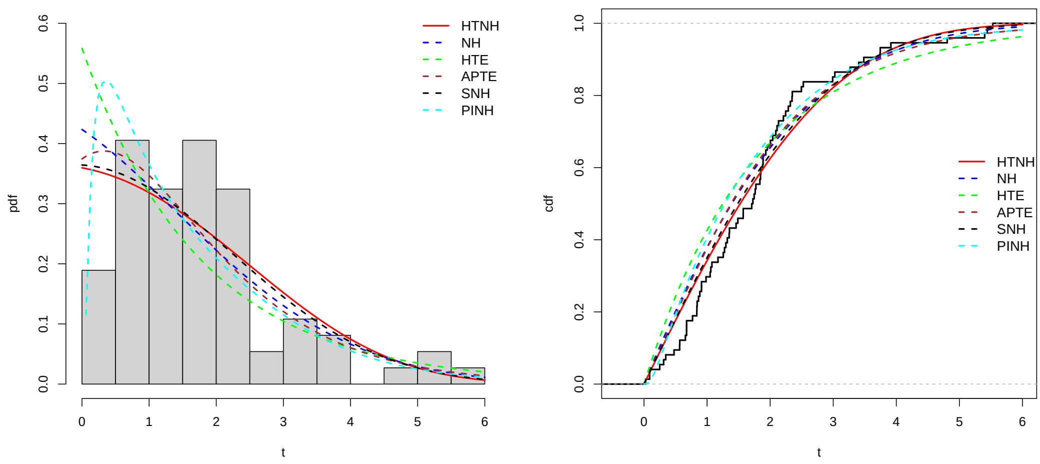

Figure 10. Estimation plots of pdf and cdf of the fitting distributions using second dataset.

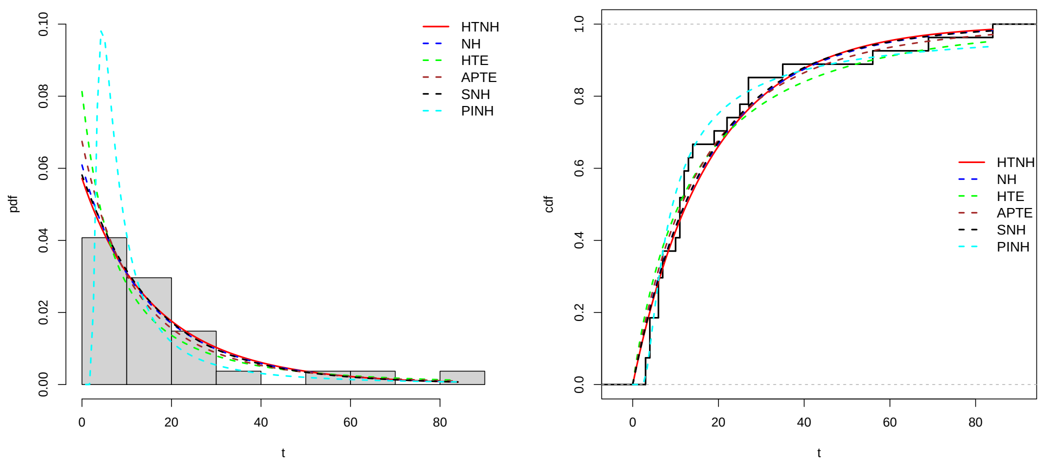

Figure 11. Estimation plots of pdf and cdf of the fitting distributions using third dataset.

8. Concluding Remarks

This research delves into a novel generating employing the heavy-tailed function and then applied it to the Nadarajah Haghighi distribution. The new and adaptable model named the Heavy Tailed Nadarajah Haghighi model, designated as HTNH. The proposed technique is very efficient and having attractive characterizations. The focus lies in establishing several mathematical properties associated with this model. The estimation of model parameters is facilitated through the employment of the MLE procedure, and for simulation analysis, we conducted some simulation experiment studies to demonstrate the performance of the proposed MLE method. Finally, three real dataset are performed for checking the applicability our proposed model. This comprehensive exploration signifies that the HTNH model offers an improved fit for the three suggested datasets, as illustrated by the collected empirical results in the simulation part.

Acknowledgments: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R735), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

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