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Keywords: Linear derivation; linear n-derivation; generalized linear n-derivation; Lie ideal; Banach algebra; C*-algebra

Article

Linear Generalized n-Derivations on C^*-Algebras

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Abstract: Let $n \geq 2$ be a fixed integer and $\mathcal{A}$ be a C^*-algebra. A permuting n-linear map $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ is known to be symmetric generalized n-derivation if there exists a symmetric n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ such that $\mathcal{G}(x_1, x_2, \ldots, x_i x_i', \ldots, x_n) = \mathcal{G}(x_1, x_2, \ldots, x_i, \ldots, x_n)x_i' + x_i\mathfrak{D}(x_1, x_2, \ldots, x_i', \ldots, x_n)$ holds for all $x_i, x_i' \in \mathcal{A}$. In this paper, we investigate the structure of C^*-algebras involving generalized linear n-derivations. Moreover, we describe the forms of traces of linear n-derivations satisfying certain functional identity.

Keywords: linear derivation; linear n-derivation; generalized linear n-derivation; lie ideal; banach algebra; C^*-algebra

MSC: 47B48; 22D25; 46L55; 16W25

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1. Introduction

Throughout the discussion, unless otherwise mentioned, $\mathcal{A}$ will denote C^*-algebra with $\mathcal{Z}(\mathcal{A})$ as its center. However, $\mathcal{A}$ may or may not have unity. The symbols $[x, y]$ and $x \circ y$ denote the commutator $xy - yx$ and the anti-commutator $xy + yx$, respectively, for any $x, y \in \mathcal{A}$. An algebra $\mathcal{A}$ is said to be prime if $x\mathcal{A}y = \{0\}$ implies that either $x = 0$ or $y = 0$, and semiprime if $x\mathcal{A}x = \{0\}$ implies that $x = 0$, where $x, y \in \mathcal{A}$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U, r \in R$. U is called a square-closed Lie ideal of R if U is a Lie ideal and $u^2 \in U$ for all $u \in U$.

A Banach algebra is a linear associate algebra which, as a vector space, is a Banach space with norm $||.||$ satisfying the multiplicative inequality; $||xy|| \leq ||x||||y||$ for all x and y in $\mathcal{A}$. An involution on an algebra $\mathcal{A}$ is a linear map $x \mapsto x^*$ of $\mathcal{A}$ into itself such that the following conditions are hold: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, and (iii) $(x + \lambda y)^* = x^* + \bar{\lambda}y^*$ for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, the field of complex numbers, where $\bar{\lambda}$ is the conjugate of λ. An algebra equipped with an involution is called a $*$-algebra or algebra with involution.

A Banach $*$-algebra is a Banach algebra $\mathcal{A}$ together with an isometric involution $||x^*|| = ||x||$ for all $x \in \mathcal{A}$. A C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra with the additional norm condition $||x^*x|| = ||x||^2$ for all $x \in \mathcal{A}$.

A linear operator $\mathcal{D}$ on a C^*-algebra $\mathcal{A}$ is called a derivation if $\mathcal{D}(xy) = \mathcal{D}(x)y + x\mathcal{D}(y)$ holds for all $x, y \in \mathcal{A}$. Consider the inner derivation δ_a implemented by an element a in $\mathcal{A}$, which is defined as $\delta_a(x) = xa - ax$ for every x in $\mathcal{A}$, as a typical example of a nonzero derivation in a noncommutative algebra.

In order to broaden the scope of derivation, Maksa [13] introduced the concept of symmetric bi-derivations. A bi-linear map $\mathfrak{D} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be a bi-derivation if

$$\mathfrak{D}(xx', y) = \mathfrak{D}(x, y)x' + x\mathfrak{D}(x', y)$$

$$\mathfrak{D}(x, yy') = \mathfrak{D}(x, y)y' + y\mathfrak{D}(x, y')$$

holds for any $x, x', y, y' \in \mathcal{A}$. The foregoing conditions are identical if $\mathfrak{D}$ is also a symmetric map, that is, if $\mathfrak{D}(x, y) = \mathfrak{D}(y, x)$ for every $x, y \in \mathcal{A}$. In this case, $\mathfrak{D}$ is referred to as a symmetric bi-derivation

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of $\mathcal{A}$. Vukman [20] investigated symmetric bi-derivations in prime and semiprime rings. Argac and Yenigul [3] and Muthana [15] obtained the similar type of results on Lie ideals of R.

In this paper we briefly discuss the various extensions of the notion of derivations on C^*-algebras. The most general and important one among them is the notion of a generalized symmetric linear n-derivations on C^*-algebras. The concept of derivation and symmetric bi-derivation was generalized by Park [16] as follows: a n-linear map $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is said to be a symmetric(permuting) linear n-derivation if $\mathfrak{D}$ is permuting and $\mathfrak{D}(x_1, x_2, \ldots, x_i x_i', \ldots, x_n) = \mathfrak{D}(x_1, x_2, \ldots, x_n)x_i' + x_i\mathfrak{D}(x_1', x_2, \ldots, x_n)$ hold for all $x_i, x_i' \in \mathcal{A}$, $i = 1, 2, \ldots, n$. A map $\mathfrak{d} : \mathcal{A} \to \mathcal{A}$ defined by $\mathfrak{d}(x) = \mathfrak{D}(x, x, \ldots, x)$ is called the trace of $\mathfrak{D}$. If $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is permuting and n-linear, then the trace $\mathfrak{d}$ of $\mathfrak{D}$ satisfies the relation

$$\mathfrak{d}(x+y) = \mathfrak{d}(x) + \mathfrak{d}(y) + \sum_{k=1}^{n-1} {}^nC_k\, h_k(x;y)$$

for all $x, y \in \mathcal{A}$, where ${}^nC_k = \binom{n}{k}$ and

$$h_k(x;y) = \mathfrak{D}(\underbrace{x, \ldots, x}_{(n-k)\text{-times}}, \underbrace{y, \ldots, y}_{k\text{-times}}).$$

Ashraf et al. [4] introduced the notion of symmetric generalized n-derivations in a ring, building upon the concept of generalized derivation. Let $n \geq 1$ be a fixed positive integer. A symmetric n-linear map $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ is known to be symmetric linear generalized n-derivation if there exists a symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ such that $\mathcal{G}(x_1, x_2, \ldots, x_i x_i', \ldots, x_n) = \mathcal{G}(x_1, x_2, \ldots, x_i, \ldots, x_n)x_i' + x_i\mathfrak{D}(x_1, x_2, \ldots, x_i', \ldots, x_n)$ holds for all $x_i, x_i' \in \mathcal{A}$.

There has been considerable interest in the structure of linear derivation and linear bi-derivation on C^*-algebras. Derivations on C^*-algebras were described in various ways by different authors. For example, in 1966, Kadison [11] proved that each linear derivation of a C^*-algebra annihilates its center. In 1989, Mathieu [14] extended the Posner's first result [17] on C^*-algebras. Basically, he proved that if the product of two linear derivations d and d' on a C^*-algebra is a linear derivation then $dd' = 0$. Very recently, Ekrami and Mirzavaziri [7] showed that "if $\mathcal{A}$ is a C^*-algebra admitting two linear derivations d and d' on $\mathcal{A}$, then there exists a linear derivation D on $\mathcal{A}$ such that $dd' + d'd = D^2$ if and only if d and d' are linearly dependent".

In [2], Ali and Khan proved that if $\mathcal{A}$ is a C^*-algebra admitting a symmetric bilinear generalized $*$-biderivation $\mathcal{H} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with an associated symmetric bilinear $*$-biderivation $\mathcal{B} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, then $\mathcal{H}$ maps $\mathcal{A} \times \mathcal{A}$ into $Z(\mathcal{A})$. In [19], Rehman and Ansari characterized the trace of symmetric bi-derivation and obtain more general results by considering various conditions on a subset of the ring R, viz. Lie ideal of R. Basically, they proved that: let R be a prime ring with $char R \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f, the trace of B. If $[f(x), x] = 0$ for all $x \in U$, then either $U \subseteq Z(R)$ or $f = 0$ (see also [1,8,12,18] for recent results).

In the prospect of above motivation, we try to prove some results based on linear generalized n-derivations of C^*-algebras. We aim to achieve broader outcomes by examining various conditions within a specific subset of the C^*-algebra $\mathcal{A}$, viz. Lie ideal of $\mathcal{A}$. Precisely, we prove that if $\mathcal{A}$ is a C^*-algebra, U is a square closed Lie ideal of $\mathcal{A}$ admitting a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition of $(g(xy) - g(yx)) \pm [x, y] \in Z(R)$ for all $x, y \in U$, then $U \subseteq Z(R)$.

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2. The Results

In order to establish the proofs of our main theorems, we first state a result which we use frequently in the proof of our main results.

Lemma 1. *[10, Corollary 2.1] Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that $U \nsubseteq Z(R)$ and $a, b \in U$.*

1. *If $aUa = \{0\}$, then $a = 0$.*
2. *If $aU = \{0\}$ ($Ua = \{0\}$), then $a = 0$.*
3. *If U is a square closed Lie ideal and $aUb = \{0\}$, then $ab = 0$ and $ba = 0$.*

Lemma 2. *[9, Lemma 1] Let R be a semiprime, 2 torsion-free ring and let U be a Lie ideal of R. Suppose that $[U, U] \subseteq Z(R)$, then $U \subseteq Z(R)$.*

Lemma 3. *[5] Let n be a fixed positive integer and R a $n!$-torsion free ring. Suppose that $y_1, y_2, \ldots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ for $\lambda = 1, 2, \ldots, n$. Then $y_i = 0$ for $i = 1, 2, \ldots, n$.*

Daif and Bell [6] proved that if a semiprime ring admits a derivation d such that either $xy - d(xy) = yx - d(yx)$ or $xy + d(xy) = yx + d(yx)$ holds for all $x, y \in R$, then R is commutative. In this section, apart from proving other results, we expand the previous result by demonstrating the following theorem for the traces of generalized linear n-derivation on certain subsets of $\mathcal{A}$.

Theorem 1. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathcal{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition of $(g(xy) - g(yx)) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. It is given that
$$(g(xy) - g(yx)) \pm [x, y] \in Z(\mathcal{A}) \text{ for all } x, y \in U.$$
Replacing y by $x + my$, where $1 \leq m \leq n-1$ in the given condition, we get
$$g(x(x+my)) - g((x+my)x) \pm [x, x+my] \in Z(\mathcal{A}) \text{ for all } x, y \in U$$
which on solving, we have
$$g(xmy) - g(myx) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2, \ldots, x^2}_{(n-t)-\text{times}}, \underbrace{xmy, \ldots, xmy}_{t-\text{times}}) - \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2, \ldots, x^2}_{(n-t)-\text{times}}, \underbrace{myx, \ldots, myx}_{t-\text{times}}) \pm [x, my] \in Z(\mathcal{A}) \text{ for all } x, y \in U. \quad (1)$$
By using hypothesis, we get
$$\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2, \ldots, x^2}_{(n-t)-\text{times}}, \underbrace{xmy, \ldots, xmy}_{t-\text{times}}) - \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2, \ldots, x^2}_{(n-t)-\text{times}}, \underbrace{myx, \ldots, myx}_{t-\text{times}}) \pm [x, my] \in Z(\mathcal{A})$$
for all $x, y \in U$. Making use of Lemma 3, we see that
$$\mathcal{G}(x^2, \ldots, x^2, xy) - \mathcal{G}(x^2, \ldots, x^2, yx) \in Z(\mathcal{A}) \text{ for all } x, y \in U. \quad (2)$$

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For $1 \leq m \leq n$, (1) can also be written as

$$m^n g(xy) - m^n g(yx) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2,\ldots,x^2}_{(n-t)-\text{times}}, \underbrace{xmy,\ldots,xmy}_{t-\text{times}}) - \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2,\ldots,x^2}_{(n-t)-\text{times}}, \underbrace{myx,\ldots,myx}_{t-\text{times}}) \pm [x,my] \in Z(\mathcal{A}) \text{ for all } x,y \in U.$$

Agai making use of Lemma 3, we have

$$n\{\mathcal{G}(x^2,\ldots,x^2,xy) - \mathcal{G}(x^2,\ldots,x^2,yx)\} \pm [x,y] \in Z(\mathcal{A}) \text{ for all } x,y \in U. \quad (3)$$

From (2) and (3), we get $[x,y] \in Z(\mathcal{A})$ for all $x,y \in U$. As every C^*-algebra is a semiprime ring, using Lemma 2, we get $U \subseteq Z(\mathcal{A})$. $\square$

Theorem 2. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $d : \mathcal{A} \to \mathcal{A}$ satisfying the condition of $(g(x) \pm g(y)) \pm x \circ y \in Z(\mathcal{A})$ for all $x,y \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(\mathcal{A})$. We have given that

$$(g(x) - g(y)) \pm x \circ y \in Z(\mathcal{A}) \text{ for all } x,y \in U.$$

Replacing y by $x + my$, where $z \in U$ and $1 \leq m \leq n-1$ in the given condition, we get

$$g(x) \pm g(x+my) \pm (x \circ x + my) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U$$

which on solving, we have

$$g(x) \pm g(x)g(my) \pm \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x,\ldots,x}_{(n-t)-\text{times}}, \underbrace{my,\ldots,my}_{t-\text{times}}) \pm x \circ x \pm x \circ my \in Z(\mathcal{A}) \text{ for all } x,y,z \in U. \quad (4)$$

Using the given condition, we get

$$g(x) \pm x^2 \pm \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x,\ldots,x}_{(n-t)-\text{times}}, \underbrace{my,\ldots,my}_{t-\text{times}}) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U.$$

Multiply the above equation by m which implies that

$$mA_1(x,y) + m^2 A_2(x,y) + \cdots + m^{n-1} A_{n-1}(x,y) \in Z(\mathcal{A})$$

for all $x,y,z \in U$ where $A_t(x,y)$ represents the term in which z appears t-times.

Making use of Lemma 3, we see that

$$\mathcal{G}(x,\ldots,x,y) \in Z(\mathcal{A}) \text{ for all } x,y \in U.$$

Replace y by x, we get

$$g(x) \in Z(\mathcal{A}) \text{ for all } x,y, \in U.$$

From hypothesis, we have $x \circ y \in Z(\mathcal{A})$ for all $x,y \in U$. Again replace x by yx, we have $y(x \circ y) \in Z(\mathcal{A})$ which imply $[y(x \circ y), z] \in Z(\mathcal{A})$. On solving, we get $[y,z](x \circ y) = 0$ for all $x,y,x \in U$. Again

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replace x by xz, we have $[y,z]x[z,y] = 0$ for all $x,y,z \in U$. By Lemma 1, we have $[z,y] = 0$ for all $y,z \in U$. Again using Lemma 2, we get $U \subseteq Z(\mathcal{A})$, which is a contradiction. □

Theorem 3. *Let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. Suppose that $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying $g(x^2) \pm x^2 = 0$ for all $x \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(\mathcal{A})$. We have given that $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ be symmetric linear generalized n-derivations associated with $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ of a C^*-algebra $\mathcal{A}$ such that $g(x^2) \pm x^2 = 0$ for all $x \in U$. Therefore, $\mathcal{A}$ is semiprime as $\mathcal{A}$ is a C^*-algebra. Now replacing x by $x + my$, $y \in U$ for $1 \le m \le n-1$ in the given condition, we get

$$g(x+my)^2 \pm (x+my)^2 = 0 \text{ for all } x,y \in U.$$

Further solving, we have

$$\begin{aligned}
g(x^2) + g(m(xy+yx)) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2,\ldots,x^2}_{(n-t)-\text{times}}, \underbrace{m(xy+yx),\ldots,m(xy+yx)}_{t-\text{times}}) + \\
g((my)^2) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x^2+m(xy+yx),\ldots,x^2+m(xy+yx)}_{(n-t)-\text{times}}, \underbrace{(my)^2,\ldots,(my)^2}_{t-\text{times}}) \\
\pm x^2 \pm (my)^2 \pm m(xy+yx) = 0 \text{ for all } x,y \in U.
\end{aligned}$$

In accordance of the given condition and Lemma 3, we get

$$n\mathcal{G}(x^2,\ldots,x^2,xy+yx) \pm (xy+yx) = 0 \text{ for all } x,y \in U.$$

Replacing y by x, we find that

$$2ng(x^2) \pm 2x^2 = 0,$$

or

$$x^2 = 0.$$

This implies that $xy + yx = 0$ for all $x,y \in U$. Replacing y by yz, where $z \in U$, we get $[x,y]z = 0$. Again replacing z by $z[x,y]$, we get $[x,y]z[x,y] = 0$ for all $x,y,z \in U$. Using the Lemma 1, we get $[x,y] = 0$ for all $x,y \in U$. By Lemma 2, we get $U \subseteq Z(\mathcal{A})$, a contradiction. □

Corollary 1. *For any fixed integer $n \ge 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying $g(x \circ y) \pm x \circ y = 0$ for all $x,y \in U$ then $U \subseteq Z(\mathcal{A})$.*

Theorem 4. *For any fixed integer $n \ge 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $[g(x), g(y)] - [x,y] \in Z(\mathcal{A})$ for all $x,y \in U$
(ii) $[g(x), g(y)] - [y,x] \in Z(\mathcal{A})$ for all $x,y \in U$.

Then $U \subseteq Z(\mathcal{A})$.

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Proof. (i) Given that

$$[g(x), g(y)] - [x, y] \in Z(\mathcal{A}) \text{ for all } x, y \in U. \tag{5}$$

Consider a positive integer m; $1 \leq m \leq n-1$. Replacing y by $y + mz$, where $z \in U$ in (5), we get

$$[g(x), g(y + mz)] - [x, y + mz] \in Z(\mathcal{A}) \text{ for all } x, y, z \in U.$$

On further solving, we get

$$[g(x), g(y)] + [g(x), g(mz)] + [g(x), \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{y, \ldots, y}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})] - [x, y] - [x, mz] \in Z(\mathcal{A}) \text{ for all } x, y, z \in U.$$

On taking account of hypothesis, we see that

$$mA_1(x, y, z) + m^2 A_2(x, y, z) + \cdots + m^{n-1} A_{n-1}(x, y, z) \in Z(\mathcal{A})$$

where $A_t(x, y, z)$ represents the term in which z appears t-times.

Using Lemma 3, we have

$$[g(x), \mathcal{G}(y, \ldots, y, z)] \in Z(\mathcal{A}) \text{ for all } x, y, z \in U.$$

In particular, for $z = y$, we get

$$[g(x), g(y)] \in Z(\mathcal{A}) \text{ for all } x, y \in U.$$

Now using the given condition, we find that

$$[x, y] \in Z(\mathcal{A}) \text{ for all } x, y \in U.$$

From Lemma 2, $U \subseteq Z(\mathcal{A})$.

(ii) Follows from the first implication with a slight modification. □

Corollary 2. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $d : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $g(x)g(y) \pm xy \in Z(\mathcal{A})$ for all $x, y \in U$
(ii) $g(x)g(y) \pm yx \in Z(\mathcal{A})$ for all $x, y \in U$.

Then $U \subseteq Z(\mathcal{A})$.

Corollary 3. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $d : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $[g(x), g(y)] = [x, y]$ for all $x, y \in U$
(ii) $[g(x), g(y)] = [y, x]$ for all $x, y \in U$.

Then $U \subseteq Z(\mathcal{A})$.

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Theorem 5. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathcal{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition $g(x \circ y) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. Replacing y by $y + mz$ for $1 \leq m \leq n-1$, $z \in U$ in the given condition, we get

$$g(x \circ (y + mz)) \pm [x, y + mz] \in Z(\mathcal{A}) \text{ for all } x, y, z \in U.$$

On further solving and using the specified condition, we get

$$\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{x \circ y, \ldots, x \circ y}_{(n-t)-\text{times}}, \underbrace{x \circ mz, \ldots, x \circ mz}_{t-\text{times}}) \in Z(\mathcal{A}) \text{ for all } x, y, z \in U$$

which implies that

$$mA_1(x, y, z) + m^2 A_2(x, y, z) + \cdots + m^{n-1} A_{n-1}(x, y, z) \in Z(\mathcal{A})$$

for all $x, y, z \in U$ where $A_t(x, y, z)$ represents the term in which z appears t-times. Using Lemma 3, we get

$$\mathcal{G}(x \circ y, \ldots, x \circ y, x \circ z) \in Z(\mathcal{A}) \text{ for all } x, y, z \in U. \tag{6}$$

For $z = y$, we get $g(x \circ y) \in Z(\mathcal{A})$ then our hypothesis reduces to $[x, y] \in Z(\mathcal{A})$. Using the Lemma 2, we get $U \subseteq Z(\mathcal{A})$. □

Corollary 4. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathcal{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition $\mathcal{d}(x \circ y) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$, then $U \subseteq Z(\mathcal{A})$.*

Theorem 6. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathcal{d} : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

- (i) $g([x, y]) \pm g(x) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$
- (ii) $g([x, y]) \pm g(y) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$.

Then $U \subseteq Z(\mathcal{A})$.

Proof. (i) Given that

$$g([x, y]) \pm g(x) \pm [x, y] \in Z(\mathcal{A}) \text{ for all } x, y \in U.$$

Replacing x by $x + mz$, where $z \in U$ and $1 \leq m \leq n-1$ in the given condition, we get

$$g([x + mz, y]) \pm g(x + mz) \pm [x + mz, y] \in Z(\mathcal{A}) \text{ for all } x, y \in U$$

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which on solving and using hypothesis, we obtain

$$\sum_{t=1}^{n-1} {}^{n}C_{t}\mathcal{G}(\underbrace{[x,y],\ldots,[x,y]}_{(n-t)-\text{times}},\underbrace{[mz,y],\ldots,[mz,y]}_{t-\text{times}})$$
$$\pm\sum_{t=1}^{n-1} {}^{n}C_{t}\mathcal{G}(\underbrace{x,\ldots,x}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}}) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U$$

which implies that

$$mA_1(x,y,z) + m^2A_2(x,y,z) + \cdots + m^{n-1}A_{n-1}(x,y,z) \in Z(\mathcal{A})$$

for all $x,y,z \in U$ where $A_t(x,y,z)$ represents the term in which z appears t-times.

Making use of Lemma 3 and torsion restriction, we see that

$$\mathcal{G}([x,y],\ldots,[x,y],[z,y]) \pm \mathcal{G}(x,\ldots,x,z) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U.$$

Replace z by x to get

$$g([x,y]) \pm g(x) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U.$$

Hence, by using the given condition, we find that $[x,y] \in Z(\mathcal{A})$. On taking account of Lemma 2, we get $U \subseteq Z(\mathcal{A})$.

(ii) Given that

$$g([x,y]) \pm g(x) \pm [x,y] \in Z(\mathcal{A}) \text{ for all } x,y \in U.$$

Replacing y by $y+mz$, where $z \in U$ and $1 \leq m \leq n-1$ in the given condition, we get

$$g([x,y+mz]) \pm g(y+mz) \pm [x,y+mz] \in Z(\mathcal{A}) \text{ for all } x,y \in U$$

which on solving and using hypothesis, we obtain

$$\sum_{t=1}^{n-1} {}^{n}C_{t}\mathcal{G}(\underbrace{[x,y],\ldots,[x,y]}_{(n-t)-\text{times}},\underbrace{[x,mz],\ldots,[x,mz]}_{t-\text{times}})$$
$$\pm\sum_{t=1}^{n-1} {}^{n}C_{t}\mathcal{G}(\underbrace{y,\ldots,y}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}}) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U$$

which implies that

$$mA_1(x,y,z) + m^2A_2(x,y,z) + \cdots + m^{n-1}A_{n-1}(x,y,z) \in Z(\mathcal{A})$$

for all $x,y,z \in U$ where $A_t(x,y,z)$ represents the term in which z appears t-times.

Making use of Lemma 3 and torsion restriction, we see that

$$\mathcal{G}([x,y],\ldots,[x,y],[x,z]) \pm \mathcal{G}(y,\ldots,y,z) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U.$$

Replace z by y to get

$$g([x,y]) \pm g(y) \in Z(\mathcal{A}) \text{ for all } x,y,z \in U.$$

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Hence, by using the given condition, we find that $[x, y] \in Z(\mathcal{A})$. On taking account of Lemma 2, we get $U \subseteq Z(\mathcal{A})$.

(ii) Follows from the first implication with a slight modification. □

Corollary 5. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $\mathscr{d}([x, y]) \pm \mathscr{d}(x) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$
(ii) $\mathscr{d}([x, y]) \pm \mathscr{d}(y) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$

Then $U \subseteq Z(\mathcal{A})$.

Theorem 7. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $g(x) \circ g(y) \pm x \circ y \in Z(\mathcal{A})$ for all $x, y \in U$
(ii) $g(x) \circ g(y) \pm [x, y] \in Z(\mathcal{A})$ for all $x, y \in U$

Then $U \subseteq Z(\mathcal{A})$.

Proof. (i) Suppose on the contrary that $U \nsubseteq Z(\mathcal{A})$. It is given that

$$g(x) \circ g(y) \pm x \circ y \in Z(\mathcal{A}) \text{ for all } x, y \in U.$$

Replacing y by $y + mz$, where $z \in U$ and $1 \leq m \leq n-1$ in the given condition, we get

$$g(x) \circ g(y + mz) \pm x \circ (y + mz) \in Z(\mathcal{A}) \text{ for all } x, y, z \in U$$

which on solving, we have

$$g(x) \circ g(y) + g(x) \circ g(mz) + g(x) \circ \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{y, \ldots, y}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})$$
$$\pm x \circ y \pm x \circ mz \in Z(\mathcal{A}) \text{ for all } x, y, z \in U.$$

By using hypothesis, we get

$$g(x) \circ \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{y, \ldots, y}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}}) \in Z(\mathcal{A}) \text{ for all } x, y, z \in U$$

which implies that

$$mA_1(x, y, z) + m^2 A_2(x, y, z) + \cdots + m^{n-1} A_{n-1}(x, y, z) \in Z(\mathcal{A})$$

for all $x, y, z \in U$ where $A_t(x, y, z)$ represents the term in which z appears t-times.

Making use of Lemma 3, we see that

$$g(x) \circ \mathcal{G}(y, \ldots, y, z) \in Z(\mathcal{A}) \text{ for all } x, y, z \in U.$$

In particular, $z = y$, we get

$$g(x) \circ g(y) \in Z(\mathcal{A}) \text{ for all } x, y \in U.$$

Hence, by using the given condition, we find that $x \circ y \in Z(\mathcal{A})$ for all $x, y \in U$. Replacing x by yx, we get $y(x \circ y) \in Z(\mathcal{A})$ for all $x, y \in U$. We can also write it as

$$[y(x \circ y), z] \text{ for all } x, y, z \in U$$

which on solving, we get $[y, z]x \circ y = 0$ for all $x, y, z \in U$. Again replace x by xz and using the same equation, we get $[y, z]x[z, y] = 0$ for all $x, y, z \in U$. Using Lemma 1, we have $[z, y] = 0$ for all $z, y \in U$. By Lemma 2, we have $U \subseteq Z(\mathcal{A})$ which is a contradiction.

(ii) Proceeding in the same way as in (i), we conclude. □

Corollary 6. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $\mathscr{d}(x) \circ \mathscr{d}(y) \pm x \circ y = 0$ for all $x, y \in U$
(ii) $\mathscr{d}(x) \circ \mathscr{d}(y) \pm [x, y] = 0$ for all $x, y \in U$

then $U \subseteq Z(\mathcal{A})$.

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