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On Approximate Variational Inequalities and Bilevel Programming Problems

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Abstract: In this paper, we consider a class of bilevel programming problems (abbreviated as, BLPP). Exploiting the generalized approximate convexity assumptions, we investigate the relations among the solutions of approximate Minty (respectively, Stampacchia) type variational inequalities (abbreviated as, AMTVI (respectively, ASTVI)), and the local ϵ-quasi solutions of the BLPP. Moreover, by employing the generalized Knaster–Kuratowski–Mazurkiewicz (abbreviated as, KKM)-Fan's lemma, we derive some existence results for the solutions of approximate variational inequalities (abbreviated as, AVI), namely, AMTVI and ASTVI. A non-trivial example is given to highlight the importance of the established results. To the best of our knowledge, there is no research paper available in the literature that establishes relationships between the AVI and the BLPP under the assumptions of generalized approximate convexity in terms of limiting subdifferentials.

Keywords: Limiting subdifferentials; ϵ-quasi solutions; Approximate convex functions; Variational inequalities

MSC: 90C34, 90C46, 90C48, 90C29

1. Introduction

Bilevel programming problems are a type of problem where one participant, called the leader, tries to make decisions that consider how another participant, called the follower, will react. The leader's decisions impact the follower's choices. The upper-level part of the problem deals with what the leader wants to achieve and the conditions they need to satisfy, while the lower-level deals with the follower's goals and the constraints associated with them. The origin of the concept of BLPP can be attributed to the influential contributions of Von Stackelberg [41]. The first mathematical bilevel model was developed in 1973 by Bracken and McGill [7]. Subsequently, numerous researchers have created fascinating theories and applications, including optimistic and pessimistic methods, single-level reformulation, optimality criteria, duality results, algorithms, etc. BLPPs have important applications across multiple fields of modern research, including engineering, medicine, and economics as well as BLPPs offer numerous benefits from both theoretical and practical standpoints see, for instance, [4,5,21] and the references cited therein.

Bilevel programming problems are hierarchical problems consisting of two decision parameters. These variables are not independent of each other but act according to a certain hierarchy. In this type of problem, the variables of the first problem act as leaders, and the variables of the second problem act as followers. Obtaining the optimal solution for the second problem is crucial for determining the objective function value of the first problem. Numerous researchers, such as Bard [1–3] and Outrata [38], have explored bilevel programming problems due to their fascinating attributes and significant relevance. Dempe [13] derived the necessary and sufficient optimality criteria for BLPP. Yezza [48] established the first-order necessary optimality criteria for general BLPPs. Moreover, Yezza [48] formulated the general multilevel programming problem and deduced the necessary conditions of optimality in the general case. Dempe [14] rectified deficiencies in [48], specifically addressing

Proposition 2.1 and Theorem 5.1. The optimistic version of BLPPs and its necessary optimality conditions are studied by Dempe [16]. For further insights into bilevel programming problems, we refer to [2,15] and the references cited therein.

The concept of variational inequality (abbreviated as, VI) was introduced by Hartman and Stampacchia [22]. Variational inequalities appear in the forms of Minty VI [30] and Stampacchia VI [42]. Variational inequalities have several applications in the fields of economics, game theory, and traffic analysis, see, [11,17,23]. Giannessi [17] introduced the notion of vector VI for finite-dimensional Euclidean spaces. VI problems have been studied by several scholars as tools for solving optimization problems, for more exposition, see, [17,18,26,44–47], and the references cited therein. Komlósi [25] derived the equivalence among the solutions of Minty and Stampacchia VI and the optimal solution of the minimization problem. Kinderlehrer and Stampacchia [23] studied the relations between the solutions of VI and minimization problems. Crespi et al. [10] investigated the relations between the solutions of Minty VI and scalar optimization problems. Kohli [24] studied the relations between variational inequalities and BLPP involving generalized convex functions in terms of convexificators.

The main motivation and objective for investigating the relationships between the solutions of AVI, namely, AMTVI, ASTVI, and the local ϵ-quasi solutions of BLPP in the notion of limiting subdifferential are threefold. Firstly, in several real-world problems, nonsmooth phenomena occur frequently. To deal with such problems, Clarke introduced the notion of subdifferential for a certain class of locally Lipschitz functions [9]. However, the convexity of the Clarke subdifferential has led to various limitations. To overcome these shortcomings stemming from its convexity, Mordukhovich proposed the concept of the limiting subdifferential [35]. Limiting subdifferential offers an improved Lagrange multiplier rule compared to the Clarke subdifferential and is recognized as the smallest subdifferential among all known robust subdifferentials. Secondly, Convexity and generalized convexity are pivotal in the domains of operations research, economics, and engineering. Moreover, within optimization theory, convexity plays a crucial role as it ensures that a stationary point serves as a global minimizer, and the first-order optimality criteria are transformed into sufficient conditions for identifying a point as a global minimizer. Mangasarian [29] generalized the notion of convex function by introducing the class of pseudoconvex functions. For a detailed study of generalized convex functions, we refer to [8,31,34]. Ngai et al. [36] introduced the notion of approximate convex function. Recently, several generalizations of approximate convex functions have been introduced, for example, Bhatia et al. [6] and Gupta et al. [20]. Thirdly, to the best of our knowledge, there is only one research paper (see, [25]) available in the literature that investigates the relationships between the solutions of BLPP and variational inequalities. However, the investigation of the relationships between the solutions of AVI and the local ϵ-quasi solutions of BLPP in the notion of limiting subdifferentials has not been explored yet. Consequently, this paper aims to address this specific research gap by establishing the results that make relationships between the solutions of AMTVI, ASTVI, and the local ϵ-quasi solutions of BLPP in the notion of limiting subdifferentials.

Motivated by the works of [6,10,24,32,43], in this paper, we consider BLPP and establish the relationships between the solutions of approximate convex functions, namely, AMTVI and ASTVI, and the local ϵ-quasi solutions of BLPP. Moreover, we deduce some existence results for the solutions of AMTVI and ASTVI, by employing the assumption of generalized KKM-Fan's lemma.

The novelty and contributions of this paper are threefold. Firstly, the results of this paper extend the analogous results in [10,25] from single-level optimization problems to more general optimization problems, namely, bilevel optimization problems. Secondly, since the limiting subdifferential is the least among all the known robust subdifferentials and offers an enhanced Lagrange multiplier rule compared to the Clarke subdifferential, therefore our findings naturally sharpen the analogous results of [6,24,32,43]. Thirdly, the established results of this paper extend the analogous results in [6,24,27,43] for a broader class of approximate convex functions.

The organization of this article is as follows. In Section 2, we recall some basic definitions and preliminaries. In Section 3, employing the potent tool of limiting subdifferential, we investigate the

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equivalence among the solutions of AMTVI and ASTVI, and the local ϵ-quasi solution of nonsmooth BLPP. In Section 4, a generalized Fan lemma has been employed to establish the existence results for the solutions of AVI.

2. Definition and Preliminaries

Throughout this paper, we use the notation $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product in the n-dimensional Euclidean space $\mathbb{R}^n$. For a nonempty subset Ω of $\mathbb{R}^n \times \mathbb{R}^m$ equipped with the Euclidean norm $\| \cdot \|$, we signify the closure and interior of Ω by clΩ and intΩ, respectively.

The definition of a convex set, provided below is from [34].

Definition 1. *A set Ω is termed as a convex set, provided for all $(\wp_1, \wp_2), (\overline{\wp}_1, \overline{\wp}_2) \in \Omega$, one has*

$$(\wp_1, \wp_2) + \mu((\overline{\wp}_1, \overline{\wp}_2) - (\wp_1, \wp_2)) \in \Omega, \forall \mu \in [0, 1].$$

Now, we recall the following definitions related to nonsmooth analysis from [35].

Definition 2. *For a continuous function $\chi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, Fréchet subdifferential of χ is defined as follows:*

$$\hat{\partial}\chi(\wp_1, \wp_2) := \Bigg\{ v \in \mathbb{R}^n \times \mathbb{R}^m : \liminf_{(\wp_1', \wp_2') \to (\wp_1, \wp_2)} \frac{\chi(\wp_1', \wp_2') - \chi(\wp_1, \wp_2) - \langle v, (\wp_1', \wp_2') - (\wp_1, \wp_2) \rangle}{\|(\wp_1', \wp_2') - (\wp_1, \wp_2)\|} \geq 0 \Bigg\}$$

Definition 3. *The limiting subdifferential of χ at $(\overline{\wp}_1, \overline{\wp}_2) \in \mathbb{R}^n \times \mathbb{R}^m$, denoted by $\partial_M \chi(\overline{\wp}_1, \overline{\wp}_2)$, is defined as*

$$\partial_M \chi(\overline{\wp}_1, \overline{\wp}_2) = \limsup_{(\wp_1, \wp_2) \to (\overline{\wp}_1, \overline{\wp}_2)} \hat{\partial}\chi(\wp_1, \wp_2),$$

where lim sup is the Painlevé-Kuratowski outer limit.

Remark 1. *For a locally Lipschitz function χ at $(\wp_1, \wp_2) \in \Omega$, the set valued map $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$, defined by $\mathcal{G}(\wp_1, \wp_2) := \partial_M \chi(\wp_1, \wp_2)$, is closed.*

Definition 4. *If χ is finite at $(\overline{\wp}_1, \overline{\wp}_2) \in \mathbb{R}^n \times \mathbb{R}^m$, then χ is lower-regular at $(\overline{\wp}_1, \overline{\wp}_2)$ if $\hat{\partial}\chi(\overline{\wp}_1, \overline{\wp}_2) = \partial_M \chi(\overline{\wp}_1, \overline{\wp}_2)$.*

Now, we consider the following bilevel programming problem:

BLPP: $\min_{\wp_1, \wp_2} \Phi(\wp_1, \wp_2)$

$$\text{subject to: } H_j(\wp_1, \wp_2) \leq 0, j \in J := \{1, \ldots, m_2\}, \wp_2 \in \psi(\wp_1),$$

$$\min_{\wp_2} \phi(\wp_1, \wp_2)$$

$$\text{subject to } h_i(\wp_1, \wp_2) \leq 0, i \in I := \{1, \ldots, m_1\}$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $H_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, j \in J$, and $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, i \in I$ are real valued functions and

$$\psi(\wp_1) = \underset{\wp_2}{\operatorname{argmin}} \{\phi(\wp_1, \wp_2) : h_i(\wp_1, \wp_2) \leq 0, i \in I\}.$$

So, the basic idea is that based on the choice of the leader, the follower minimizes his objective function and the leader then uses the obtained solution $\wp_2 = \wp_2(\wp_1)$ to minimize his objective function. BLPP is said to be well defined if we can uniquely determine the optimal solution of the lower level problem for every $\wp_1 \in \mathbb{R}^n$. In literature, two types of solution concepts have been studied for the problems having more than one optimal solutions for lower level problem, such as optimistic solution and pessimistic solution.

In the optimistic approach, the follower considers an optimal solution which is the best from the leader's perspective. Therefore, one has the following optimistic bilevel programming problem:

$$\text{OBLPP:} \quad \min_{\wp_1} \psi_\circ(\wp_1), \wp_1 \in \mathbb{R}^n$$

$$\text{where } \psi_\circ(\wp_1) = \min_{\wp_2}\{\Phi(\wp_1,\wp_2) : H_j(\wp_1,\wp_2) \leq 0, j \in J, \wp_2 \in \psi(\wp_1)\}$$

and $\psi(\wp_1)$ is the set of optimal solutions for the following lower level problem

$$\min_{\wp_2} \phi(\wp_1,\wp_2)$$
$$\text{subject to } h_i(\wp_1,\wp_2) \leq 0, i \in I.$$

Let, S be the set of all feasible solutions to the problem BLPP, that is,

$$S := \{(\wp_1,\wp_2) \in \mathbb{R}^n \times \mathbb{R}^m : H_j(\wp_1,\wp_2) \leq 0,\ j \in J,\ h_i(\wp_1,\wp_2) \leq 0,\ i \in I,\ \wp_2 \in \psi(\wp_1)\}.$$

Now, we introduce two following notations which will be used in this sequel.

$$\Omega := \{(\wp_1,\wp_2) \in \mathbb{R}^n \times \mathbb{R}^m : H_j(\wp_1,\wp_2) \leq 0,\ j \in J,\ h_i(\wp_1,\wp_2) \leq 0,\ i \in I\},$$
$$\mathcal{F}_{\wp_1}(\wp_2) = \phi(\wp_1,\wp_2),\ \forall(\wp_1,\wp_2) \in \Omega.$$

Therefore, it is evident that $S = \{(\wp_1,\wp_2) \in \Omega : \wp_2 = \text{argmin } \mathcal{F}_{\wp_1}(\wp_2)\}$.
In the rest of the paper, we will assume that Ω is a non-empty convex subset of $\mathbb{R}^n \times \mathbb{R}^m$.

The following definition from Ngai and Penot [37] represents the notion of approximate convexity of a real-valued function.

Definition 5. *A function $\chi : \Omega \to \mathbb{R}$ is termed as an approximate convex function around $(\overline{\wp}_1,\overline{\wp}_2) \in \Omega$, provided for any $\epsilon > 0, \exists d > 0$, the following inequality is satisfied:*

$$\chi(\mu(\wp_1^2,\wp_2^2) + (1-\mu)(\wp_1^1,\wp_2^1)) \leq \mu\chi(\wp_1^2,\wp_2^2) + (1-\mu)\chi(\wp_1^1,\wp_2^1) + \epsilon\mu(1-\mu)\|(\wp_1^2,\wp_2^2) - (\wp_1^1,\wp_2^1)\|,$$

for all $(\wp_1^1,\wp_2^1), (\wp_1^2,\wp_2^2) \in B((\overline{\wp}_1,\overline{\wp}_2), d) \cap \Omega$ and $\mu \in [0,1]$.

We have the following characterization for lower semicontinuous approximate convex functions from Ngai and Penot [37].

Proposition 1. *A lower semicontinuous function $\chi : \Omega \to \mathbb{R}$ is approximate convex around $(\overline{\wp}_1,\overline{\wp}_2)$, if and only if for any $\epsilon > 0, \exists d > 0$, such that*

$$\chi\left(\wp_1^2,\wp_2^2\right) - \chi\left(\wp_1^1,\wp_2^1\right) \geq \left\langle \xi, \left(\wp_1^2,\wp_2^2\right) - \left(\wp_1^1,\wp_2^1\right)\right\rangle - \epsilon\left\|\left(\wp_1^2,\wp_2^2\right) - \left(\wp_1^1,\wp_2^1\right)\right\|,$$

for any $(\wp_1^1,\wp_2^1), (\wp_1^2,\wp_2^2) \in B((\overline{\wp}_1,\overline{\wp}_2), d) \cap \Omega$ and any $\xi \in \partial_M\chi(\wp_1^1,\wp_2^1)$.

On the lines of Bhatia et al. [6] and Golestani et al. [19], in the following definitions, we define the notion of generalized approximate convex functions in terms of limiting subdifferentials.

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Definition 6. *A function $\chi : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-pseudoconvex of type I around $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$, if for all $\epsilon > 0, \exists d > 0$, such that for all $\left(\wp_1^1, \wp_2^1\right), \left(\wp_1^2, \wp_2^2\right) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, and if*

$$\left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle \geq 0, \text{ for some } \xi \in \partial_M \chi\left(\wp_1^2, \wp_2^2\right),$$

then

$$\chi\left(\wp_1^1, \wp_2^1\right) - \chi\left(\wp_1^2, \wp_2^2\right) \geq -\epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\|.$$

Definition 7. *A function $\chi : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-pseudoconvex of type II around $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$, if for all $\epsilon > 0, \exists d > 0$, such that for all $\left(\wp_1^1, \wp_2^1\right), \left(\wp_1^2, \wp_2^2\right) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, and if*

$$\left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle + \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\| \geq 0, \text{ for some } \xi \in \partial_M \chi\left(\wp_1^2, \wp_2^2\right),$$

then

$$\chi\left(\wp_1^1, \wp_2^1\right) \geq \chi\left(\wp_1^2, \wp_2^2\right)$$

Remark 2. *It is evident from the above definitions, if $\chi : \Omega \to \mathbb{R}$ is approximate ∂_M-pseudoconvex of type II around $\overline{\wp} \in \Omega$, then χ is also approximate ∂_M-pseudoconvex of type I around $\overline{\wp} \in \Omega$. But the converse may not true. For example, let $\chi : [-1, 1] \to \mathbb{R}$ be given as*

$$\chi(\wp) = \begin{cases} \wp, & \wp \leq 0 \\ \wp^2 + 1, & \wp > 0 \end{cases}$$

It can be verified that the limiting subdifferentiable of χ is:

$$\partial_M \chi(\wp) = \begin{cases} 1, & \wp < 0 \\ [1, \infty), & \wp = 0 \\ 2\wp, & \wp > 0 \end{cases}$$

Then, one can show that χ is approximate ∂_M-pseudoconvex of type I around $\overline{\wp} = 0$, but not approximate ∂_M-pseudoconvex of type II around $\overline{\wp} = 0$.

Remark 3. *The pseudoconvexity of the function χ at $\overline{\wp} \in \Omega$ implies the approximate ∂_M-pseudoconvexity of χ of type I around $\overline{\wp}$. But the converse may not be true. For example, let $\chi : [-2\pi, 2\pi] \to \mathbb{R}$ be given as*

$$\chi(\wp) = \begin{cases} -\wp^2 + 1, & \wp < 0 \\ \sin \wp + e^{\wp}, & \wp \geq 0 \end{cases}$$

The limiting subdifferential of χ is given as

$$\partial_M \chi(\wp) = \begin{cases} -2\wp, & \wp < 0, \\ [0, 2], & \wp = 0, \\ \cos \wp + e^{\wp}, & \wp > 0. \end{cases}$$

Then χ is approximate ∂_M-pseudoconvex of type I around $\overline{\wp}$, but not pseudoconvex at $\overline{\wp}$, as for $\wp > \pi$,

$$\langle \xi, \wp - \overline{\wp} \rangle \geq 0, \ \xi \in \partial_M \chi(\overline{\wp}) \text{ does not imply } \chi(\wp) \geq \chi(\overline{\wp}).$$

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Definition 8. *A function $\chi : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-quasiconvex of type I around $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$, if for all $\epsilon > 0, \exists d > 0$, such that for all $(\wp_1^1, \wp_2^1), (\wp_1^2, \wp_2^2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, and if*

$$\chi\left(\wp_1^1, \wp_2^1\right) \leq \chi\left(\wp_1^2, \wp_2^2\right)$$

then

$$\left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle - \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\| \leq 0, \forall \xi \in \partial_M \chi\left(\wp_1^2, \wp_2^2\right)$$

Definition 9. *A function $\chi : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-quasiconvex of type II around $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$, if for all $\epsilon > 0, \exists d > 0$, such that for all $(\wp_1^1, \wp_2^1), (\wp_1^2, \wp_2^2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, and if*

$$\chi\left(\wp_1^1, \wp_2^1\right) \leq \chi\left(\wp_1^2, \wp_2^2\right) + \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\|,$$

then

$$\left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle \leq 0, \forall \xi \in \partial_M \chi\left(\wp_1^2, \wp_2^2\right).$$

Remark 4. *From the above definitions, it follows that, if $\chi : \Omega \to \mathbb{R}$ is approximate ∂_M-quasiconvex of type II around $\overline{\wp} \in \Omega$, then χ is approximate ∂_M-quasiconvex of type I around $\overline{\wp} \in \Omega$. But the converse need not to be true. For example, let $\chi : [-\pi, \pi] \to \mathbb{R}$ be given as*

$$\chi(\wp) = \begin{cases} -\wp^2, & \wp < 0 \\ \sin \wp, & \wp \geq 0 \end{cases}$$

We can show that the limiting subdifferential of χ is:

$$\partial_M \chi(\wp) = \begin{cases} -2\wp, & \wp < 0 \\ [0, 1], & \wp = 0 \\ \cos \wp, & \wp > 0 \end{cases}$$

Therefore, one can conclude that, χ is approximate ∂_M-quasiconvex of type I around $\overline{\wp} = 0$, but not approximate ∂_M-quasiconvex of type II around $\overline{\wp} = 0$.

Definition 10. *A function $\chi : \Omega \to \mathbb{R}$ is termed as an approximate ∂_{M^-}-quasiconvex function around $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$, provided for each $\epsilon > 0, \exists d > 0$, such that the following implication holds:*

$$\begin{aligned} &\chi\left(\wp_1^1, \wp_2^1\right) \leq \chi\left(\wp_1^2, \wp_2^2\right) - \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\| \\ &\quad \Longrightarrow \left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle + \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\| \leq 0, \end{aligned}$$

for any $(\wp_1^1, \wp_2^1), (\wp_1^2, \wp_2^2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$ and for all $\xi \in \partial_M \chi(\wp_1^2, \wp_2^2)$.

Definition 11. *A function $\chi : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-pseudoconvex function around $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$, provided for each $\epsilon > 0, \exists d > 0$, such that the following implication holds:*

$$\begin{aligned} &\chi\left(\wp_1^1, \wp_2^1\right) < \chi\left(\wp_1^2, \wp_2^2\right) - \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\| \\ &\quad \Longrightarrow \left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle + \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\| < 0, \end{aligned}$$

for any $(\wp_1^1, \wp_2^1), (\wp_1^2, \wp_2^2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$ and for all $\xi \in \partial_M \chi(\wp_1^2, \wp_2^2)$.

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Remark 5. *It is worthwhile to mention that, the limiting subdifferential of a locally Lipschitz function at a point is included in Clarke subdifferential at that point. Therefore, if $\chi : \Omega \to \mathbb{R}$ is locally Lipschitz and exhibits generalized approximate convexity around $\overline{\wp} \in \Omega$ in terms of the Clarke subdifferential, (see, Bhatia et al. [6]), then χ is also generalized ∂_M-approximate convex function around $\overline{\wp} \in \Omega$. However, the converse may not be true. Indeed, consider the lower semicontinuous function $\chi : [-1, 1] \to \Omega$ defined as*

$$\chi(\wp) = \begin{cases} \wp, & \wp \leq 0 \\ \wp^2 + 1, & \wp > 0 \end{cases}$$

One can verify that the limiting subdifferentiable of χ is as follows:

$$\partial_M \chi(\wp) = \begin{cases} 1, & \wp < 0 \\ [1, \infty), & \wp = 0 \\ 2\wp, & \wp > 0. \end{cases}$$

This illustrates that χ is approximate ∂_M-pseudoconvex of type I around $\overline{\wp} = 0$, but not approximate pseudoconvex of type I around $\overline{\wp} = 0$, in terms of Clarke subdifferential, as χ is not locally Lipschitz at $\overline{\wp} = 0$, and hence, Clarke subdifferential may not exist at $\overline{\wp} = 0$.

Definition 12. *A multivalued mapping $G : \Omega \to 2^{\Omega}$ is said to be approximate ϵ-pseudomonotone around $(\overline{\wp}_1, \overline{\wp}_2)$, if there exists $d > 0$, such that for each $(\wp_1^1, \wp_2^1), (\wp_1^2, \wp_2^2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, if*

$$\left\langle \zeta, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle + \epsilon \left\| \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\| \geq 0$$

then

$$\left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle \geq 0$$

whenever $\xi \in G\left(\wp_1^1, \wp_2^1\right), \zeta \in G\left(\wp_1^2, \wp_2^2\right)$.

The subsequent mean value theorem for locally Lipschitz functions from [34], will be used in the sequel.

Theorem 1. *Let Φ be Lipschitz on an open set containing $\left[\left(\wp_1^2, \wp_2^2\right), \left(\wp_1^1, \wp_2^1\right)\right]$ in Ω. Moreover, if Φ is lower regular on $\left(\left(\wp_1^2, \wp_2^2\right), \left(\wp_1^1, \wp_2^1\right)\right)$. Then, one has*

$$\Phi\left(\wp_1^1, \wp_2^1\right) - \Phi\left(\wp_1^2, \wp_2^2\right) = \left\langle \xi, \left(\wp_1^1, \wp_2^1\right) - \left(\wp_1^2, \wp_2^2\right) \right\rangle,$$

for some $\xi \in \partial_M \Phi(u_1, u_2); (u_1, u_2) \in \left(\left(\wp_1^2, \wp_2^2\right), \left(\wp_1^1, \wp_2^1\right)\right)$.

The following notions of ϵ-quasi solution and local ϵ-quasi solution for BLPP are adaptations of the notions of ϵ-quasi solution and local ϵ-quasi solution for scalar optimization problems from Loridan [28].

Definition 13. *Let $\epsilon > 0$ be given. A point $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$ is said to be an ϵ-quasi solution to the BLPP if for any $(\wp_1, \wp_2) \in \Omega$, the following inequalities hold:*

$$\Phi(\wp_1, \wp_2) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2) - \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|,$$
$$\mathcal{F}_{\wp_1}(\wp_2) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2) - \epsilon \|\wp_2 - \overline{\wp}_2\|.$$

Definition 14. *Let $\epsilon > 0$ be given. A point $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$ is considered to be a local ϵ-quasi solution to the BLPP, if there exists $d > 0$ such that the following inequalities hold:*

$$\Phi(\wp_1, \wp_2) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2) - \epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|,$$
$$\mathcal{F}_{\wp_1}(\wp_2) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2) - \epsilon\|\wp_2 - \overline{\wp}_2\|,$$

for any $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$.

Now, we consider the following AMTVI and ASTVI in terms of limiting subdifferentials:

AMTVI: Find $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$ such that for an $\epsilon > 0, \exists d > 0$, such that for each $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$ and all $\xi_1 \in \partial_M \Phi(\wp_1, \wp_2)$ and $\xi_2 \in \partial_M \mathcal{F}_{\wp_1}(\wp_2)$, the following inequalities hold:

$$\langle \xi_1, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle \geq -\epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|,$$
$$\langle \xi_2, \wp_2 - \overline{\wp}_2 \rangle \geq -\epsilon\|\wp_2 - \overline{\wp}_2\|.$$

ASTVI: Find $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$ such that for an $\epsilon > 0, \exists d > 0$, such that for each $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, there exists $\zeta_1 \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2)$ and $\zeta_2 \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)$ such that the following inequalities hold:

$$\langle \zeta, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle \geq -\epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|,$$
$$\langle \zeta_2, \wp_2 - \overline{\wp}_2 \rangle \geq -\epsilon\|\wp_2 - \overline{\wp}_2\|.$$

Remark 6. *For $\epsilon = 0$, the above variational inequalities AMTVI and ASTVI reduce to local versions of nonsmooth Minty and Stampacchia VIs, respectively (see, [10,24,25]). Kohli [24] studied the Minty and Stampacchia VI problems in terms of convexificators. Nevertheless, the results in Sections 3 and 4 of the paper are more general than that of Kohli [24] in view of the fact that generalized approximate convex functions employed in this work are more general than generalized convex functions utilized by Kohli. Furthermore, the findings in Sections 3 and 4 are sharper than that of Kohli [24] as limiting subdifferential is smaller than convexificator which is employed in the work of Kohli.*

3. Relationship among BLPP, ASTVI, and AMTVI

This section is devoted to studying the equivalence relationships between the solutions of AVI, namely, AMTVI, ASTVI, and the local ϵ-quasi solutions of the BLPP within the framework of limiting subdifferential.

From now onwards, let $\epsilon > 0$ be given and Φ be a lower semicontinuous function unless otherwise specified.

Theorem 2. *Let Φ and $\mathcal{F}_{\wp_1}$ be approximate convex functions around $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$. If $(\overline{\wp}_1, \overline{\wp}_2)$ is local ϵ-quasi solution of BLPP, then $(\overline{\wp}_1, \overline{\wp}_2)$ solves AMTVI with respect to (w.r.t) 2ϵ.*

Proof. Let $(\overline{\wp}_1, \overline{\wp}_2)$ is a local ϵ-quasi solution of BLPP, but $(\overline{\wp}_1, \overline{\wp}_2)$ does not solve AMTVI w.r.t 2ϵ. Then for all $d > 0$ we can get $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$ and $\xi_1 \in \partial_M \Phi(\wp_1, \wp_2)$ and $\xi_2 \in \partial_M \mathcal{F}_{\wp_2}(\wp_1)$ such that

$$\begin{aligned} &\langle \xi_1, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + 2\epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| < 0, \text{ or} \\ &\langle \xi_2, \wp_2 - \overline{\wp}_2 \rangle\rangle + 2\epsilon\|\wp_2 - \overline{\wp}_2\| < 0. \end{aligned} \tag{1}$$

Since Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-convex around $(\overline{\wp}_1, \overline{\wp}_2)$, therefore for each $\epsilon > 0$, we can get $d > 0$, such that, for every $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$ and $\xi_1 \in \partial_M \Phi(\wp_1, \wp_2)$ and $\xi_2 \in \partial_M \mathcal{F}_{\wp_2}(\wp_1)$, one has

$$\begin{aligned} &\Phi(\overline{\wp}_1, \overline{\wp}_2) - \Phi(\wp_1, \wp_2) \geq \langle \xi_1, (\overline{\wp}_1, \overline{\wp}_2) - (\wp_1, \wp_2) \rangle - \epsilon\|(\overline{\wp}_1, \overline{\wp}_2) - (\wp_1, \wp_2)\|, \text{ and} \\ &\mathcal{F}_{\wp_1}(\overline{\wp}_2) - \mathcal{F}_{\wp_2}(\wp_2) \geq \langle \xi_2, \overline{\wp}_2 - \wp_2 \rangle - \epsilon\|\wp_2 - \overline{\wp}_2\|. \end{aligned} \tag{2}$$

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From (2) together with the fact that $\epsilon > 0$, we get

$$\begin{aligned}\Phi(\wp_1,\wp_2)-\Phi(\overline{\wp}_1,\overline{\wp}_2)+\epsilon\|(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\| \\ \leq \langle\xi_1(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\rangle+2\epsilon\|(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\|,\end{aligned} \tag{3}$$

$$\mathcal{F}_{\wp_2}(\wp_2)-\mathcal{F}_{\wp_2}(\overline{\wp}_2)+\epsilon\|\wp_2-\overline{\wp}_2\| \leq \langle\xi_2,\wp_2-\overline{\wp}_2\rangle+2\epsilon\|\wp_2-\overline{\wp}_2\|. \tag{4}$$

From (1), (3), and (4), for each $(\wp_1,\wp_2)\in B((\overline{\wp}_1,\overline{\wp}_2),d)\cap\Omega$, it follows that

$$\begin{aligned}&\Phi(\wp_1,\wp_2)-\Phi(\overline{\wp}_1,\overline{\wp}_2)+\epsilon\|(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\|<0, \text{ or} \\ &\mathcal{F}_{\wp_2}(\wp_2)-\mathcal{F}_{\wp_2}(\overline{\wp}_2)+\epsilon\|\wp_2-\overline{\wp}_2\|<0,\end{aligned}$$

this contradicts the fact that $(\overline{\wp}_1,\overline{\wp}_2)$ is a local ϵ-quasi solution of BLPP. □

Theorem 3. *Let Φ and $\mathcal{F}_{\wp_1}$ be locally Lipschitz lower-regular functions at $(\overline{\wp}_1,\overline{\wp}_2)$. Moreover, assume that $(\overline{\wp}_1,\overline{\wp}_2)$ solves AMTVI w.r.t $\epsilon'<\epsilon$ and Φ, $\mathcal{F}_{\wp_1}$ are approximate convex functions around $(\overline{\wp}_1,\overline{\wp}_2)\in\Omega$. Then, in conclusion $(\overline{\wp}_1,\overline{\wp}_2)$ is a local ϵ-quasi solution of BLPP.*

Proof. By arguing, let us suppose that $(\overline{\wp}_1,\overline{\wp}_2)\in\Omega$ solves AMTVI w.r.t $\epsilon'<\epsilon$, but $(\overline{\wp}_1,\overline{\wp}_2)$ is not a local ϵ-quasi solution of BLPP. Hence, for all $d>0$, $\exists(\wp_1,\wp_2)\in B((\overline{\wp}_1,\overline{\wp}_2),d)\cap\Omega$, such that

$$\begin{aligned}&\Phi(\wp_1,\wp_2)<\Phi(\overline{\wp}_1,\overline{\wp}_2)-\epsilon\|(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\|, \text{ or} \\ &\mathcal{F}_{\wp_1}(\wp_2)<\mathcal{F}_{\wp_1}(\overline{\wp}_2)-\epsilon\|\wp_2-\overline{\wp}_2\|.\end{aligned} \tag{5}$$

Let $(\wp_1(\mu),\wp_2(\mu))=(\overline{\wp}_1,\overline{\wp}_2)+\mu((\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2))$ for all $\mu\in[0,1]$. Since, Φ and $\mathcal{F}_{\wp_1}$ are approximate convex functions around $(\overline{\wp}_1,\overline{\wp}_2)$, hence for each $\epsilon>0$, $\exists d>0$, and for all $(\wp_1,\wp_2)\in B((\overline{\wp}_1,\overline{\wp}_2),d)\cap\Omega$, we have

$$\begin{aligned}\Phi((\overline{\wp}_1,\overline{\wp}_2)+\mu((\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)))\leq&\mu\Phi(\wp_1,\wp_2)+(1-\mu)\Phi(\overline{\wp}_1,\overline{\wp}_2)+ \\ &\epsilon\mu(1-\mu)\|(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\|,\mu\in[0,1].\end{aligned} \tag{6}$$

$$\mathcal{F}_{\wp_1}(\overline{\wp}_2+\mu(\wp_2-\overline{\wp}_2))\leq\mu\mathcal{F}_{\wp_1}(\wp_2)+(1-\mu)\mathcal{F}_{\wp_1}(\overline{\wp}_2)+\epsilon\mu(1-\mu)\|\wp_2-\overline{\wp}_2\|,\ \mu\in[0,1] \tag{7}$$

Let $\mu\in]0,1[$ be arbitrary. Now invoking the mean value theorem, there exist $\mu_1',\mu_2'\in(0,\mu)$ and $\xi_1^*\in\partial_M\Phi((\overline{\wp}_1,\overline{\wp}_2)+\ \mu_1'((\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)))$ and $\xi_2^*\in\partial_M\mathcal{F}_{\wp_1}(\overline{\wp}_2+\mu_2'(\wp_2-\overline{\wp}_2))$ such that

$$\mu\langle\xi_1^*,(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\rangle=\Phi((\overline{\wp}_1,\overline{\wp}_2)+\mu((\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)))-\Phi(\overline{\wp}_1,\overline{\wp}_2). \tag{8}$$

$$\mu\langle\xi_2^*,\wp_2-\overline{\wp}_2\rangle=\mathcal{F}_{\wp_1}(\overline{\wp}_2+\mu(\wp_2-\overline{\wp}_2))-\mathcal{F}_{\wp_1}(\overline{\wp}_2). \tag{9}$$

Exploiting (6)−(9), we have

$$\langle\xi_1^*,(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\rangle\leq\Phi(\wp_1,\wp_2)-\Phi(\overline{\wp}_1,\overline{\wp}_2)+\epsilon(1-\mu)\|(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\|, \tag{10}$$

$$\langle\xi_2^*,\wp_2-\overline{\wp}_2\rangle\leq\mathcal{F}_{\wp_1}(\wp_2)-\mathcal{F}_{\wp_1}(\overline{\wp}_2)+\epsilon\mu(1-\mu)\|\wp_2-\overline{\wp}_2\|. \tag{11}$$

From (5), (10) and (11), it follows that

$$\begin{aligned}&\langle\xi_1^*,(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\rangle<-\epsilon\mu\|(\wp_1,\wp_2)-(\overline{\wp}_1,\overline{\wp}_2)\|, \text{ or} \\ &\langle\xi_2^*,\wp_2-\overline{\wp}_2\rangle<-\epsilon\mu\|\wp_2-\overline{\wp}_2\|.\end{aligned} \tag{12}$$

Since $\mu_1', \mu_2' > 0$, from (12), it follows that

$$\begin{aligned} &\langle \xi_1^*, (\wp_1(\mu_1'), \wp_2(\mu_1')) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon' \|(\wp_1(\mu_1'), \wp_2(\mu_1')) - (\overline{\wp}_1, \overline{\wp}_2)\| < 0, \text{ or} \\ &\langle \xi_2^*, \wp_2(\mu_2') - \overline{\wp}_2 \rangle < \epsilon' \|\wp_2(\mu_2') - \overline{\wp}_2\|, \text{ where } \epsilon' = \epsilon\mu. \end{aligned} \tag{13}$$

This contradicts the fact that $(\overline{\wp}_1, \overline{\wp}_2)$ solves AMTVI. □

Theorem 4. *Let $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$ is a local ϵ-quasi solution of BLPP with Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-quasiconvex functions of type II around $(\overline{\wp}_1, \overline{\wp}_2)$. Then $(\overline{\wp}_1, \overline{\wp}_2)$ solves AMTVI w.r.t same ϵ.*

Proof. Since $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$ is a local ϵ-quasi solution of BLPP, therefore $\exists d > 0$ and for all $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, we have

$$\begin{aligned} &\Phi(\wp_1, \wp_2) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2) - \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|, \text{ and} \\ &\mathcal{F}_{\wp_1}(\wp_2) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2) - \epsilon \|\wp_2 - \overline{\wp}_2\|. \end{aligned} \tag{14}$$

Moreover, as Φ and $\mathcal{F}_{\wp_1}(\wp_2)$ are approximate ∂_M-quasiconvex functions of type II, hence for any $\epsilon > 0$, we can get $d' > 0$ and for each $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d') \cap \Omega$, if

$$\Phi(\overline{\wp}_1, \overline{\wp}_2) \leq \Phi(\wp_1, \wp_2) + \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|,$$

then

$$\langle \xi, (\overline{\wp}_1, \overline{\wp}_2) - (\wp_1, \wp_2) \rangle \leq 0, \forall \xi \in \partial_M \Phi(\wp_1, \wp_2),$$

and

$$\mathcal{F}_{\wp_1}(\overline{\wp}_2) \leq \mathcal{F}_{\wp_1}(\wp_2) + \epsilon \|\wp_2 - \overline{\wp}_2\| \implies \langle \eta, \overline{\wp}_2 - \wp_2 \rangle \leq 0, \ \forall \eta \in \partial_M \mathcal{F}_{\wp_1}(\wp_2).$$

Let $\overline{d} = \min\{d, d'\}$. Then from (14) and ∂_M-quasiconvexity of type II of Φ and $\mathcal{F}_{\wp_1}$, for every $(\wp_1, \wp_2) \in B\Big((\overline{\wp}_1, \overline{\wp}_2), \overline{d}\Big) \cap \Omega$, it follows that

$$\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle \geq 0 \geq -\epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|,$$

and

$$\langle \eta, \wp_2 - \overline{\wp}_2 \rangle \geq 0 \geq -\epsilon \|\wp_2 - \overline{\wp}_2\|,$$

that is,

$$\begin{aligned} &\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0, \forall \xi \in \partial_M \Phi(\wp_1, \wp_2) \text{ and} \\ &\langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon \|\wp_2 - \overline{\wp}_2\| \geq 0, \ \forall \eta \in \partial_M \mathcal{F}_{\wp_1}(\wp_2). \end{aligned}$$

Hence, the theorem is proved. □

Theorem 5. *Let $(\overline{\wp}_1, \overline{\wp}_2)$ is a solution of ASTVI w.r.t ϵ with Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-pseudoconvex functions around $(\overline{\wp}_1, \overline{\wp}_2)$. Then $(\overline{\wp}_1, \overline{\wp}_2)$ is a local ϵ-quasi solution of BLPP.*

Proof. By arguing, suppose that $(\overline{\wp}_1, \overline{\wp}_2)$ solves ASTVI w.r.t ϵ, but not a local ϵ-quasi solution of BLPP. Hence, for each $d > 0$, we can get $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, such that

$$\begin{aligned} &\Phi(\wp_1, \wp_2) < \Phi(\overline{\wp}_1, \overline{\wp}_2) - \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|, \text{ or} \\ &\mathcal{F}_{\wp_1}(\wp_2) < \mathcal{F}_{\wp_1}(\overline{\wp}_2) - \epsilon \|\wp_2 - \overline{\wp}_2\|. \end{aligned} \tag{15}$$

Since Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-pseudoconvex functions around $(\overline{\wp}_1, \overline{\wp}_2)$, then for every $\epsilon > 0$ there exists $d > 0$ and for any $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, (15) implies

$$\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| < 0, \ \forall \xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2)$$
$$\langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon \|\wp_2 - \overline{\wp}_2\| < 0, \ \forall \eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2),$$

this contradicts our assumption. □

Theorem 6. *Let $(\overline{\wp}_1, \overline{\wp}_2)$ solves ASTVI w.r.t ϵ, and Φ and $\mathcal{F}_{\wp_1}(\wp_2)$ are approximate ∂_M-pseudoconvex functions of type II around $(\overline{\wp}_1, \overline{\wp}_2)$. Then $(\overline{\wp}_1, \overline{\wp}_2)$ is a local ϵ-quasi solution of BLPP.*

Proof. Let $(\overline{\wp}_1, \overline{\wp}_2)$ solves ASTVI w.r.t ϵ. Then we can get a $d > 0$, such that for each $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, there exist $\xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2)$ and $\eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)$ such that

$$\begin{aligned} &\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0, \text{ and} \\ &\langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon \|\wp_2 - \overline{\wp}_2\| \geq 0. \end{aligned} \tag{16}$$

Since Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-pseudoconvex functions of type II around $(\overline{\wp}_1, \overline{\wp}_2)$, then for every $\epsilon > 0$, $\exists d' > 0$, and for any $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d') \cap \Omega$, if

$$\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0, \text{ for some } \xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2),$$
$$\langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon \|\wp_2 - \overline{\wp}_2\| \geq 0, \text{ for some } \eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)$$

then

$$\Phi(\wp_1, \wp_2) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2),$$
$$\mathcal{F}_{\wp_1}(\wp_2) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2).$$

Let $\overline{d} = \min\{d, d'\}$. Then from (16) and ∂_M-pseudoconvexity of type II of Φ and $\mathcal{F}_{\wp_1}$, for every $(\wp_1, \wp_2) \in B\Big((\overline{\wp}_1, \overline{\wp}_2), \overline{d}\Big) \cap \Omega$, it follows that

$$\Phi(\wp_1, \wp_2) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2) - \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|, \text{ and}$$
$$\mathcal{F}_{\wp_1}(\wp_2) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2) - \epsilon \|\wp_2 - \overline{\wp}_2\|.$$

Hence, $(\overline{\wp}_1, \overline{\wp}_2)$ is a local ϵ-quasi solution of BLPP. Hence, the theorem is proved. □

Theorem 7. *Let Φ and $\mathcal{F}_{\wp_1}$ be locally Lipschitz functions at $(\overline{\wp}_1, \overline{\wp}_2)$. Moreover, assume that $(\overline{\wp}_1, \overline{\wp}_2)$ be a local ϵ-quasi solution of BLPP with Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-quasiconvex functions of type II around $(\overline{\wp}_1, \overline{\wp}_2)$. Then $(\overline{\wp}_1, \overline{\wp}_2)$ is a solution of ASTVI w.r.t same ϵ.*

Proof. Let $(\overline{\wp}_1, \overline{\wp}_2) \in \Omega$ is a local ϵ-quasi solution of BLPP. Then we can get a $d > 0$, such that for each $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, we have

$$\begin{aligned} &\Phi(\wp_1, \wp_2) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2) - \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \\ &\mathcal{F}_{\wp_1}(\wp_2) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2) - \epsilon \|\wp_2 - \overline{\wp}_2\|. \end{aligned} \tag{17}$$

Since Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-quasiconvex functions of type II around $(\overline{\wp}_1, \overline{\wp}_2)$, therefore for all $\epsilon > 0$, we can get a $d' > 0$ and for all $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d') \cap \Omega$, if

$$\Phi(\overline{\wp}_1, \overline{\wp}_2) \leq \Phi(\wp_1, \wp_2)) + \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|$$
$$\mathcal{F}_{\wp_1}(\overline{\wp}_2) \leq \mathcal{F}_{\wp_1}(\wp_2) + \epsilon \|\wp_2 - \overline{\wp}_2\|,$$

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then
$$\langle \xi, (\overline{\wp}_1, \overline{\wp}_2) - (\wp_1, \wp_2) \rangle \leq 0, \forall \xi \in \partial_M \Phi(\wp_1, \wp_2)$$
$$\langle \eta, \overline{\wp}_2 - \wp_2 \rangle \leq 0, \ \forall \eta \in \partial_M \mathcal{F}_{\wp_1}(\wp_2).$$

Let $\overline{d} = \min\{d, d'\}$ and $(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)) = (\overline{\wp}_1, \overline{\wp}_2) + \mu((\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)), \mu \in (0,1)$, such that $\mu < \overline{d}$. Then from (17), it follows that

$$\begin{aligned} &\Phi(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)) \geq \Phi(\overline{\wp}_1, \overline{\wp}_2) - \epsilon\|(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)) - (\overline{\wp}_1, \overline{\wp}_2)\| \\ &\mathcal{F}_{\wp_1}(\overline{\wp}_2(\mu)) \geq \mathcal{F}_{\wp_1}(\overline{\wp}_2) - \epsilon\|\overline{\wp}_2(\mu) - \overline{\wp}_2\|. \end{aligned} \tag{18}$$

Employing (18) and approximate ∂_M-quasiconvexity of type II of the functions Φ and $\mathcal{F}_{\wp_1}$, one has

$$\left\langle \xi_\mu, (\overline{\wp}_1, \overline{\wp}_2) - (\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)) \right\rangle \leq 0, \forall \xi_\mu \in \partial_M \Phi(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)),$$
$$\langle \eta_\mu, \overline{\wp}_2 - \overline{\wp}_2(\mu) \rangle \leq 0, \ \forall \eta_\mu \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2(\mu)),$$

that is,
$$\begin{aligned} &\left\langle \xi_\mu, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \right\rangle \geq 0, \forall \xi_\mu \in \partial_M \Phi(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)), \\ &\langle \eta_\mu, \wp_2 - \overline{\wp}_2 \rangle \geq 0, \ \forall \eta_\mu \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2(\mu)). \end{aligned} \tag{19}$$

By making use of (19), we have

$$\begin{aligned} &\left\langle \xi_\mu, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \right\rangle + \epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0, \ \forall \xi_\mu \in \partial_M \Phi(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)) \\ &\langle \eta_\mu, \wp_2 - \overline{\wp}_2 \rangle + \epsilon\|\wp_2 - \overline{\wp}_2\| \geq 0, \ \forall \eta_\mu \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2(\mu)). \end{aligned} \tag{20}$$

Since, $\partial_M \Phi$ and $\partial_M \mathcal{F}_{\wp_1}$ are closed, $\xi_\mu \in \partial_M \Phi(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu))$, $\eta_\mu \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2(\mu))$, $\xi_\mu \to \xi$, $\eta_\mu \to \eta$, and $(\overline{\wp}_1(\mu), \overline{\wp}_2(\mu)) \to (\overline{\wp}_1, \overline{\wp}_2)$ as $\mu \to 0$, we have $\xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2)$ and $\eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)$. Therefore, for any $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap S$, there exist $\xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2)$ and $\eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)$ such that

$$\begin{aligned} &\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0 \\ &\langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon\|\wp_2 - \overline{\wp}_2\| \geq 0. \end{aligned} \tag{21}$$

Hence, the theorem is proved. □

Theorem 8. *Let $(\overline{\wp}_1, \overline{\wp}_2)$ is a solution of ASTVI w.r.t ϵ and $\partial_M \Phi$ and $\partial_M \mathcal{F}_{\wp_1}$ are approximate ϵ-pseudomonotone. Then $(\overline{\wp}_1, \overline{\wp}_2)$ solves AMTVI w.r.t same ϵ.*

Proof. Let $(\overline{\wp}_1, \overline{\wp}_2)$ solves ASTVI w.r.t ϵ. Then we can get a $d > 0$ and for all $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d) \cap \Omega$, there exists $\xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2)$ and $\eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)$ such that

$$\begin{aligned} &\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0, \\ &\langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon\|\wp_2 - \overline{\wp}_2\| \geq 0. \end{aligned} \tag{22}$$

Since $\partial_M \Phi$, $\partial_M \mathcal{F}_{\wp_1}$ are approximate ϵ-pseudomonotone, then there exists $0 < d' \leq d$, such that from (22), for all $(\wp_1, \wp_2) \in B((\overline{\wp}_1, \overline{\wp}_2), d') \cap \Omega$ and all $\xi \in \partial_M \Phi(\wp_1, \wp_2)$, $\eta \in \partial_M \mathcal{F}_{\wp_1}(\wp_2)$, we have

$$\begin{aligned} &\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle \geq 0 \\ &\langle \eta, \wp_2 - \overline{\wp}_2 \rangle \geq 0. \end{aligned} \tag{23}$$

Since $\epsilon > 0$, from (23), it follows that

$$\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0,$$
$$\langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon\|\wp_2 - \overline{\wp}_2\| \geq 0.$$

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Hence, $(\overline{\wp}_1, \overline{\wp}_2)$ is a solution of AMTVI w.r.t ϵ. □

The following example illustrates the importance of the established results.

Example 1. *We consider the following bilevel programming problem:*

$$\text{BLPP: } \min_{\wp_1,\wp_2} \Phi(\wp_1,\wp_2) = |\wp_1| + |\wp_2|$$

$$\text{subject to } H(\wp_1,\wp_2) = \wp_1\wp_2 \leq 0, \wp_2 \in \psi(\wp_1),$$

where $\psi(\wp_1)$ is the set of optimal solutions of the following convex optimization problem

$$\min_{\wp_2} \phi(\wp_1,\wp_2) = |\wp_2|$$

$$\text{subject to } h_1(\wp_1,\wp_2) = -\wp_1^2\wp_2 \leq 0, h_2(\wp_1,\wp_2) = \begin{cases} \wp_2^2 - \wp_2, & \wp_2 \geq 0 \\ -\wp_2, & \wp_2 < 0 \end{cases}$$

where $\Phi, \phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_1, h_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The set $\psi(\wp_1)$ of optimal solution for lower-level problem is given by

$$\psi(\wp_1) := \{0\}, \forall \wp_1 \in \mathbb{R},$$

and $\mathcal{F}_{\wp_1}(\wp_2) = |\wp_2|$. Moreover, we have

$$\Phi(\wp_1, 0) := |\wp_1|$$

Let $\mathcal{S}$ denotes the set of all feasible solutions of BLPP, that is, $\mathcal{S} = \{(\wp_1, 0) : \wp_1 \in \mathbb{R}\}$. Then, for $\epsilon = \frac{1}{5}$, it can be verified that $(0,0)$ is a local ϵ-quasi solution of the problem.

Moreover, it can be seen that

$$\partial_M \Phi(0,0) = \{(\xi_1,\xi_2) : |\xi_1| \leq \xi_2 \leq 1\} \cup \{(\xi_1,\xi_2) : \xi_1 \in [-1,1], |\xi_1| = -\xi_2 \leq 1\},$$

$$\partial_M \mathcal{F}_{\wp_1}(0) = [-1,1],$$

and Φ and $\mathcal{F}_{\wp_1}$ are approximate ∂_M-pseudoconvex and approximate ∂_M-quasiconvex around $(0,0)$.

Furthermore, we can verify that $(0,0)$ is a solution of $(AMTVI)$ w.r.t ϵ, as for all $(\wp_1,\wp_2) \in B\Big((0,0), \frac{1}{2}\Big) \cap \mathcal{S}$, we have

$$\langle \xi_1, (\wp_1,\wp_2) - (0,0)\rangle + \epsilon\|(\wp_1,\wp_2) - (0,0)\| \geq 0,$$
$$\langle \xi_2, \wp_2 - 0\rangle + \epsilon\|\wp_2 - 0\| \geq 0$$

Moreover, $(0,0)$ also solves ASTVI w.r.t same ϵ, as for all $(\wp_1,\wp_2) \in B\Big((0,0), \frac{1}{2}\Big) \cap \Omega$, there exists $\zeta_1 \in \partial_M \Phi(0,0)$ and $\zeta_2 \in \partial_M \mathcal{F}_{\wp_1}(0)$ such that

$$\langle \zeta_1, (\wp_1,\wp_2) - (0,0)\rangle + \epsilon\|(\wp_1,\wp_2) - (0,0)\| \geq 0,$$
$$\langle \zeta_2, \wp_2 - 0\rangle + \epsilon\|\wp_2 - 0\| \geq 0.$$

4. Existence Results

In this section, we employ the generalized KKM-Fan's lemma to derive certain conditions for the existence of the solution of AMTVI and ASTVI.

The following definition of KKM map is from Rezaie and Zafarani [40].

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Definition 15. *A set-valued map $B : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is termed as a KKM map, if for any subset $\{\wp_1, \wp_2, \ldots, \wp_p\}$ of $\mathbb{R}^n$, it fulfills*

$$\operatorname{conv}\{\wp_1, \wp_2, \ldots, \wp_p\} \subseteq \bigcup_{i=1}^{p} B(\wp_i),$$

where $\operatorname{conv}\{\wp_1, \wp_2, \ldots, \wp_p\}$ denotes the convex hull of $\{\wp_1, \wp_2, \ldots, \wp_p\}$.

The following KKM-Fan's lemma is from [40].

Lemma 1. *Let $B, M : \mathcal{K} \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be two set valued mappings, such that the following are satisfied:*

1. *$M(y) \subseteq B(y)$, for all $y \in \mathcal{K}$,*
2. *M is a KKM-map,*
3. *$B(y)$ is closed for all $y \in \mathcal{K}$ and is bounded for at least one $y \in \mathcal{K}$.*

Then $\bigcap_{y \in \mathcal{K}} B(y) \neq \emptyset$.

Theorem 9. *Let Ω be a nonempty, compact set and $\Phi, \mathcal{F}_{\wp_1} : \Omega \to \mathbb{R}$ be locally Lipschitz functions. Then ASTVI has a solution in Ω.*

Proof. For each $(\wp_1, \wp_2) \in \Omega$, define the set-valued mappings $B, M : \Omega \to 2^{\Omega}$ by

$$\begin{aligned} B(\wp_1, \wp_2) = M(\wp_1, \wp_2) = \{&(\overline{\wp}_1, \overline{\wp}_2) \in \Omega : \langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2) \rangle + \epsilon \|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0, \\ &\text{for some } \xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2),\ \langle \eta, \wp_2 - \overline{\wp}_2 \rangle + \epsilon \|\wp_2 - \overline{\wp}_2\| \geq 0,\ \text{for some } \eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)\}. \end{aligned}$$

Clearly, B and M are nonempty. Now, we claim that M is a KKM map. If not, there exists a finite set $\left\{(\wp_1^1, \wp_2^1), \ldots, \left(\wp_1^p, \wp_2^p\right)\right\} \subseteq \Omega$ and $\mu_k \geq 0, k = 1, \ldots, p$ with $\sum_{k=1}^{p} \mu_k = 1$, such that

$$(\wp_1^*, \wp_2^*) = \sum_{k=1}^{p} \mu_k \left(\wp_1^k, \wp_2^k\right) \notin \bigcup_{k=1}^{p} M\left(\wp_1^k, \wp_2^k\right) \tag{24}$$

From (24), it follows that $(\wp_1^*, \wp_2^*) \notin M\left(\wp_1^k, \wp_2^k\right)$ for all $k = 1, \ldots, p$, that is

$$\begin{aligned} &\left\langle \xi, \left(\wp_1^k, \wp_2^k\right) - (\wp_1^*, \wp_2^*) \right\rangle + \epsilon \left\| \left(\wp_1^k, \wp_2^k\right) - (\wp_1^*, \wp_2^*) \right\| < 0,\ \forall \xi \in \partial_M \Phi(\wp_1^*, \wp_2^*) \\ &\left\langle \eta, (\wp_2^k - \wp_2^*) \right\rangle + \epsilon \|\wp_2^k - \wp_2^*\| < 0,\ \forall \eta \in \partial_M \mathcal{F}_{\wp_1}(\wp_2^*). \end{aligned} \tag{25}$$

Multiplying (25) by μ_k and adding the resulting inequality we get

$$\begin{aligned} 0 &> \sum_{k=1}^{p} \left\langle \xi, \mu_k \left(\wp_1^k, \wp_2^k\right) - \mu_k (\wp_1^*, \wp_2^*) \right\rangle + \epsilon \sum_{k=1}^{p} \mu_k \left\| \left(\wp_1^k, \wp_2^k\right) - (\wp_1^*, \wp_2^*) \right\| \\ &\geq \left\langle \xi, \sum_{k=1}^{p} \mu_k \left(\wp_1^k, \wp_2^k\right) - (\wp_1^*, \wp_2^*) \right\rangle + \epsilon \left\| \sum_{k=1}^{p} \mu_k \left(\wp_1^k, \wp_2^k\right) - (\wp_1^*, \wp_2^*) \right\| \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} 0 &> \sum_{k=1}^{p} \left\langle \eta, \mu_k \left(\wp_2^k\right) - \mu_k (\wp_2^*) \right\rangle + \epsilon \sum_{k=1}^{p} \mu_k \left\| \left(\wp_2^k\right) - (\wp_2^*) \right\| \\ &\geq \left\langle \eta, \sum_{k=1}^{p} \mu_k \left(\wp_2^k\right) - (\wp_2^*) \right\rangle + \epsilon \left\| \sum_{k=1}^{p} \mu_k \left(\wp_2^k\right) - (\wp_2^*) \right\| \\ &= 0, \end{aligned}$$

which is a contradiction. Hence, $B = M$ is a KKM map.

Now, our aim is to show that the set-valued map B is closed for all $(\wp_1, \wp_2) \in \Omega$. Let $\{(\wp_1^n, \wp_2^n)\}$ be a sequence in B, such that $(\wp_1^n, \wp_2^n) \to (\wp_1^*, \wp_2^*) \in \Omega$. We have to show that $(\wp_1^*, \wp_2^*) \in B$. Since $(\wp_1^n, \wp_2^n) \in B$, therefore there exist $\xi_n \in \partial_M \Phi(\wp_1^n, \wp_2^n)$ and $\eta_n \in \partial_M \mathcal{F}_{\wp_1}(\wp_2^n)$, such that

$$\begin{aligned}&\langle \xi_n, (\wp_1, \wp_2) - (\wp_1^n, \wp_2^n)\rangle + \epsilon\|(\wp_1, \wp_2) - (\wp_1^n, \wp_2^n)\| \geq 0,\\&\langle \eta_n, (\wp_2) - (\wp_2^n)\rangle + \epsilon\|\wp_2 - \wp_2^n\| \geq 0.\end{aligned} \tag{26}$$

Since $\partial_M \Phi$ and $\partial_M \mathcal{F}_{\wp_1}$ are closed, $\xi_n \in \partial_M \Phi(\wp_1^n, \wp_2^n)$, $\eta_n \in \partial_M \mathcal{F}_{\wp_1}(\wp_2^n)$, $\xi_n \to \xi$, $\eta_n \to \eta$ and $(\wp_1^n, \wp_2^n) \to (\wp_1^*, \wp_2^*)$. We have $\xi \in \partial_M \Phi(\wp_1^*, \wp_2^*)$ and $\eta \in \partial_M \mathcal{F}_{\wp_1}(\wp_2^*)$,

$$\begin{aligned}&\langle \xi, (\wp_1, \wp_2) - (\wp_1^*, \wp_2^*)\rangle + \epsilon\|(\wp_1, \wp_2) - (\wp_1^*, \wp_2^*)\| \geq 0,\\&\langle \eta, \wp_2 - \wp_2^*\rangle + \epsilon\|\wp_2 - \wp_2^*\| \geq 0.\end{aligned}$$

Hence, B is closed. Since, Ω is bounded therefore B is bounded for each $(\wp_1, \wp_2) \in \Omega$. Therefore, utilizing Lemma 1, we have

$$\bigcap_{(\wp_1, \wp_2) \in \Omega} B(\wp_1, \wp_2) \neq \emptyset,$$

which implies that for some $(\wp_1, \wp_2) \in \Omega$, we get a $(\overline{\wp}_1, \overline{\wp}_2)$, such that for all $\xi \in \partial_M \Phi(\overline{\wp}_1, \overline{\wp}_2)$, $\eta \in \partial_M \mathcal{F}_{\wp_1}(\overline{\wp}_2)$ we get

$$\begin{aligned}&\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1 - \overline{\wp}_2)\rangle + \epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\| \geq 0\\&\langle \eta, \wp_2 - \overline{\wp}_2\rangle + \epsilon\|\wp_2 - \overline{\wp}_2\| \geq 0.\end{aligned}$$

Hence, ASTVI has a solution in Ω. □

Theorem 10. *Let Ω be a nonempty compact set and $\Phi, \mathcal{F}_{\wp_1} : \Omega \to \mathbb{R}$ be locally Lipschitz function. Then AMTVI has a solution in Ω.*

Proof. Let set-valued mapping $B = M : \Omega \to 2^{\Omega}$ be such that

$$\begin{aligned}B(\wp_1, \wp_2) = \Big\{&(\overline{\wp}_1, \overline{\wp}_2) \in \Omega : \forall \xi \in \partial_M \Phi(\wp_1, \wp_2),\ \forall \eta \in \partial_M \mathcal{F}_{\wp_1}(\wp_2) \text{ such that}\\&\langle \xi, (\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\rangle + \epsilon\|(\wp_1, \wp_2) - (\overline{\wp}_1, \overline{\wp}_2)\|\\&\geq 0,\ \langle \eta, \wp_2 - \overline{\wp}_2\rangle + \epsilon\|\wp_2 - \overline{\wp}_2\| \geq 0,\ \forall (\wp_1, \wp_2) \in \Omega\Big\}.\end{aligned}$$

Now, proceeding along the lines of Theorem 9, we get the required result. □

5. Conclusions and Future Directions

In this paper, we have considered BLPP, as well as AMTVI and ASTVI in terms of limiting subdifferential. We have derived the relationships among the solutions of the AMTVI and ASTVI and the local ϵ-quasi solution to the nonsmooth BLPP under the appropriate assumptions of generalized approximate convexity. Furthermore, existence results for the solution of AMTVI and ASTVI have been established by employing generalized KKM-Fan's lemma. A non-trivial example has been provided to illustrate the importance and relevance of these findings.

The results derived in this paper extend several noteworthy findings in the literature for certain classes of generalized approximate convex functions using the notion of limiting subdifferential as well as generalizing them for a wider class of optimization problems. In particular, the results of this paper extend the analogous results in [10,25] from single-level optimization problems to more general optimization problems, namely, bilevel optimization problems. Moreover, since the limiting subdifferential is the least among all the known robust subdifferentials and offers an enhanced

Lagrange multiplier rule compared to the Clarke subdifferential, therefore our findings naturally sharpen the analogous results of [6,24,32,43]. Furthermore, the established results of this paper extend the corresponding results in [6,24,27,43] for a broader class of approximate convex functions.

Considering the contributions of Deb and Sinha [12] and Oveisiha and Zafarani [39], we aim to extend the findings of this paper to multiobjective bilevel programming problems and to a broader space, such as the Asplund space, in our future research endeavors.

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