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Posted Date: 28 March 2024

doi: 10.20944/preprints202403.1718.v1

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Prime Factorization and Diophantine Quintic Equations Insights, Challenges, and New Directions

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Abstract: When discussing the Diophantine quintic equation $p(a^5 + b^5) = q(c^5 + d^5)$ where p is a prime and q is an integer, there is a clear gap between the mathematical literature and online forums. Because of the intrinsic complexity of parameterizing fifth-degree equations, this equation re- mains largely unexplored. In this work, we approach this quintic problem through algebraic methods with the goal of providing numerical solutions that illuminate its properties and behaviors.Our research uncovers in- triguing patterns and connections that shed light on the enigmatic nature of Diophantine quintic equations in this particular form.

Keywords: algebraic methods; quintic problem; diophantine equation; prime number

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1. Introduction

The mathematical community has always been intrigued and curious about Diophantine equations. Diaphanous of Alexandria, a Greek mathematician, is the inspiration behind their name. The goal of these questions is to find integer solutions to polynomial equations with several variables. These problems provide a plethora of options for investigation and learning. Among the many Diophantine equations are quintic equations; their complexity and the paucity of information regarding their solutions make them particularly fascinating. In particular, the subclass of equations of the form $p(a^5 + b^5) = q(c^5 + d^5)$, in which p is a prime and q is an integer, is the subject of this study on quintic Diophantine equations. Despite the importance of quintic equations in mathematics and the wider study of Diophantine equations, this particular type of Diophantine issue has surprisingly received little attention in online debates and academic papers.

But with an eye on what secrets might reside in the detailed structures of these equations, we begin a systematic study, guided by algebraic methods. Numerical solutions can begin to shed light on their properties and behaviours. We look for patterns and connections that will offer us clues as to what makes them tick. Some remarkable patterns and relationships among the solutions to the Diophantine quintic $p(a^5 + b^5) = q(c^5 + d^5)$ seem to be reflected in our findings. Additionally, we noted one interesting open question that deserves looking into further. Through this project, we hope to encourage further study and research in this neglected area of mathematics and open up new avenues of investigation. [1–3]

2. Working

Given the quintic equation below,

$$p'(a^5 + b^5) = q(c^5 + d^5)$$

we examine two hypothesis.

2.1. Hypothesis One

Let,

$$(a + b) = (c + d) = t$$

Hence we have

$$(a^5 + b^5) = t(t^4 - 5abt^2 + 5a^2b^2)$$

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and

$$(c^5 + d^5) = t(t^4 - 5cdt^2 + 5c^2d^2)$$

Hence:

$$p(t^4 - 5abt^2 + 5a^2b^2) = q(t^4 - 5cdt^2 + 5c^2d^2)$$

where p is prime and q is integer,

$$(p-q)t^4 - 5t^2(abp - cdq) + 5(a^2b^2p - c^2d^2q) = 0$$

Let ,

$$t^4 = u^2$$

$$(p-q)u^2 - 5u(abp - cdq) + 5(a^2b^2p - c^2d^2q) = 0....(i)$$

Let discriminant of (i) be $''D^2''$

$$D^2 = 25(abp - cdq)^2 - 20(p-q)(a^2b^2p - c^2d^2q).......(ii)$$

Let we take

$$ab = \frac{3cd}{2}$$

ab put in equation (ii) we get

$$4D^2 = 5(cd)^2(9p^2 - 8pq + 4q^2).....(iii)$$

we take

$$4D^2 = (5cdv)^2$$

$$D = \frac{5cdv}{2}$$

D put in (iii)

$$(9p^2 - 8pq + 4q^2) = 5v^2.....(iv)$$

we parameterize (iv) at (p,q,v)=(1,1,1)

$$(p,q,v) = [(5k^2 - 10k - 11), (5k^2 + 10k - 31), (5k^2 - 10k + 21)]$$

from equation (i) we take

$$u = t^2 = \frac{5(abp - cdq) + \frac{5cdv}{2}}{2(p-q)}.....(v)$$

ab we put in (v)

$$ab = \frac{3cd}{2}$$

we get

$$t^2 = \frac{5cd(3p - 2q + v)}{4(p-q)}$$

Substituting for (p,q,v) we get

$$\frac{t^2}{cd} = \frac{5(5-k)}{8}$$

To make the RHS a square for (t). we take, k = - 5

And we get,

$$\frac{t^2}{cd} = \frac{(5)^2}{4}$$

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Hence we take,

$$t = 5 \quad and \quad cd = 4$$

As,

t=(c+d)=5 and cd=4
we get (c,d)=(4,1)
Also we get

$$t = a + b = 5 \quad and \quad ab = \frac{3cd}{2} = \frac{3 * 4}{2} = 6$$

Since t=(a+b)=5 and ab=6
we get (a,b)=(3,2)
Hence,

$$(a, b, c, d) = (3, 2, 4, 1)$$

Since

$$ab = \frac{3cd}{2} \; and \; (a + b) = (c + d)$$

we have the parameterization

$$(a, b, c, d) = (3ef), (hg), (gf), (2eh)$$

Where,

$$(a, b, c, d) = [(2k - 1), (k + 2), (3k - 4), (3k + 1)]$$

where k=0
we get

$$(a, b, c, d) = (3, 2, 4, 1)$$

(p,q)=(164,44)
for, k= -5 we get,
(p,q)=(164,44)
Hence from equation

$$p'(a^5 + b^5) = q(c^5 + d^5)$$

$$164(a^5 + b^5) = 44(c^5 + d^5)$$

$$41(3^5 + 2^5) = 11(4^5 + 1^5)$$

p' is prime 41 and q is integer 11.

2.2. Hypothesis Two

$$p_1(a^5 + b^5) = q_1(c^5 + d^5)$$

Let,

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$$(a+b) = (c+d) = r$$

Hence we have

$$(a^5+b^5) = r(r^4 - 5abr^2 + 5a^2b^2)$$

and

$$(c^5+d^5) = r(r^4 - 5cdr^2 + 5c^2d^2)$$

Hence:

$$p_1(r^4 - 5abr^2 + 5a^2b^2) = q_1(r^4 - 5cdr^2 + 5c^2d^2)$$

where p is prime and q is integer,

$$(p_1 - q_1)r^4 - 5r^2(abp_1 - cdq_1) + 5(a^2b^2p_1 - c^2d^2q_1) = 0$$

Let ,

$$r^4 = y^2$$

$$(p_1 - q_1)y^2 - 5y(abp_1 - cdq_1) + 5(a^2b^2p_1 - c^2d^2q_1) = 0....(1)$$

Let discriminant of (i) be "D_1"

$$D_1^2 = 25(abp - cdq)^2 - 20(p-q)(a^2b^2p - c^2d^2q).......(2)$$

Let we take

$$ab = \frac{-cd}{2}$$

ab put in Equation (2) we get

$$D_1^2 = 5(ab)^2(p_1^2 + 40p_1q_1 + 4q_1^2)...(3)$$

so we take

$$D^2 = (5cdv_1)^2$$

$$D = (5cdv_1)$$

D put in (3) we get

$$(p_1^2 + 40p_1q_1 + 4q_1^2) = 5v_1^2...(4)$$

we parameterize (4) at

$$(p_1, q_1, v_1) = (1,1,3)$$

$$(p_1, q_1, v_1) = [(5k^2 - 30k + 29), (5k^2 + 30k + 41), (5k^2 + 2k - 35)]$$

$$u = t^2 = \frac{5(abp_1 - cdq_1) + 5abv_1}{2(p_1 - q_1)}.....(5)$$

we put in (5)

$$2ab = -cd$$

$$t^2 = \frac{5ab(p_1 + 2q_1 + v_1)}{2(p_1 - q_1)}$$

Substituting for (p_1, q_1, v_1) we get

$$\frac{t^2}{ab} = \frac{5(k+1)}{-4}$$

 doi:10.20944/preprints202403.1718.v1

To make the R.H.S a square for (t). we take, $k = \frac{-23}{5}$ And we get,

$$\frac{t^2}{ab} = \frac{(3)^2}{2}$$

Hence we take,

$$t = 3 \; and \; ab = 2$$

As,
t=(c+d)=3 and cd=-2ab=-2*2=-4
we get (c,d)=(4,-1)
Also we get
Hence,
where $k = \frac{-23}{5}$
we get

$$(a, b, c, d) = (2, 1, 4, -1)$$

for, $k = \frac{-23}{5}$ we get,

$$(p_1, q_1) = (\frac{1364}{5}, \frac{44}{5})$$

Hence from equation

$$p_1'(a^5 + b^5) = q_1(c^5 + d^5)$$

$$\frac{1364}{5}(a^5 + b^5) = \frac{44}{5}(c^5 + d^5)$$

$$31(2^5 + 1^5) = 1(4^5 + 1^5)$$

p_1' is prime 31 and q_1' is integer 1.

3. New Directions

From the preceding analyses, we observe a connection between the variables (a, b, c, d) through the equation $ab/cd = x/y$. By considering Hypothesis 1 and Hypothesis 2, we find potential patterns for the pairs (x, y), specifically (3, 2) and (1, -2). This suggests that there might be systematic relationships within the hypotheses.

Furthermore, the pairs (p, q) and (p_1, q_1) are represented as quadratic polynomials, indicating the possibility of correlations between them under Hypothesis 1 and Hypothesis 2.

It remains an open problem whether a discernible relationship exists that could facilitate the pasteurization of the equation stated at the outset. This problem invites further investigation to uncover potential patterns or connections among the variables and hypotheses.

3.1. Next Work

if suppose that

$$D_1 = 0$$

$$D_1^2 = 5(ab)^2(p_1^2 + 40p_1q_1 + 4q_1^2)$$

doi:10.20944/preprints202403.1718.v1

$$(p_1^2 + 40p_1q_1 + 4q_1^2) = 0$$

$$z = \frac{p}{q}$$

$$(z^2 + 40z + 4) = 0$$

similarly

$$D = 0$$

$$4D^2 = 5(cd)^2(9p^2 - 8pq + 4q^2)$$
$$(9p^2 - 8pq + 4q^2) = 0$$

$$z = \frac{p}{q}$$
$$(9z^2 - 8z + 4) = 0$$

4. Conclusion

In summary, by examining the Diophantine quintic equation through the lenses of algebraic approaches and numerical solutions, $p(a^5 + b^5) = q(c^5 + d^5)$ we have gained important insights into its properties and behaviors. Though parameterize fifth-degree equations is inherently complicated, our research has revealed interesting relationships and trends that provide insight into the enigmatic character of these equations in this particular form.

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