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Constructing Physics From Measurements

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Constructing Physics From Measurements

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Abstract: We present a reformulation of fundamental physics - from an enumeration of independent axioms to the solution of a single optimization problem. Any experiment begins with an initial state preparation, involves some physical operation, and ends with a final measurement. Working from this structure, we maximize the entropy of a final measurement relative to its initial preparation subject to a measurement constraint, the later defining the domain of the theory. Since we keep the structure of an experiment entirely general, solving this optimization problem identifies the unique optimal predictive theory that holds true for all realizable experiments within the domain. We then find that using the natural constraint – which spawns the most general domain supported by this optimization problem – points to a unification of fundamental physics. Rather than as separate postulates, we obtain quantum mechanics, general relativity, and the Standard Model gauge symmetries within a unified theory. Notably, mathematical consistency further restricts valid solutions to 3+1 dimensions only. This reformulation reveals that the apparent complexity of modern physics, with its various forces, symmetries, and dimensional constraints, emerges as the solution an optimization problem over all realizable experiments of nature.

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1. Introduction

Statistical mechanics (SM), in the formulation developed by E.T. Jaynes [1,2], is founded on an entropy optimization principle. Specifically, the Boltzmann entropy is maximized under the constraint of a fixed average energy $\overline{E}$:

$$\overline{E} = \sum_i \rho_i E_i \tag{1}$$

The Lagrange multiplier equation defining the optimization problem is:

$$\mathcal{L} = -k_B \sum_i \rho_i \ln \rho_i + \lambda \left(1 - \sum_i \rho_i\right) + \beta \left(\overline{E} - \sum_i \rho_i E_i\right), \tag{2}$$

where λ and β are Lagrange multipliers enforcing the normalization and average energy constraints. Solving this optimization problem yields the Gibbs measure:

$$\rho_i = \frac{1}{Z} \exp(-\beta E_i), \tag{3}$$

where $Z = \sum_i \exp(-\beta E_i)$ is the partition function.

For comparison, quantum mechanics (QM) is not formulated as the solution to an optimization problem, but rather consists of a collection of axioms[3,4]:

QM Axiom 1 of 5	**State Space:** Every physical system is associated with a complex Hilbert space, and its state is represented by a ray (an equivalence class of vectors differing by a non-zero scalar multiple) in this space.
QM Axiom 2 of 5	**Observables:** Physical observables correspond to Hermitian (self-adjoint) operators acting on the Hilbert space.

QM Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is governed by the Schrödinger equation, where the Hamiltonian operator represents the system's total energy.

QM Axiom 4 of 5 **Measurement:** Measuring an observable projects the system into an eigenstate of the corresponding operator, yielding one of its eigenvalues as the measurement result.

QM Axiom 5 of 5 **Probability Interpretation:** The probability of obtaining a specific measurement outcome is given by the squared magnitude of the projection of the state vector onto the relevant eigenstate (Born rule).

Physical theories have traditionally been constructed in two distinct ways. Some, like quantum mechanics, are defined through a set of mathematical axioms that are first postulated and then verified against experiments. Others, like statistical mechanics, emerge as solutions to optimization problems with experimentally-verified constraints.

We propose to generalize the optimization methodology of E.T. Jaynes to encompass all of physics, aiming to derive a unified theory from a single optimization problem.

To that end, we introduce the following constraint:

Axiom 1 (Nature)**.**

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i$$

where $\mathbf{M}_i$ *are* $n \times n$ *matrices, and* $\overline{\mathbf{M}}$ *is their average.*

This constraint, as it replaces the scalar E_i with the matrix $\mathbf{M}_i$, extends E.T. Jaynes' optimization method to encompass non-commutative observables and symmetry group generators required for fundamental physics.

We then construct an optimization problem:

Definition 1 (Physics)**.** *Physics is the solution to:*

$$\underbrace{\mathcal{L}}_{\substack{\text{an} \\ \text{optimization} \\ \text{problem}}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{on the entropy} \\ \text{of a measurement} \\ \text{relative to its preparation} \\ \text{over all}}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}}$$

where λ *and* τ *are Lagrange multipliers enforcing the normalization and natural constraints, respectively.*

This single definition constitutes our complete proposal for a reformulation of fundamental physics, and no additional principles will need to be introduced.

By using the relative Shannon entropy rather than the Boltzmann entropy of statistical mechanics, this optimization framework naturally extends beyond thermodynamic variables to encompass predictions for any type of experiment. This generalization is possible because the relative entropy considers a final measurement state relative to an initial preparation state, which is the basic structure of an experiment. The normalization constraint ensures we are dealing with a proper predictive theory, while the natural constraint ensures the theory occupies no states that are incompatible with the measurements. Together, these components capture the complete evolution, including initial and final states, that defines any experiment. The key insight is that since our formulation keeps the structure of experiments completely general, and our optimization considers all possible predictive theories for that structure, the resulting solution necessarily holds true for all realizable experiments within the domain. In other words, this optimization problem identifies the unique optimal predictive theory of nature that, by construction, holds true for all realizable experiments within the domain defined by the constraint.

Specifically, the solution provides a complete set of axioms that automatically satisfy the requirements of a physical theory - they are mathematically rigorous, internally consistent, optimally predictive and apply to any possible experiments within the domain defined by the constraint, reducing our reliance on the brute postulation of axioms. As we will demonstrate, when the constraint is the natural constraint, the solution to this optimization problem points toward a unification of physics. The apparently separate structures of statistical mechanics, quantum mechanics, general relativity, and the Standard Model emerge naturally without additional assumptions and without generating undesirable artifacts like extra dimensions or unobserved gauge symmetries.

Theorem 1. *The general solution of the optimization problem is:*

$$\rho_i = \frac{p_i \det\exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det\exp\left(-\tau \mathbf{M}_j\right)}$$

Proof. We solve the maximization problem by setting the derivative of the Lagrangian with respect to ρ_i to zero:

$$\frac{\partial \mathcal{L}}{\partial \rho_i} = -\ln\frac{\rho_i}{p_i} - 1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i = 0. \tag{4}$$

$$\implies \ln\frac{\rho_i}{p_i} = -1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i. \tag{5}$$

$$\implies \rho_i = p_i \exp(-1-\lambda)\exp(-\tau \operatorname{tr} \mathbf{M}_i). \tag{6}$$

Normalizing the probabilities using $\sum_i \rho_i = 1$, we find:

$$1 = \sum_i \rho_i = \exp(-1-\lambda)\sum_i p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i), \tag{7}$$

$$\implies \exp(1+\lambda) = \sum_j p_j \exp\left(-\tau \operatorname{tr} \mathbf{M}_j\right). \tag{8}$$

Substituting back, we obtain:

$$\rho_i = \frac{p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i)}{\sum_j p_j \exp\left(-\tau \operatorname{tr} \mathbf{M}_j\right)}. \tag{9}$$

Finally, using the identity $\det\exp(\mathbf{M}) = \exp\operatorname{tr}\mathbf{M}$ for square matrices $\mathbf{M}$, we get:

$$\rho_i = \frac{1}{Z} p_i \det\exp(-\tau \mathbf{M}_i). \tag{10}$$

□

where $Z = \sum_j p_j \det\exp\left(-\tau \mathbf{M}_j\right)$.

This solution encapsulates three distinct special cases:

1. **Statistical Mechanics:**
   To recover statistical mechanics from Equation 10, we consider the case where the matrices $\mathbf{M}_i$ are 1×1, i.e., scalars. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \left[E_i\right], \tag{11}$$

and take p_i to be a uniform distribution. Then, Equation 10 reduces to the Gibbs distribution:

$$\rho_i = \frac{1}{Z}\exp(-\tau E_i), \tag{12}$$

where τ corresponds to β in traditional statistical mechanics. This demonstrates that our solution generalizes SM, as it recovers it when $\mathbf{M}_i$ are scalars.

2. **Quantum Mechanics:**
   By choosing $\mathbf{M}_i$ to generate the U(1) group, we derive the axioms of quantum mechanics from entropy maximization. Specifically, we set:

   $$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}, \tag{13}$$

   where E_i are energy levels. In the results section, we will detail how this choice leads to a probability measure that includes a unitarily invariant ensemble and the Born rule, satisfying all five axioms of QM.
3. **Fundamental Physics:**
   Extending our approach, we choose $\mathbf{M}_i$ to be 4×4 matrices representing the generators of the Spin^c(3,1) group. Specifically, we consider multivectors of the form $\mathbf{u} = \mathbf{f} + \mathbf{b}$, where $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudoscalar of the 3+1D geometric algebra $\mathrm{GA}(3,1)$. The matrix representation of $\mathbf{M}_i$ is:

   $$\mathbf{M}_i = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix}, \tag{14}$$

   where $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the Spin^c(3,1) group, which includes both Lorentz transformations and U(1) phase rotations. This choice leads to a relativistic quantum probability measure:

   $$\rho_i = \frac{p_i \det\exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det\exp(-\tau \mathbf{M}_j)}, \tag{15}$$

   where τ emerges as a parameter generating boosts, rotations, and phase transformations.
   In the results section, we show that the associated Dirac current is automatically invariant under the gauge symmetries of the Standard Model, specifically SU(3), SU(2) and U(1). Furthermore, we show that the metric tensor of general relativity emerges via a double-copy mechanism applied to the Dirac current, associating to a quantum theory of gravity.
4. **Dimensional Obstructions:**
   Axiom 1 yields valid probability measures only in specific geometric cases. Beyond the instances of statistical mechanics and quantum mechanics, Axiom 1 yields a consistent solution only in 3+1 dimensions. In other dimensional configurations, various obstructions arises violating the axioms of probability theory. The following table summarizes the geometric cases and their obstructions:

Dimensions	*Optimal Predictive Theory of Nature*	
$\mathrm{GA}(0)$	**Statistical Mechanics**	(16)
$\mathrm{GA}(0,1)$	**Quantum Mechanics**	(17)
$\mathrm{GA}(1,0)$	Obstructed (Negative probabilities)	(18)
$\mathrm{GA}(2,0)$	**Quantum Mechanics**	(19)

$$\mathrm{GA}(1,1) \qquad \text{Obstructed (Negative probabilities)} \tag{20}$$
$$\mathrm{GA}(0,2) \qquad \text{Obstructed (Non-real probabilities)} \tag{21}$$
$$\mathrm{GA}(3,0) \qquad \text{Obstructed (Non-real probabilities)} \tag{22}$$
$$\mathrm{GA}(2,1) \qquad \text{Obstructed (Non-real probabilities)} \tag{23}$$
$$\mathrm{GA}(1,2) \qquad \text{Obstructed (Non-real probabilities)} \tag{24}$$
$$\mathrm{GA}(0,3) \qquad \text{Obstructed (Non-real probabilities)} \tag{25}$$
$$\mathrm{GA}(4,0) \qquad \text{Obstructed (Non-real probabilities)} \tag{26}$$
$$\mathrm{GA}(3,1) \qquad \textbf{Quantum Gravity + Standard Model} \tag{27}$$
$$\mathrm{GA}(2,2) \qquad \text{Obstructed (Negative probabilities)} \tag{28}$$
$$\mathrm{GA}(1,3) \qquad \text{Obstructed (Non-real probabilities)} \tag{29}$$
$$\mathrm{GA}(0,4) \qquad \text{Obstructed (Non-real probabilities)} \tag{30}$$
$$\mathrm{GA}(5,0) \qquad \text{Obstructed (Non-real probabilities)} \tag{31}$$
$$\vdots \qquad \vdots$$
$$\mathrm{GA}(6,0) \qquad \text{Suspected Obstructed (No observables)} \tag{32}$$
$$\vdots \qquad \vdots$$
$$\infty \tag{33}$$

where $\mathrm{GA}(p,q)$ means the geometric algebra of $p+q$ dimensions.

We will first investigate the unobstructed cases in Section 2.1, 2.2 and 2.3 and then demonstrate the obstructions in Section 2.4. These obstructions are desirable because they automatically limit the theory to 3+1D, thus providing a built-in mechanism for the observed dimensionality of our universe.

2. Results

2.1. Quantum Mechanics

In statistical mechanics (SM), the central observation is that energy measurements of a thermally equilibrated system tend to cluster around a fixed average value (Equation 1). In contrast, quantum mechanics (QM) is characterized by the presence of interference effects in measurement outcomes. To capture these features within an entropy maximization framework, we introduce the following special case of Axiom 1:

Definition 2 (U(1) Generating Constraint)**.** *We reduce the generality of Axiom 1 to the generator of the U(1) group. Specifically, we replace*

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \quad \textit{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \tag{34}$$

Here, E_i are scalar values (e.g., energy levels), ρ_i are the probabilities of outcomes, and the matrices $\mathbf{M}_i$ generate the U(1) group.

The general solution of the optimization problem reduces as follows

$$\rho_i = \frac{1}{\sum_i p_i \det\exp\left(-\tau \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right)} \det\exp\left(-\tau \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) p_i \tag{35}$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

To align our results with conventional quantum mechanical notation, we translate the matrices to complex numbers. Specifically, we consider that:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + ib. \tag{36}$$

Then, we note the following equivalence with the complex norm:

$$\begin{aligned} \det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} &= r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a & (37) \\ &= r^2(\cos^2(b) + \sin^2(b)) & (38) \\ &= \|r(\cos(b) + i\sin(b))\| & (39) \\ &= \|r\exp(ib)\| & (40) \end{aligned}$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial prepration into :

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \|\exp(-itE_i/\hbar)\|}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE_i/\hbar)\|}_{\text{Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{41}$$

The wavefunction emerges by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_i p_i \|\exp(-itE_i/\hbar)\| = Z = \langle \psi | \psi \rangle \tag{42}$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE_1/\hbar) & & \\ & \ddots & \\ & & \exp(-itE_n/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \tag{43}$$

We clarify that p_i represents the probability associated with the initial preparation of the wavefunction, where $p_i = \langle \psi_i(0) | \psi_i(0) \rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle \psi | \psi \rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates QM Axiom 1 of 5.
- In Z, an observable must satisfy:

$$\overline{O} = \sum_i p_i O_i \|\exp(-itE_i/\hbar)\| \tag{44}$$

Since $Z = \langle \psi | \psi \rangle$, then any self-adjoint operator satisfying the condition $\langle \mathbf{O}\psi | \phi \rangle = \langle \psi | \mathbf{O}\phi \rangle$ will equate the above equation, simply because $\langle \mathbf{O} \rangle = \langle \psi | \mathbf{O} | \psi \rangle$. This demonstrates QM Axiom 2 of 5.

- Upon transforming Equation 43 out of its eigenbasis through unitary operations, we find that the energy, E_i, typically transforms in the manner of a Hamiltonian operator:

  $$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \tag{45}$$

  The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

  $$\begin{aligned} \frac{\partial}{\partial t}|\psi(t)\rangle &= \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle) && (46)\\ &= -i\mathbf{H}/\hbar\exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle && (47)\\ &= -i\mathbf{H}/\hbar|\psi(t)\rangle && (48)\\ \implies \mathbf{H}|\psi(t)\rangle &= i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle && (49) \end{aligned}$$

  which is the Schrödinger equation. This demonstrates QM Axiom 3 of 5.
- From Equation 43 it follows that the possible microstates E_i of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from ρ, with the measured state being the occupied microstate i. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

  $$\rho_i(t) = \frac{1}{\langle\psi|\psi\rangle}(\psi_i(t))^\dagger\psi_i(t). \tag{50}$$

  In scenarios where the probability measure $\rho_i(\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

  $$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{51}$$

  Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates QM Axiom 4 of 5.
- Finally, since the probability measure (Equation 41) replicates the Born rule, QM Axiom 5 of 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the U(1) generating constraint is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. QM Axioms 1, 2, 3, 4, and 5 are shown to be theorems of a single more foundamental axiom.

2.2. RQM in 2D

In this section, we investigate a model, isomorphic to quantum mechanics, that lives in 2D which provides a valuable starting point before addressing the more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier Equation is:

$$\mathcal{L} = -\sum_i \rho_i \ln\frac{\rho_i}{p_i} + \lambda\left(1 - \sum_i \rho_i\right) + \frac{1}{2}\theta\mathrm{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i\mathbf{M}_i\right) \tag{52}$$

where λ and θ are the Lagrange multipliers, and where $\mathbf{M}_i$ is the 2×2 matrix representation of the multivectors of GA(2).

In general a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of GA(2), where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ a pseudo-scalar, is represented as follows:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y \tag{53}$$

This holds for any 2×2 matrix and any multivectors of GA(2).

The basis elements are defined as:

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_x \wedge \sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{54}$$

To investigate this case in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 3 (Multivector Conjugate)**.** *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* GA(2)*. The multivector conjugate is defined as:*

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{b} \tag{55}$$

The determinant of the matrix representation of a multivector can be expressed as a multivector self-product:

Theorem 2 (The Determinant in Multivector Self-Product Form)**.**

$$\mathbf{u}^{\ddagger}\mathbf{u} = \det \mathbf{M} \tag{56}$$

Proof. Let $\mathbf{u} = a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y$, and let $\mathbf{M}$ be its matrix representation $\left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right]$. Then:

$$\begin{aligned}
1: \quad & \mathbf{u}^{\ddagger}\mathbf{u} && (57)\\
& = (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y)^{\ddagger}(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (58)\\
& = (a - x\sigma_x - y\sigma_y - b\sigma_x \wedge \sigma_y)(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (59)\\
& = a^2 - x^2 - y^2 + b^2 && (60)\\
2: \quad & \det \mathbf{M} && (61)\\
& = \det \left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right] && (62)\\
& = (a+x)(a-x) - (y-b)(y+b) && (63)\\
& = a^2 - x^2 - y^2 + b^2 && (64)
\end{aligned}$$

□

This theorem establishes a connection between the determinant and a multivector self-product form. By expressing the determinant in a self-product form, we can later restrict our attention to the even subalgebra of GA(2), where this self-product form becomes positive-definite, thus yielding a proper inner product. This construction will be essential for developing quantum mechanical structures such as probability measures and observables within the geometric algebra framework.

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 4 (Multivector Conjugate Transpose)**.** *Let* $V \in (\mathrm{GA}(2))^n$*:*

$$V = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{65}$$

The multivector conjugate transpose of V is defined as first taking the transpose and then the element-wise multivector conjugate:

$$V = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \ldots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{66}$$

Definition 5 (Bilinear Form)**.** *Let V and W be two vectors valued in $\mathrm{GA}(2)$. We introduce the following bilinear form:*

$$VW = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \ldots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \tag{67}$$

Theorem 3 (Inner Product)**.** *Restricted to the even sub-algebra of $\mathrm{GA}(2)$, the bilinear form is an inner product.*

Proof.

$$VW_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \ldots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{68}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \sigma_x \wedge \sigma_y$. ☐

Let us now solve the optimization problem for the even multivectors of $\mathrm{GA}(2,0)$, whose inner product is positive-definite.

We take $a \to 0, \mathbf{x} \to 0$ then $\mathbf{M}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a\to 0, \mathbf{x}\to 0} = \mathbf{b} \implies \mathbf{M} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \tag{69}$$

The Lagrange multiplier equation can be solved as follows:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}[\rho_1, \ldots, \rho_n]}{\partial \rho_i} && (70)\\
&= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} && (71)\\
&= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} && (72)\\
\implies \ln \frac{\rho_i}{p_i} &= -p_i - \lambda - \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} && (73)\\
\implies \rho_i &= p_i \exp(-p_i - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \right) && (74)\\
&= \frac{1}{Z(\theta)} p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \right) && (75)
\end{aligned}$$

The partition function $Z(\theta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{76}$$

$$\implies (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{77}$$

$$Z(\theta) := \sum_i p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{78}$$

Consequently, the optimal probability measure that connects an initial preparation p_i to a final measurement ρ_i, in 2D is:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp\left(-\frac{1}{2}\theta \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det \exp\left(-\frac{1}{2}\theta \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right)}_{\text{Spin(2) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{79}$$

Definition 6 (Spin(2)-valued Wavefunction)**.**

$$\psi = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 \\ \vdots \\ \sqrt{\rho_2} R_2 \end{bmatrix} \tag{80}$$

where $\sqrt{\rho_i} = e^{\frac{1}{2}a_i}$ *representing the square root of the probability and* $R_i = e^{\frac{1}{2}\mathbf{b}_i}$ *representing a rotor in 2D.*

The partition function of the probability measure can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

Theorem 4 (Partition Function)**.** $Z = \psi\psi$

Proof.

$$\psi\psi = \sum_i \psi_i^\ddagger \psi_i = \sum_i \rho_i R_i^\ddagger R_i = \sum_i \rho_i = Z \tag{81}$$

□

Definition 7 (Spin(2)-valued Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \tag{82}$$

Theorem 5. *The partition function is invariant with respect to the Spin(2)-valued evolution operator.*

Proof. We note that:

$$T\mathbf{v}T\mathbf{v} = \mathbf{v}\mathbf{v} = \mathbf{v}^\ddagger T^\ddagger T\mathbf{v} \implies T^\ddagger T = I \tag{83}$$

then, since $\begin{bmatrix} e^{\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} = I$, the relation $T^\ddagger T = I$ is satisfied. □

We note that the even sub-algebra of GA(2), being closed under addition and multiplication and constituting an inner product through its bilinear form, allows for the construction of a Hilbert space. In this context, the Hilbert space is Spin(2)-valued. The primary distinction between a wavefunction in a complex Hilbert space and one in a Spin(2)-valued Hilbert space lies in the subject matter of the theory. Specifically, in the latter, the construction governs the change in orientation experienced by an observer (versus change in time), which in turn dictates the measurement basis used in the experiment, consistently with the rotational symmetry and freedom of the system.

The dynamics of observer orientation transformations are described by a variant of the Schrödinger equation, which is derived by taking the derivative of the wavefunction with respect to the Lagrange multiplier, θ:

Definition 8 (Spin(2)-valued Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix}\psi_1(\theta)\\ \vdots \\ \psi_n(\theta)\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\mathbf{b}_1 & & \\ & \ddots & \\ & & -\frac{1}{2}\mathbf{b}_n\end{bmatrix}\begin{bmatrix}\psi_1(\theta)\\ \vdots \\ \psi_n(\theta)\end{bmatrix} \tag{84}$$

Here, θ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the Spin(2), locally across the states of the Hilbert space. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction for the Spin(2) group.

Definition 9 (David Hestenes' Formulation)**.** *In 3+1D, the David Hestenes' formulation [5] of the wavefunction is* $\psi = \sqrt{\rho}Re^{ib/2}$*, where* $R = e^{\mathbf{f}/2}$ *is a Lorentz boost or rotation and where* $e^{ib/2}$ *is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to* $\psi = \sqrt{\rho}R$*, which is the form we have recovered.*

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 10 (Dirac Current)**.** *Given the basis* σ_x *and* σ_y*, the Dirac current for the 2D theory is defined as:*

$$J_x \equiv \psi^\ddagger\sigma_x\psi = \rho\underbrace{R^\ddagger\sigma_x R}_{\mathrm{SO}(2)} = \rho\tilde{\sigma}_x \tag{85}$$

$$J_y \equiv \psi^\ddagger\sigma_y\psi = \rho\underbrace{R^\ddagger\sigma_y R}_{\mathrm{SO}(2)} = \rho\tilde{\sigma}_y \tag{86}$$

where $\tilde{\sigma}_x$ *and* $\tilde{\sigma}_y$ *are a SO(2) rotated basis vectors.*

2.2.1. 1+1D Obstruction

As stated in the introduction, of the dimensional cases, only 2D and 3+1D are free of obstructions. For instance, the 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\mathbf{e}_0 \wedge \mathbf{e}_1)(a + b\mathbf{e}_0 \wedge \mathbf{e}_1)$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. This is why we had to use 2D instead of 1+1D in this two-dimensional introduction. In the following section, we will investigate the 3+1D case, then we will show why all other dimensional cases are obstructed.

2.3. RQM in 3+1D

Extending the framework to relativistic quantum mechanics begins by considering measurements relative to Spin^c(3,1) symmetries. This allows for transformations that include boosts, rotations, and

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phases, enabling relativistic consistency within the same entropy-based framework. The relevant Lagrange multiplier equation is as follows:

$$\mathcal{L} = -\sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i\right) + \frac{1}{2}\zeta \mathrm{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right) \tag{87}$$

where

$$\mathbf{M}_i = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix}, \tag{88}$$

Here, $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the Spin^c(3,1) group, which includes both Lorentz transformations and U(1) phase rotations.

The solution (proof in Annex B) is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det\exp(-\frac{1}{2}\zeta \mathbf{M}_i)}}_{\text{Spin}^c\text{(3,1) Invariant Ensemble}} \underbrace{\det\exp(-\frac{1}{2}\zeta \mathbf{M}_i)}_{\text{Spin}^c\text{(3,1) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{89}$$

where ζ is a "twisted-phase" rapidity. As we will argue in Section 2.4, this probability measure is the most sophisticated solution to the general optimization problem (Axiom 1) that is free of obstructions.

2.3.1. Preliminaries

As we did in the 2D case, our initial goal here also will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra $\mathrm{GA}(3,1)$.

Definition 11 (Multivector). *Let* **u** *be a multivector of* $\mathrm{GA}(3,1)$*. Its general form is:*

$$\mathbf{u} = a \tag{90}$$

$$+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \tag{91}$$

$$+ f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 + f_{13}\gamma_1 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 \tag{92}$$

$$+ p\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + q\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + w\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \tag{93}$$

$$+ b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \tag{94}$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ *are the basis vectors in the real Majorana representation.*

A more compact notation for **u** *is*

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{95}$$

where a is a scalar, **x** *a vector,* **f** *a bivector,* **v** *is pseudo-vector and* **b** *a pseudo-scalar.*

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation:

Definition 12 (Matrix Representation of $\mathbf{u}$)**.**

$$\mathbf{M} = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix} \tag{96}$$

To manipulate and analyze multivectors in $\mathrm{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 13 (Multivector Conjugate (in 4D))**.**

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{97}$$

Definition 14 (3,4 Blade Conjugate)**.** *The 3,4 blade conjugate of* $\mathbf{u}$ *is*

$$\lfloor \mathbf{u} \rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{98}$$

Lundholm[6] proposes a number the multivector norms, and shows that they are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 15. *The self-products associated with low-dimensional geometric algebras are:*

$$\begin{aligned} \mathrm{GA}(0,1): &\quad \varphi^{\dagger}\varphi \qquad (99) \\ \mathrm{GA}(2,0): &\quad \varphi^{\ddagger}\varphi \qquad (100) \\ \mathrm{GA}(3,0): &\quad \lfloor \varphi^{\ddagger}\varphi \rfloor_{3} \varphi^{\ddagger}\varphi \qquad (101) \\ \mathrm{GA}(3,1): &\quad \lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi \qquad (102) \\ \mathrm{GA}(4,1): &\quad (\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi)^{\dagger}(\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi) \qquad (103) \end{aligned}$$

We can now express the determinant of the matrix representation of a multivector via the self-product $\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi$. This choice is not arbitrary, but the unique choice with allows us to represent the determinant of the matrix representation of a multivector within $\mathrm{GA}(3,1)$:

Theorem 6 (Determinant as a Multivector Self-Product)**.**

$$\lfloor \mathbf{u}^{\ddagger}\mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger}\mathbf{u} = \det \mathbf{M} \tag{104}$$

Proof. Please find a computer assisted proof of this equality in Annex C. □

As can be seen from this theorem, the relationship between determinants and multivector products becomes more sophisticated in 3+1D. Unlike the 2D case where the determinant could be expressed using a product of two terms, in GA(3,1) the determinant requires two products involving four copies of the multivector. This is reflected in the structure $\lfloor \mathbf{u}^{\ddagger}\mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger}\mathbf{u}$, which cannot be reduced to a simpler self-product of two terms. However, when we restrict to the even subalgebra of GA(3,1), this quadruple product does become positive-definite as it did in GA(2), making it suitable as a probability measure despite having four terms.

With a definition of $\mathrm{GA}(3,1)$-valued vectors:

Definition 16 (GA$(3,1)$-valued Vector)**.**

$$V=\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_n\end{bmatrix}=\begin{bmatrix}a_1+\mathbf{x}_1+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_1\\ \vdots\\ a_n+\mathbf{x}_n+\mathbf{f}_n+\mathbf{v}_n+\mathbf{b}_n\end{bmatrix}\tag{105}$$

we are subsequently able to express the partition function in terms of the multivector self-product:

Definition 17 (Double-Copy Product)**.** *Instead of an inner product, we obtain what we call a double-copy product:*

$$VVVV=\sum_i\lfloor\underbrace{\psi_i^\ddagger\psi_i}_{copy\ 1}\rfloor_{3,4}\underbrace{\psi_i^\ddagger\psi_i}_{copy\ 2}\tag{106}$$

$$=\underbrace{\lfloor\begin{bmatrix}\mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n\end{bmatrix}\begin{bmatrix}\mathbf{u}_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathbf{u}_n\end{bmatrix}\rfloor_{3,4}}_{copy\ 1}\underbrace{\begin{bmatrix}\mathbf{u}_1^\ddagger & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathbf{u}_n^\ddagger\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_n\end{bmatrix}}_{copy\ 2}\tag{107}$$

Theorem 7 (Partition Function)**.** $Z=VVVV$

Proof.

$$VVVV\tag{108}$$

$$=\lfloor\begin{bmatrix}\mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n\end{bmatrix}\begin{bmatrix}\mathbf{u}_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathbf{u}_n\end{bmatrix}\rfloor_{3,4}\begin{bmatrix}\mathbf{u}_1^\ddagger & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathbf{u}_n^\ddagger\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_n\end{bmatrix}\tag{109}$$

$$=\lfloor\begin{bmatrix}\mathbf{u}_1^\ddagger\mathbf{u}_1 & \dots & \mathbf{u}_n\mathbf{u}_n\end{bmatrix}\rfloor_{3,4}\begin{bmatrix}\mathbf{u}_1^\ddagger\mathbf{u}_1\\ \vdots\\ \mathbf{u}_n^\ddagger\mathbf{u}_n\end{bmatrix}\tag{110}$$

$$=\lfloor\mathbf{u}_1^\ddagger\mathbf{u}_1\rfloor_{3,4}\mathbf{u}_1^\ddagger\mathbf{u}_1+\dots+\lfloor\mathbf{u}_n^\ddagger\mathbf{u}_n\rfloor_{3,4}\mathbf{u}_n^\ddagger\mathbf{u}_n\tag{111}$$

$$=\sum_{i=1}^n\det\mathbf{M}_{\mathbf{u}_i}\tag{112}$$

$$=Z\tag{113}$$

□

Desirable properties for the double-copy product are introduced by addressing the issue of non-positivity. First, we establish non-negativity:

Theorem 8 (Non-negativity)**.** *The double-copy product, applied to the even subalgebra of* GA$(3,1)$*, is always non-negative.*

Proof. Let $V=\begin{bmatrix}a_1+\mathbf{f}_1+\mathbf{b}_1\\ \vdots\\ a_n+\mathbf{f}_n+\mathbf{b}_n\end{bmatrix}$. Then,

$$VVVV \tag{114}$$

$$= \lfloor \begin{bmatrix} (a_1+\mathbf{f}_1+\mathbf{b}_1)^\ddagger(a_1+\mathbf{f}_1+\mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1+\mathbf{f}_1+\mathbf{b}_1)^\ddagger(a_1+\mathbf{f}_1+\mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{115}$$

$$= \lfloor \begin{bmatrix} (a_1-\mathbf{f}_1+\mathbf{b}_1)(a_1+\mathbf{f}_1+\mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1-\mathbf{f}_1+\mathbf{b}_1)(a_1+\mathbf{f}_1+\mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{116}$$

$$= \lfloor \begin{bmatrix} a_1^2 + a_1\mathbf{f}_1 + a_1\mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1\mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1\mathbf{f}_1 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{117}$$

$$= \lfloor \begin{bmatrix} a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{118}$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3) & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{119}$$

We note that the terms are now complex numbers, which we rewrite as $\Re(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\Im(z) = -4(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} z_1 & \dots & z_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{120}$$

$$= \begin{bmatrix} z_1^\dagger & \dots & z_2^\dagger \end{bmatrix} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{121}$$

$$= z_1^\ddagger z_1 + \dots + z_n^\ddagger z_n \tag{122}$$

which is always non-negative. $\square$

Finally, *positive-definiteness* is automatically achieved because solving the optimization problem exponentiates the multivector, yielding:

Definition 18 (Spin$^c(3,1)$ Wavefunction)**.**

$$\psi = \begin{bmatrix} e^{\frac{1}{2}(a_1+\mathbf{f}_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n+\mathbf{f}_n+\mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1}\, R_1\, B_1 \\ \vdots \\ \sqrt{\rho_n}\, R_n\, B_n \end{bmatrix},$$

where:

- $\sqrt{\rho_i} = e^{\frac{1}{2}a_i} > 0$ *is a positive scalar factor ensuring non-negativity and yielding a single zero element when also including* $\rho_i = 0$ *to the set of valid wavefunctions.*
- $R_i = e^{\frac{1}{2}\mathbf{f}_i}$ *is a rotor representing Lorentz transformations (rotations and boosts in spacetime).*
- $B_i = e^{\frac{1}{2}\mathbf{b}_i}$ *is a complex phase factor, as* $\mathbf{b}_i = b_i I$ *and* $e^{\frac{1}{2}\mathbf{b}_i} = e^{\frac{1}{2}b_i I} = \cosh\left(\frac{b_i}{2}\right) + I\sinh\left(\frac{b_i}{2}\right)$.

In this representation:

- The exponential map $e^{\frac{1}{2}(\mathbf{f}_i+\mathbf{b}_i)}$ maps elements of the algebra to the connected component of the identity in the spin group Spin$^c(3,1)$.

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- The wavefunction ψ captures both the amplitude (through $\sqrt{\rho_i}$) and the phase (through R_i and B_i) of the quantum state.
- The wavefunction ψ is an element of the even subalgebra of GA(3,1).

Thus, the double-copy product $\psi\psi\psi\psi$ over wavefunction ψ is positive-definite.

Now, let us turn our attention to the evolution operator, which leaves the partition function invariant:

Definition 19 (Spin$^c(3,1)$ Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1+\mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n+\mathbf{b}_n)} \end{bmatrix} \tag{123}$$

In turn, this leads to a variant of the Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ:

Definition 20 (Spin$^c(3,1)$-valued Schrödinger equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\mathbf{f}_1+\mathbf{b}_1) & & \\ & \ddots & \\ & & -\frac{1}{2}(\mathbf{f}_n+\mathbf{b}_n) \end{bmatrix} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} \tag{124}$$

In this case ζ represents a one-parameter evolution parameter akin to time, which is able to transform the measurement basis under action of the Spin$^c(3,1)$ group. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the Spin$^c(3,1)$-wavefunction.

Theorem 9 (Spin^c(3,1) invariance)**.** *Let $e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}$ be a general element of Spin^c(3,1). Then, the equality:*

$$\lfloor \psi^\ddagger \psi \rfloor_{3,4} \psi^\ddagger \psi = \lfloor (e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \rfloor_{3,4} (e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \tag{125}$$

is always satisfied.

Proof.

$$\begin{aligned} &\lfloor (e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \rfloor_{3,4} (e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi && (126) \\ &\quad = \lfloor \psi^\ddagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \rfloor_{3,4} \psi^\ddagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi && (127) \\ &\quad = \lfloor \psi^\ddagger e^{\mathbf{b}}\psi \rfloor_{3,4} \psi^\ddagger e^{\mathbf{b}}\psi && (128) \\ &\quad = \lfloor \psi^\ddagger \psi \rfloor_{3,4} e^{-\mathbf{b}}e^{\mathbf{b}}\psi^\ddagger \psi && (129) \\ &\quad = \lfloor \psi^\ddagger \psi \rfloor_{3,4} \psi^\ddagger \psi && (130) \end{aligned}$$

□

2.3.2. RQM

Definition 21 (David Hestenes' Wavefunction)**.** *The* Spin$^c(3,1)$*-valued wavefunction we have recovered is formulated identically to David Hestenes'[5] formulation of the wavefunction within GA(3,1).*

$$\psi = \underbrace{e^{\frac{1}{2}(a+\mathbf{f}+\mathbf{b})}}_{ours} = \underbrace{\sqrt{\rho}Re^{-ib/2}}_{Hestenes'} \tag{131}$$

where $e^{\frac{1}{2}a} = \sqrt{\rho}$, $e^{\frac{1}{2}\mathbf{f}} = R$ *and* $e^{\frac{1}{2}\mathbf{b}} = e^{-ib/2}$.

Before we continue the RQM investigation, let us note that the double-copy product contains two copies of a bilinear form $\psi^\ddagger\psi$:

$$\lfloor \underbrace{\psi^\ddagger\psi}_{\text{copy 1}} \rfloor_{3,4} \underbrace{\psi^\ddagger\psi}_{\text{copy 2}} \tag{132}$$

In the present and upcoming section, we will investigate the properties of each copy individually, leaving the properties specific to the double-copy for the section on quantum gravity.

Taking a single copy, the Dirac current is obtained directly from the gamma matrices, as follows:

Definition 22 (Dirac Current)**.** *The definition of the Dirac current is the same as Hestenes':*

$$J \equiv \psi^\ddagger\gamma_\mu\psi = \rho R^\ddagger B^\ddagger\gamma_\mu BR = \rho R^\ddagger\gamma_\mu B^{-1}BR = \rho \underbrace{R^\ddagger\gamma_\mu R}_{\textit{SO(3,1)}} = \rho\tilde{\gamma}_\mu \tag{133}$$

where $\tilde{\gamma}_\mu$ *is a SO(3,1) rotated basis vector.*

2.3.3. Standard Model Gauge Symmetries

Based on the results of Hestenes and Lasenby [7,8], we will now demonstrate that the double-copy product is automatically invariant under transformations corresponding to the U(1), SU(2), and SU(3) symmetries, as well as under unitary transformations satisfying $U^\dagger U = I$, all of which play fundamental roles in the Standard Model of particle physics. These symmetries constitute the set of transformations that leave the Dirac current invariant, i.e., $(T\psi)^\ddagger\gamma_0 T\psi = \psi^\ddagger\gamma_0\psi$ with T valued in $\mathrm{GA}(3,1)$.

Theorem 10 (U(1) Invariance)**.** *Let* $e^{\frac{1}{2}\mathbf{b}}$ *be a general element of U(1). Then, the equality*

$$\lfloor\psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor\underbrace{(e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi}_{\textit{copy 1}}\rfloor_{3,4}\underbrace{(e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi}_{\textit{copy 2}} \tag{134}$$

is satisfied, yielding a U(1) symmetry for each copied bilinear form.

Proof. Equation 134 is invariant if this expression is satisfied:

$$e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 \tag{135}$$

This is always satisfied simply because $e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 e^{-\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{b}} = \gamma_0$ □

Theorem 11 (SU(2) Invariance)**.** *Let* $e^{\frac{1}{2}\mathbf{f}}$ *be a general element of Spin(3,1). Then, the equality:*

$$\lfloor\psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor\underbrace{(e^{\frac{1}{2}\mathbf{f}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi}_{\textit{copy 1}}\rfloor_{3,4}\underbrace{(e^{\frac{1}{2}\mathbf{f}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi}_{\textit{copy 2}} \tag{136}$$

is satisfied for if $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$ *(which generates SU(2)), yielding a SU(2) symmetry for each copied bilinear form.*

Proof. Equation 136 is invariant if this expression is satisfied[7]:

$$e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0 \tag{137}$$

We now note that moving the left-most term to the right of the gamma matrix yields:

$$e^{-E_1\gamma_0\gamma_1-E_2\gamma_0\gamma_2-E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2}\gamma_0 e^{\frac{1}{2}\mathbf{f}} \tag{138}$$

$$= \gamma_0 e^{E_1\gamma_0\gamma_1+E_2\gamma_0\gamma_2+E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2} e^{\frac{1}{2}\mathbf{f}} \tag{139}$$

Therefore, the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $E_1 = E_2 = E_3 = 0$, leaving $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$:

Finally, we note that $e^{\theta_1\gamma_2\gamma_3+\theta_2\gamma_1\gamma_3+\theta_3\gamma_1\gamma_2}$ generates SU(2). □

Theorem 12 (SU(3))**.** *The equality*

$$\lfloor \psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor \underbrace{(\mathbf{f}\psi)^\ddagger\gamma_0\mathbf{f}\psi}_{\textit{copy 1}}\rfloor_{3,4}\underbrace{(\mathbf{f}\psi)^\ddagger\gamma_0\mathbf{f}\psi}_{\textit{copy 2}} \tag{140}$$

where **f** *is a bivetor, satisfies SU(3).*

Proof. First, we note the following action:

$$-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0 \tag{141}$$

which we can rewrite as follows:

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \tag{142}$$

The first three terms anticommute with γ_0, while the last three commute with γ_0:

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2(q)\gamma_1\gamma_3 - B_3(q)\gamma_1\gamma_2)\mathbf{f}(q) \tag{143}$$

This can be written as:

$$\gamma_0(\mathbf{E}-\mathbf{B})(\mathbf{E}+\mathbf{B}) \tag{144}$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{EB} - \mathbf{BE} - \mathbf{B}^2) \tag{145}$$

where $\mathbf{E} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3$ and $\mathbf{B} = B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$.

Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we require: 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{EB} = \mathbf{BE}$. The first requirement expands as follows:

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \tag{146}$$

which, as the norm of three complex numbers, is the defining conditions for the SU(3) symmetry group. □

We can show that **f** is invertible by showing that the norm is never 0:

$$norm(\mathbf{f}) = \mathbf{f}^2 = \mathbf{E}^2 + \mathbf{B}^2 + 2i\mathbf{E}\cdot\mathbf{B} \neq 0 \tag{147}$$

We note that for $norm(\mathbf{f}) = 0$, it is necessary but not sufficient that $\mathbf{E}\cdot\mathbf{B} = 0$, which is a dot product. However, the condition 2) $\mathbf{EB} = \mathbf{BE} \implies \mathbf{BE} - \mathbf{EB} = 0$ is a cross-product. Since it cannot be the case that the dot product and the cross-product of two vectors both be zero, it must be the case that $\mathbf{E}\cdot\mathbf{B} \neq 0$. Therefore **f** is invertible.

Finally, since **f** is invertible, there exists an exponential form [5] where $\mathbf{f} = \exp(\alpha + F + \beta)$. As such **f** can be interpretation as a wavefunction state, and its invariance with respect to the SU(3) symmetry means that the double-copy supports invariant SU(3) evolution natively.

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Theorem 13 (Unitary Invariance)**.** *Let U be $n \times n$ unitary matrices. Then unitary invariance:*

$$\psi\gamma_\mu\psi\psi\gamma_\nu\psi = U\psi\gamma_\mu U\psi U\psi\gamma_\nu U\psi \implies U^\dagger U = I \tag{148}$$

is individually satisfied for each copied bilinear form.

Proof. Equation 148 is satisfied if $U^\ddagger\gamma_\mu U = \gamma_\mu$. Since U is valued in complex numbers, then $U^\ddagger = U^T$, and since $\gamma_\mu\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\mu$, it follows that:

$$\gamma_\mu U^\dagger U = \gamma_\mu \tag{149}$$

which is satisfied when $U^\dagger U = I$. □

Finally, we note that the invariances SU(3), SU(2) and U(1) discussed above can be promoted to local symmetries using standard gauge theory construction techniques.

In conventional QM, the Born rule naturally leads to a U(1)-valued gauge theory due to the following symmetry:

$$(e^{-i\theta(x)}\psi(x))^\dagger e^{-i\theta(x)}\psi(x) = \psi(x)^\dagger\psi(x) \tag{150}$$

However, the $\mathrm{SU}(3)$ and $\mathrm{SU}(2)$ symmetries do not emerge from the probability measure in the same straightforward manner and are typically introduced by hand, justified by experimental observations. This raises the question: why these specific symmetries and not others? In contrast, within our framework, all three symmetry groups—$\mathrm{U}(1)$, $\mathrm{SU}(2)$, and $\mathrm{SU}(3)$—as well as the $\mathrm{Spin}(3,1)$ and unitary symmetries, follow naturally from the invariance of the probability measure, in the same way that the $\mathrm{U}(1)$ symmetry follows from the Born rule. This suggests a deeper underlying principle governing the symmetries in fundamental physics.

2.3.4. A Starting Point for a Theory of Quantum Gravity

In the previous section, we developed a quantum theory valued in Spin^c(3,1), which served as the arena for RQM. We then demonstrated how a single copy of this theory leads to the gauge symmetries of the standard model, the Dirac current and other features of RQM. The goal of this section is to extend this methodology to basis vectors, in which the metric tensor emerges as an observable. To achieve this, we will utilize both copies of the double-copy product.

We recall the definition of the metric tensor in terms of basis vectors of geometric algebra, as follows:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) \tag{151}$$

Then, we note that the double-copy product acts on a pair of basis element $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$, as follows:

$$\begin{aligned}
&\frac{1}{2}\Big(\lfloor\underbrace{\psi^\ddagger\mathbf{e}_\mu\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{\psi^\ddagger\mathbf{e}_\nu\psi}_{\text{copy 2}} + \lfloor\underbrace{\psi^\ddagger\mathbf{e}_\nu\varphi}_{\text{copy 2}}\rfloor_{3,4}\underbrace{\psi^\ddagger\mathbf{e}_\mu\psi}_{\text{copy 1}}\Big) && (152)\\
&= \frac{1}{2}\Big(\tilde{R}\underbrace{\rho e^{ib/2}e^{-ib/2}}_{\text{Born rule copy 1}}\mathbf{e}_\mu R\tilde{R}\underbrace{\rho e^{-ib/2}e^{ib/2}}_{\text{Born rule copy 2}}\mathbf{e}_\nu R + \tilde{R}\underbrace{\rho e^{ib/2}e^{-ib/2}}_{\text{Born rule copy 2}}\mathbf{e}_\nu R\tilde{R}\underbrace{\rho e^{-ib/2}e^{ib/2}}_{\text{Born rule copy 1}}\mathbf{e}_\mu R\Big) && (153)\\
&= \frac{1}{2}\rho^2(\tilde{R}\mathbf{e}_\mu R\tilde{R}\mathbf{e}_\nu R + \tilde{R}\mathbf{e}_\nu R\tilde{R}\mathbf{e}_\nu R) && (154)\\
&= \underbrace{\rho^2}_{\text{probability}}\underbrace{\frac{1}{2}(\tilde{\mathbf{e}}_\mu\tilde{\mathbf{e}}_\nu + \tilde{\mathbf{e}}_\nu\tilde{\mathbf{e}}_\mu)}_{\text{metric tensor}} && (155)
\end{aligned}$$

where $\tilde{\mathbf{e}}_\mu$ and $\tilde{\mathbf{e}}_\nu$ are SO(3,1) rotated basis vectors, and where ρ^2 is a probability measure.

As one can swap $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ and obtain the same metric tensor, the double-copy product guarantees that $g_{\mu\nu}$ is symmetric.

Furthermore, since $\mathbf{e}_\mu^\ddagger = -\mathbf{e}_\mu$, we get:

$$\begin{aligned} &\lfloor(\mathbf{e}_\mu\psi)^\ddagger\psi\rfloor_{3,4}(\mathbf{e}_\nu\psi)^\ddagger\psi && (156)\\ &\quad=\lfloor\psi^\ddagger(-1)\mathbf{e}_\mu^\ddagger\psi\rfloor_{3,4}\psi^\ddagger(-1)\mathbf{e}_\nu^\ddagger\psi && (157)\\ &\quad=\lfloor\psi^\ddagger\mathbf{e}_\mu\psi\rfloor_{3,4}\psi^\ddagger\mathbf{e}_\nu\psi && (158)\end{aligned}$$

which allows us to conclude that $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ are self-adjoint within the double-copy product, entailing the interpretation of $g_{\mu\nu}$ as an observable.

In the double-copy product, the metric tensor emerges as a double copy of Dirac currents. This formulation suggests that the metric tensor encodes the probabilistic structure of a quantum theory of gravity *in the form of a symmetric rank-2 tensor*, analogous to how the Dirac current encodes the probabilistic structure of a special relativistic quantum theory *in the form of a 4-vector*.

Let us now investigate the dynamics. We recall that the evolution operator (Definition 19) is:

$$T=\begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1+\mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n+\mathbf{b}_n)}\end{bmatrix} \tag{159}$$

Acting on the wavefunction, the effect of this operator cascades down to the basis vectors via the double-copy product:

$$\lfloor\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\mu T\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\nu T\psi}_{\text{copy 2}}+\lfloor\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\nu T\psi}_{\text{copy 2}}\rfloor_{3,4}\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\mu T\psi}_{\text{copy 1}} \tag{160}$$

which realizes an $\mathrm{SO}(3,1)$ transformation of the metric tensor via action of the exponential of a bivector, and a double-copy unitary invariant transformation via action of the exponential of a pseudo-scalar:

$$\underbrace{\lfloor\psi^\ddagger\underbrace{e^{\frac{1}{2}\zeta\mathbf{f}}\mathbf{e}_\mu e^{-\frac{1}{2}\zeta\mathbf{f}}}_{\text{SO(3,1) evolution}}\underbrace{e^{\frac{1}{2}\zeta\mathbf{b}}e^{-\frac{1}{2}\zeta\mathbf{b}}}_{\text{unitary evolution}}\psi\rfloor_{3,4}}_{\text{copy 1}}\underbrace{\psi^\ddagger\underbrace{e^{\frac{1}{2}\zeta\mathbf{f}}\mathbf{e}_\mu e^{-\frac{1}{2}\zeta\mathbf{f}}}_{\text{SO(3,1) evolution}}\underbrace{e^{\frac{1}{2}\zeta\mathbf{b}}e^{-\frac{1}{2}\zeta\mathbf{b}}}_{\text{unitary evolution}}\psi}_{\text{copy 2}}+\cdots \tag{161}$$

In summary, this initial investigation has identified a scenario in which the metric tensor is measured using basis vectors. The evolution operator, governed by an adapted Schrödinger equation, dynamically realizes SO(3,1) transformations on the metric tensor. Furthermore, the amplitudes associated with possible metric tensors are derived from a double-copy of unitary quantum theories acting on the basis vectors. This formulation simultaneously preserves the SO(3,1) symmetry, essential for describing spacetime structure, and the unitary symmetry, fundamental to quantum mechanics. It describes all changes of basis transformations that an observer in 3+1D spacetime can perform prior to measuring (in the quantum sense) a basis system in spacetime, and attributes a probability to the outcome (the outcome being the metric tensor).

2.3.5. The Einstein Field Equation

In the previous section, we established that the wavefunction transforms under global SO(3,1) Lorentz transformations. To promote this to a local symmetry where the transformation parameters can vary with position x^μ, we need to introduce a gauge connection ω_μ^{ab} (the spin connection) to construct a covariant derivative[9]:

$$D_\mu=\partial_\mu+\omega_\mu^{ab}S_{ab} \tag{162}$$

where S_{ab} are the generators of SO(3,1).

Under local SO(3,1) transformations, the spin connection transforms inhomogeneously:

$$\omega_\mu^{ab} \to \omega_\mu'^{ab} = \Lambda^a_c(x)\Lambda^b_d(x)\omega_\mu^{cd} + \Lambda^a_c(x)\partial_\mu \Lambda^b_c(x) \tag{163}$$

The commutator of two covariant derivatives defines the Riemann curvature tensor:

$$[D_\mu, D_\nu] = R^{ab}_{\mu\nu}(\omega)S_{ab} \tag{164}$$

where $R^{ab}_{\mu\nu}(\omega)$ is the curvature of the spin connection:

$$R^{ab}_{\mu\nu}(\omega) = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac}\omega^b_{c\nu} - \omega_\nu^{ac}\omega^b_{c\mu} \tag{165}$$

This Riemann tensor represents the field strength of the SO(3,1) gauge field, analogous to how $F_{\mu\nu}$ represents the field strength in electromagnetism. From here, we can construct the Ricci tensor through contraction of the Riemann tensor in its Lorentz indices.

Thus, gauging the SO(3,1) symmetry naturally leads to the geometric structure of curved spacetime through the Riemann curvature tensor, which encodes how vectors change under parallel transport around infinitesimal loops. The Einstein field equations then follow from varying an action constructed from these geometric objects.

2.3.6. Linearized Quantum Gravity

The usual perturbation approach to quantum gravity directly promote $h_{\mu\nu}$ from $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ to an operator $\hat{h}_{\mu\nu}$ and compute its expectation values using the Born rule. However, our probability measure assigns an expectation values to the geometric algebra basis operators $\hat{\mathbf{e}}_\mu$ and $\hat{\mathbf{e}}_\nu$, yielding a metric tensor expectation value via the double-copy. This suggests we should quantize the basis vectors rather than the metric tensor. To explore this approach, we examine linearized gravity and we express the basis vector as a perturbation of a flat spacetime metric $\mathbf{e}_\mu = \gamma_\mu + h_\mu$. The metric tensor then becomes:

$$\begin{aligned}
g_{\mu\nu} &= \frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu) && (166)\\
&= \frac{1}{2}((\gamma_\mu + h_\mu)(\gamma_\nu + h_\nu) + (\gamma_\nu + h_\nu)(\gamma_\mu + h_\mu)) && (167)\\
&= \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\mu h_\nu + h_\mu\gamma_\nu + h_\mu h_\nu + \gamma_\nu\gamma_\mu + \gamma_\nu h_\mu + h_\nu\gamma_\mu + h_\nu h_\mu) && (168)\\
&= \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) + \frac{1}{2}(\gamma_\mu h_\nu + h_\mu\gamma_\nu + \gamma_\nu h_\mu + h_\nu\gamma_\mu) + \frac{1}{2}(h_\mu h_\nu + h_\nu h_\mu) && (169)\\
&= \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) + \frac{1}{2}(\gamma_\mu h_\nu + h_\mu\gamma_\nu - h_\mu\gamma_\nu - \gamma_\mu h_\nu) + \frac{1}{2}(h_\mu h_\nu + h_\nu h_\mu) && (170)\\
&= \underbrace{\frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)}_{\eta_{\mu\nu}} + \underbrace{\frac{1}{2}(h_\mu h_\nu + h_\nu h_\mu)}_{h_{\mu\nu}} && (171)
\end{aligned}$$

It is well known that working in de Donder gauge $\partial^\alpha h_{\alpha\mu} - \frac{1}{2}\partial_\mu h = 0$, where $h = \eta^{\mu\nu}h_{\mu\nu}$, the Einstein-Hilbert action reduces to its linearized form as follows:

$$S^{(1)}_{EH}[h_{\mu\nu}] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu h_{\rho\sigma}\partial^\mu h^{\rho\sigma} - \frac{1}{4}\partial_\mu h \partial^\mu h\right) \tag{172}$$

Furthermore, varying this action with respect to $h_{\mu\nu}$ and applying the transverse-traceless gauge, the wave equation follows as the equation of motion:

$$\Box h_{\mu\nu} = 0 \tag{173}$$

This result will be our starting point, however consistently with our notation, we will express $h_{\mu\nu}$ using the geometric algebra notation:

$$\Box(h_\mu h_\nu + h_\nu h_\mu) = 0 \tag{174}$$

The solutions in terms of modes are as follows:

$$h_\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \Big(\epsilon_\mu^{(\lambda)}(\vec{k}) a_{\vec{k},\lambda} e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \epsilon_\mu^{(\lambda)*}(\vec{k}) a^*_{\vec{k},\lambda} e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)} \Big) \tag{175}$$

$$h_\nu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \Big(\epsilon_\nu^{(\lambda)}(\vec{k}) a_{\vec{k},\lambda} e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \epsilon_\nu^{(\lambda)*}(\vec{k}) a^*_{\vec{k},\lambda} e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)} \Big) \tag{176}$$

Then, we promote $h_\mu(\vec{x},t)$ and $h_\nu(\vec{x},t)$ to operators:

$$\hat{h}_\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \Big(\epsilon_\mu^{(\lambda)}(\vec{k}) \hat{a}_{\vec{k},\lambda} e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \epsilon_\mu^{(\lambda)*}(\vec{k}) \hat{a}^*_{\vec{k},\lambda} e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)} \Big) \tag{177}$$

$$\hat{h}_\nu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \Big(\epsilon_\nu^{(\lambda)}(\vec{k}) \hat{a}_{\vec{k},\lambda} e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \epsilon_\nu^{(\lambda)*}(\vec{k}) \hat{a}^*_{\vec{k},\lambda} e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)} \Big) \tag{178}$$

Finally, these operators can be used with the double-copy product to produce a metric tensor expectation value:

$$\langle \hat{h}_{\mu\nu} \rangle = \frac{1}{2} \Big(\psi | \hat{h}_\mu | \psi | \psi | \hat{h}_\nu | \psi + \psi | \hat{h}_\nu | \psi | \psi | \hat{h}_\mu | \psi \Big) \tag{179}$$

This ends our investigation of the quantum gravitational theory. The UV-behaviour of the theory will be investigated in a future work.

2.4. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to resolve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except the previously explored cases are obstructed. By obstructed, we mean that the solution to the entropy maximization problem, ρ, does not satisfy all axioms of probability theory.

Dimensions	*Optimal Predictive Theory of Nature*	
GA(0)	**Statistical Mechanics**	(180)
GA(0, 1)	**Quantum Mechanics**	(181)
GA(1, 0)	Obstructed (Negative probabilities)	(182)
GA(2, 0)	**Quantum Mechanics**	(183)
GA(1, 1)	Obstructed (Negative probabilities)	(184)
GA(0, 2)	Obstructed (Non-real probabilities)	(185)
GA(3, 0)	Obstructed (Non-real probabilities)	(186)
GA(2, 1)	Obstructed (Non-real probabilities)	(187)
GA(1, 2)	Obstructed (Non-real probabilities)	(188)
GA(0, 3)	Obstructed (Non-real probabilities)	(189)
GA(4, 0)	Obstructed (Non-real probabilities)	(190)

$$\mathrm{GA}(3,1) \qquad \textbf{Quantum Gravity + Standard Model} \tag{191}$$
$$\mathrm{GA}(2,2) \qquad \text{Obstructed (Negative probabilities)} \tag{192}$$
$$\mathrm{GA}(1,3) \qquad \text{Obstructed (Non-real probabilities)} \tag{193}$$
$$\mathrm{GA}(0,4) \qquad \text{Obstructed (Non-real probabilities)} \tag{194}$$
$$\mathrm{GA}(5,0) \qquad \text{Obstructed (Non-real probabilities)} \tag{195}$$
$$\vdots \qquad \vdots$$
$$\mathrm{GA}(6,0) \qquad \text{Suspected Obstructed (No observables)} \tag{196}$$
$$\vdots \qquad \vdots$$
$$\infty \tag{197}$$

Let us now demonstrate the obstructions mentioned above.

Theorem 14 (Non-real probabilities). *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. These geometric algebras are classified as follows:

$$\mathrm{GA}(0,2) \cong \mathbb{H} \tag{198}$$
$$\mathrm{GA}(3,0) \cong \mathbb{M}_2(\mathbb{C}) \tag{199}$$
$$\mathrm{GA}(2,1) \cong \mathbb{M}_2^2(\mathbb{R}) \tag{200}$$
$$\mathrm{GA}(1,2) \cong \mathbb{M}_2(\mathbb{C}) \tag{201}$$
$$\mathrm{GA}(0,3) \cong \mathbb{H}^2 \tag{202}$$
$$\mathrm{GA}(4,0) \cong \mathbb{M}_2(\mathbb{H}) \tag{203}$$
$$\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H}) \tag{204}$$
$$\mathrm{GA}(0,4) \cong \mathbb{M}_2(\mathbb{H}) \tag{205}$$
$$\mathrm{GA}(5,0) \cong \mathbb{M}_2^2(\mathbb{H}) \tag{206}$$

The determinant of these objects is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. □

Theorem 15 (Negative probabilities). *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable.*

Proof. This category contains three dimensional configurations:

$\mathrm{GA}(1,0)$ Let $\psi = a + be_1$, then:

$$(a+be_1)^\ddagger(a+be_1) = (a-be_1)(a+be_1) = a^2 - b^2 e_1 e_1 = a^2 - b^2 \tag{207}$$

which is valued in $\mathbb{R}$.

$\mathrm{GA}(1,1)$ Let $\psi = a + be_0e_1$, then:

$$(a+be_0e_1)^\ddagger(a+be_0e_1) = (a-be_0e_1)(a+be_0e_1) = a^2 - b^2 e_0e_1e_0e_1 = a^2 - b^2 \tag{208}$$

which is valued in $\mathbb{R}$.

GA(1,2) Let $\psi = a + b e_0 e_\varnothing e_1 e_2$, where $e_0^2 = -1, e_\varnothing^2 = -1, e_1^2 = 1, e_2^2 = 1$, then:

$$\lfloor (a+\mathbf{b})^\ddagger (a+\mathbf{b}) \rfloor_{3,4} (a+\mathbf{b})^\ddagger (a+\mathbf{b}) \tag{209}$$

$$= \lfloor a^2 + 2a\mathbf{b} + \mathbf{b}^2 \rfloor_{3,4} (a^2 + 2a\mathbf{b} + \mathbf{b}^2) \tag{210}$$

We note that $\mathbf{b}^2 = b^2 e_0 e_\varnothing e_1 e_2 e_0 e_\varnothing e_1 e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \tag{211}$$

$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \tag{212}$$

$$= (a^2 + b^2)^2 - 4a^2 b^2 \tag{213}$$

which is valued in $\mathbb{R}$.

In all of these cases the probability can be negative. □

[No observables (6D)] The multivector representation of the norm in 6D cannot satisfy any observables.

Argument. In six dimensions and above, the self-product patterns found in Definition 15 collapse. The research by Acus et al.[10] in 6D geometric algebra concludes that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{214}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[10].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{215}$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{216}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{217}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{218}$$

From Equation 214, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{219}$$

Any equality of the above type between $b_1\mathbf{O}$ and $b_2\mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) be constructible in 6D, using our method. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

[No observables (above 6D)] The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $\mathrm{GA}(1,3)$ even if in relativity, the signature of the metric $(+,-,-,-)$ versus $(-,-,-,+)$ does not influence the physics. However, in geometric algebra, $\mathrm{GA}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

3. Discussion

When asked to define what a physical theory is, an informal answer may be that it is a predictive framework of measurements that applies to all possible experiments realizable within a domain, with the whole of nature being the most general domain. While physicists have expressed these theories through sets of axioms, we propose a more direct approach - mathematically realizing this fundamental definition itself. This definition is realized as an optimization problem (Definition 1). The solution to this optimization problem then yields precisely those axioms that realize the physical theory over said domain. Succinctly, physics is the solution to:

$$\underbrace{\mathcal{L}}_{\substack{\text{an}\\\text{optimization}\\\text{problem}}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{on the entropy}\\\text{of a measurement}\\\text{relative to its preparation}\\\text{over all}}} + \lambda\underbrace{\left(1-\sum_i \rho_i\right)}_{\text{predictive theories}} + \tau \operatorname{tr}\underbrace{\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}} \tag{220}$$

The relative Shannon entropy represents the basic structure of any experiment, quantifying the difference between its initial preparation and its final measurement.

We now justify the matrix structure of the natural constraint. The natural constraint is specifically chosen to be the most general structure that admits a solution to this optimization problem. This generality follows from several key mathematical requirements. The constraint must involve quantities that form an algebra, as the solution requires taking exponentials:

$$\exp X = 1 + X + \frac{1}{2}X^2 + \dots \tag{221}$$

which involves addition, powers, and scalar multiplication of X. Furthermore, the use of the trace operation necessitates that X must be represented by square $n \times n$ matrices. Thus Axiom 1 involves $n \times n$ matrices:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \tag{222}$$

The trace operation is utilized because the constraint must be converted back to a scalar for use in the Lagrange multiplier equation; while any function that maps an algebra to a scalar would achieve that, specifically picking the trace recovers quantum mechanics in the $\mathrm{GA}(0,1)$ case.

These mathematical requirements demonstrate that the natural constraint, as formulated in Axiom 1, represents the most general structure for this optimization problem. More precisely, Axiom 1, as it admits the minimal mathematical structure required to solve an arbitrary entropy maximization problem, can be understood as the most general extension of the statistical mechanics average energy constraint which contains QM (as induced by the trace) as a specific solution.

Thus, having established both the mathematical structure and its generality, we can understand how this minimal ontology operates. Since our formulation keeps the structure of experiments completely general, and our optimization considers all possible predictive theories for that structure, and the constraint is the most general constraint possible for that structure, the resulting optimal physical theory necessarily applies to all possible experiments of nature.

This ontology is both operational, being grounded in the basic structure of experiments rather than abstract entities, and constructive, showing how physical laws emerge from optimization over all possible predictive theories subject to the natural constraint. Physics is encapsulated not as a predefined collection of fundamental axioms but as the optimal solution to a well-defined optimization problem over all experiments realizable within a domain. This represents a significant philosophical shift from traditional physical ontologies where laws are typically taken as primitive.

The next step in our derivation is to represent the determinant of the $n \times n$ matrices through a self-product of multivectors involving various conjugate structures. This induces an algebra of observables via these conjugates while maintaining an equivalence with the matrix representation of the natural constraint. By examining the various dimensional configurations of geometric algebras, we find that GA(3,1), representing 4×4 real matrices, admits a sub-algebra whose determinant is positive-definite. All other dimensional configurations fail to admit such a positive-definite structure, with only three exceptions: GA(0) yielding statistical mechanics, GA(0,1) and a sub-algebra of GA(2,0) yielding quantum mechanics.

The solution reveals that the 3+1D case harbours a new type of probability amplitude structure analogous to complex amplitudes, one that exhibits the characteristic elements of a quantum mechanical theory. Instead of complex-valued amplitudes, we have amplitudes valued in the even sub-algebra of GA(3,1). This probability amplitude is identical to the David Hestenes' wavefunction, but comes with an extended Born rule and rather than a complex Hilbert space structure, it lives in a double-copy structure. This double-copy structure naturally incorporates a quantum theory of gravity - the mere construction of the Dirac current automatically yields a metric tensor along with U(1) and SO(3,1) symmetries. Moreover, the double-copy structure exhibits invariance with the SU(3) group, the SU(2) and the U(1) groups.

Interpretation:

This framework presents a novel interpretation of quantum mechanics where the quantum state and its evolution emerge purely from optimizing the relative entropy between initial and final experimental states, under the natural constraint. Unlike traditional interpretations that begin by postulating entities like wavefunctions or multiple worlds, this approach starts only with measurement outcomes and derives quantum mechanics as the optimal predictive framework that applies to all experiments realizable within this domain.

In this interpretation, the wavefunction isn't a fundamental physical entity but rather emerges as a mathematical tool - it's the solution to an optimization problem. The Born rule and even unitary evolution emerges naturally in the same way.

This interpretation aligns quantum mechanics more closely with statistical mechanics, where probability distributions aren't physical entities but rather optimal descriptions of our knowledge given certain constraints. Just as the Gibbs distribution emerges from maximizing entropy subject to energy constraints, the quantum formalism emerges from maximizing relative entropy subject to the natural constraint.

This provides a more economical interpretation that avoids additional ontological commitments beyond what's directly supported by measurement, while still recovering the full predictive power of quantum mechanics.

Time:

Just as temperature emerges in statistical mechanics as a Lagrange multiplier β that characterizes how energy states are distributed, time emerges as a Lagrange multiplier $\tau = t/\hbar$ that characterizes how measurement outcomes changes over time. It's not an external parameter or fundamental entity, but rather arises naturally from the optimization framework as a Lagrange multiplier that describes how an observer experiences changes in measurement outcomes.

Measurements:

In this interpretation, measurements are the primary empirical foundation - they are what is actually observed, not derived. The quantum formalism itself is inferred from these measurement outcomes through entropy maximization. This reverses the usual interpretational direction: rather than starting with a wavefunction that "collapses" during measurement (creating the measurement problem), we start with measurements in the form of the natural constraint and derive the quantum formalism as the optimal way to predict future measurements from past ones. The wavefunction and its evolution are mathematical tools derived from entropy optimization over possible measurement outcomes, rather than physical entities that somehow "collapse" during measurement.

4. Conclusion

This work presents a simple reformulation of fundamental physics inspired by E.T. Jaynes' formulation of statistical mechanics. An optimization problem is defined that considers the space of all possible experiments, the space of all predictive theories, while being constrained by the space of all possible measurements. Its resolution then automatically selects the relevant physics - encompassing quantum mechanics, general relativity, and the Standard Model gauge symmetries - as the optimal solution.

The power of this reformulation lies in its explanatory reach: it suggests that these particular theories are in fact the optimal predictive frameworks for our universe, and further provides a mechanism for why spacetime exhibits 3+1 dimensionality. By framing physics as the solution to a single entropy optimization problem, constrained only by the measurements nature allows, this work presents a significant philosophical shift. Physical laws are no longer taken as primitive, but rather arise from the interplay between the natural constraint and the drive to maximize predictive capacity over all realizable experiments.

Statements and Declarations

- Funding: This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.
- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.
- Data Availability Statement: No datasets were generated or analyzed during the current study.
- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

Appendix A SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} = \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann Entropy}} + \underbrace{\lambda\left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta\left(\overline{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \tag{A1}$$

We solve the maximization problem as follows:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}(\rho_i, \ldots, \rho_n)}{\partial \rho_i} && \text{(A2)}\\
&= -\ln \rho_i - 1 - \lambda - \beta E_i && \text{(A3)}\\
&= \ln \rho_i + 1 + \lambda + \beta E_i && \text{(A4)}\\
\implies \ln \rho_i &= -1 - \lambda - \beta E_i && \text{(A5)}\\
\implies \rho_i &= \exp(-1-\lambda)\exp(-\beta E_i) && \text{(A6)}\\
&= \frac{1}{Z(\tau)}\exp(-\beta E_i) && \text{(A7)}
\end{aligned}
$$

The partition function, is obtained as follows:

$$
\begin{aligned}
1 &= \sum_i \exp(-1-\lambda)\exp(-\beta E_i) && \text{(A8)}\\
\implies (\exp(-1-\lambda))^{-1} &= \sum_i \exp(-\beta E_i) && \text{(A9)}\\
Z(\tau) &:= \sum_i \exp(-\beta E_i) && \text{(A10)}
\end{aligned}
$$

Finally, the probability measure is:

$$
\rho_i = \frac{1}{\sum_i \exp(-\beta E_i)}\exp(-\beta E_i) \qquad \text{(A11)}
$$

Appendix B RQM in 3+1D

$$
\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \lambda \underbrace{\left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\frac{1}{2}\zeta \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{Universal Measurement Constraint}} \qquad \text{(A12)}
$$

The solution is obtained using the same step-by-step process as the 2D case, and yields:

$$
\rho_i = \underbrace{\frac{1}{\sum_i p_i \det\exp(-\frac{1}{2}\zeta\mathbf{M}_i)}}_{\text{Spin}^c(3,1)\text{ Invariant Ensemble}} \underbrace{\det\exp\left(-\frac{1}{2}\zeta\mathbf{M}_i\right)}_{\text{Spin}^c(3,1)\text{ Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \qquad \text{(A13)}
$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}(\rho_1, \ldots, \rho_n)}{\partial \rho_i} && \text{(A14)}\\
&= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i && \text{(A15)}\\
&= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i && \text{(A16)}\\
\implies \ln \frac{\rho_i}{p_i} &= -p_i - \lambda - \frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i && \text{(A17)}\\
\implies \rho_i &= p_i \exp(-p_i - \lambda)\exp\left(-\frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i\right) && \text{(A18)}\\
&= \frac{1}{Z(\zeta)} p_i \exp\left(-\frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i\right) && \text{(A19)}
\end{aligned}
$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$\begin{aligned} 1 &= \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i\right) && \text{(A20)} \\ \implies \quad (\exp(-p_i - \lambda))^{-1} &= \sum_i p_i \exp\left(-\frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i\right) && \text{(A21)} \\ Z(\zeta) &:= \sum_i p_i \exp\left(-\frac{1}{2}\zeta \operatorname{tr} \mathbf{M}_i\right) && \text{(A22)} \end{aligned}$$

□

Appendix C SageMath program showing $\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M}_\mathbf{u}$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4
```

```
I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
                 [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
                 [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])

y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())
```

The program outputs

```
True
```

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-copy product: $\det \mathbf{M_u} = \lfloor \mathbf{u}^{\ddagger}\mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger}\mathbf{u}$.

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