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Generalized Definition of the Derivative

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Abstract-The classical limit definition of a derivative is expressed in a more general form. The general form includes two arbitrary functions of the parameter for which the limit is calculated. A special case of the general form, which includes scaling and translational symmetry transformations of the limiting parameter, is also discussed. The errors in using the classical definition and the generalized form are calculated for small values of the limiting parameter. The derivatives of some known functions are proven using the new definition. For some well-known functions, a suitable selection of the generalized form may introduce simplicity in calculating the derivatives.

Keywords: limit definition; derivative; error analysis; scaling transformations; translational transformations

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Introduction

Calculating derivatives is one of the most important and elementary topics in calculus. The geometric meaning of the derivative is given as the slope of the function at a given point. To calculate the slope of a function $f(x)$, the approximate slope is first written as a ratio of $\Delta f/\Delta x$, where $\Delta f = f(x+h) - f(x)$ and $\Delta x = h$. Then, letting $h \to 0$, the error in the approximate slope expression is reduced until the exact slope of the function $f(x)$ is calculated at point x (Thomas & Finney, 1984; Strang, 1991).

First, the abovementioned basic definition of the derivative is expressed in a more general form. Δx and Δf are expressed as functional forms of the limiting parameter. The properties of the generalized form are discussed. The error introduced without taking the limit is calculated for the classical and generalized definitions. A special case of the generalized definition that covers scaling and translational transformations is also given. Finally, for some of the well-known functions, the derivatives are determined using the generalized form. The generalized definition may introduce some simplicity in calculating the derivatives of some of the functions.

Generalized Definition of the Derivative

The classical definition of derivative, which can be traced in any calculus textbook, is

$$f'(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)-f(x)}{\varepsilon} \qquad (1)$$

Usually, in calculus textbooks, instead of the parameter ε, the more common notation of h is employed. The term $\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$ is the approximate slope of a line passing through points $(x, f(x))$ and $(x+\varepsilon, f(x+\varepsilon))$. As ε approaches zero, the approximate slope coincides with the exact slope of the function at point x. The following generalization of the derivative expression is proposed in this work for the first time:

$$f'(x) = \lim_{\varepsilon\to 0}\frac{(f(g(\varepsilon)x+h(\varepsilon))-f(x)}{g(\varepsilon)x+h(\varepsilon)-x} \qquad (2)$$

where the functions $g(\varepsilon)$ and $h(\varepsilon)$ are arbitrary functions of the limiting parameter on the condition that

$$\lim_{\varepsilon\to 0} g(\varepsilon) = 1, \qquad \lim_{\varepsilon\to 0} h(\varepsilon) = 0. \qquad (3)$$

The generalized definition can be expressed in a more compact form

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$$f'(x) = \lim_{x^* \to x} \frac{f(x^*) - f(x)}{x^* - x} \quad (4)$$

where $x^* = g(\varepsilon)x + h(\varepsilon)$ with the functions satisfying (3).

In an interesting video in YouTube (see the website given in the references), the alternative definition of the derivative is discussed in detail.

$$f'(x) = \lim_{t \to 1} \frac{f(tx) - f(x)}{tx - x} \quad . \quad (5)$$

In our generalized form, the above expression corresponds to the special case of $g(\varepsilon) = 1 + \varepsilon$ and $h(\varepsilon) = 0$ with $t = 1 + \varepsilon$. It is shown in the video that the alternative definition introduces some simplicity in determining the derivatives of some of the well-known functions. The quantum derivative, which is similar to (5), is used by physicists with the expression being $\frac{f(qx) - f(x)}{qx - x}$ without the limiting process (Kunt et al., 2022). Another generalization of the usual derivative is the fractal derivatives in which the derivative operation consists of fractional repetitions rather than integers. See Deppman et al. (2023) for a review of such fractional generalizations. For applications of fractional derivatives to boundary value problems, see He (2020).

Another subcase of the general definition (2) may also be proposed

$$f'(x) = \lim_{\varepsilon \to 0} \frac{(f(e^{\varepsilon a}x + b\varepsilon) - f(x)}{e^{\varepsilon a}x + b\varepsilon - x} \quad \text{(a, b constant parameters)} \quad (6)$$

where the transformation $x^* = e^{\varepsilon a}x + b\varepsilon$ is indeed a special Lie group of transformations covering scaling and translational transformations as special cases (Pakdemirli & Yürüsoy, 1998).

Error Analysis

If the function is unknown, as in the case of differential equations, usually the approximate form of the derivative is substituted. For example, for the first-order differential equation

$$y' = F(x, y), \quad (7)$$

the derivative of the classical version is approximated as

$$y' = \frac{y_{n+1} - y_n}{h}, n = 0,1,2, \ldots, \quad (8)$$

and substituted yielding

$$y_{n+1} = y_n + hF(x_n, y_n), \; n = 0,1,2, \ldots \quad . \quad (9)$$

The above recursive relation is the famous Euler method used in numerical analysis (O'Neil, 1991). The errors introduced by approximate definitions such as (8) are of technical importance. For the classical version given in (1), if

$$f(x + \varepsilon) = f(x) + \varepsilon f'(x) + \frac{1}{2}\varepsilon^2 f''(x) + \cdots \quad (10)$$

is substituted into (1) without the limit

$$\frac{\Delta f}{\Delta x} \cong f'(x) + \varepsilon \frac{f''(x)}{2} + \cdots \quad , \quad (11)$$

and the error in the slope is

$$e \cong \frac{\Delta f}{\Delta x} - f'(x) = \varepsilon \frac{f''(x)}{2} + \cdots. \quad (12)$$

If $\left|\frac{f''(x)}{2}\right| \leq M$, where M is a real number, then the error is of order ε.

To calculate the error in the generalized version, substitute

$$f(x^*) = f(x) + f'(x)(x^* - x) + \frac{1}{2}f''(x)(x^* - x)^2 + \cdots \quad (13)$$

to the right-hand side of (4) without taking the limit

$$\frac{\Delta f}{\Delta x} \cong f'(x) + (x^* - x)\frac{f''(x)}{2} + \cdots \quad , \quad (14)$$

and the error is

$$e \cong \frac{\Delta f}{\Delta x} - f'(x) = (x^* - x)\frac{f''(x)}{2} + \cdots \cong [(g(\varepsilon) - 1)x + h(\varepsilon)]\frac{f''(x)}{2} + \cdots. \quad (15)$$

For $\left|\frac{f''(x)}{2}\right| \leq M$, in order not to depend the error on x, $g(\varepsilon) = 1$, that is, for functions $g(\varepsilon) \neq 1$, the error depends on x and may become large for large values of x.

For $g(\varepsilon) = 1$, the error is from (15):

$$e \cong h(\varepsilon)\frac{f''(x)}{2} + \cdots \quad (16)$$

and hence, the error is of order $h(\varepsilon)$. If $h(\varepsilon)\sim\varepsilon^n$, then one can say that the error is of order ε^n in terms of the parameter ε. Theoretically speaking, more precise calculations of slope are available in terms of the parameter ε if $n > 1$.

Calculation of the Derivatives

In this section, the derivatives of some of the functions are calculated.

Example 1. The derivative of lnx is difficult to determine in the classical version

$$f'(x) = \lim_{\varepsilon\to 0}\frac{ln(x+\varepsilon)-ln(x)}{\varepsilon}, \tag{17}$$

since there are difficulties in evaluating the above limit. Taylor expansions and/or l'Hopital's rule cannot be used since they require knowledge of the derivative, which is unknown. In the generalized version, if one defines $g(\varepsilon) = 1 + \varepsilon,\ h(\varepsilon) = 0$, then the limit is

$$f'(x) = (lnx)' = \lim_{\varepsilon\to 0}\frac{ln((1+\varepsilon)x)-ln(x)}{(1+\varepsilon)x-x}, \tag{18}$$

Using the property $ln((1 + \varepsilon)x) = \ln(1 + \varepsilon) + \ln(x)$ above and simplifying

$$f'(x) = \frac{1}{x}\lim_{\varepsilon\to 0}\frac{ln(1+\varepsilon)}{\varepsilon} = \frac{1}{x}\lim_{\varepsilon\to 0} ln(1+\varepsilon)^{1/\varepsilon}, \tag{19}$$

If one defines $\varepsilon = \frac{1}{n}$, then the limit is

$$\lim_{\varepsilon\to 0} ln(1+\varepsilon)^{1/\varepsilon} = \lim_{n\to\infty} ln\left(1+\frac{1}{n}\right)^n = lne = 1 \tag{20}$$

Hence, it is proven that the derivative of $ln(x)$ is

$$f'(x) = \frac{1}{x} \tag{21}$$

The derivative of $ln(x)$ is calculated by using the integral definition $ln(x) = \int_1^x \frac{1}{t}dt$ and differentiating both sides (Thomas & Finney, 1984) or by employing the properties of the exponential function (Strang, 1991) but not directly from (17). According to the generalized definition (18), this task becomes simpler and more straightforward.

Example 2.

To calculate the derivative of the exponential function, take $g(\varepsilon) = 1$, $h(\varepsilon) = ln\,(1 + \varepsilon)$, i.e., $x^* = x + ln\,(1 + \varepsilon)$

$$f'(x) = (e^x)' = \lim_{\varepsilon\to 0}\frac{e^{x+ln\,(1+\varepsilon)}-e^x}{x+ln\,(1+\varepsilon)-x} = e^x\lim_{\varepsilon\to 0}\frac{(1+\varepsilon)-1}{ln\,(1+\varepsilon)} = e^x\lim_{\varepsilon\to 0}\frac{\varepsilon}{ln\,(1+\varepsilon)}, \tag{22}$$

and the limit can easily be calculated

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{ln\,(1+\varepsilon)} = \lim_{\varepsilon\to 0}\frac{1}{ln\,(1+\varepsilon)^{1/\varepsilon}} = \lim_{n\to\infty}\frac{1}{ln\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty}\frac{1}{ln\ e} = 1 \quad , \tag{23}$$

proving that the derivative of the function is

$$(e^x)' = e^x \quad . \tag{24}$$

In the classical definition, one needs two consecutive transformations instead of the one employed above.

$$f'(x) = (e^x)' = \lim_{\varepsilon\to 0}\frac{e^{x+\varepsilon}-e^x}{x+\varepsilon-x} = e^x\lim_{\varepsilon\to 0}\frac{e^\varepsilon-1}{\varepsilon}, \tag{25}$$

The first transformation is $\varepsilon = ln\,(1 + t)$

$$\lim_{t\to 0}\frac{e^{ln\,(1+t)}-1}{ln\,(1+t)} = \lim_{t\to 0}\frac{t}{ln\,(1+t)} = \lim_{t\to 0}\frac{1}{ln\,(1+t)^{1/t}} \tag{26}$$

and the second transformation is $t = \frac{1}{n}$, which proves that the limit equals unity and that the derivative of the exponential function is itself.

Example 3.

To calculate the derivative of the hyperbolic sine function from the generalized definition, take $g(\varepsilon) = 1,\ h(\varepsilon) = ln\,(1 + \varepsilon)$, i.e., $x^* = x + ln\,(1 + \varepsilon)$,

$$(sinhx)' = \lim_{\varepsilon\to 0}\frac{sinh(x + l\,n(1+\varepsilon)) - sinhx}{x + ln(1+\varepsilon) - x}$$

$$= \lim_{\varepsilon\to 0}\frac{sinh\,x\,cosh(l\,n(1+\varepsilon)) + cosh\,x\,sinh(l\,n(1+\varepsilon)) - sinhx}{ln\,(1+\varepsilon)}$$

$$= sinhx\lim_{\varepsilon\to 0}\frac{cosh(l\,n(1+\varepsilon))-1}{ln\,(1+\varepsilon)} + coshx\lim_{\varepsilon\to 0}\frac{sinh(l\,n(1+\varepsilon))}{ln\,(1+\varepsilon)} \tag{27}$$

The first limit is

$$\lim_{\varepsilon\to 0}\frac{cosh(l\,n(1+\varepsilon))-1}{ln(1+\varepsilon)} = \lim_{\varepsilon\to 0}\frac{e^{ln(1+\varepsilon)}+e^{-\,ln(1+\varepsilon)}-2}{2\,ln(1+\varepsilon)} = \lim_{\varepsilon\to 0}\frac{1+\varepsilon+\frac{1}{1+\varepsilon}-2}{2\,ln(1+\varepsilon)}$$

$$= \lim_{\varepsilon \to 0} \frac{\varepsilon}{2(1+\varepsilon) ln\,(1+\varepsilon)^{1/\varepsilon}}. \quad (28)$$

Since $\lim_{\varepsilon \to 0} ln(1+\varepsilon)^{\frac{1}{\varepsilon}} = 1$, the result is

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{2(1+\varepsilon)} = 0 \quad (29)$$

Proceeding in a similar way, the second limit is

$$\lim_{\varepsilon \to 0} \frac{\sinh(l\,n(1+\varepsilon))}{ln\,(1+\varepsilon)} = \lim_{\varepsilon \to 0} \frac{2+\varepsilon}{2(1+\varepsilon)} = 1 \quad (30)$$

Hence, it is proven that

$$(sinhx)' = coshx. \quad (31)$$

For trigonometric functions, the generalized definition might not introduce simplicities, and the classical definition may be employed in proving the derivatives.

Concluding Remarks

The classical limit definition of a derivative is proposed in a more general form. Functional relationships are used in the generalized version. A subversion of the most general form in which scaling and translational transformations are used is also proposed. The error analysis of the approximate definition is given. For errors not to depend on the independent variable, scaling transformations are not allowed. The new definition introduces some simplicity in proving the derivatives of some functions, such as exponential, logarithmic and hyperbolic functions.

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