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Posted Date: 27 March 2024

doi: 10.20944/preprints202403.1687.v1

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Existences Results for Some Nonsingular p-Kirchhoff Problems with ψ-Hilfer Fractional Derivative

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Abstract: In this paper, we prove the existence of three solutions for a p-Kirchhoff problem with ψ-Hilfer fractional derivative. To be more precise, we use the variational method and we prove that the associated functional energy admits a critical point in each of the three constructed sets, these critical points are weak solutions for a studied problem. Moreover, by definition of these sets, one of these solutions is positive, the second is negative, and the third one change sign. At the end of this work, we present an example to validate our main results.

Keywords: fractional calculus; variational methods; ψ-Hilfer operators; existence of solution

1. Introduction

In recent years, the notion of fractional calculus has attracted the attention of several authors. Indeed, fractional calculus has become one of the most interesting tools in many fields, for example in mechanics as studied in Purohit and Kalla [22], in oncolytic virotherapy as mentioned in Kumar et al. [9], in motion of beam on nanowire for the interested reader we cite the paper of Erturk et al. [10], in image processing one can see the monograph of Zhang et al. [44], and in viscoelasticity which is developed in the article of Mainardi [39]. Other important applications like physics, epidemiology, and engineering can be found in the papers [4,11,24].

Because fractional differential operators are very important, several papers involving different derivatives, we cite for instance the papers of Ben Ali et al. [5], Chamekh et al. [29], Ghanmi and Horrigue [31,32], Horrigue [34], Torres [25], and Wang et al. [43].

Very recently, several papers developed new derivatives like the derivative with respect to another function ψ which is one of the interesting derivatives see [11,24]. This operator generalizes some classical ones in the literature see [11,37,38].

In the last few years, several authors have concentrated on the study of problems involving the ψ-Riemann fractional derivative, we cite for examples the papers of Alsaedi and Ghanmi [1], Da Sousa et al. [26–28], Almeida [3], Nouf et al. [41], Horrigue [17]. More precisely, Nouf et al. [41] used the mountain pass theorem to prove that the following problem

$$\begin{cases} hf(\chi(t))^{p-1}\, {}_tD_T^{\theta,\psi}\left(\Lambda_p\left({}_0D_t^{\theta,\psi}u(t)\right)\right) = \lambda g(t,u(t)) + h(t,u(t)),\ 0<t<T), \\ I_{0^+}^{\theta(\theta-1);\psi}(0) = I_T^{\theta(\theta-1);\psi}(T) = 0, \end{cases}$$

admits a nontrivial weak solution, where ${}_tD_T^{\theta,\psi}$, ${}_0D_t^{\theta,\psi}$, $I_{0^+}^{\theta(\theta-1);\psi}$ and $I_T^{\theta(\theta-1);\psi}$ are the operators in the sense of ψ-Riemann.

Alsaedi and Ghanmi [1] studied the following problem

$$\begin{cases} M(\chi(s))\, \mathcal{D}_T^{\mu,\theta,\psi}\left(\Lambda_p\left(\mathcal{D}_{0^+}^{\mu,\theta,\psi}\chi(s)\right)\right) = \lambda H(s,\chi(s))) + G(s,\chi(s)), \quad s\in(0,T), \\ I_{0^+}^{\theta(\theta-1);\psi}(0) = I_T^{\theta(\theta-1);\psi}(T) = 0, \end{cases} \tag{1}$$

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where $0 < \frac{1}{p} < \mu \leq 1, 0 \leq \theta \leq 1, \lambda > 0, \Lambda_p(t) = |t|^{p-2}t$ is the p-Laplace operator, $\mathcal{D}_T^{\mu,\theta,\psi}$ and $\mathcal{D}_{0^+}^{\mu,\theta,\psi}$ are the operators in the sense of ψ-Hilfer which are introduce later in Section 2. The authors justified that the mountain pass theorem ensures the existence of a solution for (2), moreover, by the use of the $\mathbb{Z}_2$-symmetric version of this theorem, the existence of infinitely many solutions is proved.

In this work, we shall study a Kirchhoff problem of the following form:

$$\begin{cases} h\left(\int_0^T |\mathcal{D}_{0^+}^{\mu,\theta,\psi}\chi(t)|^p\,dt\right)\mathcal{D}_T^{\mu,\theta,\psi}\left(\Lambda_p\left(\mathcal{D}_{0^+}^{\mu,\theta,\psi}\chi(t)\right)\right) = \lambda g(t,\chi(t))) + f(t,\chi(t)), \\ I_{0^+}^{\theta(\theta-1);\psi}(0) = I_T^{\theta(\theta-1);\psi}(T) = 0, \end{cases} \tag{2}$$

where $0 < t < T, \lambda > 0, 0 < \frac{1}{p} < \mu \leq 1$ and for some $r > 1, m > 0$ the function h is defined by

$$h(s) = ms^{r-1},$$

While, the functions $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$, are such that the following condition holds:

($\mathbf{H}$) There exist r, q_1 and q_2 such that

$$1 < p < rp < q_1 < q_2, \tag{3}$$

moreover, for each $(s,t,\chi) \in (0,\infty) \times [0,T] \times \mathbb{R}$, we have

$$f(t,s\chi) = s^{q_1-1}f(t,\chi) \text{ and } g(t,s\chi) = s^{q_2-1}g(t,\chi). \tag{4}$$

Remark 1. *Let F, G are the antiderivatives of the functions f, g respectively with have the value zero at zero, and let H be the antiderivative of the function h with value zero at zero. If ($\mathbf{H}$) holds, then for each $(s,t,\chi) \in (0,\infty) \times [0,T] \times \mathbb{R}$, we have*

$$F(t,s\chi) = s^{q_1}F(t,\chi) \text{ and } G(t,s\chi) = s^{q_2}G(t,\chi).$$

Moreover, there exists $C_1, C_1', C_2, C_2' > 0$, such that

$$\chi f(s,\chi) = q_1 F(s,\chi) \quad \text{and} \quad C_1'|\chi|^{q_1} \leq F(s,\chi) \leq C_1|\chi|^{q_1} \tag{5}$$
$$\chi g(s,\chi) = q_2 G(s,\chi) \quad \text{and} \quad C_2'|\chi|^{q_2} \leq G(s,\chi)| \leq C_2|\chi|^{q_2}. \tag{6}$$

Theorem 2. *Under hypothesis ($\mathbf{H}$), there exists $\lambda_* > 0$ such that if λ is large enough to satisfy $\lambda > \lambda_*$, then (2) admits three nontrivial solutions. Moreover, one of these solutions is negative, the second one is positive, and the third one change.*

2. Preliminaries

This section is devoted to introducing some important results that will be used in the proof of Theorem 2. All results introduced in this section and other related results can be found in [11] and [24]. To this end, hereafter, the Euler gamma function will be denoted by Γ. a and b will denote real numbers such that $-\infty \leq a < b \leq \infty$. The function ψ will denote a C^1 function on $[a,b]$ which is positive and satisfy $\psi'(s) > 0$ for all $x \in [a,b]$. For a given x, y, we will adopt the following notation $\psi(x) - \psi(y) = \psi_x(y)$. Finally, if $1 \leq \delta \leq \infty$, then $L^\delta(a,b)$ will denote the set of all measurable function χ on $[a,b]$, such that $\int_a^b |\chi(t)|^\delta\,dt < \infty$.

Definition 3. *([11,24]) Let $\chi \in L^1([a,b])$. then the left fractional integral of χ with respect ψ is defined by*

$$I_{a^+}^{\theta,\psi}\chi(x) = \frac{1}{\Gamma(\theta)}\int_a^x \psi'(t)\psi_x(t)^{\theta-1}\chi(t)dt,$$

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and the right fractional integral of χ with respect ψ, is defined by

$$I_{b^-}^{\theta,\psi}\chi(x) = \frac{1}{\Gamma(\theta)}\int_x^b \psi'(t)\psi_t(x)^{\theta-1}\chi(t)dt.$$

Definition 4. *([26,28]) Let $\chi \in L^1([a,b])$. Assume that $0 \le \theta \le 1$, and the integer n is such that $n-1 < \mu \le n$, then the left-sided ψ-Hilfer fractional derivatives of order μ and of type θ is defined by*

$$\mathcal{D}_{a^+}^{\mu,\theta,\psi}\chi(x) \quad = \quad I_{a^+}^{\theta(n-\mu),\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{(1-\theta)(n-\mu),\psi}\chi(x),$$

and the right-sided ψ-Hilfer fractional derivatives of order μ and of type θ is defined by

$$\mathcal{D}_{b^-}^{\mu,\theta,\psi}\chi(x) \quad = \quad I_{b^-}^{\theta(n-\mu),\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b^-}^{(1-\theta)(n-\mu),\psi}\chi(x),$$

We note that the first key in the manipulation of the weak formulation of an equation in the variational approach is the integration by parts, the following formula can be found in equation (16) in [27]

$$\int_a^b I_{a^+}^{\theta,\psi}\chi(s)\xi(s)\,ds = \int_a^b \chi(s)\psi'(s)I_{b^-}^{\theta,\psi}\left(\frac{\xi(s)}{\psi'(s)}\right)ds.$$

Also, there is an analog formula for the derivative which is presented in the following lemma.

Lemma 5. *[[27]] If the function χ is absolutely countinous on $[a,b]$ and if the function α is of class C^1 on $[a,b]$ with $\alpha(a) = \alpha(b) = 0$. Then for each $0 < \mu \le 1$ and each $0 \le \theta \le 1$, we have*

$$\int_a^b \mathcal{D}_{a^+}^{\mu,\theta,\psi}\chi(s)\alpha(s)\,ds = \int_a^b \chi(s)\psi'(s)\ \mathcal{D}_{b^-}^{\mu,\theta,\psi}\left(\frac{\alpha(s)}{\psi'(s)}\right)ds. \tag{7}$$

The following remark plays an important role in manipulating the inequalities in Section 3.

Remark 6. *([19,27]) Assume that $0 < \mu \le 1$ and $r \ge 1$. Let q such that $\frac{1}{q} + \frac{1}{r} = 1$, then for all $\chi \in L^r(a,b)$, we have*

(i) $I_{a^+}^{\mu,\psi}\chi$ *is bounded in $L^r(a,b)$, in addition, we get*

$$\|I_{a^+}^{\mu,\psi}\chi\|_{L^r(a,b)} \le \frac{\psi_b(a) - \psi(a))^\mu}{\Gamma(\mu+1)}\|\chi\|_{L^r(a,b)},$$

where

$$\|\chi\|_{L^r(a,b)} = \left(\int_a^b |\chi(t)|^r dt\right)^{\frac{1}{r}}.$$

(ii) *If $\mu \in (\frac{1}{r}, 1)$, then we have $\lim\limits_{t\to a} I_{a^+}^{\mu,\psi}\chi(t) = 0$, moreover*

$$\|I_{a^+}^{\mu,\psi}\chi\|_\infty \le \frac{\psi_b(a)^{\mu-\frac{1}{r}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\|\chi\|_{L^r(a,b)},$$

where

$$\|\chi\|_\infty = ess\sup_{a\le\delta\le b}|\chi(\delta)|.$$

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3. Proof of Theorem 2

This section is devoted to proving the main result of this work, our main tools are based on variational methods, to be more precise, we construct three disjoint sets and prove that the energy functional admits one critical point in each set, after that, we prove that every critical point is a weak solution for the main problem. We begin by defining the space $\digamma$ as the closure of the space $C_0^\infty([0,T],\mathbb{R})$ according to the following norm

$$\|\chi\|_\digamma = \left(\|\chi\|^p_{L^p(0,T)} + \|{}_0\mathcal{D}_t^{\mu,\theta,\psi}\chi\|^p_{L^p(0,T)}\right)^{\frac{1}{p}}.$$

We recall from [27] that the space $\digamma$ can be equivalently defined by:
$\digamma = \left\{\chi \in L^p([0,T]) : \mathcal{D}_{0^+}^{\mu,\theta,\psi}\chi \in L^p([0,T]), I_{0^+}^{\theta(\theta-1),\psi}(0) = I_T^{\theta(\theta-1);\psi}(T) = 0\right\}.$

Remark 7. *([19,27])The following properties hold:*

(i) $\digamma$ is a Banach space which is also reflexive and separable.
(ii) If $1-\mu > \frac{1}{p}$ or if $1 \geq \mu > \frac{1}{p}$, then, for all $\chi \in \digamma$, we get

$$\|\chi\|_{L^p(0,T)} \leq \frac{\psi_T(0)^\mu}{\Gamma(\mu+1)}\|\mathcal{D}_{0^+}^{\mu,\theta,\psi}\|_{L^p(0,T)}.$$

(ii) If $\frac{1}{p} < \mu \leq 1$, then for each $\chi \in \digamma$, we have

$$\|\chi\|_\infty \leq \frac{(\psi(T)-\psi(a))^{\mu-\frac{1}{r}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\|\mathcal{D}_{0^+}^{\mu,\theta,\psi}\|_{L^r(0,T)},$$

where q is such that $\frac{1}{q}+\frac{1}{p}=1$.

It is not difficult to see that if we combine the inequalities in Remark 7 with the definition of the norm $\|.\|_\digamma$, one has

$$\|\chi\|_\infty \leq \frac{\psi_T(a)^{\mu-\frac{1}{r}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\|\chi\|_\digamma. \tag{8}$$

Next, we introduce the energy functional $\Phi : \digamma \to \mathbb{R}$, as follows:

$$\Phi(\chi) = \frac{m}{p}\|\chi\|^{pr}_{\mu,\theta,\psi} - \lambda\int_0^T G(\delta,\chi(\delta))d\delta - \int_0^T F(\delta,\chi)d\delta,$$

where

$$\|\chi\|_{\mu,\theta,\psi} = \|{}_0\mathcal{D}_t^{\mu,\theta,\psi}\chi\|_{L^p(0,T)}.$$

It is not difficult to see that the functional Φ is of class C^1, and for each $\chi,\xi \in \digamma$, one has

$$\begin{aligned}\prec \Phi'(\chi),\xi \succ \;&=\; m\|\chi\|^{p(r-1)}_{\mu,\theta,\psi}\int_0^T \left|\mathcal{D}_{0^+}^{\mu,\theta,\psi}\chi(s)\right|^{p-2}\mathcal{D}_{0^+}^{\mu,\theta,\psi}\chi(s)\,\mathcal{D}_{0^+}^{\mu,\theta,\psi}\xi(s)\,ds\\ &\quad -\lambda\int_0^T g(s,\chi(s))\xi(s)\,ds + \int_0^T f(s,\chi(s))\xi(s)\,ds.\end{aligned}$$

So critical points of $\Phi'(\chi)$ are weak solutions for problem (2).
Now, we will adopt the method used in [30] to prove the existence of solutions. To this aim, let us introduce the following sets:

$$\widetilde{S}_\pm = \left\{\chi \in \digamma : \int_0^T \chi^\pm(s)\,ds > 0\right\}, \tag{9}$$

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$$S_{\pm} = \left\{ \chi \in \widetilde{S}_{\pm} : m\|\chi^{\pm}\|^{pr}_{\mu,\theta,\psi} = \lambda \int_0^T g(s,\chi(s))\chi^{\pm}(s)\, ds + \int_0^T f(s,\chi(s))\chi^{\pm}(s)\, ds \right\},$$

and

$$S = S_+ \cap S_-,$$

where χ^+ and χ^- are given by:

$$\chi^+ = \max(0,\chi), \text{ and } \chi^- = \max(0,-\chi).$$

Lemma 8. *For all $w_1, w_2 \in \digamma$ with $w_1 > 0$ and $w_2 < 0$, there exist $t_{1,\lambda}, t_{2,\lambda} > 0$ such that $t_{1,\lambda} w_1 \in S_+$ and $t_{2,\lambda} w_2 \in S_2$. Moreover, $\lim_{\lambda\to 0} t_{1,\lambda} = \lim_{\lambda\to 0} t_{2,\lambda} = 0$. In particular, if w_1 and w_2 are with disjoint supports, then $t_{1,\lambda} w_1 + t_{2,\lambda} w_2 \in S$.*

Proof. To prove Lemma 8, we will only prove the result for S_+, because the other cases can be proved analogously.
Let $w \in \digamma$ with $w > 0$, and put

$$L(w) = \langle \Phi'(w), w \rangle.$$

Let $s > 0$, then from hypothesis $(\mathbf{H})$ and Remark 1, we have

$$\begin{aligned} L(sw) &= ms^{pr}\|w\|^{pr}_{\mu,\theta,\psi} - \lambda \int_0^T g(t,sw(t))sw(t)dt - \int_0^T f(t,sw(t))sw(t)dt \\ &= ms^{pr}\|w\|^{pr}_{\mu,\theta,\psi} - \frac{\lambda}{q_2}\int_0^T G(t,sw(t)) - \frac{1}{q_1}\int_0^T F(t,sw(t))dt \\ &\geq s^{pr}\left[m\|w\|^{pr}_{\mu,\theta,\psi} - \frac{C_1\lambda\|w\|^{q_1}_{\mu,\theta,\psi}}{q_1} s^{q_1-pr} - \frac{C_2\|w\|^{q_2}_{\mu,\theta,\psi}}{q_2} s^{q_2-pr} \right]. \end{aligned}$$

Since $pr < q_1 < q_2$, we can find $s > 0$ small enough such that $L(sw) > 0$.
Again, from hypothesis $(\mathbf{H})$ and Remark 1, we get

$$\begin{aligned} L(sw) &\leq m\|w\|^{pr}_{\mu,\theta,\psi} s^{pr} - \frac{C_1'\lambda\|w\|^{q_1}_{\mu,\theta,\psi}}{q_1} s^{q_1} - \frac{C_2'\|w\|^{q_2}_{\mu,\theta,\psi}}{q_2} s^{q_2} \\ &\leq m\|w\|^{pr}_{\mu,\theta,\psi} s^{pr} - \frac{C_1'\lambda\|w\|^{q_1}_{\mu,\theta,\psi}}{q_1} s^{q_1} \\ &\leq \frac{C_1'\lambda\|w\|^{q_1}_{\mu,\theta,\psi}}{q_1} s^{pr}\left(\frac{mq_1}{\lambda C_1'}\|w\|^{pr-q_1}_{\mu,\theta,\psi} - s^{q_1-pr} \right) \end{aligned} \tag{10}$$

Since $pr < q_1$, then for s large enough, we have $L(sw) < 0$. Hence, by the Bolzano theorem, we deduce the existence of s_λ satisfying $L(s_\lambda w) = 0$.
Finally, from equation (10) and the fact that $L(s_\lambda w) = 0$, we conclude that

$$0 < s_\lambda < \left(\frac{mq_1}{\lambda C_1'}\|w\|^{pr-q_1}_{\mu,\theta,\psi} \right)^{\frac{1}{q_1-pr}},$$

which yields to $\lim_{\lambda\to\infty} s_\lambda = 0$, as we wanted to prove. □

Put

$$K_+ = \{\chi \in S_+ : \chi \geq 0\}, \; K_- = \{\chi \in S_- : \chi \leq 0\}, \quad \text{and } K = S.$$

Then we have the following result.

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Lemma 9. *For each $\chi \in K_\pm$ or $\chi \in K$, there exists $\beta_j > 0\,(j = 1:4)$, such that*

$$\beta_1 \|\chi\|^{pr}_{\mu,\theta,\psi} \leq \beta_2 \left[\int_0^T f(t,\chi(t)) + \lambda g(t,\chi(t))\right] dt \leq \beta_3 \Phi(\chi) \leq \beta_4 \|\chi\|^{pr}_{\mu,\theta,\psi}.$$

Proof. Since the proofs are similar for the three cases. So, we will give the proof the result for K_+. For this aim, let $\chi \in K_+$, then from the definition of S_+, the first inequality holds for $\beta_1 = m$ and $\beta_2 = 1$. On the other hand, from Remark 1 and equations (3), (5) and (6), we get

$$\begin{aligned} \Phi(\chi) &= m\|\chi\|^{pr}_{\mu,\theta,\psi} - \int_0^T F(s,\chi(s))ds - \lambda \int_0^T G(s,\chi(s))ds \\ &= \frac{1}{q_1}\left(mq_1\|\chi\|^{pr}_{\mu,\theta,\psi} - \int_0^T f(s,\chi(s))\chi(s)ds - \lambda \int_0^T g(s,\chi(s))\chi(s)ds\right) \\ &\geq \frac{q_1 - 1}{q_1}\left(\int_0^T f(s,\chi(s))\chi(s)ds + \lambda \int_0^T g(s,\chi(s))\chi(s)ds\right). \end{aligned}$$

Hence, the result follows immediately if we take

$$\beta_3 = \frac{q_1}{q_1 - 1}\beta_2, \text{ and } \beta_4 = \beta_3 m.$$

□

Lemma 10. *There exists a constant D such that $\|\chi^\pm\|_{\mu,\theta,\psi} \geq D$, for all $\chi \in K_\pm$ or for all $\chi \in K$.*

Proof. Since the proofs are similar for the three cases, then, we will prove the result for K, and omit it for $K_\pm$. Let $\chi \in K$, then by definition of K, we have

$$\begin{aligned} m\|\chi\|^{pr}_{\mu,\theta,\psi} &= \int_0^T f(s,\chi(s))\chi(s)ds + \lambda \int_0^T g(s,\chi(s))\chi(s)ds \\ &= q_1 \int_0^T F(s,\chi(s))ds + q_2\lambda \int_0^T G(s,\chi(s))ds. \end{aligned}$$

It follows from Equations (5) and (6), that

$$\begin{aligned} m\|\chi\|^{rp}_{\mu,\theta,\psi} &\leq q_1 C_1 \|\chi\|^{q_1}_{\mu,\theta,\psi} + \lambda q_2 C_2 \|\chi\|^{q_2}_{\mu,\theta,\psi} \\ &\leq \tilde{C}\|\chi\|^{\delta}_{\mu,\theta,\psi}, \end{aligned} \tag{11}$$

where $\delta = q_1$ if $\|\chi\|_{\mu,\theta,\psi} \leq 1$, and $\delta = q_2$ if $\|\chi\|_{\mu,\theta,\psi} \geq 1$. Since $\delta > pr$, then the result follows from equation (11) and by putting $D = \left(\frac{m}{\tilde{C}}\right)^{\frac{1}{\delta - rp}}$. □

Next, we will present some important properties related to the manifolds $S_\pm$ and S.

Lemma 11. *$S_\pm$ and S are sub-manifolds of $\digamma$ of codimension respectively 1 and 2. The sets $K_\pm$ and K are complete. Moreover, for every $\chi \in S_\pm$ or $\chi \in S$, we have $T_\chi \digamma = T_\chi S_\pm \oplus T_\chi \oplus span\{\chi^+, \chi^-\}$. Moreover, we have a uniform continuity of the projection to the first coordinate on $S_\pm$ or S, where $T_\chi S$ denoted the tangent space at χ of S.*

Proof. We begin by observing that

$$S_\pm \subset \widetilde{S}_\pm \text{ and } S \subset \widetilde{S},$$

where $\widetilde{S}_\pm$ is given byequation (9) and $\widetilde{S} = \widetilde{S}_+ \cap \widetilde{S}_-$.
From the fact that the sets $\widetilde{S}_+$, $\widetilde{S}_-$ and $\widetilde{S}$ are open, it suffice to prove that the sets $S_\pm$ and S are

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regular sub-manifold of $\mathcal{F}$. To this end, we introduce the following C^1 functions $\varrho_\pm : M_\pm \to \mathbb{R}$ and $\varrho : M \to \mathbb{R}^2$, which are defined respectively by:

$$\varrho_\pm(\chi) = m\|\chi^\pm\|^{pr}_{\mu,\theta,\psi} - \int_0^T f(s,\chi(s))\chi^\pm(s)ds - \lambda \int_0^T g(s,\chi(s))\chi^\pm(s)ds,$$

and

$$\varrho(\chi) = (\varrho_+(\chi^+), \varrho_-(\chi^-)).$$

Since we have $S_\pm = \varrho_\pm^{-1}(0)$ and $S = \varrho^{-1}(0)$, so we need to prove that 0 is a regular value of $\varrho_\pm$ and ϱ in order to finish the proof. To this aim, let $\chi \in S_+$, then we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\varrho_+(\chi + \varepsilon\chi^+) &= \frac{d}{d\varepsilon}\Big(m(1+\epsilon)^{pr}\|\chi^+\|^{pr}_{\mu,\theta,\psi} - (1+\varepsilon)\int_0^T f(s,\chi+\varepsilon\chi^+)\chi^+(s)ds\Big) \\
&\quad -\lambda\frac{d}{d\varepsilon}\Big((1+\varepsilon)\int_0^T g(s,\chi+\varepsilon\chi^+)\chi^+(s)ds\Big) \\
&= mpr(1+\epsilon)^{pr-1}\|\chi^+\|^{pr}_{\mu,\theta,\psi} \\
&\quad -\int_0^T f(s,\chi+\varepsilon\chi^+)\chi^+ + \frac{\partial f}{\partial u}(s,\chi+\varepsilon\chi^+)(\chi^+)^2\,ds \\
&\quad -\lambda\int_0^T g(s,\chi+\varepsilon\chi^+)\chi^+ + \frac{\partial g}{\partial u}(s,\chi+\varepsilon\chi^+)(\chi^+)^2\,ds.
\end{aligned}$$

From equation (4), we have

$$\frac{\partial f}{\partial u}(s,\chi)\chi = (q_1-1)f(s,\chi) \text{ and } \frac{\partial g}{\partial u}(s,\chi)\chi = (q_2-1)f(s,\chi).$$

So, using the fact that $\chi^+\chi^- = 0$, we get

$$\frac{\partial f}{\partial u}(s,\chi)(\chi^+)^2 = (q_1-1)f(s,\chi)\chi^+ \text{ and } \frac{\partial g}{\partial u}(s,\chi)(\chi^+)^2 = (q_2-1)f(s,\chi)\chi^+.$$

Therefore, from the facts that $\chi \in S^+$ and $pr < q_1$, we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon}\varrho_+(\chi+\varepsilon\chi^+)_{|=0} &= mpr\|\chi^+\|^{pr}_{\mu,\theta,\psi} - \int_0^T f(s,\chi)\chi^+ + \frac{\partial f}{\partial u}(s,\chi)(\chi^+)^2\,ds \\
&\quad -\lambda\int_0^T g(s,\chi)\chi^+ + \frac{\partial g}{\partial u}(s,\chi)(\chi^+)^2\,ds \\
&= mpr\|\chi^+\|^{pr}_{\mu,\theta,\psi} - q_1\int_0^T f(s,\chi)\chi^+\,ds - \lambda q_2\int_0^T g(s,\chi)\chi^+\,ds \\
&\le (pr-q_1)\Big(\int_0^T f(s,\chi)\chi^+\,ds - \lambda\int_0^T g(s,\chi)\chi^+\,ds\Big) < 0.
\end{aligned}$$

Hence, $\langle\nabla\varrho_+(\chi),\chi^+\rangle < 0$, which implies that $\nabla\varrho_+(\chi) \neq 0$. So, S_+ is a regular submanifold of $\mathcal{F}$. The proof for S_- is very similar to the first one and we omit it.

Now, for the case of S, we begin by observing that since $\langle\nabla\varrho_+(\chi),\chi^+\rangle < 0$ and $\langle\nabla\varrho_-(\chi),\chi^-\rangle < 0$., then it suffice to prove that $\langle\nabla\varrho_+(\chi),\chi^-\rangle = \langle\nabla\varrho_-(\chi),\chi^+\rangle = 0$ for $\chi \in S$. This ensure that $\langle\nabla\varrho(\chi),\chi\rangle \neq 0$ for each $\chi \in M$.

Next, we aim to proving that $\langle\nabla\varrho_+(\chi),\chi^-\rangle = 0$. to this end, let $\chi \in S$, then we have

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$$\begin{aligned}
\frac{d}{d\varepsilon}\varrho_+(\chi+\varepsilon\chi^-) &= \frac{d}{d\varepsilon}\left(m\|\chi^+\|^{pr}_{\mu,\theta,\psi} - \int_0^T f(s,\chi+\varepsilon\chi^-)\chi^+(s)ds\right) \\
&\quad -\lambda\frac{d}{d\varepsilon}\left(\int_0^T g(s,\chi+\varepsilon\chi^-)\chi^+(s)ds\right) \\
&= -\int_0^T \partial f\partial\chi(s,\chi+\varepsilon\chi^-)\chi^+(s)\chi^-(s)ds \\
&\quad -\lambda\int_0^T \partial g\partial\chi(s,\chi+\varepsilon\chi^-)\chi^+(s)\chi^-(s)ds \\
&= 0.
\end{aligned}$$

Analogous, we can obtain that $\langle\nabla\varrho_-(\chi),\chi^+\rangle = 0$, which implies that S is a regular submanifold. Moreover, by classical arguments, we deduce The completeness of $K_\pm$ and K. Next, we shall prove that

$$T_\chi F = T_\chi S_+ \oplus span\{\chi^+\},$$

where $T_\chi S_+ = \{v : \langle\nabla\varrho_+(\chi),v\rangle = 0\}$. In fact, let v be a unit tangential vector in $T_\chi F$. Put

$$v_2 = \frac{\langle\nabla\varrho_+(\chi),v\rangle}{\langle\nabla\varrho_+(\chi),\chi^+\rangle}\chi^+ \text{ and } v_1 = v - v_2.$$

It is clear that $v_2 \in T_\chi S_+$, $v_1 \in span\{\chi^+\}$ and $v = v_1 + v_2$.

Similarly, we can prove that $T_\chi F = T_\chi S_- \oplus span\{\chi^-\}$ and $T_\chi F = T_\chi S \oplus span\{\chi^+,\chi^-\}$.

Finally, if we combine the above equations with the estimates given in the first part of the proof, we conclude the uniform continuity of the projections onto $T_\chi S_\pm$ and $T_\chi S$. □

Now, we define the notion of the Palais–Smale geometry.

Definition 12. *We say that Φ satisfies the Palais–Smale condition at level c if any sequence $\{\chi_k\}$ that satisfies*

$$\Phi(\chi_k) \to c \text{ and } \phi'_{\lambda,\theta}(\chi_k) \to 0$$

admits sub-sequence that converges strongly.

Lemma 13. *If $c > 0$ is small enough, then Φ satisfies the Palais–Smale condition at c.*

Proof. Since the proof is very similar to the proof of Lemma 3.5 in the paper of Alsaedi and Ghanmi [1], then we omit it. □

Next, in the following lemma, we will prove the Palais–Smale condition for the restricted functionals.

Lemma 14. *The functionals $\Phi_{|K_\pm}$ and $\Phi_{|K}$ satisfy the Palais–Smale condition for energy level c, provided that $c > 0$ is small enough.*

Proof. Let $\{\chi_j\} \subset K_+$ be a Palais–Smale sequence, so we have $\Phi_{|K_+}$ is uniformly bounded and $\nabla\Phi_{|K_+}(\chi_j)\longrightarrow 0$ as j tends to infinity. We need to show that there exists a subsequence still denoted by $\{\chi_j\}$ that converges strongly in K_+.
Let $v_j \in T_{\chi_j}F$ be a unit tangential vector such that

$$\langle\nabla\Phi(\chi_j),v_j\rangle = \|\nabla\Phi(\chi_j)\|_{(F)^{-1}}.$$

Now, by Lemma 11, $v_j = x_j + y_j$ with $x_j \in T_{\chi_j}S_+$ and $y_j \in span\{(\chi_j)_+\}$.

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Since $\Phi(\chi_j)$ is uniformly bounded, then by Lemma 9, χ_j is uniformly bounded in $\digamma$ and hence x_j is uniformly bounded in $\digamma$. Therefore, we get

$$\|\nabla\Phi(\chi_j)\|_{(\digamma)^{-1}} = \langle\nabla\Phi(\chi_j), v_j\rangle = \langle\nabla\Phi_{K_+}(\chi_j), v_j\rangle \to 0. \tag{12}$$

As v_j is uniformly bounded and $\nabla\Phi_{K_+}(\chi_j)$ strongly, the convergence to zero in equation (12) is strongly. Finally, the result follows immediately from Lemma 13. □

As a consequence of Lemma 14, we have the following result.

Lemma 15. *If $\chi \in K_\pm$ or $\chi \in K$ is a critical point of the restricted functionals $\Phi_{K_\pm}$ or Φ_K. Then, χ is also a critical point of the unrestricted functional Φ and hence is a weak solution to* (2).

3.1. Proof of Theorem 2

From Lemma 15, to prove the Theorem 2, we shall prove that the functionals $\Phi_{K_\pm}$ and Φ_K have critical points. Since the same arguments are used for $\Phi_{K_\pm}$ and Φ_K, we will give the proof for Φ_{K_+}. It is clear from the definition of K_+, that Φ is bounded below over K_+. Then using Variational Principle due to Ekeland, there exists $\{v_j\} \subset K_+$, such that

$$\Phi(v_j) \to \inf_{v \in K_+} \Phi(v) := c_{+,\lambda} \text{ and } \left(\nabla\Phi_{K_+}\right)(v_j) \to 0.$$

From Lemma 8 and the estimate given in Lemma 9, there exists $w_0 \geq 0$ such that

$$c_{+,\lambda} \leq \Phi(s_\lambda w_0) \leq m\frac{q_1 - 1}{q_1} s_\lambda^{pr} \|w_0\|_{\mu,\theta,\psi}^{pr}.$$

Now, from Lemma 8, we deduce that $c_{+,\lambda} \to 0$ as $\lambda \to 0$. Then for λ large enough, we have that $c_{+,\lambda}$ is small enough. So by Lemma 13, v_j has a convergent subsequence, that we still call v_j. Therefore Φ has a critical point in K_+ denoted by $v_{+,\lambda}$.

In the same way, Φ has a critical point in K_- denoted by $v_{-,\lambda}$. Put $v_\lambda = v_{+,\lambda} v_{-,\lambda}$, then from Lemma 15, $v_{+,\lambda}$, $v_{-,\lambda}$ and v_λ are weak solutions for problem (2). Moreover, by construction, $v_{+,\lambda}$ is positive, $v_{-,\lambda}$ is negative and v_λ changes sign.

Conclusion: In this paper we have investigated the existence and the multiplicity of solutions, moreover, we have introduced three-manifolds and proved that in each of these sets, the energy functional admits a critical point which is a nontrivial solution for the studied problem. So by definition of these manifolds, these solutions are one positive, one negative, and the other change sign. In the case when $\theta \to 0$, our problem is reduced to the one studied by Nouf et al. [41], and in the case when $\psi(x) = x$, our problem is reduced to the one studied by Ghanmi and Zhang [15]. We hope to develop other works by considering the singular double-phase problem.

Author Contributions: Writing—review & editing, H. Z. A., W. S., and A. G. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. UJ-23-DR-80. The authors, therefore, the authors thank the University of Jeddah for its technical and financial support.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alsaedi, R.; Ghanmi, A, Variational approach for the Kirchhoff problem involving the p-Laplace operator and the p-Hilfer derivative, Math. Methods Appl. Sci. 46(8)(2023), 9286-9297.
2. Almeida, R. Further properties of Osler's generalized fractional integrals and derivatives with respect to another function, Rocky Mountain J. Math., 49(8)(2019), 2459-2493.
3. Almeida, R. A Caputo fractional derivative of a function with respect to another function, Commun. Nonl. Sc. Num. Simul., 44(2017), 460–481.
4. Herrmann, R. Fractional calculus: an introduction for physicists, World Scientific, (2011).
5. Ben Ali, K.; Ghanmi, A.; Kefi, K. Existence of solutions for fractional differential equations with Dirichlet boundary conditions, Electron. J. Differ. Equ. 2016(2016), 1-11.
6. Caisheng, C.; Zonghu, X.; Jincheng, H. The Nehari manifold and the existence of multiple solutions for a singular quasilinear elliptic equation, J. Math. Anal. Appl. 393(2) (2012), 671–679.
7. Chen, W.; Pang, G. A new definition of fractional Laplacian with application to modeling three-dimensional nonlocal heat conduction, J. Comput. Phys. 309 (2016), 350-367.
8. Cho, Y.; Kim, I.; Sheen, D. A fractional-order model for MINMOD millennium, Math. Biosci. 262 (2015), 36-45.
9. Kumar, P.; Erturk, V. S.; Yusuf, S.; Kumar, A. Fractional time-delay mathematical modeling of oncolytic virotherapy, Chaos Solitons Fractals 150 (2021), 111-123.
10. Erturk, V.; Godwe, E.; Baleanu, D.; Kumar, P.; Asad, J.; Jajarmi, A. Novel fractional-order Lagrangian to describe motion of beam on nanowire, Acta Phys. Pol. A 140(3)(2021), 265-272.
11. Kilbas, A. A.; Srivastava, H. M.; Trujillo, J. J. Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 207, Elsevier Science B.V., Amsterdam, 2006.
12. Drábek, P.; Pohozaev, S. I. Positive solutions for the p-Laplacian: application of the fibrering method, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 127(4) (1997), 703–726.
13. Ghanmi, A.; Althobaiti, M. Existence results involving fractional Liouville derivative, Bol. Soc. Parana. Mat. 39(5)(2021), 93-102.
14. Ghanmi, A.; Horrigue, S. Existence of positive solutions for a coupled system of nonlinear fractional differential equations, Ukr. Math. J. 71 (1)(2019), 39-49.
15. Ghanmi, A.; Zhang, Z. Nehari manifold and multiplicity results for a class of fractional boundary value problems with p-Laplacian, Bull. Korean Math. Soc. 56(5)(2019), 1297-1314.
16. Grahovac, N. M.; Žigiċ, M. M. Modelling of the hamstring muscle group by use of fractional derivatives, Comput. Math. Appl. 59 (5)(2010), 1695-1700.
17. Horrigue, S.; Almuashi, H.; Alnashry, A. A. Existence results for some ψ-Hilfer iterative approximation, Math. found. comput. (2023), http://dx.doi.org/10.3934/mfc.2023029.
18. Jiao, F.; Zhou, Y. Existence results for fractional boundary value problem via critical point theory, Intern. Journal of Bif. and Chaos, 22(4)(2012), 1-17.
19. Ledesma, C. T. Mountain pass solution for a fractional boundary value problem, J. Fract. Calc. Appl. 5(1)(2014), 1–10.
20. Magin, R. L.; Ovadia, M. Modeling the cardiac tissue electrode interface using fractional calculus, J. Vib. Control 14 (9)(2008), 1431-1442.
21. Miller, K.; Ross, B. An introduction to the fractional calculus and fractional differential equations, Wiley and Sons, New York, 1993.
22. Purohit, S.; Kalla, S. On fractional partial differential equations related to quantum mechanics, J. Phys. A, Math. Theor. 44 (2011), 1-8.
23. Rossikhin, Y. A.; Shitikova, M. V. Analysis of two colliding fractionally damped spherical shells in modelling blunt human head impacts, Cent. Eur. J. Phys. 11(6) (2013), 760-778.
24. Samko, S. G. Kilbas, A. A.; Marichev, O. I. Fractional integrals and derivatives, theory and functions, (1993) , Gordon and Breach, Yverdon.
25. Torres, C. Mountain pass solution for a fractional boundary value problem, J. Fract. Calculus Appli., 5(1) (2014), 1-10.
26. Da Sousa, J. V.; de Oliveira, E. C. On the ψ-fractional integral and applications, Comp. Appl. Math. 38(4)(2019), https://doi.org/10.1007/s40314-019-0774-z.

27. Da Sousa, E. C.; Zuo, J. Donal O'Regand, The Nehari manifold for a ψ-Hilfer fractional p-Laplacian, Appl. Anal. 101(14)(2022), 5076-5106.
28. Da Sousa, J. V.; Tavares, L. S.; Ledesma, C. E. T. A variational approach for a problem involving a ψ-Hilfer fractional operator, J. Appl. Anal. Comput., 11(3)(2021), 1610-1630.
29. Chamekh, M.; Ghanmi, A.; Horrigue, S. Iterative approximation of positive solutions for fractional boundary value problem on the half-line, Filomat, 32(18)(2018), 6177–6187.
30. Cantizano, N. A.; Silva, A. Three solutions for a nonlocal problem with critical growth, J. Math. Anal. Appl. 469(2)(2019), 841–851.
31. Ghanmi, A.; Horrigue, S. Existence results for nonlinear boundary value problems, Filomat, 32(2)(2018), 609–618.
32. Ghanmi, A.; Horrigue, S. Existence of positive solutions for a coupled system of nonlinear fractional differential equations, Ukr. Math. J., 71(1)(2019), 39–49.
33. Han, X.; Yang, X. Existence and multiplicity of positive solutions for a system of fractional differential equation with parameters, Bound. Value Probl., 2017(1)(2017), 1–12.
34. Horrigue, S. Existence Results for a Class of Nonlinear Hadamard Fractional with p-Laplacian Operator Differential Equations, J. Math. Stat., 17(2021), 61-72.
35. Il'yasov, Y. On nonlocal existence results for elliptic equations with convex-concave nonlinearities, Nonlinear Anal. Theory Methods Appl., 61(1-2) (2005), 211-236. pp. 211–236.
36. Jarad, F.; Abdeljawad, T. Generalized fractional derivatives and Laplace transform, Discrete Contin. Dyn. Syst. - S, 13(3) (2020), 709-722.
37. Kassim, M. D.; Tatar, N. E. Stability of logarithmic type for a Hadamard fractional differential problem, J. Pseudo-Differ. Oper. Appl., 11(2020), 447–466.
38. Luchko, Y. Trujillo, J. Caputo-type modification of the Erdélyi-Kober fractional derivative, Fract. Calc. Appl. Anal., 10(3)(2007), 249–267.
39. Mainardi, F. Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models, Imperial College Press, London, World Scientific (2020), Singapore.
40. Nyamoradi, N.; Shekarbigi, M.; Razlansari, A.; Yavari, M. The Nehari manifold and application to a quasilinear elliptic equation with multiple Hardy-Type terms, J. Part. Diff. Eq, 26(3) (2013), 193–216.
41. Nouf, A.; Shammakh, W. M.; Ghanmi, A. Existence of solutions for a class of Boundary value problems involving Riemann Liouville derivative with respect to a function, FILOMAT 37(4) (2023), 1261–1270.
42. Oldham, K. Spanier, J. The fractional calculus theory and applications of differentiation and integration to arbitrary order,(1974), Elsevier, New York, Academic Press, New York.
43. Wang, G.; Ghanmi, A.; Horrigue, S.; Madian, S. Existence result and uniqueness for some fractional problem, Mathematics, 7(6)(2019), 516.
44. Zhang, X.; Boutat, D.; Liu, D. Applications of fractional operator in image processing and stability of control systems, Fractal Fract. 7(5) (2023), 359.