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Constructing Physics From Measurements

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Constructing Physics from Measurements

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Abstract: We present a reformulation of fundamental physics from an enumeration of independent axioms into the solution of a single optimization problem. Any experiment begins with an initial state preparation, involves some physical operation, and ends with a final measurement. Working from this structure, we maximize the entropy of a final measurement relative to its initial preparation subject to a measurement constraint. Solving this optimization problem for the *natural constraint* –the most permissive constraint compatible with said problem– identifies an optimal physical theory. Rather than existing as a collection of postulates, quantum mechanics, general relativity, and Yang-Mills emerge within a unified theory. Notably, mathematical consistency further restricts valid solutions to 3+1 dimensions only. This reformulation reveals that the apparent complexity of modern physics, with its various forces, symmetries, and dimensional constraints, emerges as the solution to an optimization problem constructed over all experiments realizable within the constraint of nature.

Keywords: foundations of physics

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1. Introduction

Statistical mechanics (SM), in the formulation developed by E.T. Jaynes [1,2], is founded on an entropy optimization principle. Specifically, the Boltzmann entropy is maximized under the constraint of a fixed average energy $\overline{E}$:

$$\overline{E} = \sum_i \rho_i E_i \tag{1}$$

The Lagrange multiplier equation defining the optimization problem is:

$$\mathcal{L} = -k_B \sum_i \rho_i \ln \rho_i + \lambda \left(1 - \sum_i \rho_i\right) + \beta \left(\overline{E} - \sum_i \rho_i E_i\right), \tag{2}$$

where λ and β are Lagrange multipliers enforcing the normalization and average energy constraints. Solving this optimization problem yields the Gibbs measure:

$$\rho_i = \frac{1}{Z} \exp(-\beta E_i), \tag{3}$$

where $Z = \sum_i \exp(-\beta E_i)$ is the partition function.

For comparison, quantum mechanics (QM) is not formulated as the solution to an optimization problem, but rather consists of a collection of axioms[3,4]:

QM Axiom 1 of 5 — **State Space:** Every physical system is associated with a complex Hilbert space, and its state is represented by a ray (an equivalence class of vectors differing by a non-zero scalar multiple) in this space.

QM Axiom 2 of 5 — **Observables:** Physical observables correspond to Hermitian (self-adjoint) operators acting on the Hilbert space.

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QM Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is governed by the Schrödinger equation, where the Hamiltonian operator represents the system's total energy.

QM Axiom 4 of 5 **Measurement:** Measuring an observable projects the system into an eigenstate of the corresponding operator, yielding one of its eigenvalues as the measurement result.

QM Axiom 5 of 5 **Probability Interpretation:** The probability of obtaining a specific measurement outcome is given by the squared magnitude of the projection of the state vector onto the relevant eigenstate (Born rule).

Physical theories have traditionally been constructed in two distinct ways. Some, like QM, are defined through a set of mathematical axioms that are first postulated and then verified against experiments. Others, like SM, emerge as solutions to optimization problems with experimentally-verified constraints.

We propose to generalize the optimization methodology of E.T. Jaynes to encompass all of physics, aiming to derive a unified theory from a single optimization problem.

To that end, we introduce the following constraint:

Axiom 1 (Nature)**.**

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i$$

where $\mathbf{M}_i$ are $n \times n$ matrices, and $\overline{\mathbf{M}}$ is their average.

This constraint, as it replaces the scalar E_i with the matrix $\mathbf{M}_i$, extends E.T. Jaynes' optimization method to encompass non-commutative observables and symmetry group generators required for fundamental physics.

We then construct an optimization problem:

Definition 1 (Physics)**.** *Physics is the solution to:*

$$\underbrace{\mathcal{L}}_{\substack{\text{an}\\\text{optimization}\\\text{problem}}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{on the entropy}\\\text{of a measurement}\\\text{relative to its preparation}\\\text{over all}}} + \underbrace{\lambda\left(1-\sum_i \rho_i\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr}\left(\overline{\mathbf{M}}-\sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}}$$

where λ and τ are Lagrange multipliers enforcing the normalization and natural constraints, respectively.

This single definition constitutes our complete proposal for reformulating fundamental physics—no additional principles will be introduced. By replacing the Boltzmann entropy with the relative Shannon entropy, the optimization problem extends beyond thermodynamic variables to encompass any type of experiment. This generalization occurs because relative entropy captures the essence of any experiment: the relationship between a final measurement state and its initial preparation state.

Two key constraints shape our framework. The normalization constraint ensures we are working with a proper predictive theory, while the natural constraint spawns the domain of applicability of the theory. Together, they capture the complete evolution—from initial to final states—that defines any experiment. The crucial insight is that because our formulation maintains complete generality in the structure of experiments while optimizing over all possible predictive theories, the resulting solution holds true, by construction, for all realizable experiments within its domain.

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The solution provides a complete structure that automatically satisfy the requirements of a physical theory valid for the constraint: mathematical rigour, internal consistency, optimal predictive power, and automatic validity with all realizable experiments in its domain. This approach reduces our reliance on postulating axioms through trial and error, and simplifies the foundations of physics. Specifically, when we employ the *natural constraint* (Axiom 1) –the most permissive constraint for this problem (see Discussion for proof)–, the solution spawns its largest domain, pointing towards a unified physics where fundamental theories emerge naturally—e.g. SM when $\mathbf{M} \cong \mathbb{R}$, QM when $\mathbf{M} \cong \mathfrak{u}(1)$, and general relativity (on spacetime) + Yang-Mills (in its internal spaces) when $\mathbf{M} \cong \mathbb{R} \oplus \mathfrak{spinc}(3,1)$. These three solutions are the only possible ones, as those entailed by other algebras encounter obstructions such as violating the axioms of probability theory.

Theorem 1. *The general solution of the optimization problem is:*

$$\rho_i = \frac{p_i \det \exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det \exp(-\tau \mathbf{M}_j)}$$

Proof. We solve the maximization problem by setting the derivative of the Lagrangian with respect to ρ_i to zero:

$$\frac{\partial \mathcal{L}}{\partial \rho_i} = -\ln \frac{\rho_i}{p_i} - 1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i = 0. \tag{4}$$

$$\implies \ln \frac{\rho_i}{p_i} = -1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i. \tag{5}$$

$$\implies \rho_i = p_i \exp(-1 - \lambda) \exp(-\tau \operatorname{tr} \mathbf{M}_i). \tag{6}$$

Normalizing the probabilities using $\sum_i \rho_i = 1$, we find:

$$1 = \sum_i \rho_i = \exp(-1 - \lambda) \sum_i p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i), \tag{7}$$

$$\implies \exp(1 + \lambda) = \sum_j p_j \exp\big(-\tau \operatorname{tr} \mathbf{M}_j\big). \tag{8}$$

Substituting back, we obtain:

$$\rho_i = \frac{p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i)}{\sum_j p_j \exp\big(-\tau \operatorname{tr} \mathbf{M}_j\big)}. \tag{9}$$

Finally, using the identity $\det \exp(\mathbf{M}) = \exp \operatorname{tr} \mathbf{M}$ for square matrices $\mathbf{M}$, we get:

$$\rho_i = \frac{1}{Z} p_i \det \exp(-\tau \mathbf{M}_i). \tag{10}$$

where $Z = \sum_j p_j \det \exp\big(-\tau \mathbf{M}_j\big)$. □

As we will see in the results section, this solution encapsulates three distinct special cases:

1. **Statistical Mechanics:**
   To recover SM from Equation 10, we consider the case where the matrices $\mathbf{M}_i$ are 1×1, i.e., real scalars. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = E_i, \tag{11}$$

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and take p_i to be a uniform distribution. Then, Equation 10 reduces to the Gibbs distribution:

$$\rho_i = \frac{1}{Z}\exp(-\tau E_i), \tag{12}$$

where τ corresponds to the β of SM. This demonstrates that our solution generalizes SM, as it recovers it when $\mathbf{M}_i$ are scalars.

2. **Quantum Mechanics:**
By choosing $\mathbf{M}_i$ to generate the U(1) group, we derive the axioms of QM from entropy maximization. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}, \tag{13}$$

where E_i are energy levels. In the results section, we will detail how this choice leads to the the Born rule in lieu of the Gibbs measure, and that the partition function is unitary invariant—the solution is shown to uniquely satisfy all five axioms of QM.

3. **Fundamental Physics:**
Extending our approach, we choose $\mathbf{M}_i$ to be 4×4 matrices representing the $\mathbb{R} \oplus \mathfrak{spinc}(3,1)$ algebra. Specifically, we consider multivectors of the form $\mathbf{u} = a + \mathbf{f} + \mathbf{b}$, where a is a scalar, where $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudoscalar of the 3+1D geometric algebra $\text{GA}(3,1)$. This constitute its even sub-algebra. The matrix representation of $\mathbf{M}_i$ is:

$$\mathbf{M}_i = \begin{bmatrix} a + f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & a + f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & a - f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & a - f_{02} \end{bmatrix}, \tag{14}$$

where $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the $\text{Spinc}(3,1)$ group, which includes both Lorentz boosts/rotations and the four-volume orientation, and where a is the generator of the group $\mathbb{R}^+$. Solving the optimization problem with this choice leads to a relativistic quantum probability measure extending the Born rule from $\mathbb{C}$ to $\mathbb{R}^+ \times \text{Spinc}(3,1)$. The solution is shown to uniquely satisfy both general relativity (on spacetime) and Yang-Mills (within its internal spaces).

4. **Dimensional Obstructions:**
Definition 1 yields valid probability measures only in specific cases of Axiom 1. Beyond the instances of statistical mechanics and quantum mechanics, Axiom 1 produces a consistent solution only in 3+1 dimensions. In other dimensional configurations, various obstructions arises violating the axioms of probability theory. The following table summarizes the geometric cases and their obstructions:

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Dimensions	*Optimal Predictive Theory of Nature*	
$\mathrm{GA}(0)$	**Statistical Mechanics**	(15)
$\mathrm{GA}(0,1)$	**Quantum Mechanics**	(16)
$\mathrm{GA}(1,0)$	Obstructed (Negative probabilities)	(17)
$\mathrm{GA}(2,0)$	**Quantum Mechanics**	(18)
$\mathrm{GA}(1,1)$	Obstructed (Negative probabilities)	(19)
$\mathrm{GA}(0,2)$	Obstructed (Non-real probabilities)	(20)
$\mathrm{GA}(3,0)$	Obstructed (Non-real probabilities)	(21)
$\mathrm{GA}(2,1)$	Obstructed (Non-real probabilities)	(22)
$\mathrm{GA}(1,2)$	Obstructed (Non-real probabilities)	(23)
$\mathrm{GA}(0,3)$	Obstructed (Non-real probabilities)	(24)
$\mathrm{GA}(4,0)$	Obstructed (Non-real probabilities)	(25)
$\mathrm{GA}(3,1)$	**Gravity + Yang-Mills**	(26)
$\mathrm{GA}(2,2)$	Obstructed (Negative probabilities)	(27)
$\mathrm{GA}(1,3)$	Obstructed (Non-real probabilities)	(28)
$\mathrm{GA}(0,4)$	Obstructed (Non-real probabilities)	(29)
$\mathrm{GA}(5,0)$	Obstructed (Non-real probabilities)	(30)
$\vdots$	$\vdots$	
$\mathrm{GA}(6,0)$	Suspected Obstructed (No observables)	(31)
$\vdots$	$\vdots$	
		(32)

where $\mathrm{GA}(p,q)$ means the geometric algebra of $p+q$ dimensions, where p is the number of positive signature dimensions and q of negative signature dimensions. QM shows up twice because both $\mathrm{GA}(0,1)$ and the even-subalgebra of $\mathrm{GA}(2,0)$ are isomorphic to $\mathbb{C}$.
We will first investigate the unobstructed cases in Section 2.1, 2.2 and 2.3 and then demonstrate the obstructions in Section 2.4. These obstructions are desirable because they automatically limit the theory to 3+1D, thus providing a built-in mechanism for the observed dimensionality of our universe.

2. Results

2.1. $\mathfrak{u}(1)$-Constraint: Quantum Mechanics

In SM, the central observation is that energy measurements of a thermally equilibrated system tend to cluster around a fixed average value (Equation 1). In contrast, QM is characterized by the presence of interference effects in measurement outcomes. To capture these features, we introduce the following special case of Axiom 1:

Definition 2 ($\mathfrak{u}(1)$ constraint)**.** *We reduce the generality of Axiom 1 to the generator of the* $\mathrm{U}(1)$ *group. Specifically, we replace*

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \quad \textit{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \tag{33}$$

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where E_i are scalar values (e.g., energy levels), ρ_i are the probabilities of outcomes, and the matrices $\mathbf{M}_i$ generate the $\mathrm{U}(1)$ *group.*

The general solution of the optimization problem (Theorem 1) reduces as follows

$$\rho_i = \frac{1}{\sum_i p_i \det\exp\left(-\tau\begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right)} \det\exp\left(-\tau\begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) p_i \tag{34}$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of QM, as we will demonstrate.

To align our results with conventional QM notation, we translate the matrices to complex numbers. Specifically, we consider that:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + ib. \tag{35}$$

Then, we note the following equivalence with the complex norm:

$$\det\exp\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right] = r^2 \det\left[\begin{smallmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{smallmatrix}\right], \text{ where } r = \exp a \tag{36}$$

$$= r^2(\cos^2(b) + \sin^2(b)) \tag{37}$$

$$= \|r(\cos(b) + i\sin(b))\| \tag{38}$$

$$= \|r\exp(ib)\| \tag{39}$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial prepration into :

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \|\exp(-itE_i/\hbar)\|}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE_i/\hbar)\|}_{\text{Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{40}$$

This equation describes what occurs at the instant of measurement. Essentially, its states that unitary dynamics are erased under measurement, yielding a classical probability measure.

Since the equation still contains the terms that are being erased by the measurement (i.e. the unitary dynamics), we can recover the definition of the wavefunction merely by inspection. Specifically, we decompose the complex norm into a complex number and its conjugate. The erased structure is then identified as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_i p_i \|\exp(-itE_i/\hbar)\| = Z = \langle\psi|\psi\rangle \tag{41}$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE_1/\hbar) & & \\ & \ddots & \\ & & \exp(-itE_n/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \tag{42}$$

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We clarify that p_i represents the probability associated with the initial preparation of the wavefunction, where $p_i = \langle\psi_i(0)|\psi_i(0)\rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of QM are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle\psi|\psi\rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates QM Axiom 1 of 5.
- In Z, an observable must satisfy:

$$\overline{O} = \sum_i p_i O_i \| \exp(-itE_i/\hbar) \| \tag{43}$$

  Since $Z = \langle\psi|\psi\rangle$, then any self-adjoint operator satisfying the condition $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$ will equate the above equation, simply because $\langle\mathbf{O}\rangle = \langle\psi|\mathbf{O}|\psi\rangle$. This demonstrates QM Axiom 2 of 5.
- Upon transforming Equation 42 out of its eigenbasis through unitary operations, we find that the energy, E_i, typically transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \tag{44}$$

  The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle) \tag{45}$$
$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \tag{46}$$
$$= -i\mathbf{H}/\hbar|\psi(t)\rangle \tag{47}$$
$$\implies \mathbf{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \tag{48}$$

  which is the Schrödinger equation. This demonstrates QM Axiom 3 of 5.
- From Equation 42 it follows that the possible microstates E_i of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from ρ, with the measured state being the occupied microstate i. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

$$\rho_i(t) = \frac{1}{\langle\psi|\psi\rangle}(\psi_i(t))^\dagger\psi_i(t). \tag{49}$$

  In scenarios where the probability measure $\rho_i(\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{50}$$

  Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observable, this demonstrates QM Axiom 4 of 5.
- Finally, since the probability measure (Equation 40) replicates the Born rule, QM Axiom 5 of 5 is also demonstrated.

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Revisiting QM with this perspective offers a coherent and unified narrative. Specifically, the $U(1)$ generating constraint is sufficient to entail the foundations of QM through the principle of entropy maximization—in this formulation, QM Axioms 1, 2, 3, 4, and 5 are not fundamental, but the solution to an optimization problem.

2.2. $\mathbb{R} \oplus \mathfrak{spin}(2)$-*Constraint: Euclidean QM in 2D*

In this section, we investigate a model, isomorphic to QM, that lives in 2D—it provides a valuable starting point before addressing the more complex 3+1D case. Before we solve the optimization problem, we will introduce tools to express a bilinear form over $\mathbf{M}$ using the multivectors of $GA(2,0)$ in the following section.

2.2.1. Bilinear Form

In general a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of $GA(2,0)$, where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ a pseudo-scalar, is represented as follows:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y \tag{51}$$

The basis elements are defined as:

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_x \wedge \sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{52}$$

We now introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 3 (Multivector Conjugate)**.** *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* $GA(2,0)$*. The multivector conjugate is defined as:*

$$\mathbf{u}^\ddagger = a - \mathbf{x} - \mathbf{b} \tag{53}$$

The determinant of the matrix representation of a multivector can be expressed as a multivector self-product:

Theorem 2 (Multivector Determinant in 2D)**.**

$$\mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M} \tag{54}$$

Proof. Let $\mathbf{u} = a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y$, and let $\mathbf{M}$ be its matrix representation $\left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right]$. Then:

$$\begin{aligned}
1: \quad & \mathbf{u}^\ddagger \mathbf{u} && (55)\\
& = (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y)^\ddagger (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (56)\\
& = (a - x\sigma_x - y\sigma_y - b\sigma_x \wedge \sigma_y)(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (57)\\
& = a^2 - x^2 - y^2 + b^2 && (58)\\
2: \quad & \det \mathbf{M} && (59)\\
& = \det \left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right] && (60)\\
& = (a+x)(a-x) - (y-b)(y+b) && (61)\\
& = a^2 - x^2 - y^2 + b^2 && (62)
\end{aligned}$$

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□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 4 (Multivector Conjugate Transpose)**.** *Let* $|V\rangle\rangle \in (\text{GA}(2,0))^n$*:*

$$|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{63}$$

The multivector conjugate transpose of $|V\rangle\rangle$ *is defined as first taking the transpose and then the element-wise multivector conjugate:*

$$\langle\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{64}$$

Definition 5 (Bilinear Form)**.** *Let* $|V\rangle\rangle$ *and* $|W\rangle\rangle$ *be two vectors valued in* $\text{GA}(2,0)$*:*

$$|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \qquad |W\rangle\rangle = \begin{bmatrix} a'_1 + \mathbf{x}'_1 + \mathbf{b}'_1 \\ \vdots \\ a'_n + \mathbf{x}'_n + \mathbf{b}'_n \end{bmatrix} \tag{65}$$

We introduce the following bilinear form:

$$\langle\langle V|W\rangle\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a'_1 + \mathbf{x}'_1 + \mathbf{b}'_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a'_n + \mathbf{x}'_n + \mathbf{b}'_n) \tag{66}$$

Theorem 3 (Inner Product)**.** *Restricted to the even sub-algebra of* $\text{GA}(2,0)$*, the bilinear form is an inner product.*

Proof.

$$\langle\langle V|W\rangle\rangle_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{67}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \sigma_x \wedge \sigma_y$—we note that $(\sigma_x \wedge \sigma_y)^2 = -1$. □

2.2.2. 1+1D Obstruction

The reader may wonder why we are using 2D instead of the more physically relevant 1+1D for the lower dimensional example. As stated in the introduction the 1+1D is obstructed. Specifically, the 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\mathbf{e}_0 \wedge \mathbf{e}_1)(a + b\mathbf{e}_0 \wedge \mathbf{e}_1)$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. As such, neither 1+1D nor any of its sub-algebras satisfy all axioms of probability theory, hence it is obstructed.

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2.2.3. $\mathfrak{spin}(2)$-Constraint: $\cong$ Quantum Mechanics

Let us first investigate the $\mathfrak{spin}(2)$-constraint, then we will investigate the more general $\mathbb{R} \oplus \mathfrak{spin}(2)$-constraint. This constraint is recovered by posing $a \to 0$ and $\mathbf{x} \to 0$ then $\mathbf{M}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a\to 0, \mathbf{x}\to 0} = \mathbf{b} \implies \mathbf{M} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \tag{68}$$

The fundamental Lagrange Multiplier Equation:

$$\mathcal{L} = -\sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i\right) + \frac{1}{2}\theta \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right) \tag{69}$$

where

1. λ and θ are the Lagrange multipliers
2. $\mathbf{M}_i$ is the 2×2 matrix representation of the multivectors of $\mathrm{GA}(2,0)$, reduced by $a \to 0$ and $\mathbf{x} \to 0$
3. the factor $(1/2)$ is there to regularize the adjoint action on a vector $e^{-(1/2)\mathbf{b}_i}\mathbf{v}e^{(1/2)\mathbf{b}_i} = \mathbf{v}'$

It yields the following solution:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det\exp\left(-\frac{1}{2}\theta\begin{bmatrix} 0 & -b_i \\ b_i & 0\end{bmatrix}\right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det\exp\left(-\frac{1}{2}\theta\begin{bmatrix} 0 & -b_i \\ b_i & 0\end{bmatrix}\right)}_{\text{Spin(2) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{70}$$

As with the $\mathfrak{u}(1)$-constraint case, this equation describes what the Born rule erases from the solution space at the instant of measurement, to yield a classical probability measure. By inspection, we find that the Spin(2) dynamics operating on a Spin(2)-valued wavefunction are the erased structures:

Definition 6 (Spin(2)-valued Wavefunction)**.**

$$\begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \begin{bmatrix} \exp\left(\frac{1}{2}\theta\mathbf{b}_1\right) & & \\ & \ddots & \\ & & \exp\left(\frac{1}{2}\theta\mathbf{b}_n\right) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \tag{71}$$

The dynamics are described by a variant of the Schrödinger equation, which is derived by taking the derivative of the wavefunction with respect to the Lagrange multiplier, θ:

Definition 7 (Spin(2)-valued Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{b}_1 & & \\ & \ddots & \\ & & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} \tag{72}$$

where

1. *θ represents a global one-parameter evolution parameter akin to time*
2. *$\mathbf{b}_i$ is the generator of* $\mathrm{Spin}(2)$ *transformations.*

Since $\mathrm{Spin}(2) \cong \mathrm{U}(1)$, then it should come to no surprise that the theory resulting from the $\mathfrak{spin}(2)$-constraint is of the same mathematical form as QM, obtained from the $\mathfrak{u}(1)$-constraint.

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2.2.4. $\mathbb{R} \oplus \mathfrak{spin}(2)$-Constraint: Euclidean QM in 2D

Now, we solve the optimization problem for the curvilinear case where $\frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu) = g_{\mu\nu}$. We will also move to the continuum $\sum \to \int$. The optimization problem is described as follows:

$$\mathcal{L} = -\int_L \rho(\theta,r) \ln \rho(\theta,r) \sqrt{h_\theta} \mathrm{d}r + \lambda \left(1 - \int_L \rho(\theta,r) \sqrt{h_\theta} \mathrm{d}r\right) + \frac{1}{2}\theta \left(\overline{\mathbf{M}} - \int_L \rho(\theta,r) \mathbf{M}(r) \sqrt{h_\theta} \mathrm{d}r\right) \quad (73)$$

where

1. λ and θ are the Lagrange multipliers
2. where $\sqrt{h_\theta}$ describes the metric of the 2D space foliated in slices of constant θ. This foliation results in radial lines from the origin to infinity with angle θ. A less sophisticated way of saying this is that we use polar coordinates in curved space (specifically, for flat space $\sqrt{h_\theta} = r$).
3. where L is the integration length for the slices.

Probability, normalized for a given initial slice, will be conserved as the system evolves along θ. In GA$(2,0)$, the algebra $\mathbb{R} \oplus \mathfrak{spin}(2)$ corresponds to its even sub-algebra:

$$\mathbf{M} \cong a + b\mathbf{e}_1\mathbf{e}_2 = a + bI = a + \mathbf{b} \quad (74)$$

In this representation, a generates dilation (corresponding to translations on the θ-foliated slices) and $\mathbf{b}$ generates rotations (corresponding to circular motion on the θ-foliated slices). However, this expression is only valid in flat spacetime. In curved spacetime, the action of this algebra generalizes to the covariant derivative:

$$\mathbf{M} \cong \sigma_1 \mathbf{e}^\mu \left(\partial_\mu + \frac{1}{2}\omega_\mu^{12} \mathbf{e}_1 \mathbf{e}_2\right) = \sigma_1 \mathbf{e}^\mu D_\mu \quad (75)$$

which includes ∂_μ, the partial derivative, and $\frac{1}{2}\omega_\mu^{12}\mathbf{e}_1\mathbf{e}_2$, the Spin$(2)$ connection—acting on the wavefunction to transform its components in curved spacetime. The term $\mathbf{e}^\mu$ is required to contract the indices of ∂_μ and ω_μ, producing an odd multivector—and the term σ_1 converts it back to an even multivector. The term σ_1 also selects a preferred frame—the laboratory frame. Its role is similar to γ_0 in the Dirac Lagrangian.

Solving the optimization problem for this value of $\mathbf{M}$, yields:

$$\rho(\theta,r) = \frac{1}{\int_L p(r) \det\exp\left(-\frac{1}{2}\theta\sigma_1\mathbf{e}^\mu D_\mu\right)\sqrt{h_\theta}\mathrm{d}r} \det\exp\left(-\frac{1}{2}\theta\sigma_1\mathbf{e}^\mu D_\mu\right) p(r) \quad (76)$$

Again as before, this probability identifies the structure from the solution space that measurements must erase to yield a classical probability measure. This structure is the wavefunction, given as follows:

$$\psi(\theta,r) = \exp\left(\frac{1}{2}\theta\sigma_1\mathbf{e}^\mu D_\mu\right)\psi(0,r) \quad (77)$$

Then, taking the derivative with respect to θ yields the Schrödinger equation:

$$\frac{\partial}{\partial\theta}\psi(\theta,r) = \frac{1}{2}\sigma_1\mathbf{e}^\mu D_\mu \psi(\theta,r) \quad (78)$$

Revealing that the Hamiltonian, curiously expressed as multivector–not a scalar (more on that in Theorem 4), is $H = \frac{1}{2}\sigma_1\mathbf{e}^\mu D_\mu$.

Definition 8 (David Hestenes' Formulation). *In 3+1D, the David Hestenes' formulation [5] of the wavefunction is $\psi = \sqrt{\rho}Re^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as*

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the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho}R$*, where* ρ *is a probability density and* R *is a rotor—this is the form we have recovered.*

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 9 (Dirac Current)**.** *The Dirac current for the 2D theory is defined as:*

$$J \equiv \psi^\ddagger \mathbf{e}_\mu \psi = \rho \underbrace{R^\ddagger \mathbf{e}_\mu R}_{\mathrm{SO}(2)} = \rho \mathbf{e}'_\mu \tag{79}$$

where $\mathbf{e}'_\mu$ *is a* $\mathrm{SO}(2)$*-rotated basis vector.*

We recall that in QM, the Hamiltonian relate to the Lagrangian as follows:

$$L = \operatorname{tr}\left(\frac{\psi^\ddagger H \psi}{\psi^\ddagger \psi}\right) \tag{80}$$

With that, we can prove the following:

Theorem 4 (Dirac equation)**.** *The equation of motion of the Schrödinger equation (Equation 78), is the Dirac equation.*

Proof. Since $H = \frac{1}{2}\sigma_1 \mathbf{e}^\mu D_\mu$ (Equation 78), we write:

$$\frac{\delta}{\delta\psi^\ddagger}\int_{\mathcal{M}} \operatorname{tr}\left(\frac{\psi^\ddagger \sigma_1 \mathbf{e}^\mu D_\mu \psi}{2\psi^\ddagger\psi}\right) \mathrm{d}\theta \mathrm{d}r = 0 \tag{81}$$

We can already recognize the numerator term $\psi^\ddagger \sigma_1 \mathbf{e}^\mu D_\mu \psi$ as the Dirac Lagrangian (in 2D). Nonetheless, let us show explicitly:

$$\begin{aligned}
&\Longrightarrow \frac{\delta}{\delta\psi^\ddagger}\operatorname{tr}\left(\frac{\psi^\ddagger \sigma_1 \mathbf{e}^\mu D_\mu \psi}{2\psi^\ddagger\psi}\right) = 0 && (82)\\
&\Longrightarrow \frac{\sigma_1 \mathbf{e}^\mu D_\mu \psi 2\psi^\ddagger \psi}{2\psi^\ddagger\psi}\delta\psi^\ddagger + \frac{\psi^\ddagger \sigma_1 \mathbf{e}^\mu D_\mu \psi 2\psi}{2\psi^\ddagger\psi}\delta\psi^\ddagger = 0 && (83)\\
&\Longrightarrow \sigma_1 \mathbf{e}^\mu D_\mu \psi \delta\psi^\ddagger + \sigma_1 \mathbf{e}^\mu D_\mu \psi \delta\psi^\ddagger = 0 && (84)\\
&\Longrightarrow \mathbf{e}^\mu D_\mu \psi = 0 && (85)
\end{aligned}$$

which is the Dirac equation in 2D. □

Concluding Remarks: One might be tempted to simply extend $\mathbf{e}^\mu D_\mu \psi = 0$ to 3+1D by adding the basis vectors $\mathbf{e}_0$ and $\mathbf{e}_3$, and call it a day. This approach would produce the conventional Dirac equation of quantum field theory in spacetime. However, when we solve the optimization problem directly in 3+1D, we discover that the Dirac equation represents only a subsolution of a more comprehensive probabilistic structure. This complete structure naturally incorporates both gravity (acting on spacetime) and Yang-Mills (acting on internal spaces). These results, presented in the following section, indicate that the conventional Dirac equation in 3+1D, in a sense, represents an extension of 2D quantum theory that captures only part of the full probabilistic structure available in 3+1D spacetime.

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2.3. $\mathbb{R} \oplus \mathfrak{spinc}(3,1)$-*Constraint: Gravity + Yang-Mills*

Extending the framework to relativistic quantum mechanics begins by considering a measurement constraint having the $\mathbb{R}^+ \times \text{Spinc}(3,1)$ symmetry. This allows for transformations that include dilations, boosts/rotations, and re-orientations (David Hestene describes "re-orientation" as representing the changing orientation of the spin plane due to Zitterbewegung).

Another necessary change regards the interpretation of ψ from a probability amplitude, to that of a field amplitude ϕ. As such, and consistently with usual quantum field theory (QFT) interpretation, the notion of charge conservation will replace that of probability conservation. The notation will be changed as follows:

$$\psi \to \phi \tag{86}$$

$$\sqrt{\rho} R e^{-ib/2} \to \sqrt{\chi} R e^{-ib/2} \tag{87}$$

$\mathbf{M}$ will represent the algebra of $\mathbb{R} \oplus \mathfrak{spinc}(3,1)$:

$$\mathbf{M} = \begin{bmatrix} a + f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & a + f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & a - f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & a - f_{02} \end{bmatrix} \tag{88}$$

Using $\text{GA}(3,1)$ notation, $\mathbf{M}$ can be equivalently represented as:

$$\mathbf{M} \cong a + \mathbf{f} + \mathbf{b} \tag{89}$$

where a is a scalar, $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudo-scalar.

As we did for the $\mathbb{R} \oplus \mathfrak{spin}(2)$-constraint, we will develop the framework for the continuum case $\sum \to \int$ which can be obtained by solving the following Lagrange multiplier equation:

$$\mathcal{L} = -\int_V \chi(x,\zeta) \ln \frac{\chi(x,\zeta)}{p(x)} \sqrt{h_\zeta} \mathrm{d}^3 x + \frac{1}{2} \zeta \operatorname{tr}\left(\overline{\mathbf{M}} - \int_V \chi(x,\zeta) \mathbf{M}(x) \sqrt{h_\zeta} \mathrm{d}^3 x \right) \tag{90}$$

where

1. ζ is the twisted-rapidity acting on $\mathbb{R} \oplus \mathfrak{spinc}(3,1)$ to form a one-parameter group via the exponential map.
2. h_ζ is the determinant of the induced spatial metric on the hypersurface of constant twisted-rapidity ζ. Is is a general curvature version of Rindler's coordinates.
3. V represents the causally accessible region
4. the normalization constraint $\lambda(1 - \int_V \chi(x,\zeta) \sqrt{h_\zeta} \mathrm{d}^3 x)$ has been dropped, consistently with a conserved charge interpretation (which will come from the Lagrangian) replacing probability conservation (which comes from a constraint in the optimization problem).

The integration is performed over a foliation of spacetime by surfaces of constant ζ, where twisted-rapidity is measured relative to a specified reference frame. This ensures the solution exists only over events that are causally accessible to observers characterized by a specific rapidity, respecting both quantum principles and relativistic causal structure.

Using the same technique as Theorem 1, solving the optimization problem here yields:

$$\chi(x,\zeta) = \underbrace{\det \exp\left(-\frac{1}{2} \zeta \mathbf{M}(x) \right)}_{\text{Spin}^c(3,1) \text{ Born rule}} \underbrace{p(x)}_{\text{Initial preparation}} \tag{91}$$

The partition function is absent because we dropped the normalization constraint.

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As before this equation describes the structure that must erased from the solution space at the instant of measurement to yield a classical probability measure. In what follows, we will describe this structure. But first, in the following section, we will express the determinant of $\mathbf{M}$ using the multivectors of $\mathrm{GA}(3,1)$.

2.3.1. The Multivector Determinant

As we did in the beginning of the 2D case, our goal here will be to express $\det \mathbf{M}$ as a self-product of elements of the vector space. To achieve that, we begin by defining a general multivector in the geometric algebra $\mathrm{GA}(3,1)$.

Definition 10 (Multivector). *Let* $\mathbf{u}$ *be a multivector of* $\mathrm{GA}(3,1)$. *Its general form is:*

$$\begin{aligned}
\mathbf{u} = {} & a && (92)\\
& + t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 && (93)\\
& + f_{01}\gamma_0\gamma_1 + f_{02}\gamma_0\gamma_2 + f_{03}\gamma_0\gamma_3 + f_{12}\gamma_1\gamma_2 + f_{13}\gamma_1\gamma_3 + f_{23}\gamma_2\gamma_3 && (94)\\
& + p\gamma_1\gamma_2\gamma_3 + q\gamma_0\gamma_2\gamma_3 + v\gamma_0\gamma_1\gamma_3 + w\gamma_0\gamma_1\gamma_2 && (95)\\
& + b\gamma_0\gamma_1\gamma_2\gamma_3 && (96)
\end{aligned}$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ *are the basis vectors in the real Majorana representation.*

A more compact notation for $\mathbf{u}$ *is*

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{97}$$

where a is a scalar, $\mathbf{x}$ *a vector,* $\mathbf{f}$ *a bivector,* $\mathbf{v}$ *is pseudo-vector and* $\mathbf{b}$ *a pseudo-scalar.*

A general multivector of $\mathrm{GA}(3,1)$ can be represented by a 4×4 real matrix (and vice-versa) using the real Majorana representation:

Definition 11 (Matrix Representation of $\mathbf{u}$).

$$\mathbf{M} = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix} \tag{98}$$

To manipulate and analyze multivectors in $\mathrm{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the pseudo-blade conjugate, and the multivector determinant.

Definition 12 (Multivector Conjugate (in 4D)).

$$\mathbf{u}^\ddagger = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{99}$$

Definition 13 (Pseudo-Blade Conjugate (in 4D)). *The pseudo-blade conjugate of* $\mathbf{u}$ *is*

$$\mathbf{u}^\dagger = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{100}$$

Lundholm[6] proposes a number the multivector norms, and shows that they are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

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Definition 14. *The self-products associated with low-dimensional geometric algebras are:*

$$\mathrm{GA}(0,1): \quad \varphi^*\varphi \tag{101}$$
$$\mathrm{GA}(2,0): \quad \varphi^\ddagger\varphi \tag{102}$$
$$\mathrm{GA}(3,0): \quad (\varphi^\ddagger\varphi)^*\varphi^\ddagger\varphi \tag{103}$$
$$\mathrm{GA}(3,1): \quad (\varphi^\ddagger\varphi)^\dagger\varphi^\ddagger\varphi \tag{104}$$
$$\mathrm{GA}(4,1): \quad ((\varphi^\ddagger\varphi)^\dagger\varphi^\ddagger\varphi)^*((\varphi^\ddagger\varphi)^\dagger\varphi^\ddagger\varphi) \tag{105}$$

where φ^* *is a conjugate that reverses the sign of pseudo-scalar blade (i.e. the highest degree blade of the algebra).*

We can now express the determinant of the matrix representation of a multivector via a self-product. This choice is unique:

Theorem 5 (The Multivector Determinant)**.**

$$(\mathbf{u}^\ddagger\mathbf{u})^\dagger\mathbf{u}^\ddagger\mathbf{u} = \det \mathbf{M} \tag{106}$$

Proof. Please find a computer assisted proof of this equality in Annex B. □

As can be seen from this theorem, the relationship between determinants and multivector products becomes more sophisticated in 3+1D. Unlike the 2D case where the determinant could be expressed using a product of two terms, in $\mathrm{GA}(3,1)$ the determinant requires two products involving four copies of the multivector. This is reflected in the structure $(\mathbf{u}^\ddagger\mathbf{u})^\dagger\mathbf{u}^\ddagger\mathbf{u}$, which cannot be reduced to a simpler self-product of two terms.

Theorem 6 (Positive-Definiteness over $\mathbb{R}^+ \times \mathrm{Spinc}(3,1)$)**.** *Let* $\mathbf{u} = \exp(\frac{1}{2}(a+\mathbf{f}+\mathbf{b}))$ *be a general invertible element of the even-subalgebra of* $\mathrm{GA}(3,1)$*. As such,* $\mathbf{u}$ *is in* $\mathbb{R}^+ \times \mathrm{Spinc}(3,1)$*. Then the multivector determinant* $(\mathbf{u}^\ddagger\mathbf{u})^\dagger\mathbf{u}^\ddagger\mathbf{u}$ *is positive-definite.*

Proof. Since scalars, bivectors and pseudoscalars commute, we have:

$$\exp(\frac{1}{2}(a+\mathbf{f}+\mathbf{b})) = e^{a/2}e^{\mathbf{f}/2}e^{\mathbf{b}/2} \tag{107}$$

Using this convenient form, the proof is as follows:

$$(\mathbf{u}^\ddagger\mathbf{u})^\dagger\mathbf{u}^\ddagger\mathbf{u} \tag{108}$$
$$= e^{a/2}e^{-\mathbf{f}/2}e^{-\mathbf{b}/2}e^{a/2}e^{\mathbf{f}/2}e^{-\mathbf{b}/2}e^{a/2}e^{-\mathbf{f}/2}e^{\mathbf{b}/2}e^{a/2}e^{\mathbf{f}/2}e^{\mathbf{b}/2} \tag{109}$$
$$= e^{2a} \tag{110}$$

which is positive-definite—the exponential of a real number a is in $\mathbb{R}^+$. □

2.3.2. The $\mathbb{R}^+ \times \mathrm{Spin}^c(3,1)$-Valued Field

By inspection, the solution to the optimisation problem for the $\mathbb{R} \oplus \mathfrak{spinc}(3,1)$-constraint, identifies the following field:

Definition 15 ($\mathbb{R}^+ \times \mathrm{Spin}^c(3,1)$-valued Field)**.**

$$\phi(\zeta) = \exp\left(\frac{1}{2}\zeta(a+\mathbf{f}+\mathbf{b})\right)\phi(0) \tag{111}$$

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Theorem 7 (David Hestenes' Wavefunction)**.** *The $\mathbb{R}^+ \times \mathrm{Spin}^c(3,1)$-valued field is formulated using the same geometric structure as David Hestenes'[5] formulation of the wavefunction within GA(3,1). Specifically, David Hestenes' wavefunction is a special case of our result, where the field magnitude sums to 1.*

Proof.

$$\underbrace{e^{\frac{1}{2}(a(x)+\mathbf{f}(x)+\mathbf{b}(x))}}_{\text{ours}} \propto \underbrace{\sqrt{\rho(x)}R(x)e^{-ib(x)/2}}_{\text{Hestenes'}} \tag{112}$$

where $e^{\frac{1}{2}a(x)} \propto \sqrt{\rho(x)}$, $e^{\frac{1}{2}\mathbf{f}(x)} = R(x)$ and $e^{\frac{1}{2}\mathbf{b}(x)} = e^{-ib(x)/2}$. Here, $\rho(x)$ is a probability density (versus a field magnitude), $R(x)$ is a rotor and $e^{-ib(x)/2}$ describes the four-volume orientation. Adding the normalisation constraint to the optimisation problem forces the field magnitude to sum to 1, which recovers David Hestenes' wavefunction as a special case. □

This field leads to a variant of the Schrödinger equation obtained by taking its derivative with respect to the Lagrange multiplier ζ:

Definition 16 ($\mathbb{R}^+ \times \mathrm{Spin}^c(3,1)$-valued Schrödinger equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(\zeta) = \frac{1}{2}(a + \mathbf{f} + \mathbf{b})\phi(\zeta) \tag{113}$$

This Schrödinger equation is able to act on the wavefunction to dilate its probability measure, rotate or boost its rotor and to jiggle its orientation in flat spacetime. In curved spacetime, it generalizes to:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(\zeta,x) = \frac{1}{2}\gamma_0\mathbf{e}^\mu(\partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} + IV_\mu)\phi(\zeta,x) \tag{114}$$

$$= \frac{1}{2}\gamma_0\mathbf{e}^\mu D_\mu\phi(\zeta,x) \tag{115}$$

which includes ∂_μ, the partial derivative, $\frac{1}{2}\omega_\mu^{ab}\gamma_{ab}$ the $\mathrm{Spinc}(3,1)$ connection and IV_μ a $\mathrm{U}(1)$ connection acting on the four-volume orientation—acting on the wavefunction to transform its components in curved spacetime. Likewise to the 2D case, $\mathbf{e}^\mu$ is used to contract with D_μ, leaving no free indices. But since it produces an odd-multivector in the process, the term γ_0 is added converting the result back into an even-multivector. It also picks a preferred frame—the laboratory frame. Its effect is similar to the presence of γ_0 in the Dirac Lagrangian.

2.3.3. Geometry

Definition 17 (Dirac Current)**.** *Using a single-copy of the multivector determinant, the definition of the Dirac current is the same as Hestenes':*

$$J \equiv \overbrace{\phi^\ddagger\mathbf{e}_0\phi}^{\text{one copy}} \tag{116}$$

$$= \rho R^\ddagger e^{-ib/2}\mathbf{e}_0 e^{-ib/2}R \tag{117}$$

$$= \rho R^\ddagger\mathbf{e}_0 e^{ib/2}e^{-ib/2}R \tag{118}$$

$$= \rho R^\ddagger\mathbf{e}_0 R \tag{119}$$

$$= \rho\mathbf{e}_0' \tag{120}$$

where $\mathbf{e}_0'$ is a SO(3,1) rotated basis vector.

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Theorem 8 (Metric Tensor). *Taking advantage of the multivector determinant formulation, we utilize both copies to obtain the metric tensor as basis vector measurement:*

$$g_{\mu\nu} = \text{tr}\left(\frac{\overbrace{(\phi^\ddagger \mathbf{e}_\mu \phi)^\dagger}^{\text{copy 1}} \overbrace{\phi^\ddagger \mathbf{e}_\nu \phi}^{\text{copy 2}}}{\underbrace{(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi}_{\chi^2}}\right) \tag{121}$$

Proof.

$$\begin{aligned}
\text{tr}\left(\frac{(\phi^\ddagger \mathbf{e}_\mu \phi)^\dagger \phi^\ddagger \mathbf{e}_\nu \phi}{(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi}\right) &= \text{tr}\left(R^\ddagger e^{-ib/2} \mathbf{e}_\mu e^{-ib/2} R R^\ddagger e^{-ib/2} \mathbf{e}_\nu e^{-ib/2} R\right) && (122)\\
&= \text{tr}\left(R^\ddagger \mathbf{e}_\mu e^{ib/2} e^{-ib/2} R R^\ddagger \mathbf{e}_\nu e^{ib/2} e^{-ib/2} R\right) && (123)\\
&= \text{tr}\left(R^\ddagger e_\mu R R^\ddagger e_\nu R\right) && (124)\\
&= \text{tr}\left(\mathbf{e}'_\mu \mathbf{e}'_\nu\right) && (125)\\
&= \text{tr}\left(\mathbf{e}'_\mu \cdot \mathbf{e}'_\nu + \mathbf{e}'_\nu \wedge \mathbf{e}'_\mu\right) && (126)\\
&= \text{tr}\left(g_{\mu\nu} + \mathbf{e}'_\nu \wedge \mathbf{e}'_\mu\right) && (127)\\
&= g_{\mu\nu} && (128)
\end{aligned}$$

□

2.3.4. Dynamics

In the $\mathbb{R} \oplus \mathfrak{spin}(2)$-constraint section we utilized the correspondance between the Hamiltonian of the Schrodinger equation and the Lagrangian (Equation 80):

$$L = \text{tr}\left(\frac{\psi^\ddagger H \psi}{\psi^\ddagger \psi}\right) \tag{129}$$

This definition generalizes to the multivector determinant in 3+1D as follows:

$$L = \text{tr}\left(\frac{(\phi^\ddagger H \phi)^\dagger \phi^\ddagger H \phi}{(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi}\right) \tag{130}$$

which contains two copies.

Definition 18 (Kinetic Energy). *Applying the Hamiltonian $H = \frac{1}{2}\gamma_0 \mathbf{e}^\mu D_\mu$ to each bilinear copy, yields:*

$$T = \text{tr}\left(\frac{\overbrace{(\phi^\ddagger \gamma_0 \mathbf{e}^\mu D_\mu \phi)^\dagger}^{\text{copy 1}} \overbrace{\phi^\ddagger \gamma_0 \mathbf{e}^\nu D_\nu \phi}^{\text{copy 2}}}{4\underbrace{(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi}_{\chi^2}}\right) \tag{131}$$

Theorem 9 (Dirac Equation). *Varying the action yields the Dirac equation as a sufficient (but not necessary) equation of motion:*

$$\delta \int_{\mathcal{M}} \text{tr}\left(\frac{(\phi^\ddagger \gamma_0 \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\ddagger \gamma_0 \mathbf{e}^\nu D_\nu \phi}{4(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi}\right) \sqrt{-|g|}\, \mathrm{d}^4 x = 0 \implies \underbrace{\mathbf{e}^\mu D_\mu \phi = 0}_{\text{Dirac Equation}} \tag{132}$$

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Proof.

$$\frac{\delta}{\delta\phi^{\ddagger\dagger}}\operatorname{tr}\left(\frac{(\phi^{\ddagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{4(\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi}\right)=0 \tag{133}$$

$$\implies \operatorname{tr}\frac{\delta}{\delta\phi^{\ddagger\dagger}}\left(\frac{(\phi^{\ddagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{4(\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi}\right)=0 \tag{134}$$

$$\implies \operatorname{tr}\left(\frac{(\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi(\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi}{((\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi)^2}\delta\phi^{\ddagger\dagger}+\frac{(\phi^{\ddagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi\phi^{\dagger}\phi^{\ddagger}\phi}{((\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi)^2}\delta\phi^{\ddagger\dagger}\right)=0 \tag{135}$$

$$\implies \operatorname{tr}\left(\frac{(\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{(\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi}\delta\phi^{\ddagger\dagger}+\frac{(\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi}{(\phi^{\ddagger}\phi)^{\dagger}\phi^{\ddagger}\phi}\delta\phi^{\ddagger\dagger}\right)=0 \tag{136}$$

$$\implies (\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi=0 \tag{137}$$

For the condition to be satisfied, it is sufficient but not necessary that $\mathbf{e}^{\mu}D_{\mu}\phi=0$, which is the Dirac equation. A second condition $\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi=0$ also reduces to the Dirac equation because $\phi^{\ddagger}$ is invertible by definition. □

The multivector determinant formulation thus contains the solutions that satisfy the Dirac equation. However, broader solutions where the trace condition $\operatorname{tr}\big((\phi^{\ddagger}\gamma_0\mathbf{e}^{\mu}D_{\mu}\phi)^{\dagger}\phi^{\ddagger}\gamma_0\mathbf{e}^{\nu}D_{\nu}\phi\big)=0$ holds without $\mathbf{e}^{\mu}D_{\mu}\phi=0$, also exist. We will now investigate these broader solutions.

2.3.5. Gravity

Theorem 10 (Quantum Action)**.** *Let us investigate a subspace of the field where $R=1$ and $e^{-ib/2}=1$, such that $\phi=\sqrt{\chi}$. Due to its non-linearity, the kinetic energy produces a quantum potential in addition to the usual kinetic energy term:*

$$\operatorname{tr}\left(\frac{\left(\sqrt{\chi}\mathbf{e}^{\mu}D_{\mu}\chi\right)^{\dagger}\mathbf{e}^{\nu}D_{\nu}\sqrt{\chi}}{\chi^2}\right)=\underbrace{\frac{1}{2\chi^2}(\partial\chi)^2}_{\textit{Quantum Kinetics}}-\underbrace{\left(\frac{1}{4\chi^2}(\partial\chi)^2-\frac{\partial^2\chi}{2\chi}\right)}_{\textit{Quantum Potential}} \tag{138}$$

The quantum potential herein described is the relativistic version of the quantum potential found in the Bohm-Broglie reformulation of QM, whereas the quantum kinetics can be understood as a scalar field kinetic term. When integrated, they define a quantity that we refer to as the quantum action:

$$S=\underbrace{\int\left(\frac{1}{4\chi^2}(\partial\chi)^2+\frac{\partial^2\chi}{2\chi}\right)\sqrt{-|g|}\mathrm{d}^4x}_{\textit{Quantum Action}} \tag{139}$$

Proof.

$$\operatorname{tr}\left(\left(\chi^{-2}\sqrt{\chi}\mathbf{e}^{\mu}D_{\mu}\chi\right)^{\dagger}\mathbf{e}^{\nu}D_{\nu}\sqrt{\chi}\right) \tag{140}$$
$$= -\operatorname{tr}\left(\chi^{-2}\sqrt{\chi}(\mathbf{e}^{\mu}\partial_{\mu}\chi)\mathbf{e}^{\nu}\partial_{\nu}\sqrt{\chi} + \chi^{-2}\sqrt{\chi}\chi\mathbf{e}^{\mu}\partial_{\mu}\mathbf{e}^{\nu}\partial_{\nu}\sqrt{\chi}\right) \tag{141}$$
$$= -\operatorname{tr}\left(\chi^{-2}2^{-1}(\mathbf{e}^{\mu}\partial_{\mu}\chi)(\mathbf{e}^{\nu}\partial_{\nu}\chi)\right) + \operatorname{tr}\left(\chi^{-1}\sqrt{\chi}4^{-1}\chi^{-3/2}\mathbf{e}^{\mu}\partial_{\mu}\mathbf{e}^{\nu}\partial_{\nu}\chi\right) - \operatorname{tr}\left(\chi^{-1}\sqrt{\chi}2^{-1}\chi^{-1/2}\mathbf{e}^{\mu}\partial_{\mu}\mathbf{e}^{\nu}\partial_{\nu}\chi\right) \tag{142}$$
$$= -\operatorname{tr}\left(\frac{(\mathbf{e}^{\mu}\partial_{\mu}\chi)(\mathbf{e}^{\nu}\partial_{\nu}\chi)}{2\chi^2} - \frac{(\mathbf{e}^{\mu}\partial_{\mu}\chi)(\mathbf{e}^{\nu}\partial_{\nu}\chi)}{4\chi^2} + \frac{\mathbf{e}^{\mu}\partial_{\mu}\mathbf{e}^{\nu}\partial_{\nu}\chi}{2\chi}\right) \tag{143}$$
$$= \frac{1}{2\chi^2}(\partial\chi)^2 - \frac{1}{4\chi^2}(\partial\chi)^2 + \frac{\partial^2\chi}{2\chi} \tag{144}$$
$$= \frac{1}{4\chi^2}(\partial\chi)^2 + \frac{\partial^2\chi}{2\chi} \tag{145}$$

□

Theorem 11 (Equation of Motion)**.** *Varying the quantum action:*

$$S = \int \left(\frac{1}{4\chi^2}(\partial\chi)^2 + \frac{\partial^2\chi}{2\chi}\right)\sqrt{-|g|}\mathrm{d}^4x \tag{146}$$

produces:

$$\partial^2\chi = \chi\Box\chi \tag{147}$$

as the equation of motion.

Proof.

$$\delta\left(\frac{1}{4\chi^2}(\partial\chi)^2 + \frac{\partial^2\chi}{2\chi}\right) = 0 \tag{148}$$
$$\implies -\frac{(\partial\chi)^2}{2\chi^3}\delta\chi - \partial_{\mu}\left(\frac{\partial^{\mu}\chi}{2\chi^2}\right)\delta\chi + \frac{\partial^2(\delta\chi)}{2\chi} - \frac{\partial^2\chi}{2\chi^2}\delta\chi = 0 \tag{149}$$
$$\implies -\frac{(\partial\chi)^2}{2\chi^3}\delta\chi + \frac{(\partial\chi)^2}{\chi^3}\delta\chi - \frac{\partial^2\chi}{2\chi^2}\delta\chi + \frac{\partial^2(\delta\chi)}{2\chi} - \frac{\partial^2\chi}{2\chi^2}\delta\chi = 0 \tag{150}$$
$$\implies \frac{(\partial\chi)^2}{2\chi^3}\delta\chi - \frac{\partial^2\chi}{\chi^2}\delta\chi + \frac{\partial^2(\delta\chi)}{2\chi} = 0 \tag{151}$$

To proceed further, we are required to do integration by part for the last term:

$$\int \frac{\partial^2(\delta\chi)}{2\chi}\mathrm{d}^4x = -\int \frac{(\partial\chi)\partial(\delta\chi)}{2\chi^2}\mathrm{d}^4x \tag{152}$$

Then, a second integration by part, yields:

$$-\int \frac{(\partial\chi)\partial(\delta\chi)}{2\chi^2}\mathrm{d}^4x = -\int \delta\chi\left(\frac{\partial^2\chi}{2\chi^2} - \frac{(\partial\chi)^2}{\chi^3}\right)\mathrm{d}^4x \tag{153}$$

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$$\implies \frac{(\partial\chi)^2}{2\chi^3}\delta\chi - \frac{\partial^2\chi}{\chi^2}\delta\chi - \frac{\partial^2\chi}{2\chi^2}\delta\chi + \frac{(\partial\chi)^2}{\chi^3}\delta\chi = 0 \tag{154}$$

$$\implies \frac{3(\partial\chi)^2}{2\chi^3}\delta\chi - \frac{3\partial^2\chi}{2\chi^2}\delta\chi = 0 \tag{155}$$

$$\implies \partial^2\chi = \frac{(\partial\chi)^2}{\chi} \tag{156}$$

$$\implies \chi\Box\chi = (\partial\chi)^2 \tag{157}$$

□

To interpret this action and resulting equation of motion, let us now introduce the surprisal field and associated definitions.

Definition 19 (Surprisal Field)**.** *We define a change of variable:*

$$\varphi := -\ln\chi \tag{158}$$

We call φ the surprisal field.

Definition 20 (Surprisal Equation of Motion)**.** *We note that the change of variable $\varphi := -\ln\chi$, changes the equation of motion as follows:*

$$\chi\Box\chi = (\partial\chi)^2 \underbrace{\rightarrow}_{\varphi:=-\ln\chi} \Box\varphi = 0 \tag{159}$$

which is the Klein-Gordon equation in curved spacetime, applied to the surprisal field.

Definition 21 (Surprisal Conservation)**.** *The following current:*

$$\nabla_\mu(\partial^\mu\varphi) = 0 \tag{160}$$

identifies the surprisal as the conserved charge of this action.

Definition 22 (Surprisal Expectation Value)**.** *The surprisal expectation value is merely the entropy H of a region V of the manifold:*

$$\underbrace{\langle\ln\chi\rangle}_{\text{expectation value}} = \underbrace{-\int_V \chi(x)\underbrace{\ln\chi(x)}_{\text{observable}}\sqrt{h_\zeta}\mathrm{d}^3x}_{\text{Definition of Entropy}} \tag{161}$$

Interpretation:

In information theory, the **surprisal** of an event with probability density $\rho(x)$ is defined as $-\ln\rho(x)$, and the entropy $H = -\int\rho\ln\rho\, d^4x$ represents its expectation value. In our framework, however, the field χ replaces ρ but differs critically:

- χ is **not a probability density**—it lacks a conserved current ($\nabla_\mu(\chi u^\mu) \neq 0$) and is not normalized—but it is positive-definite.
- Instead, χ is interpreted as an **information density**, encoding spacetime's local information content.

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The **surprisal** is defined as $\varphi = -\ln\chi$, which in this theory satisfies the Klein-Gordon equation $\Box\varphi = 0$. This ensures:

1. **Conservation**: The current $j^\mu = \partial^\mu\varphi$ is conserved ($\nabla_\mu j^\mu = 0$), making $Q = \int_V j^\mu \sqrt{h_\zeta}\, d^3x$ a conserved charge.
2. **Causal Propagation**: Surprisal propagates at light speed, enforcing that the quantity φ of information χ cannot spread superluminally—a core tenet of relativity.

Before we continue the interpretation of this theory, let us introduce a few more theorems.

Theorem 12 (Ricci Scalar)**.** *Let us investigate another subspace of the field where $\sqrt{\chi} = 1$ and $e^{-ib/2} = 1$, such that $\phi = R$. Then the kinetic energy T reduces to the Ricci scalar $\mathcal{R}$.*

$$\mathrm{tr}\Big((\tilde{R}\mathbf{e}^\mu D_\mu R)^\dagger \tilde{R}\mathbf{e}^\nu D_\nu R\Big) = \mathcal{R} \tag{162}$$

Proof.

$$\begin{aligned}
&\mathrm{tr}\Big((\tilde{R}\mathbf{e}^\mu D_\mu R)^\dagger \tilde{R}\mathbf{e}^\nu D_\nu R\Big) && (163)\\
&= -\mathrm{tr}(\tilde{R}\mathbf{e}^\mu D_\mu \mathbf{e}^\nu D_\nu R) \qquad \text{via } \tilde{R}R = 1 && (164)\\
&= -\mathrm{tr}(\tilde{R}D^2 R) && (165)\\
&= -\mathrm{tr}(R\tilde{R}D^2) && (166)\\
&= \mathrm{tr}(D^2) && (167)\\
&= \mathcal{R} \qquad \text{via Lichnerowicz-Weitzenböck identity} && (168)
\end{aligned}$$

which is the Ricci scalar. □

Definition 23 (Gravity)**.** *Let us now consider the full space of the wavefunction $\psi = \sqrt{\rho}Re^{-ib/2}$. We are automatically lead into a theory of gravity:*

$$S = \int_{\mathcal{M}} \mathrm{tr}\left(\frac{(\phi^\ddagger\mathbf{e}^\mu D_\mu\phi)^\dagger \phi^\ddagger \mathbf{e}^\nu D_\nu\phi}{(\phi^\ddagger\phi)^\dagger\phi^\ddagger\phi^\ddagger}\right)\sqrt{-|g|}\mathrm{d}^4x \tag{169}$$

which expands, via Theorem 10 and 12, as follows:

$$S = \int_{\mathcal{M}} \left(\mathcal{R} + \textit{cross-terms} + \frac{1}{4\chi^2}(\partial\chi)^2 + \frac{\partial^2\chi}{2\chi}\right)\sqrt{-|g|}\mathrm{d}^4x \tag{170}$$

We note the following equations of motion which must be simultaneously satisfied:

1. *Varying with respect to $g_{\mu\nu}$ yields the EFE with the Einstein tensor from $\mathcal{R}$, and is sourced by the quantum action variation yielding the stress-energy tensor.*
2. *Varying with respect to χ gives equations of motion that define the flow of χ in spacetime.*

Interpretation (cont'd):

Thus, while quantum mechanics relies on probabilistic amplitudes ψ, our formulation recasts general relativity as a **deterministic theory of information dynamics**, where spacetime geometry and surprisal flux are dual aspects of $\mathcal{R}$ and χ. The distribution of surprisal in spacetime dictates its geometric structure, which in turns dictates how it propagates. General relativity is to information, what quantum mechanics is to probability.

Revisiting General Relativity with this perspective shows that the natural constraint is sufficient to entail the theory through the principle of entropy maximization—in this formulation, the speed of light as a limit on the propagation of the quantity of information (via the surprisal obeying the Klein-Gordon equation), and even the Einstein field equations are not fundamental, but the solution to an optimization problem on entropy.

2.3.6. Yang-Mills

In QFT, the standard method to identify a local gauge symmetry is to start with a global symmetry of the action or probability measure and then localize it by introducing gauge fields. For example, the U(1) gauge symmetry arises naturally in electromagnetism as the group preserving the probability density (Born rule) under local phase transformations. However, the non-Abelian SU(2) and SU(3) gauge symmetries of the Standard Model are not derived from first principles in this way; their inclusion is empirically motivated by particle physics experiments.

Improvement via Multivector Determinant Formulation:

Our framework demonstrates that Yang-Mills theories emerge naturally from constraints on the wavefunction's probability measure and Dirac current. Specifically:

1. **Probability Measure:** The quadratic form $(\phi^\ddagger\phi)^\dagger\phi^\ddagger\phi = \chi^2$ enforces rotor invariance $\phi \to R\phi$, restricting transformations to those satisfying $R^\ddagger R = 1$, for some rotor R of a geometric algebra of n dimensions:

$$(\phi^\ddagger R^\ddagger R\phi)^\dagger\phi^\ddagger R^\ddagger R\phi = (\phi^\ddagger\phi)^\dagger\phi^\ddagger\psi \implies R^\ddagger R = 1. \tag{171}$$

   Solutions to $R^\ddagger R = 1$ are rotor transformations generated by bivectors in the Clifford algebra. For a $2n$-dimensional algebra, these generate $\mathrm{Spin}(2n)$, whose subgroups include $\mathrm{SU}(n)$.
2. **Dirac Current:** The spacetime current $\phi^\ddagger\mathbf{e}_0\phi = \mathbf{e}_0$ requires gauge generators to commute with $\mathbf{e}_0$, confining them to an internal space. This implies:

$$\phi^\ddagger e^{-\theta^i f_i}\mathbf{e}_0 e^{\theta^i f_i}\phi = \phi^\ddagger\mathbf{e}_0\phi \implies [f_i, \mathbf{e}_0] = 0, \tag{172}$$

   where f_i are bivector generators. Thus, f_i act only on **internal degrees of freedom**, orthogonal to spacetime.
3. **Spacetime:** The origin of the multivector determinant from STA, defines the resulting internal space againts spacetime.

These constraints limit the allowable symmetry to groups generated by bivector exponentials (which are compact Lie groups), and acting on the internal spaces of spacetime. Since $\mathrm{SU}(n) \subset \mathrm{Spin}(2n)$, this framework inherently includes the Standard Model within its landscape but also generalizes to larger symmetries such as those found in condensed matter systems with emergent $\mathrm{SU}(n)$ symmetries.

Wavefunction and Symmetry Structure:

The total wavefunction is a tensor product of spacetime (STA) and internal space components:

1. For $\mathrm{SU}(n)$ Yang-Mills:

$$\phi_{\mathrm{STA}} \otimes \phi_{\mathbb{C}^n}. \tag{173}$$

2. For the Standard Model $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$:

$$\phi_{\mathrm{STA}} \otimes \phi_{\mathbb{C}} \otimes \phi_{\mathbb{C}^2} \otimes \phi_{\mathbb{C}^3}. \tag{174}$$

Covariant Derivative and Action (Ex. Standard Model):

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The covariant derivative incorporates spacetime curvature (gravity) and gauge fields:

$$D_\mu = \begin{pmatrix} \partial_\mu + \frac{\omega_\mu^{ab}}{2}\gamma_{ab} + ig'YB_\mu + ig\frac{\sigma^a}{2}W_\mu^a + ig_s\frac{\lambda^a}{2}G_\mu^a & \Phi \\ \Phi^\dagger & \partial_\mu + \frac{\omega_\mu^{ab}}{2}\gamma_{ab} + ig'YB_\mu + ig_s\frac{\lambda^a}{2}G_\mu^a \end{pmatrix}, \quad (175)$$

where:

1. γ_{ab}: Generators of $\mathrm{Spin}(3,1)$ (gravitational spin connection).
2. B_μ, W_μ^a, G_μ^a: $\mathrm{U}(1)$, $\mathrm{SU}(2)$, and $\mathrm{SU}(3)$ gauge fields.
3. Φ: Higgs field (SU(2) doublet).

It acts on the left/right split of the field.

Our previous gravitational action is reconstructed with a spectral function f:

$$S = \int_{\mathcal{M}} \mathrm{tr}\left(f\left(\frac{1}{\Lambda^2} \frac{(\phi^\ddagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\ddagger \mathbf{e}^\mu D_\mu \phi}{(\phi^\ddagger \phi)^\dagger \phi^\ddagger \phi} \right)\right) \sqrt{-|g|}\, \mathrm{d}^4 x. \quad (176)$$

Expanding f yield the field strength term $\mathrm{tr}(f(D^2/\Lambda^2))$ which via the Heat kernel further yields the Standard Model + gravity (see A. H. Chamseddine and Alain Connes [7] for method):

1. **Leading Terms**:
   (a) Cosmological constant: $\propto \Lambda^4 \int \sqrt{-|g|}\, \mathrm{d}^4 x$.
   (b) Einstein-Hilbert term: $\propto \Lambda^2 \int R\sqrt{-|g|}\, \mathrm{d}^4 x$.
2. **Yang-Mills and Higgs**:
   (a) Gauge kinetic terms: $\propto \int \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \sqrt{-|g|}\, \mathrm{d}^4 x$.
   (b) Higgs kinetic and potential terms:

$$\propto \int \left(|D_\mu \Phi|^2 + \Lambda^2 |\Phi|^2 + \frac{1}{\Lambda^2}|\Phi|^4 \right) \sqrt{-|g|}\, \mathrm{d}^4 x. \quad (177)$$

3. **Yukawa Couplings** (from matter fields):

$$\propto \int y_{ij}\bar{\phi}_i \Phi \phi_j \sqrt{-|g|}\, \mathrm{d}^4 x. \quad (178)$$

Key Notes:

1. **Higher-Order Terms:** Higher order field strength terms appear but are suppressed by Λ^{-2}, making them negligible at low energies.
2. **Uniqueness:** The Standard Model is not uniquely selected but resides within the landscape of allowed Yang-Mills theories.
3. **Experimental Consistency:** The framework ressembles Connes' spectral action (see A. H. Chamseddine and Alain Connes [7]), recovering the Standard Model and general relativity while allowing for testable extensions (e.g., higher-curvature gravity).

This formulation unifies gauge symmetries and gravity within the double-product structure.

2.3.7. Yang-Mills Axioms as Theorems

In Section 2.1, we demonstrated that all 5 axioms of quantum mechanics are derivable from the solution to the optimization problem in $\mathrm{GA}(0,1)$. Here, our aim is to do the same but for the axioms of Yang-Mills theory which occur in $\mathrm{GA}(3,1)$. First, let us list the axioms:

1. **Compact Gauge Group**: The symmetry group is a compact Lie group G.
2. **Local Gauge Invariance**: Fields transform under spacetime-dependent (local) group elements $T(x) \in G$.

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3. **Gauge Connections**: Gauge fields A_μ are introduced as connections in the covariant derivative $D_\mu = \partial_\mu + A_\mu$.
4. **Field Strength**: The curvature $F_{\mu\nu} = [D_\mu, D_\nu]$ defines the dynamics.
5. **Yang-Mills Action**: The action depends on $F_{\mu\nu}$, e.g., $\int \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$.

Now for the theorems.

Theorem 13 (Compact Gauge Group)**.** *The allowed symmetries form a compact Lie group $G \subset \mathrm{Spin}(2n)$.*

Proof. :

1. **Constraint**: $(\phi^\ddagger\phi)^\dagger\phi^\ddagger\phi = \chi^2$ implies invariance of arbitrary n-dimentional rotors: $R^\ddagger R = 1$.
2. **Structure of Solutions**: Rotor transformations in finite-dimensional Clifford algebras are generated by bivectors. These generate $\mathrm{Spin}(2n)$ and its subgroups, which are compact Lie groups.

Thus, the gauge group G is inherently compact and derived from the algebra structure. □

Theorem 14 (Local Gauge Invariance)**.** *The theory is invariant under spacetime-dependent $T(x) \in G$.*

Proof. :

1. **Wavefunction Transformation**: $\phi \to R(x)\phi$, where $R(x) = e^{\theta^i(x) f_i}$ (exponentials of spacetime-dependent bivectors).
2. **Probability Measure**: $(\phi^\ddagger\phi)^\dagger\phi^\ddagger\phi \to (\phi^\ddagger R^\ddagger R\phi)^\dagger\phi^\ddagger R^\ddagger R\phi = \chi^2$.
3. **Dirac Current**: $\phi^\ddagger \mathbf{e}_0\phi \to \phi^\ddagger R^\ddagger \mathbf{e}_0 R\phi = \phi^\ddagger\mathbf{e}_0\phi$, since $[f_i, \mathbf{e}_0] = 0$.

□

Theorem 15 (Gauge Connections)**.** *The covariant derivative $D_\mu = \partial_\mu + A_\mu$ emerges to maintain invariance under local $R(x)$.*

Proof. :

1. **Minimal Coupling**: To preserve $D_\mu\phi \to R(x)D_\mu\phi$, the derivative must transform as $\partial_\mu \to \partial_\mu + A_\mu$, where $A_\mu = f_i A^i_\mu(x)$.
2. **Gauge Field Definition**: Let $\partial_\mu R(x) = A_\mu R(x)$, then: $D_\mu\phi = \partial_\mu\phi + A_\mu\phi \implies D_\mu(R\phi) = RD_\mu\phi$.
3. **Clifford Algebra Embedding**: The A_μ are bivector fields in $\mathcal{C}\ell(2n)$, ensuring $A_\mu \in \mathfrak{g}$ (the Lie algebra of G)).

□

Theorem 16 (Field Strength)**.** *The commutator $F_{\mu\nu} = [D_\mu, D_\nu]$ defines the field strength.*

Proof. :

1. **Kinetic Energy**: The kinetic energy expands to include the field strength tensor:

$$\frac{(\phi^\ddagger\gamma_0\mathbf{e}^\mu D_\mu\phi)^\dagger\phi^\ddagger\gamma_0\mathbf{e}^\mu D_\mu\phi}{(\phi^\ddagger\phi)^\dagger\phi^\ddagger\phi} = \text{kinetic terms} + F_{\mu\nu} \tag{179}$$

where $F_{\mu\nu}$ is the field strength (Shown in Definition 23). □

Theorem 17 (Yang-Mills Action)**.** *The spectral action over the kinetic energy includes the kinetic term $\int \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$.*

Proof. :

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1. **Heat Kernel Expansion**: As shown in Equation 176 (see A. H. Chamseddine and Alain Connes [7] for method), the field strength term of the spectral action $S = \mathrm{tr}\big(f(D^2/\Lambda^2)\big)$ expands as: $S \sim \int \Big(\cdots + F^a_{\mu\nu}F^{a\mu\nu} + \cdots\Big)\sqrt{-|g|}\,\mathrm{d}^4x$.

□

Revisiting Yang-Mills with this perspective shows that the natural constraint is sufficient to entail the theory through the principle of entropy maximization—in this formulation, Yang-Mills axioms 1, 2, 3, 4, and 5 are not fundamental, but the solution to an optimization problem.

2.4. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to resolve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except the previously explored cases are obstructed. By obstructed, we mean that the solution to the entropy maximization problem, ρ, does not satisfy all axioms of probability theory.

Dimensions	*Optimal Predictive Theory of Nature*	
$\mathrm{GA}(0)$	**Statistical Mechanics**	(180)
$\mathrm{GA}(0,1)$	**Quantum Mechanics**	(181)
$\mathrm{GA}(1,0)$	Obstructed (Negative probabilities)	(182)
$\mathrm{GA}(2,0)$	**Quantum Mechanics**	(183)
$\mathrm{GA}(1,1)$	Obstructed (Negative probabilities)	(184)
$\mathrm{GA}(0,2)$	Obstructed (Non-real probabilities)	(185)
$\mathrm{GA}(3,0)$	Obstructed (Non-real probabilities)	(186)
$\mathrm{GA}(2,1)$	Obstructed (Non-real probabilities)	(187)
$\mathrm{GA}(1,2)$	Obstructed (Non-real probabilities)	(188)
$\mathrm{GA}(0,3)$	Obstructed (Non-real probabilities)	(189)
$\mathrm{GA}(4,0)$	Obstructed (Non-real probabilities)	(190)
$\mathrm{GA}(3,1)$	**Gravity + Yang-Mills**	(191)
$\mathrm{GA}(2,2)$	Obstructed (Negative probabilities)	(192)
$\mathrm{GA}(1,3)$	Obstructed (Non-real probabilities)	(193)
$\mathrm{GA}(0,4)$	Obstructed (Non-real probabilities)	(194)
$\mathrm{GA}(5,0)$	Obstructed (Non-real probabilities)	(195)
$\vdots$	$\vdots$	
$\mathrm{GA}(6,0)$	Suspected Obstructed (No observables)	(196)
$\vdots$	$\vdots$	

Let us now demonstrate the obstructions mentioned above.

Theorem 18 (Non-real probabilities)**.** *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

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Proof. These geometric algebras are classified as follows:

$$\mathrm{GA}(0,2) \cong \mathbb{H} \quad (197)$$
$$\mathrm{GA}(3,0) \cong \mathbb{M}_2(\mathbb{C}) \quad (198)$$
$$\mathrm{GA}(2,1) \cong \mathbb{M}_2^2(\mathbb{R}) \quad (199)$$
$$\mathrm{GA}(1,2) \cong \mathbb{M}_2(\mathbb{C}) \quad (200)$$
$$\mathrm{GA}(0,3) \cong \mathbb{H}^2 \quad (201)$$
$$\mathrm{GA}(4,0) \cong \mathbb{M}_2(\mathbb{H}) \quad (202)$$
$$\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H}) \quad (203)$$
$$\mathrm{GA}(0,4) \cong \mathbb{M}_2(\mathbb{H}) \quad (204)$$
$$\mathrm{GA}(5,0) \cong \mathbb{M}_2^2(\mathbb{H}) \quad (205)$$

The determinant of these objects is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. □

Theorem 19 (Negative probabilities)**.** *The even sub-algebra of these dimensional configurations allows for negative probabilities, making them unsuitable.*

Proof. This category contains three dimensional configurations:
$\mathrm{GA}(1,0)$: Let $\psi = a + be_1$, then:

$$(a+be_1)^\ddagger(a+be_1) = (a-be_1)(a+be_1) = a^2 - b^2e_1e_1 = a^2 - b^2 \quad (206)$$

which is valued in $\mathbb{R}$.
$\mathrm{GA}(1,1)$: Let $\psi = a + be_0e_1$, then:

$$(a+be_0e_1)^\ddagger(a+be_0e_1) = (a-be_0e_1)(a+be_0e_1) = a^2 - b^2e_0e_1e_0e_1 = a^2 - b^2 \quad (207)$$

which is valued in $\mathbb{R}$.
$\mathrm{GA}(2,2)$: Let $\psi = a + be_0e_\emptyset e_1e_2$, where $e_0^2 = -1, e_\emptyset^2 = -1, e_1^2 = 1, e_2^2 = 1$, then:

$$((a+\mathbf{b})^\ddagger(a+\mathbf{b}))^\dagger(a+\mathbf{b})^\ddagger(a+\mathbf{b}) \quad (208)$$
$$= (a^2 + 2a\mathbf{b} + \mathbf{b}^2)^\dagger(a^2 + 2a\mathbf{b} + \mathbf{b}^2) \quad (209)$$

We note that $\mathbf{b}^2 = b^2e_0e_\emptyset e_1e_2e_0e_\emptyset e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \quad (210)$$
$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \quad (211)$$
$$= (a^2 + b^2)^2 - 4a^2b^2 \quad (212)$$

which is valued in $\mathbb{R}$.
In all of these cases the probability can be negative. □

Conjecture 1 (No observables (6D))**.** *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 14 collapse. The research by Acus et al.[8] in 6D geometric algebra concludes that the determinant, so far defined

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through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{213}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[8].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{214}$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{215}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{216}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{217}$$

From Equation 213, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{218}$$

Any equality of the above type between $b_1\mathbf{O}$ and $b_2\mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) be constructible in 6D, using our method. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No observables (above 6D))**.** *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $\text{GA}(1,3)$ even if in relativity, the signature of the metric $(+,-,-,-)$ versus $(-,-,-,+)$ does not influence the physics. However, in geometric algebra, $\text{GA}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\text{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

3. Discussion

When asked to define what a physical theory is, an informal answer may be that it is a predictive framework of measurements that applies to all possible experiments realizable within a domain, with nature as a whole being the most general domain. While physicists have expressed these theories through sets of axioms, we propose a more direct approach—mathematically realizing this fundamental definition itself. This definition is realized as an optimization problem (Definition 1) that can be solved

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directly. The solution to this optimization problem yields precisely those structures that realize the physical theory over said domain. Succinctly, physics is the solution to:

$$\underbrace{\mathcal{L}}_{\substack{\text{an}\\ \text{optimization}\\ \text{problem}}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{on the entropy}\\ \text{of a measurement}\\ \text{relative to its preparation}\\ \text{over all}}} \quad + \underbrace{\lambda\left(1-\sum_i \rho_i\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr}\left(\overline{\mathbf{M}}-\sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}} \tag{219}$$

The relative Shannon entropy represents the basic structure of any experiment, quantifying the informational difference between its initial preparation and its final measurement.

The natural constraint is chosen to be the most general structure that admits a solution to this optimization problem. This generality follows from key mathematical requirements. The constraint must involve quantities that form an algebra, as the solution requires taking exponentials:

$$\exp X = 1 + X + \frac{1}{2}X^2 + \dots \tag{220}$$

which involves addition, powers, and scalar multiplication of X. The use of the trace operation further necessitates that X must be represented by square $n \times n$ matrices. Thus Axiom 1 involves $n \times n$ matrices:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \tag{221}$$

The trace operation is utilized because the constraint must be converted back to a scalar for use in the Lagrange multiplier equation; while any function that maps an algebra to a scalar would achieve that, picking the trace recovers quantum mechanics in the $\mathrm{GA}(0,1)$ case.

These mathematical requirements demonstrate that the natural constraint, as it admits the minimal mathematical structure required to solve an arbitrary entropy maximization problem, can be understood as the most general extension of the statistical mechanics average energy constraint which contains QM (as induced by the trace) as a specific solution.

Thus, having established both the mathematical structure and its generality, we can understand how this minimal ontology operates. Since our formulation keeps the structure of experiments completely general, our optimization considers all possible predictive theories for that structure, and the constraint is the most general constraint possible for that structure, the resulting optimal physical theory applies, by construction, to all realizable experiments within its domain.

This ontology is both operational, being grounded in the basic structure of experiments rather than abstract entities, and constructive, showing how physical laws emerge from optimization over all possible predictive theories subject to the natural constraint. Physics is encapsulated not as a predefined collection of fundamental axioms but as the optimal solution to a well-defined optimization problem over all experiments realizable within the domain. This represents a significant philosophical shift from traditional physical ontologies where laws are typically taken as primitive.

The next step in our derivation is to represent the determinant of the $n \times n$ matrices through a self-product of multivectors involving various conjugate structures. By examining the various dimensional configurations of geometric algebras, we find that GA(3,1), representing 4×4 real matrices, admits a sub-algebra whose determinant is positive-definite for its invertible members. All other dimensional configurations fail to admit such a positive-definite structure, with two exceptions: statistical mechanics (found in GA(0)) and quantum mechanics (found in GA(0,1) and in a sub-algebra of GA(2,0)).

The solution reveals that the 3+1D case harbours a new type of field amplitude structure analogous to complex amplitudes, one that exhibits the characteristic elements of a quantum mechanical theory.

Instead of complex-valued amplitudes, we have amplitudes valued in the invertible subset of the even sub-algebra of GA(3,1). When normalized, this amplitude is identical to David Hestenes' wavefunction, but comes with an extended Born rule represented by the determinant, and rather than a complex Hilbert space, it lives in a "double-product structure". This double-product structure automatically incorporates gravity via the Spin(3,1) connection and local gauge theories as Yang-Mills theories. The square of the Dirac operator, automatically generated by the Lagrangian, then generates the invariants of gravity and of the Yang-Mills theory via a heat kernel expansion, along with the matter fields quantifying the system's information via surprisal and limiting its propagation speed.

Interpretation: This framework establishes quantum mechanics as the emergent solution to entropy optimization constrained by measurement outcomes, rather than a set of axiomatic entities. The wavefunction arises as a non-fundamental calculational tool, akin to statistical mechanics' probability distributions. Since all spacetime and quantum structures (i.e. the five axioms of QM, gravity, the Yang-Mills axioms, 3+1D for spacetime, etc.) are in the wavefunction, they are thus revealed as epistemic necessities for describing experiments within the constraint of nature, and are void of ontological commitments.

Measurements: In our interpretation, initial preparations and final measurements constitute the irreducible physical boundaries of an experiment. These empirical endpoints define reality, with the quantum formalism emerging as the least-biased bridge between them through relative entropy maximization under the natural constraint. Traditional interpretations problematically treat measurement as a physical process transforming wavefunctions, necessitating the ad hoc "collapse" postulate. We invert this ontology: the Schrödinger equation and Born rule describe not physical evolution but an inferential relationship connecting preparation to measurement outcomes. The wavefunction exists solely as an epistemic construct encoding this relationship. This eliminates the measurement problem by excising its root cause: the reification of intermediate states (i.e. states between initial preparation and final measurement) as ontological entities.

4. Conclusion

This work suggests that the formal essence of fundamental physics—statistical mechanics, quantum mechanics, general relativity, Yang-Mills and the dimensionality of spacetime—emerges not from axiomatic constructs but as necessity. By reframing physics as the solution to a single optimization problem on relative entropy, constrained only by what measurements nature permits, we derive these structures from first principles. This reformulation dissolves the artificial divide between quantum and classical frameworks: statistical mechanics, quantum theory, gravity and gauge theory all become special cases of information optimization under progressively richer constraints.

The conceptual pivot is ontological inversion. Traditional approaches postulate physical entities (wavefunction, fields) and retroactively justify them through measurement. Our framework begins and ends with measurement outcomes, with the *natural constraint* acting as the keystone. Applied to all possible experiments, this constraint uniquely compels the emergence of Yang-Mills, spacetime, and gravity. These structures dominate not by nature's preference but by mathematical inevitability—they comprise the only solution satisfying the natural constraint for all realizable experiments.

This work suggests a radical re-envisioning of physical law. Concepts we reify as fundamental—Hilbert spaces, gravitational curvature—are revealed as epistemic artifacts: the observer's efficient calculus for experimental prediction under the constraint of nature. Just as entropy maximization can derive thermodynamics from atomic motions, entropy maximization can derive quantum mechanics from the difference between initial preparations and final measurements. The set of all realizable experiments replaces atoms as the fundamental building block of physics. Future work must test this paradigm's predictive limits, but its mere viability invites reconsideration of long-held ontological assumptions about what physics ultimately describes.

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- Data Availability Statement: No datasets were generated or analyzed during the current study.

- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

A SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} = \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \tag{222}$$

We solve the maximization problem as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_i, \dots, \rho_n)}{\partial \rho_i} \tag{223}$$

$$= -\ln \rho_i - 1 - \lambda - \beta E_i \tag{224}$$

$$= \ln \rho_i + 1 + \lambda + \beta E_i \tag{225}$$

$$\implies \ln \rho_i = -1 - \lambda - \beta E_i \tag{226}$$

$$\implies \rho_i = \exp(-1-\lambda)\exp(-\beta E_i) \tag{227}$$

$$= \frac{1}{Z(\tau)} \exp(-\beta E_i) \tag{228}$$

The partition function, is obtained as follows:

$$1 = \sum_i \exp(-1-\lambda)\exp(-\beta E_i) \tag{229}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_i \exp(-\beta E_i) \tag{230}$$

$$Z(\tau) := \sum_i \exp(-\beta E_i) \tag{231}$$

Finally, the probability measure is:

$$\rho_i = \frac{1}{\sum_i \exp(-\beta E_i)} \exp(-\beta E_i) \tag{232}$$

B SageMath Program Showing $\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M}_\mathbf{u}$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
```

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```
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4


I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
                 [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
```

```
                        [0, 0, 0, -1],
                        [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                        [0, 0, -1, 0],
                        [0, -1, 0, 0],
                        [1, 0, 0, 0]])

y3 = Matrix(SR, [[-1, 0, 0, 0],
                        [0, 1, 0, 0],
                        [0, 0, -1, 0],
                        [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())
```

The program outputs

```
True
```

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-product: $\det \mathbf{M_u} = \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}$.

References

1. Jaynes, E.T. Information theory and statistical mechanics. *Physical review* **1957**, *106*, 620.
2. Jaynes, E.T. Information theory and statistical mechanics. II. *Physical review* **1957**, *108*, 171.
3. Dirac, P.A.M. *The principles of quantum mechanics*; Number 27, Oxford university press, 1981.
4. Von Neumann, J. *Mathematical foundations of quantum mechanics: New edition*; Vol. 53, Princeton university press, 2018.
5. Hestenes, D. Spacetime physics with geometric algebra (Page 6). *American Journal of Physics* **2003**, *71*, 691–714.
6. Lundholm, D. Geometric (Clifford) algebra and its applications. *arXiv preprint math/0605280* **2006**.
7. Chamseddine, A.H.; Connes, A. The spectral action principle. *Communications in Mathematical Physics* **1997**, *186*, 731–750.
8. Acus, A.; Dargys, A. Inverse of multivector: Beyond p+ q= 5 threshold. *arXiv preprint arXiv:1712.05204* **2017**.