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Existence and Nonexistence of Positive Solutions for Semilinear Elliptic Equations Involving Hardy-Sobolev Critical Exponents

Lin-Lin Wang and Yong-Hong Fan *

School of Mathematics and Statistics Sciences, Ludong University, Yantai 264025, China
* Correspondence: yhfan@ldu.edu.cn

Abstract: In this paper, a class of semi-linear elliptic equations involving Hardy-Sobolev critical exponents has been investigated. This problem comes from the consideration of standing waves in the anisotropic Schrödinger equation, also it is very important in the field of hydrodynamics, glaciology, quantum field theory and statistical mechanics. By a detailed estimation for the extremum function and using Mountain Pass Lemma with $(PS)_c$ conditions, the existence of positive solutions has been obtained. On the other hand, by establishing Pohozaev-type identity and using the properties of Bessel function, the nonexistence of positive solution also has been obtained. These results are extensions of E. Jannelli's research ([1], Theorem 1.A–1.C).

Keywords: semilinear elliptic equation; Hardy-Sobolev critical exponents; Mountain Pass Lemma; $(PS)_c$ condition

MSC: 35B09; 35J20; 35J60; 35J75

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1. Introduction

In [2], the authors investigated the following semilinear elliptic problem

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s} u + g(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

Where Ω is an open bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary $\partial\Omega$ and $0 \in \Omega$, $0 \leq \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$, $0 \leq s < 2$, $2^*(s) = 2(N-s)/(N-2)$ is the Hardy-Sobolev critical exponent. This problem comes from the consideration of standing waves in the anisotropic Schrödinger equation, also it is very important in the field of hydrodynamics, glaciology, quantum field theory and statistical mechanics (see [3–6]).

Assume that $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $G(x,t) = \int_0^t g(x,s)\,ds$ such that

(g_1) There exist constants c_1, $c_2 > 0$ and $p \in (2, 2^*)$ (here $2^* = 2^*(0) = 2N/(N-2)$ is the Sobolev critical exponent) such that

$$|g(x,t)| \leq c_1|t| + c_2|t|^{p-1}, \text{ for any } (x,t) \in \Omega \times \mathbb{R}.$$

(g_2) There exists a constant $K > 0$ big enough such that

$$G(x,t) \geq Kt^2, \text{ for any } (x,t) \in \Omega \times \mathbb{R}.$$

(g_3) There are constants $\rho > 2$ and $\nu > 0$ such that

$$\rho G(x,t) \leq g(x,t)t + \nu t^2, \text{ for any } (x,t) \in \Omega \times \mathbb{R}.$$

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Theorem 1. *([2], **Theorem 1.1.**) Suppose that $N \geq 3, 0 \leq \mu < \overline{\mu} - 1, 0 \leq s < 2, g(x,t)$ satisfies (g_1), (g_2), (g_3), then (1) has at least one nonnegative solution.*

From the proof of Theorem 1, one can conclude that it also need c_1 to be sufficiently small (see page 222, line -8 in [2]). So we may wonder how small should c_1 be, this problem is one of our objects to be solved in this paper.

Many authors investigated problem (1) when $g(x,u)$ is a special function, such as [1,7–14]. And some authors concentrated on the case when $g(x,u)$ is a general function that satisfies some definite conditions, such as [2,15,16] and so on. For the study on problem (1), the classical method is variational method (see [17–19]), we should point out that the classical Mountain Pass Lemma can not be applied directly to (1), because (1) contains the Hardy-Sobolev critical exponent $\frac{|u|^{2^*(s)-2}}{|x|^s}u$. As we all know, the essential reason is the embedding from $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ is continuous, but not compact. In order to overcome the difficulty caused by this non-compact embedding, one can use the principle of concentrated compactness proposed by P.L. Lions ([20–23]), also we could use the Mountain Pass Lemma with $(PS)_c$ conditions proposed by H. Brezis and L. Nirenberg (one can refer to [7,19]), and so on. These theoretical methods has greatly promoted the development of nonlinear analysis, and many excellent results have been obtained, for convenience, we list some which are useful for our study.

The following two theorems describe the existence of positive solution for (1) when the function $g(x,t)$ grows linearly at $t = 0$.

Theorem 2. *([1], Theorem 1.A–1.C). In (1), let $g(x,u) = \lambda u$ and $s = 0$.*

(B1) If $\mu \leq \overline{\mu} - 1$, then when $\lambda < \lambda_1(\mu)$, problem (1) has at least a positive solution in $H_0^1(\Omega)$, where

$$\lambda_1(\mu) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx}{\int_\Omega |u|^2 dx}. \tag{2}$$

(B2) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, then when $\lambda_*(\mu) < \lambda < \lambda_1(\mu)$, problem (1) has at least a positive solution in $H_0^1(\Omega)$, where

$$\beta = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu} \text{ and } \lambda_*(\mu) = \min_{\varphi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \varphi|^2 / |x|^{2\beta} dx}{\int_\Omega \varphi^2 / |x|^{2\beta} dx}. \tag{3}$$

(B3) If $\overline{\mu} - 1 < \mu < \overline{\mu}$ and $\Omega = B(0,R)$ (i.e., the ball centered at $x = 0$ with radius R), then (1) has no positive solution for $\lambda \leq \lambda_*(\mu)$.

One can easily see that when $g(x,u) = \lambda u$, $g(x,u)$ satisfies (g_1), but when $\lambda < 2K$, (g_2) is invalid.

In [15], D.S. Kang and Y.B. Deng obtained the existence of the solution to problem

$$-\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2} u + a(x) |u|^{r-2} u, \ x \in \mathbb{R}^N, \tag{4}$$

where $a(x)$ is a nonnegative function and locally bounded in $\mathbb{R}^N \setminus \{0\}$, $a(x) = O\left(|x|^{-s}\right)$ in the bounded neighborhood of the origin, $a(x) = O\left(|x|^{-t}\right)$ as $|x| \to \infty$, $0 \leq s < t < 2$, $2^*(t) < r < 2^*(s)$. They obtained: If $r > \max\left\{ \frac{N-s}{\sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}}, \frac{N-s-2\sqrt{\overline{\mu}-\mu}}{\sqrt{\overline{\mu}}} \right\}$, then (4) has at least one solution.

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On the other hand, when $g(x,u) = \lambda u$ and $s = 0$, as E. Jannelli [1] said, "The space dimension N plays a fundamental role when one seeks the positive solutions of (1)". In [24], the authors studied

$$\begin{aligned} -\Delta u &= \lambda u + u^{\frac{N+2}{N-2}} \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{5}$$

and obtained

Theorem 3. *(a)When $N \geq 4$, problem (5) has a solution for every $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition; moreover it has no solution if $\lambda \notin (0, \lambda_1)$ and Ω is starshaped; (b)when $N = 3$ and Ω is a ball, problem (5) has a solution if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$.*

Motivated by the above mentioned references, we naturally proposed the following problems:

(P1) In Theorem 1, what would happen when $\overline{\mu} - 1 \leq \mu < \overline{\mu}$.

(P2) Can we further weaken the conditions (g_1), (g_2) and (g_3)?

(P3) For problem (1), if g is a general function, whether the space dimension N still play an important role.

To solve these problems, we tried our best, and obtained

Theorem 1.1. *Suppose that $N \geq 3$, $0 \leq \mu < \bar{\mu}$, $0 \leq s < 2$, $a(x)$ is nonnegative and continuous in $\overline{\Omega} \setminus \{0\}$ (for short, set Ω^0), there exists a neighborhood of the origin $U(0) \subset \Omega$ and $0 \leq q < 2$ such that $a(x) = O(|x|^{-q})$ for $x \in U(0)$, and $g(x,t)$ satisfy*

(g_1') *$g \in C(\Omega^0 \times \mathbb{R}^+, \mathbb{R}^+)$ and there exist constants $\lambda > 0$ and $2 < p < 2^*(1 - \frac{q}{N})$ such that*

$$\lambda t \leq g(x,t) \leq \lambda t + a(x)t^{p-1}, \text{ for any } (x,t) \in \Omega^0 \times \mathbb{R}^+.$$

(g_2') *Assume that there exist two nonnegative constants*

$$\rho > 2 \text{ and } 0 \leq \nu \leq \frac{(\theta - 2)\rho\lambda}{2\theta} (\theta = \min\{\rho, 2^*(s)\})$$

such that

$$\rho G(x,t) \leq g(x,t)t + \nu t^2, \text{ for any } (x,t) \in \Omega^0 \times \mathbb{R}^+.$$

We can get

(i) If $0 \leq \mu \leq \overline{\mu} - 1$, then when $\lambda < \lambda_1(\mu)$, problem (1) has at least one positive solution in $H_0^1(\Omega)$.

(ii) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, then when $\lambda_*(\mu) < \lambda < \lambda_1(\mu)$, problem (1) has at least one positive solution in $H_0^1(\Omega)$.

Theorem 1.2. *Suppose that $N \geq 3$, $\overline{\mu} - 1 < \mu < \bar{\mu}$, $0 \leq s < 2$, $\Omega = B(0,R)$, $g(x,t)$ satisfy*

(g_3') *$g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $g(x,t) = g(|x|,t)$ for $(x,t) \in \overline{\Omega} \times R$ and $g(r,t)$ is decreasing in r, where $r = |x|$;*

(g_4') *There exist two positive constants*

$$c_1' \leq \frac{N-2}{2N}, \ c_2' \leq \frac{1}{N}$$

such that

$$G(x,t) \leq c_1' g(x,t)t + c_2' \lambda t^2, \text{ for any } x \in \overline{\Omega} \times \mathbb{R}^+,$$

then when $\lambda \leq \lambda_*(\mu)$, problem (1) has no positive solution in $H_0^1(\Omega)$.

Remark 1. *Comparing the above two theorems with Theorem 1, 2, one can easily see that (g_1) and (g_1') are exactly the same. Here we don't need the condition (g_2). And comparing with (g_3), (g_2') only constricts the*

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range of parameter ν. For Theorem A, the conclusion (ii) in Theorem 1.1 is new. We can also see that all the conclusions in Theorem B are included in Theorem 1.1 and Theorem 1.2.

As applications of Theorem 1.1 and Theorem 1.2, we give an example.

Example 1. *Consider the following elliptic problem*

$$\begin{cases} -\Delta u-\mu\dfrac{u}{|x|^2}=\dfrac{|u|^{2^*(s)-2}}{|x|^s}u+\lambda u+a(|x|)u^{\alpha}, & x\in\Omega\setminus\{0\}, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{6}$$

where $a(r)\in C^1([0,+\infty),[0,+\infty))$ and $a'(r)\le 0$, $N\ge 3$, $0\le s<2$, $\Omega=B(0,R)$, by Theorem 1.1 and and Theorem 1.2, we have

(i) If $0\le\mu\le\overline{\mu}-1$, $1<\alpha<\frac{N+2}{N-2}$, then when $\lambda<\lambda_1(\mu)$, problem (6) has at least one positive solution in $H_0^1(\Omega)$.

(ii) If $\overline{\mu}-1<\mu<\overline{\mu}$, $1<\alpha<\frac{N+2}{N-2}$, then when $\lambda_*(\mu)<\lambda<\lambda_1(\mu)$, problem (6) has at least one positive solution in $H_0^1(\Omega)$.

(iii) If $\overline{\mu}-1<\mu<\overline{\mu}$, $\alpha\ge\frac{N+2}{N-2}$ and $\lambda\le\lambda_*(\mu)$, then (6) has no positive solution in $H_0^1(\Omega)$.

Corollary 1. *For the following elliptic problem*

$$\begin{cases} -\Delta u-\mu\dfrac{u}{|x|^2}=\dfrac{|u|^{2^*(s)-2}}{|x|^s}u+\lambda u+a(|x|)u^{\frac{N+2}{N-2}}, & x\in\Omega\setminus\{0\}, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{7}$$

where $a(r)\in C^1([0,+\infty),[0,+\infty))$ and $a'(r)\le 0$, $N\ge 3$, $0\le s<2$, $\Omega=B(0,R)$. If $\overline{\mu}-1<\mu<\overline{\mu}$ and $\lambda\le\lambda_*(\mu)$, then (7) has no positive solution in $H_0^1(\Omega)$. Especially, for any $a\ge 1$, $\overline{\mu}-1<\mu<\overline{\mu}$ and $\lambda\le\lambda_*(\mu)$, the equations

$$\begin{cases} -\Delta u-\mu\dfrac{u}{|x|^2}=\lambda u+au^{\frac{N+2}{N-2}}, & x\in\Omega\setminus\{0\}, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{8}$$

has no positive solution in $H_0^1(\Omega)$.

Remark 2. *To the best of our knowledge, the conclusions in Corollary 1.1 are new. The conclusion (b) in Theorem C is included in Corollary 1.1 and the conclusion (a) in Theorem C is included in Example 1.1.*

Corollary 2. *For the following elliptic problem*

$$\begin{cases} -\Delta u=\dfrac{|u|^{2^*(s)-2}}{|x|^s}u+\lambda u+a(|x|)u^{\alpha}, & x\in\Omega\setminus\{0\}, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{9}$$

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where $a(r) \in C^1([0,+\infty),[0,+\infty))$ and $a'(r) \leq 0$, $N \geq 4$, $0 \leq s < 2$, if $1 < \alpha < \frac{N+2}{N-2}$, then when $\lambda < \lambda_1(0)$, problem (9) has at least one positive solution in $H_0^1(\Omega)$. Especially, for any $a \geq 0$, $N \geq 4$, $1 < \alpha < \frac{N+2}{N-2}$ and $\lambda < \lambda_1(0)$, the equations

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{N+2}{N-2}} + a u^{\alpha}, & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{10}$$

has at least one positive solution in $H_0^1(\Omega)$.

Remark 3. *From Corollary 1.2, we can easily see that the existence of a positive solution for the equation (10) is independent of the subcritical terms au^{α} $(a > 0, 1 < \alpha < \frac{N+2}{N-2})$.*

We organized the rest paper as follows. In section 2, we give some preliminaries about Hardy inequality, the properties of variational functional corresponding to equation (1) and the properties of extremal functions. In section 3, by using the Mountain Pass Lemma with $(PS)_c$ conditions, we give a detailed proof of Theorem 1.1. In section 4, by establishing Pohozaev-type identity and using the properties of Bessel function, we give a detailed proof of Theorem 1.2.

2. Preliminaries

In this section, we give some lemmas which will be useful for our study, for more details, one can refer to the references and cited therein.

Lemma 1. *([25]). Assume that $1 < p < N$ and $u \in W_0^{1,p}(\Omega)$. Then*

$$\int_{\Omega} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{N-p}\right)^p \int_{\Omega} |\nabla u|^p \, dx.$$

By Lemma 2, we can define equivalent norm and inner product in $H_0^1(\Omega)$ as following for $0 \leq \mu < \bar{\mu}$:

$$\|u\| := \left[\int_{\Omega}\left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)dx\right]^{\frac{1}{2}}, \ (u,v) := \int_{\Omega}\left(\nabla u \nabla v - \mu\frac{uv}{|x|^2}\right)dx, \ \forall u,v \in H_0^1(\Omega).$$

Notice that the values of $g(x,t)$ are irrelevant for $t < 0$ in Theorem 1.1, so we define

$$g(x,t) = 0, \text{ for } (x,t) \in \Omega^0 \times (-\infty, 0).$$

To study the existence of positive solution for (1), we first consider the existence of nontrivial solutions to the problem

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = \frac{(u^+)^{2^*(s)-1}}{|x|^s} + g(x,u), & x \in \Omega^0, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{11}$$

where

$$u^+ = \max\{u, 0\}.$$

Obviously, the existence of positive solution for (1) is equivalent to the existence of positive solution for (11).

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The energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ corresponding to (11) is given by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega G(x,u)\,dx,\ u \in H_0^1(\Omega), \tag{12}$$

$J(u)$ is well defined with $J \in C^1\left(H_0^1(\Omega), \mathbb{R}\right)$ and for any $v \in H_0^1(\Omega)$,

$$\langle J'(u), v\rangle = (u,v) - \int_\Omega \frac{(u^+)^{2^*(s)-1}}{|x|^s} v dx - \int_\Omega g(x,u)\, v dx.$$

For $0 \le \mu < \bar{\mu}$, define the best constant (see [13,26])

$$A_{\mu,s}(\Omega) \triangleq \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}}. \tag{13}$$

The following two lemmas could be found in [13].

Lemma 2. *([13]). Suppose $0 \le s < 2$ and $0 \le \mu < \bar{\mu}$. Then we have*
(i) $A_{\mu,s}(\Omega)$ is independent of Ω.
(ii) $A_{\mu,s}(\Omega)$ is attained when $\Omega = \mathbb{R}^N$ by the functions

$$y_\varepsilon(x) = \frac{\left(2\varepsilon(\bar{\mu}-\mu)(N-s)/\sqrt{\bar{\mu}}\right)^{\sqrt{\bar{\mu}}/(2-s)}}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\left(\varepsilon + |x|^{(2-s)\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}}}\right)^{(N-2)/(2-s)}},$$

for all $\varepsilon > 0$. Moreover, the extremal functions $y_\varepsilon(x)$ solve the equation

$$-\Delta u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}u,\ x \in \mathbb{R}^N\setminus\{0\}$$

and satisfy

$$\int_{\mathbb{R}^N}\left(|\nabla y_\varepsilon|^2 - \mu\frac{|y_\varepsilon|^2}{|x|^2}\right)dx = \int_{\mathbb{R}^N}\frac{|y_\varepsilon|^{2^*(s)}}{|x|^s}dx = A_{\mu,s}^{\frac{N-s}{2-s}}. \tag{14}$$

Let

$$C_\varepsilon = \left(\frac{2\varepsilon(\bar{\mu}-\mu)(N-s)}{\sqrt{\bar{\mu}}}\right)^{\sqrt{\bar{\mu}}/(2-s)},\ U_\varepsilon(x) = \frac{y_\varepsilon(x)}{C_\varepsilon}, \tag{15}$$

and define a cut-off function $\varphi(x) \in C_0^\infty(\Omega)$ such that

$$\varphi(x) = \begin{cases} 1, & |x| \le R, \\ 0, & |x| \ge 2R, \end{cases}$$

where $B_{2R}(0) \subset \Omega$, $0 \le \varphi(x) \le 1$, for $R < |x| < 2R$ ($R \le R_0$, R_0 will be defined later), set

$$u_\varepsilon(x) = \varphi(x)U_\varepsilon(x),\ \ v_\varepsilon(x) = u_\varepsilon(x)/\left(\int_\Omega |u_\varepsilon(x)|^{2^*(s)}|x|^{-s}dx\right)^{\frac{1}{2^*(s)}}, \tag{16}$$

then $\int_\Omega \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s}dx = 1$.

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Lemma 3. *([13]). Let $v_\varepsilon(x)$ be defined as in (16), then $v_\varepsilon(x)$ satisfies the following estimates:*

$$\|v_\varepsilon\|^2 = A_{\mu,s} + O\left(\varepsilon^{\frac{N-2}{2-s}}\right), \tag{17}$$

$$\int_\Omega |v_\varepsilon|^q \, dx = \begin{cases} O\left(\varepsilon^{\frac{\sqrt{\bar\mu}q}{2-s}}\right), & 1 \le q < \frac{N}{\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}, \\ O\left(\varepsilon^{\frac{\sqrt{\bar\mu}q}{2-s}} |\ln\varepsilon|\right), & q = \frac{N}{\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}, \\ O\left(\varepsilon^{\frac{\sqrt{\bar\mu}(N-q\sqrt{\bar\mu})}{(2-s)\sqrt{\bar\mu-\mu}}}\right), & \frac{N}{\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} < q < 2^*. \end{cases} \tag{18}$$

Lemma 4. *([1]). Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $N \ge 3$, $0 \in \Omega$. Then $\lambda_*(\mu)$ is attained for a positive $\overline{\varphi} \in H_0^1(\Omega)$, and $0 < \lambda_*(\mu) < \lambda_1(\mu)$.*

Remark 4. *Lemma 4 shows that the interval $(\lambda_*(\mu), \lambda_1(\mu))$ is not empty.*

Lemma 5. *Let $\|u_\varepsilon\|$, $A_{\mu,s}$, C_ε be defined as above, then we have*

$$\|u_\varepsilon\|^2 = C_\varepsilon^{-2} A_{\mu,s}^{\frac{N-s}{2-s}} + D, \quad \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx = C_\varepsilon^{-2^*(s)} A_{\mu,s}^{\frac{N-s}{2-s}} + E,$$

where

$$D = \int_{R \le |x| \le 2R} \left(|\nabla u_\varepsilon|^2 - \mu \frac{u_\varepsilon^2}{|x|^2}\right) dx - \int_{|x| \ge R} \left(|\nabla U_\varepsilon|^2 - \mu \frac{U_\varepsilon^2}{|x|^2}\right) dx,$$

$$E = -\int_{|x| \ge R} \frac{|U_\varepsilon|^{2^*(s)}}{|x|^2} dx + \int_{R \le |x| \le 2R} \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx.$$

Moreover, let $\xi = \varepsilon^{\frac{\sqrt{\bar\mu}}{2-s}}$, then

$$\lim_{\xi \to 0^+} \xi \frac{\partial D}{\partial \xi} = 0, \quad \lim_{\xi \to 0^+} \xi^{\frac{N-s}{N-2}} \frac{\partial E}{\partial \xi} = 0.$$

And there exists $R_0 > 0$, when $R \le R_0$,

$$\overline{\lim_{\varepsilon \to 0^+}} D < \int_\Omega \frac{|\nabla \varphi(x)|^2}{|x|^{2\beta}} dx. \tag{19}$$

Proof. By (14) and (15), we know

$$\begin{aligned}
\|u_\varepsilon\|^2 &= \int_\Omega \left(|\nabla u_\varepsilon|^2 - \mu \frac{u_\varepsilon^2}{|x|^2}\right) dx \\
&= \int_{|x| \le R} \left(|\nabla U_\varepsilon|^2 - \mu \frac{U_\varepsilon^2}{|x|^2}\right) dx + \int_{R \le |x| \le 2R} \left(|\nabla u_\varepsilon|^2 - \mu \frac{u_\varepsilon^2}{|x|^2}\right) dx \\
&= \int_{\mathbb{R}^N} \left(|\nabla U_\varepsilon|^2 - \mu \frac{U_\varepsilon^2}{|x|^2}\right) dx + D \\
&= C_\varepsilon^{-2} \int_{\mathbb{R}^N} \left(|\nabla y_\varepsilon|^2 - \mu \frac{|y_\varepsilon|^2}{|x|^2}\right) dx + D \\
&= C_\varepsilon^{-2} A_{\mu,s}^{\frac{N-s}{2-s}} + D.
\end{aligned}$$

□

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It follows from $\varphi \in C_0^\infty(\Omega)$ and $0 \le \varphi(x) \le 1$ that there exists $M > 0$, for any $x \in \Omega$,

$$|\varphi(x) \cdot \nabla\varphi(x)| \le M. \tag{20}$$

By $\lim_{\varepsilon \to 0^+} U_\varepsilon = \frac{1}{|x|^\beta}$, $\int_{|x| \ge 2R} \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}} dx \ge 0$, $\beta^2 - \mu > 0$ and (20), we have

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} D &= \int_{R \le |x| \le 2R} \left(\left| \nabla \left(\frac{\varphi(x)}{|x|^\beta} \right) \right|^2 - \mu \frac{|\varphi(x)|^2}{|x|^{2\beta+2}} \right) dx - \int_{|x| \ge R} \left(\left| \nabla \left(\frac{1}{|x|^\beta} \right) \right|^2 - \mu \frac{1}{|x|^{2\beta+2}} \right) dx \\
&= \int_{R \le |x| \le 2R} \left| \frac{\nabla\varphi(x)}{|x|^\beta} + \varphi(x) \nabla \left(\frac{1}{|x|^\beta} \right) \right|^2 dx \\
&\quad - \mu \int_{R \le |x| \le 2R} \frac{|\varphi(x)|^2}{|x|^{2\beta+2}} dx - \left(\beta^2 - \mu\right) \int_{|x| \ge R} \frac{1}{|x|^{2\beta+2}} dx \\
&= \int_{R \le |x| \le 2R} \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}} dx + \left(\beta^2 - \mu\right) \int_{R \le |x| \le 2R} \frac{|\varphi(x)|^2}{|x|^{2\beta+2}} dx \\
&\quad + 2\beta \int_{R \le |x| \le 2R} \frac{x\varphi(x) \cdot \nabla\varphi(x)}{|x|^{2\beta+2}} dx - \left(\beta^2 - \mu\right) \int_{|x| \ge R} \frac{1}{|x|^{2\beta+2}} dx \\
&= \int_{R \le |x| \le 2R} \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}} dx + \left(\beta^2 - \mu\right) \int_{R \le |x| \le 2R} \frac{|\varphi(x)|^2 - 1}{|x|^{2\beta+2}} dx \\
&\quad + 2\beta \int_{R \le |x| \le 2R} \frac{x\varphi(x) \cdot \nabla\varphi(x)}{|x|^{2\beta+2}} dx - \left(\beta^2 - \mu\right) \int_{|x| \ge 2R} \frac{1}{|x|^{2\beta+2}} dx \\
&\le \int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}} dx - \left(\beta^2 - \mu\right) \int_{|x| \ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta \int_{R \le |x| \le 2R} \frac{|\varphi(x)| \cdot |\nabla\varphi(x)|}{|x|^{2\beta+1}} dx \\
&\le \int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}} dx - \left(\beta^2 - \mu\right) \int_{|x| \ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R \le |x| \le 2R} \frac{1}{|x|^{2\beta+1}} dx.
\end{aligned}$$

Let $r = |x|$ and make an N-dimensional spherical coordinate transformation, then

$$\begin{aligned}
& -\left(\beta^2 - \mu\right) \int_{|x| \ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R \le |x| \le 2R} \frac{1}{|x|^{2\beta+1}} dx \\
= \; & -\left(\beta^2 - \mu\right) S_{N-1} \int_{2R}^{+\infty} \frac{1}{r^{2\beta+3-N}} dr + 2\beta M S_{N-1} \int_R^{2R} \frac{1}{r^{2\beta+2-N}} dr,
\end{aligned}$$

where S_{N-1} denotes the N-dimensional unit spherical surface area. Since $2\beta + 3 - N = 1 + 2\sqrt{\overline{\mu} - \mu} > 1$, we have

$$\int_{2R}^{+\infty} \frac{1}{r^{2\beta+3-N}} dr = \frac{(2R)^{N-2-2\beta}}{2\beta + 2 - N}.$$

If $2\beta + 2 - N = 1$, then $\int_R^{2R} \frac{1}{r^{2\beta+2-N}} dr = \ln 2$ and $\lim_{R \to 0^+} \frac{(2R)^{N-2-2\beta}}{2\beta+2-N} = +\infty$. Thus, there exists $R_1 > 0$, such that $-\left(\beta^2 - \mu\right) \int_{|x| \ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R \le |x| \le 2R} \frac{1}{|x|^{2\beta+1}} dx < 0$, for $R \le R_1$.

If $2\beta + 2 - N < 1$, then $\lim_{R \to 0^+} \int_R^{2R} \frac{1}{r^{2\beta+2-N}} dr = \lim_{R \to 0^+} \frac{R^{N-1-2\beta} - (2R)^{N-1-2\beta}}{2\beta+1-N} = 0$, thus there exists a positive constant $R_2 > 0$, such that $-\left(\beta^2 - \mu\right) \int_{|x| \ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R \le |x| \le 2R} \frac{1}{|x|^{2\beta+1}} dx < 0$, for $R \le R_2$.

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If $2\beta+2-N>1$, then $\lim\limits_{R\to 0^+}\int_R^{2R}\frac{1}{r^{2\beta+2-N}}dr=\lim\limits_{R\to 0^+}\frac{R^{N-1-2\beta}-(2R)^{N-1-2\beta}}{2\beta+1-N}=+\infty$, and $N-2-2\beta<N-1-2\beta<0$ implies that there exists some positive constant $R_3>0$, such that $-\left(\beta^2-\mu\right)\int_{2R}^{+\infty}\frac{1}{r^{2\beta+3-N}}dr+2\beta M\int_R^{2R}\frac{1}{r^{2\beta+2-N}}dr<0$, for $R\le R_3$.

As mentioned above, when $R\le R_0=\min\{R_1,R_2,R_3\}$,

$$\lim_{\varepsilon\to 0+} D<\int_\Omega\frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}}dx.$$

Furthermore, by (14) and (15),

$$\begin{aligned}\int_\Omega\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx&=\int_{|x|\le R}\frac{|U_\varepsilon|^{2^*(s)}}{|x|^s}dx+\int_{R\le|x|\le 2R}\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx\\&=\int_{\mathbb{R}^N}\frac{|U_\varepsilon|^{2^*(s)}}{|x|^s}dx+E\\&=C_\varepsilon^{-2^*(s)}\int_{\mathbb{R}^N}\frac{|y_\varepsilon|^{2^*(s)}}{|x|^s}dx+E\\&=C_\varepsilon^{-2^*(s)}A_{\mu,s}^{\frac{N-s}{2-s}}+E.\end{aligned}$$

Definition 1. *([27]). Let E be a Banach space. Given $c\in\mathbb{R}$, we will say that $I\in C^1(E,\mathbb{R})$ satisfies the $(PS)_c$ condition if any sequence $\{u_n\}\subset E$ such that $I(u_n)\to c$ and $I'(u_n)\to 0$ as $n\to\infty$ possesses a convergent subsequence.*

Lemma 6. *Assume that $a(x)$ is nonnegative and continuous in Ω^0, there exists a neighborhood of the origin $U(0)\subset\Omega$ and $0\le q<2$ such that $a(x)=O(|x|^{-q})$ for $x\in U(0)$, then for any $2<p<2^*(1-\frac{q}{N})$ and $\frac{pN}{N-q}<\gamma<2^*$, $a(x)\in L_{\frac{\gamma}{\gamma-p}}(\Omega)$.*

Proof. Without loss of generality, we only need to prove $a(x)\in L_{\frac{\gamma}{\gamma-p}}(U(0))$, notice that $a(x)=O(|x|^{-q})$ for $x\in U(0)$, thus there exists a positive constant C_1 such that □

$$a(x)\le C_1|x|^{-q}\text{ for }x\in U(0),$$

therefore

$$\int_{U(0)}a(x)^{\frac{\gamma}{\gamma-p}}dx\le C_2\int_0^\delta\frac{dr}{r^{\frac{\gamma q}{\gamma-p}-N+1}},$$

where

$$C_2=C_1^{\frac{\gamma}{\gamma-p}}\omega_N,$$

notice that $\gamma>\frac{pN}{N-q}$ implies that

$$\frac{\gamma q}{\gamma-p}<N,$$

thus

$$\int_{U(0)}a(x)^{\frac{\gamma}{\gamma-p}}dx<+\infty.$$

This completes the proof.

3. The Existence of Positive Solutions for (1)

In this section, we give the proof of Theorem 1.1. First, we have

Lemma 7. *Suppose (g_1'), (g_2') and $\lambda<\lambda_1(\mu)$ hold, then for any*

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$$c \in \left(0, \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}\right),$$

J defined as in (12) satisfies the $(PS)_c$ condition.

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be any sequence such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$, first we prove that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Arguing by contradiction, without loss of generality, suppose that $\|u_n\| \to \infty$. From $J(u_n) \to c$, $J'(u_n) \to 0$ and (g_2'), also notice that for any $(x,t) \in \Omega^0 \times \mathbb{R}$, $\rho G(x,t) \le g(x,t)t + \nu t^2$, we have

$$\begin{aligned} c + 1 + o(1)\|u_n\| &\ge J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \int_\Omega \left(\frac{1}{\theta} g(x,u_n)u_n - G(x,u_n)\right)dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{2^*(s)}\right)\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \frac{\nu}{\rho}\int_\Omega u_n^2 dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\theta}\right)\frac{\lambda}{\lambda_1(\mu)}\|u_n\|^2 \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(1 - \frac{\lambda}{\lambda_1(\mu)}\right)\|u_n\|^2. \end{aligned}$$

Which is a contradiction. Hence $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega)$ and there exists u such that $u_n \rightharpoonup u (n \to \infty)$, up to a subsequence. Hereafter, without loss of generality, we say $u_n \rightharpoonup u (n \to \infty)$ or $u_n \to u (n \to \infty)$, it means maybe one of the subsequences $\{u_{n_k}\}$ of $\{u_n\}$ satisfies $u_{n_k} \rightharpoonup u (k \to \infty)$ or $u_{n_k} \to u (k \to \infty)$. Furthermore, by the weak continuity of J', $J'(u) = 0$. From $u_n \in H_0^1(\Omega)$, $u_n \rightharpoonup u (n \to \infty)$, by the embedding theorem, we have $u_n \to u (n \to \infty)$ in $L^\gamma(\Omega)$, for any $1 < \gamma < 2^*$. Let $g_1(x,t) = g(x,t)t$, from (g_1'), we have $|g_1(x,t)| \le \lambda t^2 + a(x)|t|^p$, by Hŏlder inequality, when $\gamma > p$, we have

$$\int_\Omega a(x)|\varphi(x)|^p dx \le \left(\int_\Omega a^{\frac{\gamma}{\gamma-p}}(x)dx\right)^{(\gamma-p)/\gamma} \left(\int_\Omega |\varphi(x)|^\gamma dx\right)^{p/\gamma}, \tag{21}$$

if we choose $\frac{pN}{N-q} < \gamma < 2^*$, then by Lemma 6, we have $a(x) \in L_{\frac{\gamma}{\gamma-p}}(\Omega)$, which implies that $g_1 : L^\gamma(\Omega) \to L^1(\Omega)$ is a continuous and bounded operator. Therefore, we have

$$\lim_{n\to\infty}\int_\Omega (g_1(x,u_n) - g_1(x,u))\,dx = 0,$$

that is,

$$\lim_{n\to\infty}\int_\Omega g(x,u_n)u_n dx = \int_\Omega g(x,u)\,u dx. \tag{22}$$

Similarly, we have

$$\lim_{n\to\infty}\int_\Omega G(x,u_n)\,dx = \int_\Omega G(x,u)\,dx. \tag{23}$$

Thus

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega G(x,u)\,dx = c + o(1) \tag{24}$$

and

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega g(x,u)\,u dx = o(1), \tag{25}$$

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which implies that

$$\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx = \frac{2(N-s)}{2-s}\left[\int_\Omega G(x,u)\,dx - \frac{1}{2}\int_\Omega g(x,u)\,udx + c\right] + o(1), \tag{26}$$

therefore

$$\|u_n\|^2 = \frac{2(N-s)}{2-s}\left[\int_\Omega G(x,u)\,dx - \frac{1}{2}\int_\Omega g(x,u)\,udx + c\right] + \int_\Omega g(x,u)\,udx + o(1). \tag{27}$$

We claim that

$$\|u\|^2 = \frac{2(N-s)}{2-s}\left[\int_\Omega G(x,u)\,dx - \frac{1}{2}\int_\Omega g(x,u)\,udx + c\right] + \int_\Omega g(x,u)\,udx, \tag{28}$$

in fact, notice that $J(u) = c$ and $J'(u) = 0$, repeat the above derivation, (28) obviously holds true. By (27),

$$\|u_n\| \to \|u\|\,,\ n \to \infty. \tag{29}$$

By (29) and $u_n \rightharpoonup u(n \to \infty)$, we have

$$u_n \to u \text{ in } H_0^1(\Omega)\,,\ n \to \infty.$$

□

Lemma 8. *Suppose* (g_1'), (g_2'), $\lambda < \lambda_1(\mu)$, *and*

$$c \in \left(0, \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right),$$

then u *defined in the proof of Lemma 7 is a positive solution of (11).*

Proof. From the definition of u, also notice the relationship between the functional J and problem (11), we can easily know that u is a solution of (11).

If $u \equiv 0$ in Ω, from $\langle J'(u_n), u_n\rangle = o(1)$ and (22), we have

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx = o(1)\,. \tag{30}$$

By the definition of $A_{\mu,s}$,

$$\|u_n\|^2 \geq A_{\mu,s}\left(\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}. \tag{31}$$

From (30) and (31), we have

$$o(1) \geq \|u_n\|^2\left(1 - A_{\mu,s}^{-\frac{2^*(s)}{2}}\|u_n\|^{2^*(s)-2}\right).$$

If $\|u_n\| \to 0$, then $J(u_n) \to 0$, which contradicts $c > 0$. Therefore,

$$\|u_n\|^2 \geq A_{\mu,s}^{\frac{N-s}{2-s}} + o(1)\,. \tag{32}$$

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By (g_2'), (30) and (32), we have

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx + o(1) \\ &= \frac{2-s}{2(N-s)}\|u_n\|^2 + o(1) \\ &\geq \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}} + o(1), \end{aligned}$$

which contradicts $c < \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}$. Thus, $u \not\equiv 0$.

Notice that for any $v \in H_0^1(\Omega)$,

$$\langle J'(u), v\rangle = (u,v) - \int_\Omega \frac{(u^+)^{2^*(s)-1}}{|x|^s}vdx - \int_\Omega g(x,u)\,vdx = 0.$$

Therefore

$$(u,u^-) - \int_\Omega \frac{(u^+)^{2^*(s)-1}}{|x|^s}u^-dx - \int_\Omega g(x,u)\,u^-dx = 0$$

and

$$(u,u^+) - \int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega g(x,u)\,u^+dx = 0,$$

where $u^- = \max\{-u,0\}$.

According to the definition of g, u^+ and u^-, we have

$$\int_\Omega g(x,u)\,u^+dx = \int_\Omega g(x,u)\,udx,\ \int_\Omega \frac{(u^+)^{2^*(s)-1}}{|x|^s}u^-dx = 0,\ \int_\Omega g(x,u)\,u^-dx = 0,$$

so

$$||u^-||^2 = 0,$$

that is $u \geq 0$. Moreover by the strong maximum principle, $u > 0$. Which completes the proof. □

Lemma 9. *Assume that (g_1'), (g_2') and $\lambda < \lambda_1(\mu)$ hold, then the functional J admits a $(PS)_c$ sequence at level*

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1],\, H_0^1(\Omega)\right);\ \gamma(0) = 0,\ J(\gamma(1)) < 0\right\}.$$

Proof. By (g_1') and (21), for $\frac{pN}{N-q} < \gamma < 2^*$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega G(x,u)\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|u\|^{2^*(s)} - \frac{1}{p}||a(x)||_{\frac{\gamma}{\gamma-p}}\|u\|_\gamma^p - \frac{\lambda}{2}\|u\|_2^2 \\ &\geq \frac{1-\lambda/\lambda_1(\mu)}{2}\|u\|^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|u\|^{2^*(s)} - \frac{1}{p}||a(x)||_{\frac{\gamma}{\gamma-p}}\|u\|_\gamma^p, \end{aligned}$$

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here $||\cdot||_r$ represents the usual norm of space $L^r(\Omega)$. Notice that $\gamma < 2^*$, then by Sobolev Embedding Theorem, there exists a positive constant C_3 such that

$$\|u\|_\gamma^p \le C_3 \|u\|^p,$$

thus

$$J(u) \ge \frac{1-\lambda/\lambda_1(\mu)}{2}\|u\|^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|u\|^{2^*(s)} - C_4\|u\|^p, \tag{33}$$

where

$$C_4 = \frac{C_3}{p}||a(x)||_{\frac{\gamma}{\gamma-p}}.$$

Notice that $\lambda < \lambda_1(\mu)$ implies that $\frac{1-\lambda/\lambda_1(\mu)}{2} > 0$, and $p > 2$, thus there exist some positive constants α, r, such that

$$J(u) \ge \alpha > 0,\ \forall u \in \left\{u \in H_0^1(\Omega) \mid \|u\| = r\right\}.$$

Furthermore, from the nonnegativity of $G(x,u)$, we have

$$\begin{aligned} J(tv_\varepsilon) &= \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \int_\Omega G(x, tv_\varepsilon)\,dx \\ &\le \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)}, \end{aligned}$$

as $t \to +\infty$, $\lim\limits_{t\to+\infty} J(tv_\varepsilon) = -\infty$, thus there exists $t_0 > 0$, such that $\|t_0 v_\varepsilon\| > r$ and $J(t_0 v_\varepsilon) < 0$. By the Mountain Pass Lemma with $(PS)_c$ conditions, we infer that J admits a $(PS)_c$ sequence at the level c, that is, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$J(u_n) \to c \ge \alpha \text{ and } J'(u_n) \to 0.$$

This completes the proof. □

Lemma 10. *Suppose* $0 \le s < 2$, (g_1') *and* (g_2') *hold. If one of the following conditions*

(i) $0 \le \mu \le \overline{\mu} - 1, 0 < \lambda < \lambda_1(\mu)$.

(ii) $\overline{\mu} - 1 < \mu < \overline{\mu},\ \lambda_*(\mu) < \lambda < \lambda_1(\mu)$.

is true, then

$$0 < c < \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}.$$

Proof. Consider the functions

$$g(t) := J(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \int_\Omega G(x, tv_\varepsilon)\,dx,\ t \ge 0$$

and

$$\bar{g}(t) := \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)},\ t \ge 0,$$

where v_ε is defined as in (16). We have $g(0) = 0$, $\lim\limits_{t\to+\infty} g(t) = -\infty$.

Notice that (g_1') and (g_2') implies (33), therefore

$$J(tv_\varepsilon) \ge \frac{1-\lambda/\lambda_1(\mu)}{2}\|v_\varepsilon\|^2 t^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|v_\varepsilon\|^{2^*(s)} t^{2^*(s)} - C_4\|v_\varepsilon\|^p t^p, \tag{34}$$

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$\lambda < \lambda_1(\mu)$, $2^*(s) > 2$, $p > 2$ shows that $g(t) > 0$ when t is small enough, thus there exists some $t_\varepsilon > 0$ such that $g(t_\varepsilon) = \sup_{t\geq 0} g(t) > 0$, thus $c > 0$ and $g'(t_\varepsilon) = 0$, that is

$$g'(t_\varepsilon) = t_\varepsilon \|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-1} - \int_\Omega g(x, t_\varepsilon v_\varepsilon) v_\varepsilon dx = 0, \tag{35}$$

which implies that

$$\|v_\varepsilon\|^2 = t_\varepsilon^{2^*(s)-2} + \frac{1}{t_\varepsilon}\int_\Omega g(x, t_\varepsilon v_\varepsilon) v_\varepsilon dx \geq t_\varepsilon^{2^*(s)-2}. \tag{36}$$

Therefore, if we set $\bar{t}_\varepsilon := \|v_\varepsilon\|^{\frac{2}{2^*(s)-2}}$, then

$$t_\varepsilon \leq \bar{t}_\varepsilon. \tag{37}$$

Obviously, the function $\bar{g}(t)$ reaches its maximum at $\bar{t}_\varepsilon = \|v_\varepsilon\|^{\frac{2}{2^*(s)-2}}$ and is increasing in the interval $[0, \bar{t}_\varepsilon]$, then from (17), (18), (37) and (g_1'), we have

$$\begin{aligned} g(t_\varepsilon) = \bar{g}(t_\varepsilon) - \int_\Omega G(x, t_\varepsilon v_\varepsilon) dx &\leq \bar{g}(\bar{t}_\varepsilon) - \int_\Omega G(x, t_\varepsilon v_\varepsilon) dx \\ &\leq \frac{2-s}{2(N-s)} \|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \int_\Omega G(x, t_\varepsilon v_\varepsilon) dx \\ &\leq \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) - \frac{\lambda}{2} t_\varepsilon^2 ||v_\varepsilon||_2^2. \end{aligned}$$

Here we use the fact

$$\lim_{x\to 0} \frac{(1+x)^\vartheta - 1}{\vartheta x} = 1.$$

Under the case (i), by (g_1'), we have

$$g(x, t_\varepsilon v_\varepsilon) v_\varepsilon \leq \lambda v_\varepsilon^2 t_\varepsilon + a(x) v_\varepsilon^p t_\varepsilon^{p-1}.$$

Thus, for $\frac{pN}{N-q} < \gamma < 2^*$, by (36), we have

$$\left| \|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-2} \right| \leq \lambda \|v_\varepsilon\|_2^2 + \bar{t}_\varepsilon^{p-2} ||a||_{\frac{\gamma}{\gamma-p}} \|v_\varepsilon\|_\gamma^p. \tag{38}$$

By Lemma 3, (38) implies that

$$\lim_{\varepsilon\to 0^+} t_\varepsilon = A_{\mu,s}^{\frac{1}{2^*(s)-2}}. \tag{39}$$

From Lemma 3, when $\mu < \overline{\mu} - 1$,

$$\|v_\varepsilon\|_2^2 = O(\varepsilon^{\frac{N-2}{(2-s)\sqrt{\overline{\mu}-\mu}}}), \tag{40}$$

when $\mu = \overline{\mu} - 1$,

$$\|v_\varepsilon\|_2^2 = O\left(\varepsilon^{\frac{N-2}{2-s}} |ln\varepsilon|\right), \tag{41}$$

Thus for $\lambda > 0$,

$$g(t_\varepsilon) \leq \begin{cases} \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) - O(\varepsilon^{\frac{N-2}{(2-s)\sqrt{\overline{\mu}-\mu}}}) \text{ for } \mu < \overline{\mu} - 1, \\ \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) - O\left(\varepsilon^{\frac{N-2}{2-s}} |ln\varepsilon|\right) \text{ for } \mu = \overline{\mu} - 1. \end{cases}$$

The above inequalities shows that if we choose ε small enough, then

$$c \leq g(t_\varepsilon) < \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}.$$

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Under the case (ii), from (38) and (g_1'), we can get

$$g\left(t_{\varepsilon}\right) \leq \frac{2-s}{2\left(N-s\right)}\left\|v_{\varepsilon}\right\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda}{2}t_{\varepsilon}^{2}||v_{\varepsilon}||_{2}^{2}. \tag{42}$$

By (39) and (42), we have

$$g\left(t_{\varepsilon}\right) \leq \frac{2-s}{2\left(N-s\right)}\left\|v_{\varepsilon}\right\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda}{2}A_{\mu,s}^{\frac{2}{2^{*}(s)-2}}||v_{\varepsilon}||_{2}^{2} + o\left(\varepsilon^{\frac{N-2}{2-s}}\right).$$

Notice that $\lambda > \lambda_{*}\left(\mu\right)$, then by Lemma 3, we can choose ε small enough such that

$$g\left(t_{\varepsilon}\right) < \frac{2-s}{2\left(N-s\right)}\left\|v_{\varepsilon}\right\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda_{*}\left(\mu\right)}{2}A_{\mu,s}^{\frac{2}{2^{*}(s)-2}}||v_{\varepsilon}||_{2}^{2}.$$

We claim that when ε is small enough, then

$$\frac{2-s}{2\left(N-s\right)}\left\|v_{\varepsilon}\right\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda_{*}\left(\mu\right)}{2}A_{\mu,s}^{\frac{2}{2^{*}(s)-2}}||v_{\varepsilon}||_{2}^{2} \leq \frac{2-s}{2\left(N-s\right)}A_{\mu,s}^{\frac{N-s}{2-s}}.$$

In fact, by Lemma 5, we have

$$\begin{aligned}
&\lim_{\varepsilon\to0^{+}}\frac{\left\|v_{\varepsilon}\right\|^{\frac{2(N-s)}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\int_{\Omega}|v_{\varepsilon}|^{2}dx} \\
=\ &\lim_{\varepsilon\to0^{+}}\frac{\left\|u_{\varepsilon}\right\|^{\frac{2(N-s)}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}\left(\int_{\Omega}\frac{|u_{\varepsilon}|^{2^{*}(s)}}{|x|^{s}}dx\right)^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\left(\int_{\Omega}\frac{|u_{\varepsilon}|^{2^{*}(s)}}{|x|^{s}}dx\right)^{\frac{N-2}{2-s}\frac{2}{2^{*}(s)}}\int_{\Omega}|u_{\varepsilon}|^{2}dx} \\
=\ &\lim_{\varepsilon\to0^{+}}\frac{\left(D+C_{\varepsilon}^{-2}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-s}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}\left(E+C_{\varepsilon}^{-2^{*}(s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\left(E+C_{\varepsilon}^{-2^{*}(s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}\frac{2}{2^{*}(s)}}\int_{\Omega}|u_{\varepsilon}|^{2}dx} \\
=\ &\lim_{\varepsilon\to0^{+}}\frac{\left(C_{\varepsilon}^{2}D+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-s}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}\left(EC_{\varepsilon}^{2^{*}(s)}+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}}{C_{\varepsilon}^{2}A_{\mu,s}^{\frac{N-2}{2-s}}\left(EC_{\varepsilon}^{2^{*}(s)}+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}\frac{2}{2^{*}(s)}}\int_{\Omega}|u_{\varepsilon}|^{2}dx} \qquad (43) \\
=\ &\lim_{\xi\to0^{+}}\frac{\left(C^{2}D\xi+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-s}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}\left(EC^{2^{*}(s)}\xi^{\frac{N-s}{N-2}}+A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}}{C^{2}\xi A_{\mu,s}^{\frac{(N-s)(N-2)}{(2-s)^{2}}}\int_{\Omega}\frac{|\varphi(x)|^{2}}{|x|^{2\beta}}dx} \\
=\ &\frac{N-s}{2-s}\lim_{\xi\to0^{+}}\frac{Q_{1}\left(D+\xi\frac{\partial D}{\partial\xi}\right) - A_{\mu,s}^{\frac{N-s}{2-s}}Q_{2}C^{2^{*}(s)-2}\left(E\xi^{\frac{2-s}{N-2}}+\frac{N-2}{N-s}\frac{\partial E}{\partial\xi}\xi^{\frac{N-s}{N-2}}\right)}{A_{\mu,s}^{\frac{(N-s)(N-2)}{(2-s)^{2}}}\int_{\Omega}\frac{|\varphi(x)|^{2}}{|x|^{2\beta}}dx} \\
=\ &\frac{N-s}{2-s}\frac{\lim_{\xi\to0^{+}}D}{\int_{\Omega}\frac{|\varphi(x)|^{2}}{|x|^{2\beta}}dx},
\end{aligned}$$

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where $C = \left(\frac{2(\overline{\mu}-\mu)(N-s)}{\sqrt{\overline{\mu}}}\right)^{\frac{\sqrt{\overline{\mu}}}{2-s}}$, $Q_1 = \left(C^2 D\xi + A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}$, $Q_2 = \left(EC^{2^*(s)}\xi^{\frac{N-s}{N-2}} + A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-4+s}{2-s}}$.

By (19) and (43), we have

$$\lim_{\varepsilon\to 0^+}\frac{\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\int_\Omega |v_\varepsilon|^2\,dx} < \frac{N-s}{2-s}\frac{\int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}}dx}{\int_\Omega \frac{|\varphi(x)|^2}{|x|^{2\beta}}dx},$$

so, when we choose ε small enough, $c \le g(t_\varepsilon) < \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}$.
□

Proof of Theorem 1.1. By Lemmas 1, 2 and 3, the conclusion is obvious.

4. The Nonexistence of Positive Solution for (1)

In this section, we consider the nonexistence of solution for (1). To this end, we assume $\overline{\mu}-1 < \mu < \overline{\mu}$, $g(x,t) = g(|x|,t)$ for $(x,t) \in \overline{\Omega}\times\mathbb{R}$, $g(r,t)$ is decreasing in r, $\Omega = B(0,R)$, $\lambda \le \lambda_*(\mu)$, and (g_4') hold, that is, all the conditions in Theorem 1.2 hold true.

The following lemma could be found in [28].

Lemma 11. *Let $J_\tau(z)$ be the Bessel function*

$$J_\tau(z) = \sum_{i=0}^{\infty}\frac{(-1)^i (z/2)^{\tau+2i}}{i!\Gamma(i+\tau+1)}.$$

Then
(a) $z^2 J_\tau''(z) + zJ_\tau'(z) + (z^2-\tau^2)J_\tau(z) = 0$;
(b) $\forall\tau > -1, \exists\, z_\tau > 0$ *such that* $J_\tau(z) > 0$ *for* $z \in (0, z_\tau)$ *and* $J_\tau(z_\tau) = 0$;
(c) if $-1 < \tau' < \tau''$, *then* $0 < z_{\tau'} < z_{\tau''}$;
(d) $J_\tau'(z) = \frac{\tau}{z}J_\tau(z) - J_{\tau+1}(z)$;
(e) $J_{\tau+1}(z) = \frac{2\tau}{z}J_\tau(z) - J_{\tau-1}(z)$.

The proof of Theorem 1.2. From [28], we can easily see that under the condition (g_3'), any solution of (1) must be spherically symmetric. The radial equation for (1) is

$$u'' + \frac{N-1}{r}u' + \mu\frac{u}{r^2} + \frac{|u|^{2*(s)-2}}{r^s}u + g(r,u) = 0. \tag{44}$$

If (1) has a positive solution u, then $u(R) = u'(0) = 0$ and

$$u'' + \frac{N-1}{r}u' + \mu\frac{u}{r^2} + \frac{u^{2*(s)-1}}{r^s} + g(r,u) = 0. \tag{45}$$

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Let $\psi(r)$ and $w(r)$ be two smooth functions such that $\psi(0) = 0$, $\psi'(0) > 0$, $\psi''(0) = 0$. Multiply both sides of equation (45) by $r^{N-1}u'(r)\psi(r)$ and $-r^{N-1}u(r)w(r)$ respectively, then sum them and integrate them on $[0,1]$ (without loss of generality, assume $R = 1$), we have

$$\begin{aligned}
& \int_0^1 r^{N-1}(u')^2\left[\frac{\psi'}{2} - \frac{(N-1)\psi}{2r} - w\right]dr \\
& +\frac{1}{2}\int_0^1 r^{N-1}u^2\left[w'' - \frac{(N-1)w'}{r} + \frac{\mu}{r^2}\left(2w + \psi' + \frac{(N-3)\psi}{r}\right)\right]dr \\
& +\frac{\lambda}{2}\int_0^1 r^{N-1}u^2\left[2w + \psi' + \frac{(N-1)\psi}{r}\right]dr \\
& +\frac{1}{2^*(s)}\int_0^1 r^{N-s-1}u^{2^*(s)}\left[2^*(s)w + \psi' + \frac{(N-s-1)\psi}{r}\right]dr \\
& -\int_0^1 r^{N-1}\left[g(r,u) - \lambda u\right]u'\psi dr + \int_0^1 r^{N-1}\left[g(r,u) - \lambda u\right]uwdr \\
= & \ \frac{1}{2}\psi(1)\left[u'(1)\right]^2.
\end{aligned} \tag{46}$$

Choosing

$$w = \frac{\psi'}{2} - \frac{(N-1)\psi}{2r},$$
$$\psi(r) = \varphi(\sqrt{\lambda}r),$$

where the function $\varphi(r)$ is the solution of the following Cauchy problem

$$\begin{cases} \varphi''' + [(N-1)(3-N) + 4\mu]\left(\frac{\varphi'}{r^2} - \frac{\varphi}{r^3}\right) + 4\varphi' = 0; \\ \varphi(0) = 0; \varphi'(0) = 1; \varphi''(0) = 0. \end{cases}$$

From the works in [1], we know

$$\varphi(r) = rJ_{-\zeta}(r)J_{\zeta}(r),\ \zeta = \sqrt{\bar{\mu} - \mu}, \tag{47}$$

and $\psi(r)$ satisfies

$$\psi''' + [(N-1)(3-N) + 4\mu]\left(\frac{\psi'}{r^2} - \frac{\psi}{r^3}\right) + 4\lambda\psi' = 0.$$

Which implies that the Pohozaev-type identity (46) can be simplified as

$$\begin{aligned}
& \frac{2N-s-2}{2(N-s)}\int_0^1 r^{N-s-1}u^{2^*(s)}\left[\psi' - \frac{\psi}{r}\right]dr \\
& +\int_0^1 r^{N-1}\left[G(r,u) - \frac{\lambda u^2}{2}\right]\left[\psi' + \frac{(N-1)\psi}{r}\right]dr \\
& +\frac{1}{2}\int_0^1 r^{N-1}\left[g(r,u) - \lambda u\right]u\left[\psi' - \frac{(N-1)\psi}{r}\right]dr \\
= & \ \frac{1}{2}\psi(1)\left[u'(1)\right]^2.
\end{aligned}$$

By Lemma 11, if $\lambda \leq \lambda_*(\mu) = z_{-\zeta}^2$, then

$$\psi(r) = \varphi\left(\sqrt{\lambda}r\right) \geq 0, \text{ for } 0 \leq r \leq 1 \tag{48}$$

and
$$\psi'(r) - \frac{\psi}{r} < 0 \text{ on } [0,1). \tag{49}$$
Notice that
$$\begin{aligned} & \left[G(r,u) - \frac{\lambda u^2}{2}\right]\left[\psi' + \frac{(N-1)\psi}{r}\right] \\ & + \frac{1}{2}\left[g(r,u) - \lambda u\right] u \left[\psi' - \frac{(N-1)\psi}{r}\right] \\ = \; & (H_1 + H_2)\psi' + (H_1 - H_2)(N-1)\frac{\psi}{r} \\ = \; & (H_1 + H_2)\sqrt{\lambda}\varphi'(\sqrt{\lambda}r) + (H_1 - H_2)(N-1)\sqrt{\lambda}J_{-\zeta}(\sqrt{\lambda}r)J_\zeta(\sqrt{\lambda}r), \end{aligned}$$
where
$$H_1 = \left[G(r,u) - \frac{\lambda u^2}{2}\right] \text{ and } H_2 = \frac{1}{2}\left[g(r,u) - \lambda u\right] u.$$
From (47) and Lemma 11, for any $y \in (0, z_{-\zeta})$, we have
$$\begin{aligned} \varphi'(y) &= J_{-\zeta}(y)J_\zeta(y) + yJ'_{-\zeta}(y)J_\zeta(y) + yJ_{-\zeta}(y)J'_\zeta(y) \\ &= J_{-\zeta}(y)J_\zeta(y) - yJ_{1-\zeta}(y)J_\zeta(y) - yJ_{-\zeta}(y)J_{\zeta+1}(y) \\ &\le J_{-\zeta}(y)J_\zeta(y), \end{aligned}$$
thus
$$\begin{aligned} & (H_1 + H_2)\sqrt{\lambda}\varphi'(\sqrt{\lambda}r) + (H_1 - H_2)(N-1)\sqrt{\lambda}J_{-\zeta}(\sqrt{\lambda}r)J_\zeta(\sqrt{\lambda}r) \\ \le \; & \left[H_1 + H_2 + (N-1)(H_1 - H_2)\right]\sqrt{\lambda}J_{-\zeta}(\sqrt{\lambda}r)J_\zeta(\sqrt{\lambda}r). \end{aligned} \tag{50}$$
From (g_4'), (46) and (50), we have
$$\frac{1}{2}\psi(1)\left[u'(1)\right]^2 < 0.$$
Which contradicts (48). This completes the proof.

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