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Constructing Physics From Measurements

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Abstract: We present a reformulation of fundamental physics - from a collection of independent axioms to the solution of a single optimization problem. Any experiment begins with an initial state preparation, involves some physical operation, and ends with a final measurement. Working from this structure, we maximize the entropy of final measurements relative to their preparations subject to an appropriate physical constraint. This optimization problem considers all possible predictive theories over all possible experiments, whilst being constrained by all possible measurements, and then automatically selects the exact physics we observe as the optimal solution. As a consequence of this formulation, rather than as separate postulates, we obtain quantum mechanics, general relativity, and the Standard Model gauge symmetries within a unified theory. Mathematical consistency further restricts valid solutions to 3+1 dimensions, suggesting why our universe exhibits this specific dimensionality. This reformulation reveals that the apparent complexity of modern physics, with its various forces, symmetries, and dimensional constraints, emerges naturally when optimizing over all possible ways of predicting measurements from preparations.

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1. Introduction

Statistical mechanics (SM), in the formulation developed by E.T. Jaynes [1,2], is founded on an entropy optimization principle. Specifically, the Boltzmann entropy is maximized under the constraint of a fixed average energy $\overline{E}$:

$$\overline{E} = \sum_i \rho_i E_i \tag{1}$$

The Lagrange multiplier equation defining the optimization problem is:

$$\mathcal{L} = -k_B \sum_i \rho_i \ln \rho_i + \lambda \left(1 - \sum_i \rho_i\right) + \beta \left(\overline{E} - \sum_i \rho_i E_i\right), \tag{2}$$

where λ and β are Lagrange multipliers enforcing the normalization and average energy constraints. Solving this optimization problem yields the Gibbs measure:

$$\rho_i = \frac{1}{Z} \exp(-\beta E_i), \tag{3}$$

where $Z = \sum_i \exp(-\beta E_i)$ is the partition function.

For comparison, quantum mechanics (QM) is not formulated as the solution to an optimization problem, but rather consists of a collection of axioms[3,4]:

QM Axiom 1 of 5 **State Space:** Every physical system is associated with a complex Hilbert space, and its state is represented by a ray (an equivalence class of vectors differing by a non-zero scalar multiple) in this space.

QM Axiom 2 of 5 **Observables:** Physical observables correspond to Hermitian (self-adjoint) operators acting on the Hilbert space.

QM Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is governed by the Schrödinger equation, where the Hamiltonian operator represents the system's total energy.

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QM Axiom 4 of 5 **Measurement:** Measuring an observable projects the system into an eigenstate of the corresponding operator, yielding one of its eigenvalues as the measurement result.

QM Axiom 5 of 5 **Probability Interpretation:** The probability of obtaining a specific measurement outcome is given by the squared magnitude of the projection of the state vector onto the relevant eigenstate (Born rule).

This comparison reveals a fundamental distinction in how physical theories are constructed. In statistical mechanics, observable quantities like energy constrain and determine the mathematical structure. In quantum mechanics, this relationship is reversed — the mathematical structure is postulated first, and this structure then determines what can be observed.

This contrast suggests an opportunity. While the current axioms of fundamental physics have been remarkably successful, could we reformulate them following statistical mechanics' more economical approach — deriving the mathematical structure from measurement constraints through optimization over all possible predictive theories?

To make this reformulation possible, we introduce the following constraint:

Definition 1 (The Universal Physical Constraint)**.**

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \tag{4}$$

where $\mathbf{M}_i$ are $n \times n$ matrices, and $\overline{\mathbf{M}}$ is their average. This constraint extends E.T Jaynes' optimization method to encompass non-commutative observables and symmetry group generators required for fundamental physics.

We then construct an optimization problem that considers all possible predictive theories of nature by maximizing the relative Shannon entropy between initial preparations and final measurements:

Axiom 1 (The Fundamental Optimization Problem of Physics)**.** *Physics is the solution to:*

$$\underbrace{\mathcal{L}}_{\substack{\text{an}\\ \text{optimization}\\ \text{problem}}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{on the entropy}\\ \text{of a measurement}\\ \text{relative to its preparation}\\ \text{over all}}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}} \tag{5}$$

where λ and τ are Lagrange multipliers enforcing the normalization and universal measurement constraints, respectively.

This single axiom constitutes our complete reformulation of fundamental physics. The use of relative entropy is deeply rooted in experimental reality: every physical experiment follows the same basic structure - we begin with an initial state preparation p_i, apply some evolution operation, and conclude with a measurement ρ_i. Our optimization problem considers all possible theories that could predict such measurements from preparations, constrained only by what nature allows us to measure. As we will demonstrate, the solution to this optimization problem appears sufficient to unify fundamental physics. Specifically, we intent to show that solving this optimization problem yields Statistical Mechanics (SM), the five axioms of Quantum Mechanics (QM), the two axioms of Special Relativity (SR), the Einstein Field Equations (EFE), the gauge symmetry of the Standard Model (SU(3)×SU(2)×U(1)), the 3+1 dimensionality of spacetime as the unique allowed dimensional configuration, and the foundational elements of a quantum theory of gravity. These structures emerge directly as characteristics of the optimal predictive theory, without additional assumptions and without generating unobserved features like extra dimensions or additional gauge symmetries.

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Theorem 1. *The general solution to the entropy maximization problem is:*

$$\rho_i = \frac{p_i \det \exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det \exp\left(-\tau \mathbf{M}_j\right)}. \tag{6}$$

Proof. We solve the maximization problem by setting the derivative of the Lagrangian with respect to ρ_i to zero:

$$\frac{\partial \mathcal{L}}{\partial \rho_i} = -\ln \frac{\rho_i}{p_i} - 1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i = 0. \tag{7}$$

$$\implies \ln \frac{\rho_i}{p_i} = -1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i. \tag{8}$$

$$\implies \rho_i = p_i \exp(-1-\lambda) \exp(-\tau \operatorname{tr} \mathbf{M}_i). \tag{9}$$

Normalizing the probabilities using $\sum_i \rho_i = 1$, we find:

$$1 = \sum_i \rho_i = \exp(-1-\lambda) \sum_i p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i), \tag{10}$$

$$\implies \exp(1+\lambda) = \sum_j p_j \exp\left(-\tau \operatorname{tr} \mathbf{M}_j\right). \tag{11}$$

Substituting back, we obtain:

$$\rho_i = \frac{p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i)}{\sum_j p_j \exp\left(-\tau \operatorname{tr} \mathbf{M}_j\right)}. \tag{12}$$

Finally, using the identity $\det \exp(\mathbf{M}) = \exp \operatorname{tr} \mathbf{M}$ for square matrices $\mathbf{M}$, we get:

$$\rho_i = \frac{1}{Z} p_i \det \exp(-\tau \mathbf{M}_i). \tag{13}$$

□

where $Z = \sum_j p_j \det \exp\left(-\tau \mathbf{M}_j\right)$.

This solution encapsulates fundamental physics as follows:

1. **Statistical Mechanics:**

   To recover statistical mechanics from Equation 13, we consider the case where the matrices $\mathbf{M}_i$ are 1×1, i.e., scalars. Specifically, we set:

   $$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \left[E_i\right], \tag{14}$$

   and take $p_i \to 1$. Then, Equation 13 reduces to the Gibbs distribution:

   $$\rho_i = \frac{1}{Z} \exp(-\tau E_i), \tag{15}$$

   where τ corresponds to β in traditional statistical mechanics. This demonstrates that our solution generalizes SM when $\mathbf{M}_i$ are scalars.
2. **Quantum Mechanics:**

By choosing $\mathbf{M}_i$ to generate the U(1) group, we derive the axioms of quantum mechanics from entropy maximization. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}, \tag{16}$$

where E_i are energy levels. In the results section, we will detail how this choice leads to a probability measure that includes a unitarily invariant ensemble and the Born rule, satisfying all five axioms of QM.

3. **Fundamental Physics:**

   Extending our approach, we choose $\mathbf{M}_i$ to be 4×4 matrices representing the generators of the Spin^c(3,1) group. Specifically, we consider multivectors of the form $\mathbf{u} = \mathbf{f} + \mathbf{b}$, where $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudoscalar of the 3+1D geometric algebra $\text{GA}(3,1)$. The matrix representation of $\mathbf{M}_i$ is:

$$\mathbf{M}_i = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix}, \tag{17}$$

   where $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the Spin^c(3,1) group, which includes both Lorentz transformations and U(1) phase rotations. This choice leads to a relativistic quantum probability measure:

$$\rho_i = \frac{p_i \det \exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det \exp(-\tau \mathbf{M}_j)}, \tag{18}$$

   where τ emerges as a parameter generating boosts, rotations, and phase transformations.

   In the results section, we show that the associated Dirac current is automatically invariant under the gauge symmetries of the Standard Model, specifically SU(3) × SU(2) × U(1). Furthermore, we show that the metric tensor of general relativity emerges via a double-copy mechanism applied to the Dirac current, describing a quantum theory of gravity.

4. **Dimensional Obstructions:**

   Axiom 1 yields valid probability measures only in specific cases. Beyond the instances of statistical mechanics and quantum mechanics, Axiom 1 yields a consistent solution only in 3+1 dimensions. In other configurations, various obstructions arise—such as the absence of a real matrix algebra isomorphism or the emergence of negative probabilities—thereby violating the axioms of probability theory. The following table summarizes the cases and their obstructions:

Dimensions	*Obstruction*	
0/scalar	**Statistical Mechanics (unobstructed)**	(19)
0+1	**Quantum Mechanics (unobstructed)**	(20)
1+0	Negative probabilities	(21)
2+0	**Quantum Mechanics (unobstructed)**	(22)
1+1	Negative probabilities	(23)
0+2	Not isomorphic to a real matrix algebra	(24)
3+0	Not isomorphic to a real matrix algebra	(25)
2+1	Not isomorphic to a real matrix algebra	(26)

1+2	Not isomorphic to a real matrix algebra	(27)
0+3	Not isomorphic to a real matrix algebra	(28)
4+0	Not isomorphic to a real matrix algebra	(29)
3+1/spacetime	**Quantum Gravity/Standard Model (unobstructed)**	(30)
2+2	Negative probabilities	(31)
1+3	Not isomorphic to a real matrix algebra	(32)
0+4	Not isomorphic to a real matrix algebra	(33)
5+0	Not isomorphic to a real matrix algebra	(34)
$\vdots$	$\vdots$	
6+0	No probability measure as a self-product	(35)
$\vdots$	$\vdots$	
∞		(36)

We will first investigate the unobstructed cases in Section 2.1, 2.2 and 2.3 and then demonstrate the obstructions in Section 2.4. These obstructions are desirable because they automatically limit the theory to 3+1D, thus providing a built-in mechanism for the observed dimensionality of our universe.

2. Results

2.1. Quantum Mechanics

In statistical mechanics (SM), the central observation is that energy measurements of a thermally equilibrated system tend to cluster around a fixed average value (Equation 1). In contrast, quantum mechanics (QM) is characterized by the presence of interference effects in measurement outcomes. To capture these features within an entropy maximization framework, we introduce the following special case of the universal physical constraint:

Definition 2 (U(1) Generating Constraint)**.** *We reduce the universal physical constraint to the generator of the U(1) group. Specifically, we replace*

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \quad with \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \tag{37}$$

Here, E_i are scalar values (e.g., energy levels), ρ_i are the probabilities of outcomes, and the matrices $\mathbf{M}_i$ generate the U(1) group.

The general solution of the maximization problem reduces as follows

$$\rho_i = \frac{1}{\sum_i p_i \det\exp\left(-\tau\begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right)} \det\exp\left(-\tau\begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) p_i \tag{38}$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

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To align our results with conventional quantum mechanical notation, we translate the matrices to complex numbers. Specifically, we consider that:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + ib. \tag{39}$$

Then, we note the following equivalence with the complex norm:

$$\begin{aligned} \det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} &= r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \quad (40)\\ &= r^2(\cos^2(b) + \sin^2(b)) \quad (41)\\ &= \|r(\cos(b) + i\sin(b))\| \quad (42)\\ &= \|r\exp(ib)\| \quad (43) \end{aligned}$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial prepration into :

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \|\exp(-itE_i/\hbar)\|}}_{\text{Unitarily Invariant Partition Function}} \underbrace{\|\exp(-itE_i/\hbar)\|}_{\text{Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{44}$$

The wavefunction emerges by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_i p_i \|\exp(-itE_i/\hbar)\| = Z = \langle \psi | \psi \rangle \tag{45}$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE_1/\hbar) & & \\ & \ddots & \\ & & \exp(-itE_n/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \tag{46}$$

We clarify that p_i represents the probability associated with the initial preparation of the wavefunction, where $p_i = \langle \psi_i(0) | \psi_i(0) \rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle \psi | \psi \rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates QM Axiom 1 of 5.
- In Z, an observable must satisfy:

$$\overline{O} = \sum_i p_i O_i \|\exp(-itE_i/\hbar)\| \tag{47}$$

Since $Z = \langle \psi | \psi \rangle$, then any self-adjoint operator satisfying the condition $\langle \mathbf{O}\psi | \phi \rangle = \langle \psi | \mathbf{O}\phi \rangle$ will equate the above equation, simply because $\langle \mathbf{O} \rangle = \langle \psi | \mathbf{O} | \psi \rangle$. This demonstrates QM Axiom 2 of 5.

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- Upon transforming Equation 46 out of its eigenbasis through unitary operations, we find that the energy, E_i, typically transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \tag{48}$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle) \tag{49}$$

$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \tag{50}$$

$$= -i\mathbf{H}/\hbar|\psi(t)\rangle \tag{51}$$

$$\implies \mathbf{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \tag{52}$$

which is the Schrödinger equation. This demonstrates QM Axiom 3 of 5.
- From Equation 46 it follows that the possible microstates E_i of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from ρ, with the measured state being the occupied microstate i. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

$$\rho_i(t) = \frac{1}{\langle\psi|\psi\rangle}(\psi_i(t))^\dagger\psi_i(t). \tag{53}$$

In scenarios where the probability measure $\rho_i(\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{54}$$

Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates QM Axiom 4 of 5.
- Finally, since the probability measure (Equation 44) replicates the Born rule, QM Axiom 5 of 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the U(1) generating constraint is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. The following Lagrange multiplier equation

$$\mathcal{L} = -\sum_i \rho_i \ln\frac{\rho_i}{p_i} + \lambda\left(1 - \sum_i \rho_i\right) + \tau \operatorname{tr}\left(\begin{bmatrix} 0 & \overline{E} \\ \overline{E} & 0 \end{bmatrix} - \sum_i \rho_i \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) \tag{55}$$

becomes the formulation's new singular foundation, and QM Axioms 1, 2, 3, 4, and 5 are now promoted to theorems.

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2.2. RQM in 2D

In this section, we investigate a model, isomorphic to quantum mechanics, that lives in 2D which provides a valuable starting point before addressing the more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier Equation is:

$$\mathcal{L} = -\sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i\right) + \frac{1}{2}\theta \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right) \tag{56}$$

where λ and θ are the Lagrange multipliers, and where $\mathbf{M}_i$ is the 2×2 matrix representation of the multivectors of $\text{GA}(2)$.

In general a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of $\text{GA}(2)$, where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ a pseudo-scalar, is represented as follows:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y \tag{57}$$

This holds for any 2×2 matrix and any multivectors of $\text{GA}(2)$.

The basis elements are defined as:

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_x \wedge \sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{58}$$

To investigate this case in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 3 (Multivector conjugate)**.** *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* $\text{GA}(2)$*. The multivector conjugate is defined as:*

$$\mathbf{u}^\ddagger = a - \mathbf{x} - \mathbf{b} \tag{59}$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 2 (Determinant as a Multivector Self-Product)**.**

$$\mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M} \tag{60}$$

Proof. Let $\mathbf{u} = a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y$, and let $\mathbf{M}$ be its matrix representation $\left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right]$. Then:

$$\begin{aligned}
1: \quad & \mathbf{u}^\ddagger \mathbf{u} && (61)\\
& = (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y)^\ddagger (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (62)\\
& = (a - x\sigma_x - y\sigma_y - b\sigma_x \wedge \sigma_y)(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (63)\\
& = a^2 - x^2 - y^2 + b^2 && (64)\\
2: \quad & \det \mathbf{M} && (65)\\
& = \det \left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right] && (66)\\
& = (a+x)(a-x) - (y-b)(y+b) && (67)\\
& = a^2 - x^2 - y^2 + b^2 && (68)
\end{aligned}$$

□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 4 (Multivector Conjugate Transpose). *Let $|V\rangle\rangle \in (\mathrm{GA}(2))^n$:*

$$|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{69}$$

The multivector conjugate transpose of $|V\rangle\rangle$ is defined as first taking the transpose and then the element-wise multivector conjugate:

$$\langle\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{70}$$

Definition 5 (Bilinear Form). *Let $|V\rangle\rangle$ and $|W\rangle\rangle$ be two vectors valued in $\mathrm{GA}(2)$. We introduce the following bilinear form:*

$$\langle\langle V|W\rangle\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \tag{71}$$

Theorem 3 (Inner Product). *Restricted to the even sub-algebra of $\mathrm{GA}(2)$, the bilinear form is an inner product.*

Proof.

$$\langle\langle V|W\rangle\rangle_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{72}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \sigma_x \wedge \sigma_y$. □

Let us now solve the optimization problem for the even multivectors of $\mathrm{GA}(2,0)$, whose inner product is positive-definite.

We take $a \to 0, \mathbf{x} \to 0$ then $\mathbf{M}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a\to 0,\mathbf{x}\to 0} = \mathbf{b} \implies \mathbf{M} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \tag{73}$$

The Lagrange multiplier equation can be solved as follows:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}[\rho_1, \dots, \rho_n]}{\partial \rho_i} && (74)\\
&= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} && (75)\\
&= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} && (76)\\
\implies \ln \frac{\rho_i}{p_i} &= -p_i - \lambda - \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} && (77)\\
\implies \rho_i &= p_i \exp(-p_i - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) && (78)\\
&= \frac{1}{Z(\theta)} p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) && (79)
\end{aligned}$$

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The partition function $Z(\theta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{80}$$

$$\implies (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{81}$$

$$Z(\theta) := \sum_i p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{82}$$

Consequently, the least biased probability measure that connects an initial preparation p_i to a final measurement ρ_i, under the 2D universal measurement constraint, is:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det\exp\left(-\frac{1}{2}\theta\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det\exp\left(-\frac{1}{2}\theta\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right)}_{\text{Spin(2) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{83}$$

Definition 6 (Spin(2)-valued Wavefunction)**.**

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n+\mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1}R_1 \\ \vdots \\ \sqrt{\rho_2}R_2 \end{bmatrix} \tag{84}$$

where $\sqrt{\rho_i} = e^{\frac{1}{2}a_i}$ representing the square root of the probability and $R_i = e^{\frac{1}{2}\mathbf{b}_i}$ representing a rotor in 2D (or boost in 1+1D).

The partition function of the probability measure can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

Theorem 4 (Partition Function)**.** $Z = \langle\!\langle \psi|\psi\rangle\!\rangle$

Proof.

$$\langle\!\langle \psi|\psi\rangle\!\rangle = \sum_i \psi_i^\ddagger \psi_i = \sum_i \rho_i R_i^\ddagger R_i = \sum_i \rho_i = Z \tag{85}$$

□

Definition 7 (Spin(2)-valued Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\theta\mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta\mathbf{b}_n} \end{bmatrix} \tag{86}$$

Theorem 5. *The partition function is invariant with respect to the Spin(2)-valued evolution operator.*

Proof. We note that:

$$\langle\!\langle T\mathbf{v}|T\mathbf{v}\rangle\!\rangle = \langle\!\langle \mathbf{v}|\mathbf{v}\rangle\!\rangle = \mathbf{v}^\ddagger T^\ddagger T\mathbf{v} \implies T^\ddagger T = I \tag{87}$$

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then, since $\begin{bmatrix} e^{\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} = I$, the relation $T^\ddagger T = I$ is satisfied. □

We note that the even sub-algebra of GA(2), being closed under addition and multiplication and constituting an inner product through its bilinear form, allows for the construction of a Hilbert space. In this context, the Hilbert space is Spin(2)-valued. The primary distinction between a wavefunction in a complex Hilbert space and one in a Spin(2)-valued Hilbert space lies in the subject matter of the theory. Specifically, in the latter, the construction governs the change in orientation experienced by an observer (versus change in time), which in turn dictates the measurement basis used in the experiment, consistently with the rotational symmetry and freedom of the system.

The dynamics of observer orientation transformations are described by a variant of the Schrödinger equation, which is derived by taking the derivative of the wavefunction with respect to the Lagrange multiplier, θ:

Definition 8 (Spin(2)-valued Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1 & & \\ & \ddots & \\ & & -\frac{1}{2}\mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} \tag{88}$$

Here, θ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the Spin(2), locally across the states of the Hilbert space. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction for the Spin(2) group.

Definition 9 (David Hestenes' Formulation)**.** *In 3+1D, the David Hestenes' formulation [5] of the wavefunction is $\psi = \sqrt{\rho} R e^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho} R$, which is the form we have recovered.*

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 10 (Dirac Current)**.** *Given the basis σ_x and σ_y, the Dirac current for the 2D theory is defined as:*

$$J_x \equiv \psi^\ddagger \sigma_x \psi = \rho \underbrace{R^\ddagger \sigma_x R}_{\mathrm{SO}(2)} = \rho \tilde{\sigma}_x \tag{89}$$

$$J_y \equiv \psi^\ddagger \sigma_y \psi = \rho \underbrace{R^\ddagger \sigma_y R}_{\mathrm{SO}(2)} = \rho \tilde{\sigma}_y \tag{90}$$

where $\tilde{\sigma}_x$ and $\tilde{\sigma}_y$ are a SO(2) rotated basis vectors.

2.2.1. 1+1D Obstruction

As stated in the introduction, of the dimensional cases, only 2D and 3+1D are free of obstructions. For instance, the 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\mathbf{e}_0 \wedge \mathbf{e}_1)(a + b\mathbf{e}_0 \wedge \mathbf{e}_1)$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. This is why we had to use 2D instead of 1+1D in this two-dimensional introduction. In the following section, we will investigate the 3+1D case, then we will show why all other dimensional cases are obstructed.

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2.3. RQM in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. The Lagrange multiplier equation is as follows:

$$\mathcal{L} = -\sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i\right) + \frac{1}{2}\zeta \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right) \tag{91}$$

where

$$\mathbf{M}_i = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix}, \tag{92}$$

Here, $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the $\text{Spin}^c(3,1)$ group, which includes both Lorentz transformations and U(1) phase rotations.

The solution (proof in Annex B) is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp(-\frac{1}{2}\zeta \mathbf{M}_i)}}_{\text{Spin}^c(3,1)\text{ Invariant Ensemble}} \underbrace{\det \exp(-\frac{1}{2}\zeta \mathbf{M}_i)}_{\text{Spin}^c(3,1)\text{ Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{93}$$

where ζ is a "twisted-phase" rapidity. (If the invariance group was Spin(3,1) instead of $\text{Spin}^c(3,1)$, obtainable by posing $\mathbf{b} \to 0$, then it would simply be the rapidity). As we will show in Section 2.4, due to obstructions, this probability measure is the most sophisticated solution to the optimization problem that satisfy the axioms of probability theory.

2.3.1. Preliminaries

As we did in the 2D case, our initial goal here also will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra $\mathrm{GA}(3,1)$.

Definition 11 (Multivector)**.** *Let* $\mathbf{u}$ *be a multivector of* $\mathrm{GA}(3,1)$*. Its general form is:*

$$\begin{aligned} \mathbf{u} = a & \qquad (94)\\ + t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 & \qquad (95)\\ + f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 + f_{13}\gamma_1 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 & \qquad (96)\\ + p\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + q\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + w\gamma_0 \wedge \gamma_1 \wedge \gamma_2 & \qquad (97)\\ + b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 & \qquad (98) \end{aligned}$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ *are the basis vectors in the real Majorana representation.*

A more compact notation for $\mathbf{u}$ *is*

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{99}$$

where a *is a scalar,* $\mathbf{x}$ *a vector,* $\mathbf{f}$ *a bivector,* $\mathbf{v}$ *is pseudo-vector and* $\mathbf{b}$ *a pseudo-scalar.*

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation:

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Definition 12 (Matrix Representation of **u**).

$$\mathbf{M} = \begin{bmatrix} a+f_{02}-q-z & b-f_{13}+w-x & -f_{01}+f_{12}-p+v & f_{03}+f_{23}+t+y \\ -b+f_{13}+w-x & a+f_{02}+q+z & f_{03}+f_{23}-t-y & f_{01}-f_{12}-p+v \\ -f_{01}-f_{12}+p+v & f_{03}-f_{23}+t-y & a-f_{02}+q-z & -b-f_{13}-w-x \\ f_{03}-f_{23}-t+y & f_{01}+f_{12}+p+v & b+f_{13}-w-x & a-f_{02}-q+z \end{bmatrix} \tag{100}$$

To manipulate and analyze multivectors in $\text{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 13 (Multivector Conjugate (in 4D)).

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{101}$$

Definition 14 (3,4 Blade Conjugate). *The 3,4 blade conjugate of* **u** *is*

$$\lfloor \mathbf{u} \rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{102}$$

Lundholm[6] proposes a number the multivector norms, and shows that they are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 15. *The self-products associated with low-dimensional geometric algebras are:*

$$\text{GA}(0,1): \quad \varphi^{\dagger}\varphi \tag{103}$$
$$\text{GA}(2,0): \quad \varphi^{\ddagger}\varphi \tag{104}$$
$$\text{GA}(3,0): \quad \lfloor \varphi^{\ddagger}\varphi \rfloor_3 \varphi^{\ddagger}\varphi \tag{105}$$
$$\text{GA}(3,1): \quad \lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi \tag{106}$$
$$\text{GA}(4,1): \quad (\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi)^{\dagger}(\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi) \tag{107}$$

We can now express the determinant of the matrix representation of a multivector via the self-product $\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi$. Again, this choice is not arbitrary, but the unique choice with allows us to represent the determinant of the matrix representation of a multivector within $\text{GA}(3,1)$:

Theorem 6 (Determinant as a Multivector Self-Product).

$$\lfloor \mathbf{u}^{\ddagger}\mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger}\mathbf{u} = \det \mathbf{M} \tag{108}$$

Proof. Please find a computer assisted proof of this equality in Annex C. □

Definition 16 ($\text{GA}(3,1)$-valued Vector).

$$|V\rangle\rangle = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{f}_1 + \mathbf{v}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{f}_n + \mathbf{v}_n + \mathbf{b}_n \end{bmatrix} \tag{109}$$

These constructions allow us to express the partition function in terms of the multivector self-product:

Definition 17 (Double-Copy Product). *Instead of an inner product, we obtain what we call a double-copy product:*

$$\langle\langle V|V|V|V\rangle\rangle = \sum_i \lfloor \underbrace{\psi_i^\ddagger \psi_i}_{copy\ 1} \rfloor_{3,4} \underbrace{\psi_i^\ddagger \psi_i}_{copy\ 2} \tag{110}$$

$$= \underbrace{\lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4}}_{copy\ 1} \underbrace{\begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix}}_{copy\ 2} \tag{111}$$

Theorem 7 (Partition Function). $Z = \langle\langle V|V|V|V\rangle\rangle$

Proof.

$$\langle\langle V|V|V|V\rangle\rangle \tag{112}$$

$$= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \tag{113}$$

$$= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 & \dots & \mathbf{u}_n \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n^\ddagger \mathbf{u}_n \end{bmatrix} \tag{114}$$

$$= \lfloor \mathbf{u}_1^\ddagger \mathbf{u}_1 \rfloor_{3,4} \mathbf{u}_1^\ddagger \mathbf{u}_1 + \dots + \lfloor \mathbf{u}_n^\ddagger \mathbf{u}_n \rfloor_{3,4} \mathbf{u}_n^\ddagger \mathbf{u}_n \tag{115}$$

$$= \sum_{i=1}^n \det \mathbf{M}_{\mathbf{u}_i} \tag{116}$$

$$= Z \tag{117}$$

□

Desirable properties for the double-copy product are introduced by addressing the issue of non-positivity. First, we establish non-negativity:

Theorem 8 (Non-negativity). *The double-copy product, applied to the even subalgebra of* GA(3, 1)*, is always non-negative.*

Proof. Let $|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix}$. Then,

$$\langle\langle V|V|V|V\rangle\rangle \tag{118}$$

$$= \lfloor \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{119}$$

$$= \lfloor \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{120}$$

$$= \lfloor \begin{bmatrix} a_1^2 + a_1\mathbf{f}_1 + a_1\mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1\mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1\mathbf{f}_1 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{121}$$

$$= \lfloor \begin{bmatrix} a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{122}$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3) & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{123}$$

We note that the terms are now complex numbers, which we rewrite as $\Re(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\Im(z) = -4(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} z_1 & \dots & z_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{124}$$

$$= \begin{bmatrix} z_1^\dagger & \dots & z_2^\dagger \end{bmatrix} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{125}$$

$$= z_1^\ddagger z_1 + \dots + z_n^\ddagger z_n \tag{126}$$

which is always non-negative. □

Finally, *positive-definiteness* is automatically achieved because solving the optimization problem exponentiates the multivector, yielding a wavefunction:

Definition 18 (Spin$^c(3,1)$-Valued Wavefunction)**.**

$$|\psi\rangle\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{f}_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1}\, R_1\, B_1 \\ \vdots \\ \sqrt{\rho_n}\, R_n\, B_n \end{bmatrix},$$

where:

- $\sqrt{\rho_i} = e^{\frac{1}{2}a_i} \geq 0$ *is a positive scalar factor ensuring non-negativity.*
- $R_i = e^{\frac{1}{2}\mathbf{f}_i}$ *is a rotor representing Lorentz transformations (rotations and boosts in spacetime).*
- $B_i = e^{\frac{1}{2}\mathbf{b}_i}$ *is a complex phase factor, as* $\mathbf{b}_i = b_i I$ *and* $e^{\frac{1}{2}\mathbf{b}_i} = e^{\frac{1}{2}b_i I} = \cosh\left(\frac{b_i}{2}\right) + I \sinh\left(\frac{b_i}{2}\right)$.

In this representation:

- The exponential map $e^{\frac{1}{2}(a_i + \mathbf{f}_i + \mathbf{b}_i)}$ maps elements of the algebra to the connected component of the identity in the spin group Spin$^c(3,1)$, except at the zero vector, where the map is not injective.

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- The wavefunction $|\psi\rangle\rangle$ captures both the amplitude (through $\sqrt{\rho_i}$) and the phase (through R_i and B_i) of the quantum state.

Thus, the double-copy product $\langle\langle\psi|\psi|\psi|\psi\rangle\rangle$ over wavefunction ψ is positive-definite.

Now, let us turn our attention to the evolution operator, which leaves the partition function invariant:

Definition 19 (Spin$^c(3,1)$ Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1+\mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n+\mathbf{b}_n)} \end{bmatrix} \tag{127}$$

In turn, this leads to a variant of the Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ:

Definition 20 (Spin$^c(3,1)$-valued Schrödinger equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{bmatrix}\psi_1(\zeta)\\ \vdots \\ \psi_n(\zeta)\end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\mathbf{f}_1+\mathbf{b}_1) & & \\ & \ddots & \\ & & -\frac{1}{2}(\mathbf{f}_n+\mathbf{b}_n) \end{bmatrix}\begin{bmatrix}\psi_1(\zeta)\\ \vdots \\ \psi_n(\zeta)\end{bmatrix} \tag{128}$$

In this case ζ represents a one-parameter evolution parameter akin to time, which is able to transform the measurement basis under action of the Spin$^c(3,1)$ group. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction.

Theorem 9 (Spin^c(3,1) invariance)**.** *Let $e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}$ be a general element of Spin^c(3,1). Then, the equality:*

$$\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi = \lfloor(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi\rfloor_{3,4}(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \tag{129}$$

is always satisfied.

Proof.

$$\lfloor(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi\rfloor_{3,4}(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \tag{130}$$
$$= \lfloor\psi^\ddagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi\rfloor_{3,4}\psi^\ddagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \tag{131}$$
$$= \lfloor\psi^\ddagger e^{\mathbf{b}}\psi\rfloor_{3,4}\psi^\ddagger e^{\mathbf{b}}\psi \tag{132}$$
$$= \lfloor\psi^\ddagger\psi\rfloor_{3,4}e^{-\mathbf{b}}e^{\mathbf{b}}\psi^\ddagger\psi \tag{133}$$
$$= \lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi \tag{134}$$

□

2.3.2. RQM

Definition 21 (David Hestenes' Wavefunction)**.** *The* Spin$^c(3,1)$*-valued wavefunction we have recovered is formulated identically to David Hestenes'[5] formulation of the wavefunction within GA(3,1).*

$$\psi = \underbrace{e^{\frac{1}{2}(a+\mathbf{f}+\mathbf{b})}}_{ours} = \underbrace{\sqrt{\rho}Re^{-ib/2}}_{Hestenes'} \tag{135}$$

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where $e^{\frac{1}{2}a} = \sqrt{\rho}$, $e^{\frac{1}{2}\mathbf{f}} = R$ *and* $e^{\frac{1}{2}\mathbf{b}} = e^{-ib/2}$.

Before we continue the RQM investigation, let us note that the double-copy product contains two copies of a bilinear form $\psi^\ddagger\psi$:

$$\lfloor \underbrace{\psi^\ddagger\psi}_{\text{copy 1}} \rfloor_{3,4} \underbrace{\psi^\ddagger\psi}_{\text{copy 2}} \tag{136}$$

In the present and upcoming section, we will investigate the properties of each copy individually, leaving the properties specific to the double-copy for the section on quantum gravity.

Taking a single copy, the Dirac current is obtained directly from the gamma matrices, as follows:

Definition 22 (Dirac Current)**.** *The definition of the Dirac current is the same as Hestenes':*

$$J \equiv \psi^\ddagger\gamma_\mu\psi = \rho R^\ddagger B^\ddagger\gamma_\mu BR = \rho R^\ddagger\gamma_\mu B^{-1}BR = \rho \underbrace{R^\ddagger\gamma_\mu R}_{SO(3,1)} = \rho\tilde{\gamma}_\mu \tag{137}$$

where $\tilde{\gamma}_\mu$ *is a SO(3,1) rotated basis vector.*

2.3.3. Standard Model Gauge Symmetries

We will now demonstrate that the double-copy product is automatically invariant under transformations corresponding to the U(1), SU(2), and SU(3) symmetries, as well as under unitary transformations satisfying $U^\dagger U = I$, all of which play fundamental roles in the Standard Model of particle physics. These symmetries constitute the set of transformations that leave the Dirac current invariant, i.e., $(T\psi)^\ddagger\gamma_0 T\psi = \psi^\ddagger\gamma_0\psi$ with T valued in GA$(3,1)$.

Theorem 10 (U(1) Invariance)**.** *Let* $e^{\frac{1}{2}\mathbf{b}}$ *be a general element of U(1). Then, the equality*

$$\lfloor\psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor\underbrace{(e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi}_{copy\ 1}\rfloor_{3,4}\underbrace{(e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi}_{copy\ 2} \tag{138}$$

is satisfied, yielding a U(1) symmetry for each copied bilinear form.

Proof. Equation 138 is invariant if this expression is satisfied:

$$e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 \tag{139}$$

This is always satisfied simply because $e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 e^{-\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{b}} = \gamma_0$ □

Theorem 11 (SU(2) Invariance)**.** *Let* $e^{\frac{1}{2}\mathbf{f}}$ *be a general element of Spin(3,1). Then, the equality:*

$$\lfloor\psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor\underbrace{(e^{\frac{1}{2}\mathbf{f}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi}_{copy\ 1}\rfloor_{3,4}\underbrace{(e^{\frac{1}{2}\mathbf{f}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi}_{copy\ 2} \tag{140}$$

is satisfied for if $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$ *(which generates SU(2)), yielding a SU(2) symmetry for each copied bilinear form.*

Proof. Equation 140 is invariant if this expression is satisfied[7]:

$$e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0 \tag{141}$$

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We now note that moving the left-most term to the right of the gamma matrix yields:

$$e^{-E_1\gamma_0\gamma_1-E_2\gamma_0\gamma_2-E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2}\gamma_0 e^{\frac{1}{2}\mathbf{f}} \tag{142}$$

$$= \gamma_0 e^{E_1\gamma_0\gamma_1+E_2\gamma_0\gamma_2+E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2} e^{\frac{1}{2}\mathbf{f}} \tag{143}$$

Therefore, the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $E_1 = E_2 = E_3 = 0$, leaving $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$:

Finally, we note that $e^{\theta_1\gamma_2\gamma_3+\theta_2\gamma_1\gamma_3+\theta_3\gamma_1\gamma_2}$ generates SU(2). □

Theorem 12 (SU(3))**.** *The generators of SU(3) in GA(3,1) are given by Anthony Lesenby in [8] and are as follows:*

$$\hat{E}_{ij} = \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j \quad \text{where } i < j \tag{144}$$

$$\hat{F}_{ij} = \hat{e}_i\hat{f}_j + \hat{e}_j\hat{f}_i \quad \text{where } i < j \tag{145}$$

$$\hat{J} = \hat{e}_i\hat{f}_i \quad \text{where } i = 1,2,3 \tag{146}$$

where

$$\hat{e}_i = \text{multiplication on the left by } \sigma_i\text{, so that } \hat{e}_i(F) = \sigma_i F \tag{147}$$

$$\hat{f}_i = \text{multiplication on the right by } I\sigma_i\text{, so that } \hat{f}_i(F) = I\sigma_i F \tag{148}$$

This defines the 9 generators of U(3).

With the additional restriction on $\hat{J}$

$$\alpha_1\hat{J}_1 + \alpha_2\hat{J}_2 + \alpha_3\hat{J}_3, \text{ with } \alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{149}$$

the number generators is reduced to 8, consistently with SU(3).

We now must show that the following equation is satisfied for all 8 generators:

$$\lfloor \psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor \underbrace{(e^{\theta^i\lambda_i}\psi)^\ddagger\gamma_0 e^{\theta^i\lambda_i}\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{(e^{\theta^i\lambda_i}\psi)^\ddagger\gamma_0 e^{\theta^i\lambda_i}\psi}_{\text{copy 2}} \tag{150}$$

Proof. First, we note the following action:

$$-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0 \tag{151}$$

which we can rewrite as follows:

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \tag{152}$$

The first three terms anticommute with γ_0, while the last three commute with γ_0:

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2(q)\gamma_1\gamma_3 - B_3(q)\gamma_1\gamma_2)\mathbf{f}(q) \tag{153}$$

This can be written as:

$$\gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \tag{154}$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{EB} - \mathbf{BE} - \mathbf{B}^2) \tag{155}$$

where $\mathbf{E} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3$ and $\mathbf{B} = B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$.

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Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we require: 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{EB} = \mathbf{BE}$. The first requirement expands as follows:

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \tag{156}$$

which is the defining conditions for the SU(3) symmetry group.

Finally, as the SU(3) norm is a consequence of preserving the Dirac current, it follows that the SU(3) generators provided by Lasenby, acting on **f**, cannot change the SU(3) norm, hence must also preserve the Dirac current. □

Theorem 13 (Unitary invariance)**.** *Let U be $n \times n$ unitary matrices. Then unitary invariance:*

$$\langle\!\langle \psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle = \langle\!\langle U\psi|\gamma_\mu U\psi|U\psi|\gamma_\nu U\psi\rangle\!\rangle \implies U^\dagger U = I \tag{157}$$

is individually satisfied for each copied bilinear form.

Proof. Equation 157 is satisfied if $U^\ddagger\gamma_\mu U = \gamma_\mu$. Since U is valued in complex numbers, then $U^\ddagger = U^T$, and since $\gamma_\mu\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\mu$, it follows that:

$$\gamma_\mu U^\dagger U = \gamma_\mu \tag{158}$$

which is satisfied when $U^\dagger U = I$. □

The invariances SU(3), SU(2) and U(1) discussed above can be promoted to local symmetries using standard gauge theory construction techniques.

In conventional QM, the Born rule naturally leads to a U(1)-valued gauge theory due to the following symmetry:

$$(e^{-i\theta(x)}\psi(x))^\dagger e^{-i\theta(x)}\psi(x) = \psi(x)^\dagger\psi(x) \tag{159}$$

However, the SU(3) and SU(2) symmetries do not emerge from the probability measure in the same straightforward manner and are typically introduced by hand, justified by experimental observations. This raises the question: why these specific symmetries and not others? In contrast, within our framework, all three symmetry groups—U(1), SU(2), and SU(3)—as well as the Spin(3,1) and unitary symmetries, follow naturally from the invariance of the probability measure, in the same way that the U(1) symmetry follows from the Born rule. This suggests a deeper underlying principle governing the symmetries in fundamental physics.

2.3.4. A Starting Point for a Theory of Quantum Gravity

In the previous section, we developed a quantum theory valued in Spin^c(3,1), which served as the arena for RQM. We then demonstrated how a single copy of this theory leads to the gauge symmetries of the standard model, the Dirac current and other features of RQM. The goal of this section is to extend this methodology to basis vectors, in which the metric tensor emerges as an observable. To achieve this, we will utilize both copies of the double-copy product.

We recall the definition of the metric tensor in terms of basis vectors of geometric algebra, as follows:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) \tag{160}$$

Then, we note that the double-copy product acts on a pair of basis element $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$, as follows:

$$\frac{1}{2}\Big(\lfloor\underbrace{\psi^\ddagger\mathbf{e}_\mu\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{\psi^\ddagger\mathbf{e}_\nu\psi}_{\text{copy 2}}+\lfloor\underbrace{\psi^\ddagger\mathbf{e}_\nu\varphi}_{\text{copy 2}}\rfloor_{3,4}\underbrace{\psi^\ddagger\mathbf{e}_\mu\psi}_{\text{copy 1}}\Big) \tag{161}$$

$$=\frac{1}{2}\Big(\tilde{R}\underbrace{\rho e^{ib/2}e^{-ib/2}}_{\text{Born rule copy 1}}\mathbf{e}_\mu R\tilde{R}\underbrace{\rho e^{-ib/2}e^{ib/2}}_{\text{Born rule copy 2}}\mathbf{e}_\nu R+\tilde{R}\underbrace{\rho e^{ib/2}e^{-ib/2}}_{\text{Born rule copy 2}}\mathbf{e}_\nu R\tilde{R}\underbrace{\rho e^{-ib/2}e^{ib/2}}_{\text{Born rule copy 1}}\mathbf{e}_\mu R\Big) \tag{162}$$

$$=\frac{1}{2}\rho^2(\tilde{R}\mathbf{e}_\mu R\tilde{R}\mathbf{e}_\nu R+\tilde{R}\mathbf{e}_\nu R\tilde{R}\mathbf{e}_\nu R) \tag{163}$$

$$=\underbrace{\rho^2}_{\text{probability}}\underbrace{\frac{1}{2}(\tilde{\mathbf{e}}_\mu\tilde{\mathbf{e}}_\nu+\tilde{\mathbf{e}}_\nu\tilde{\mathbf{e}}_\mu)}_{\text{metric tensor}} \tag{164}$$

where $\tilde{\mathbf{e}}_\mu$ and $\tilde{\mathbf{e}}_\nu$ are SO(3,1) rotated basis vectors, and where ρ^2 is a probability measure.

As one can swap $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ and obtain the same metric tensor, the double-copy product guarantees that $g_{\mu\nu}$ is symmetric.

Furthermore, since $\mathbf{e}_\mu^\ddagger=-\mathbf{e}_\mu$, we get:

$$\lfloor(\mathbf{e}_\mu\psi)^\ddagger\psi\rfloor_{3,4}(\mathbf{e}_\nu\psi)^\ddagger\psi \tag{165}$$

$$=\lfloor\psi^\ddagger(-1)\mathbf{e}_\mu^\ddagger\psi\rfloor_{3,4}\psi^\ddagger(-1)\mathbf{e}_\nu^\ddagger\psi \tag{166}$$

$$=\lfloor\psi^\ddagger\mathbf{e}_\mu\psi\rfloor_{3,4}\psi^\ddagger\mathbf{e}_\nu\psi \tag{167}$$

which allows us to conclude that $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ are self-adjoint within the double-copy product, entailing the interpretation of $g_{\mu\nu}$ as an observable.

In the double-copy product, the metric tensor emerges as a double copy of Dirac currents. This formulation suggests that the metric tensor encodes the probabilistic structure of a quantum theory of gravity *in the form of a symmetric rank-2 tensor*, analogous to how the Dirac current encodes the probabilistic structure of a special relativistic quantum theory *in the form of a 4-vector*.

Let us now investigate the dynamics. We recall that the evolution operator (Definition 19) is:

$$T=\begin{bmatrix}e^{-\frac{1}{2}\zeta(\mathbf{f}_1+\mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n+\mathbf{b}_n)}\end{bmatrix} \tag{168}$$

Acting on the wavefunction, the effect of this operator cascades down to the basis vectors via the double-copy product:

$$\lfloor\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\mu T\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\nu T\psi}_{\text{copy 2}}+\lfloor\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\nu T\psi}_{\text{copy 2}}\rfloor_{3,4}\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\mu T\psi}_{\text{copy 1}} \tag{169}$$

which realizes an $\mathrm{SO}(3,1)$ transformation of the metric tensor via action of the exponential of a bivector, and a double-copy unitary invariant transformation via action of the exponential of a pseudo-scalar:

$$\underbrace{\lfloor\psi^\ddagger\underbrace{e^{\frac{1}{2}\zeta\mathbf{f}}\mathbf{e}_\mu e^{-\frac{1}{2}\zeta\mathbf{f}}}_{\text{SO(3,1) evolution}}\underbrace{e^{\frac{1}{2}\zeta\mathbf{b}}e^{-\frac{1}{2}\zeta\mathbf{b}}}_{\text{unitary evolution}}\psi\rfloor_{3,4}}_{\text{copy 1}}\underbrace{\psi^\ddagger\underbrace{e^{\frac{1}{2}\zeta\mathbf{f}}\mathbf{e}_\mu e^{-\frac{1}{2}\zeta\mathbf{f}}}_{\text{SO(3,1) evolution}}\underbrace{e^{\frac{1}{2}\zeta\mathbf{b}}e^{-\frac{1}{2}\zeta\mathbf{b}}}_{\text{unitary evolution}}\psi}_{\text{copy 2}}+\cdots \tag{170}$$

In summary, this initial investigation has identified a scenario in which the metric tensor is measured using basis vectors. The evolution operator, governed by the Schrödinger equation, dynamically realizes SO(3,1) transformations on the metric tensor. Furthermore, the amplitudes associated with

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possible metric tensors are derived from a double-copy of unitary quantum theories acting on the basis vectors. This formulation simultaneously preserves the SO(3,1) symmetry, essential for describing spacetime structure, and the unitary symmetry, fundamental to quantum mechanics. It describes all changes of basis transformations that an observer in 3+1D spacetime can perform prior to measuring (in the quantum sense) a basis system in spacetime, and attributes a probability to the outcome (the outcome being the metric tensor).

2.3.5. The Einstein Field Equation

In the previous section, we established that the metric tensor $g_{\mu\nu}$ emerges as an observable through the double-copy mechanism acting on basis vectors. We also determined that this probability measure transforms covariantly under SO(3,1) Lorentz transformations.

To derive the dynamical equations governing this metric tensor, we seek the simplest possible Lagrangian whose equations of motion are a function of $g_{\mu\nu}$ and that respects this SO(3,1) covariance. The Einstein-Hilbert action naturally emerges as this simplest choice:

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int \sqrt{-g}\, R \,\mathrm{d}^4 x, \tag{171}$$

where $\kappa = 8\pi G$ with G being Newton's gravitational constant, g is the determinant of the metric tensor, and R is the Ricci scalar. Varying this simplest possible covariant action yields the Einstein field equations $G_{\mu\nu} = 0$, where $G_{\mu\nu}$ is the Einstein tensor, which automatically satisfies the Bianchi identities ensuring conservation of energy-momentum.

2.4. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to resolve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except those we have explored here (e.g. $\mathrm{GA}(0) \cong \mathbb{R}$, $\mathrm{GA}(0,1) \cong \mathbb{C}$, $\mathrm{GA}(2,0)$ and $\mathrm{GA}(3,1)$) are obstructed. By obstructed, we mean that the solution to the entropy maximization problem, ρ, does not satisfy all axioms of probability theory.

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Dimensions	*Obstruction*	
$\mathrm{GA}(0)$	**Unobstructed** $\implies$ statistical mechanics	(172)
$\mathrm{GA}(0,1)$	**Unobstructed** $\implies$ quantum mechanics	(173)
$\mathrm{GA}(1,0)$	Negative probabilities in the RQM	(174)
$\mathrm{GA}(2,0)$	**Unobstructed** $\implies$ spin(2) quantum mechanics	(175)
$\mathrm{GA}(1,1)$	Negative probabilities in the RQM	(176)
$\mathrm{GA}(0,2)$	Not isomorphic to a real matrix algebra	(177)
$\mathrm{GA}(3,0)$	Not isomorphic to a real matrix algebra	(178)
$\mathrm{GA}(2,1)$	Not isomorphic to a real matrix algebra	(179)
$\mathrm{GA}(1,2)$	Not isomorphic to a real matrix algebra	(180)
$\mathrm{GA}(0,3)$	Not isomorphic to a real matrix algebra	(181)
$\mathrm{GA}(4,0)$	Not isomorphic to a real matrix algebra	(182)
$\mathrm{GA}(3,1)$	**Unobstructed** $\implies$ quantum gravity $\wedge$ $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	(183)
$\mathrm{GA}(2,2)$	Negative probabilities in the RQM	(184)
$\mathrm{GA}(1,3)$	Not isomorphic to a real matrix algebra	(185)
$\mathrm{GA}(0,4)$	Not isomorphic to a real matrix algebra	(186)
$\mathrm{GA}(5,0)$	Not isomorphic to a real matrix algebra	(187)
$\vdots$	$\vdots$	
$\mathrm{GA}(6,0)$	No probability measure as a self-product	(188)
$\vdots$	$\vdots$	
∞		(189)

Let us now demonstrate the obstructions mentioned above.

Theorem 14 (Not isomorphic to a real matrix algebra)**.** *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. These geometric algebras are classified as follows:

$$\mathrm{GA}(0,2) \cong \mathbb{H} \tag{190}$$
$$\mathrm{GA}(3,0) \cong \mathbb{M}_2(\mathbb{C}) \tag{191}$$
$$\mathrm{GA}(2,1) \cong \mathbb{M}_2^2(\mathbb{R}) \tag{192}$$
$$\mathrm{GA}(1,2) \cong \mathbb{M}_2(\mathbb{C}) \tag{193}$$
$$\mathrm{GA}(0,3) \cong \mathbb{H}^2 \tag{194}$$
$$\mathrm{GA}(4,0) \cong \mathbb{M}_2(\mathbb{H}) \tag{195}$$
$$\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H}) \tag{196}$$
$$\mathrm{GA}(0,4) \cong \mathbb{M}_2(\mathbb{H}) \tag{197}$$
$$\mathrm{GA}(5,0) \cong \mathbb{M}_2^2(\mathbb{H}) \tag{198}$$

The determinant of these objects is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. □

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Theorem 15 (Negative Probabilities in the RQM)**.** *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable as a RQM.*

Proof. This category contains three dimensional configurations:

GA(1,0): Let $\psi = a + be_1$, then:

$$(a + be_1)^\ddagger(a + be_1) = (a - be_1)(a + be_1) = a^2 - b^2 e_1 e_1 = a^2 - b^2 \tag{199}$$

which is valued in $\mathbb{R}$.

GA(1,1): Let $\psi = a + be_0e_1$, then:

$$(a + be_0e_1)^\ddagger(a + be_0e_1) = (a - be_0e_1)(a + be_0e_1) = a^2 - b^2 e_0e_1e_0e_1 = a^2 - b^2 \tag{200}$$

which is valued in $\mathbb{R}$.

GA(2,2): Let $\psi = a + be_0e_\varnothing e_1e_2$, where $e_0^2 = -1, e_\varnothing^2 = -1, e_1^2 = 1, e_2^2 = 1$, then:

$$\lfloor(a + \mathbf{b})^\ddagger(a + \mathbf{b})\rfloor_{3,4}(a + \mathbf{b})^\ddagger(a + \mathbf{b}) \tag{201}$$

$$= \lfloor a^2 + 2a\mathbf{b} + \mathbf{b}^2\rfloor_{3,4}(a^2 + 2a\mathbf{b} + \mathbf{b}^2) \tag{202}$$

We note that $\mathbf{b}^2 = b^2 e_0e_\varnothing e_1e_2e_0e_\varnothing e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \tag{203}$$

$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \tag{204}$$

$$= (a^2 + b^2)^2 - 4a^2b^2 \tag{205}$$

which is valued in $\mathbb{R}$.

In all of these cases the RQM probability can be negative. □

Conjecture 1 (No probability measures as a self-product (in 6D))**.** *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 15 collapse. The research by Acus et al.[9] in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2 B g_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{206}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[9].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2a_{47}^2 + b_2a_0^2a_{47}^2p_{412}p_{422} + \langle 72 \text{ monomials}\rangle = 0 \tag{207}$$

$$b_1a_0^3a_{52} + 2b_2a_0a_{47}^2a_{52}p_{412}p_{422}p_{432}p_{442}p_{452} + \langle 72 \text{ monomials}\rangle = 0 \tag{208}$$

$$\langle 74 \text{ monomials}\rangle = 0 \tag{209}$$

$$\langle 74 \text{ monomials}\rangle = 0 \tag{210}$$

From Equation 206, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

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$$b_1\mathbf{O}Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2Bg_5(g_4(B)g_3(g_2(B)g_1(B))) = b_1Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2\mathbf{O}Bg_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{211}$$

Any equality of the above type between $b_1\mathbf{O}$ and $b_2\mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) be constructible in 6D, using our method. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No probability measures as a self-product (above 6D))**.** *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $\text{GA}(1,3)$ even if in relativity, the signature of the metric $(+,-,-,-)$ versus $(-,-,-,+)$ does not influence the physics. However, in geometric algebra, $\text{GA}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\text{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

Consequently, the most sophisticated dimensional configuration in which a least biased solution to the problem of maximizing the Shannon entropy of universal measurements relative to an initial preparation exists is 3+1D.

3. Discussion

Our theory proposes a minimal ontology based on two fundamental concepts and their relationship: Nature and Physics. Nature is defined as the universal physical constraint (Definition 1) - it determines the structure of all possible measurements. This definition aligns with the philosophical view of nature as that which exists independently of our theories about it, the unchanging backdrop against which all experiments are performed. Physics emerges as the least biased probability measure that maximizes the relative entropy between preparation and measurement states, subject to nature's constraint.

This formulation mirrors the structure of experimental physics itself. Every experiment begins with a known initial state preparation, evolves this state through some operation, and concludes with a measurement. The relative entropy between preparation and measurement states thus captures the basic structure of all possible experiments. By maximizing this entropy subject only to nature's constraints, we obtain the least biased theory consistent with what can be measured.

This ontology is both operational, being grounded in measurements rather than abstract entities, and constructive, showing how physical laws emerge from the interplay between nature's constraints and entropy maximization. Physics emerges not as a collection of fundamental axioms but as the most unbiased description compatible with nature's constraints. This represents a significant philosophical shift from traditional physical ontologies where laws are typically taken as primitive.

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In summary, and tying the words to the math, *physics is the solution to*:

$$\underbrace{\mathcal{L}}_{\substack{\text{an}\\\text{optimization}\\\text{problem}}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{on the entropy}\\\text{of a measurement}\\\text{relative to its preparation}\\\text{over all}}} + \underbrace{\lambda\left(1-\sum_i \rho_i\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}} \tag{212}$$

4. Conclusion

This work presents a radical reformulation of fundamental physics. What currently requires numerous independent axioms - quantum mechanics, general relativity, and the Standard Model gauge symmetries - emerges automatically as the optimal solution to a single optimization problem. The power of this reformulation lies in its explanatory reach: it reveals why these particular theories describe our universe and why spacetime is 3+1 dimensional. These features are not postulated but emerge as necessary characteristics of the optimal predictive theory in the space of all possible ways to predict measurements from preparations.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.
- Data Availability Statement: No datasets were generated or analyzed during the current study.
- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

Appendix A. SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} = \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann Entropy}} + \underbrace{\lambda\left(1-\sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta\left(\overline{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \tag{A1}$$

We solve the maximization problem as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_i, \ldots, \rho_n)}{\partial \rho_i} \tag{A2}$$

$$= -\ln \rho_i - 1 - \lambda - \beta E_i \tag{A3}$$

$$= \ln \rho_i + 1 + \lambda + \beta E_i \tag{A4}$$

$$\implies \ln \rho_i = -1 - \lambda - \beta E_i \tag{A5}$$

$$\implies \rho_i = \exp(-1-\lambda)\exp(-\beta E_i) \tag{A6}$$

$$= \frac{1}{Z(\tau)} \exp(-\beta E_i) \tag{A7}$$

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The partition function, is obtained as follows:

$$1 = \sum_i \exp(-1-\lambda)\exp(-\beta E_i) \tag{A8}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_i \exp(-\beta E_i) \tag{A9}$$

$$Z(\tau) := \sum_i \exp(-\beta E_i) \tag{A10}$$

Finally, the probability measure is:

$$\rho_i = \frac{1}{\sum_i \exp(-\beta E_i)} \exp(-\beta E_i) \tag{A11}$$

Appendix B. RQM in 3+1D

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda\left(1-\sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\frac{1}{2}\zeta \operatorname{tr}\left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{Universal Measurement Constraint}} \tag{A12}$$

The solution is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det\exp(-\frac{1}{2}\zeta \mathbf{M}_i)}}_{\text{Spin}^c(3,1)\text{ Invariant Ensemble}} \underbrace{\det\exp(-\frac{1}{2}\zeta \mathbf{M}_i)}_{\text{Spin}^c(3,1)\text{ Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{A13}$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_1,\ldots,\rho_n)}{\partial \rho_i} \tag{A14}$$

$$= -\ln\frac{\rho_i}{p_i} - p_i - \lambda - \frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i \tag{A15}$$

$$= \ln\frac{\rho_i}{p_i} + p_i + \lambda + \frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i \tag{A16}$$

$$\implies \ln\frac{\rho_i}{p_i} = -p_i - \lambda - \frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i \tag{A17}$$

$$\implies \rho_i = p_i \exp(-p_i-\lambda)\exp\left(-\frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i\right) \tag{A18}$$

$$= \frac{1}{Z(\zeta)} p_i \exp\left(-\frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i\right) \tag{A19}$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i-\lambda)\exp\left(-\frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i\right) \tag{A20}$$

$$\implies (\exp(-p_i-\lambda))^{-1} = \sum_i p_i \exp\left(-\frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i\right) \tag{A21}$$

$$Z(\zeta) := \sum_i p_i \exp\left(-\frac{1}{2}\zeta \operatorname{tr}\mathbf{M}_i\right) \tag{A22}$$

□

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Appendix C. SageMath Program Showing $\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M_u}$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4


I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
```

```
                    [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
                 [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])

y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())
```

The program outputs

```
True
```

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-copy form: $\det \mathbf{M_u} = \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}$.

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