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Weighing the Pros and Cons of Newton's Sum in Quadratic Root Function Computations Analysis

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Article

Weighing the Pros and Cons of Newton's Sum in Quadratic Root Function Computations Analysis

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Abstract: This study serves as an indispensable guide, skillfully navigating readers through the nuanced landscape of Newton's Sum method, revealing both its commendable advantages and inherent drawbacks. In the realm of benefits, Newton's Sum stands out by liberating students from the need to memorize various formulas for quadratic root function computations. However, as the narrative unfolds, we draw parallels between Newton's Sum and the Pascal triangle method, both historical calculation methods reliant on prior results for subsequent calculations. This shared historical foundation reveals a limitation, particularly when addressing higher-order quadratic root functions involving exponents like α^25 + β^25. Furthermore, the Newton Sum method proves not suitable for asymmetric quadratic root function computations, adding an extra layer of complexity to its application. "This study' goes beyond traditional approaches, offering practical guidance and fostering a profound understanding of when and where to effectively apply this method. Dive into the complexities of quadratic roots, navigate challenges with confidence, and uncover the key to unlocking the full potential of Newton's Sum in your mathematical exploration." Looking towards the future, as we acknowledge the evolutionary leap from the Pascal triangle method to the binomial theorem for expanding algebraic terms, a compelling need arises to develop a new method that surmounts the limitations of Newton's Sum. This call for innovation becomes the guiding compass, propelling readers toward future endeavors, inspiring the exploration and creation of methods that elevate the understanding and application of quadratic roots to unprecedented heights.

Keywords: Newton's Sum; Quadratic Root Function Computations

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1. Introduction

1.1. Algebra Formula: Symmetric Function Of A Quadratic's Roots

i) i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Prove: $\alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta$

ii) $\alpha^3 + \beta^3 = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$

Prove:

$\alpha^3 + \beta^3 = (\alpha + \beta)[\alpha^2 + \beta^2 - \alpha\beta]$

$= (\alpha + \beta)[\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta - \alpha\beta]$

$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$

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Or $\boldsymbol{\alpha^3+\beta^3=(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}$

iii) $\alpha^4 + \beta^4 = [(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - 2(\alpha\beta)^2$

Prove: $\alpha^4 + \beta^4 = \alpha^4 + \beta^4 + 2\alpha^2\beta^2 - 2\alpha^2\beta^2$

$= (\boldsymbol{\alpha^2 + \beta^2})^2 - 2\alpha^2\beta^2$

$=[(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$ [From ii]

iv) $\alpha^5 + \beta^5$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\{(\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]\} - (\alpha\beta)^2(\alpha+\beta)$

Prove:

$\alpha^5 + \beta^5$

$= (\alpha^2+\beta^2)(\alpha^3+\beta^3) - \alpha^2\beta^3 - \beta^2\alpha^3$

$= (\alpha^2+\beta^2)(\alpha^3+\beta^3) - \alpha^2\beta^2(\beta+\alpha)$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\{(\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]\} - (\alpha\beta)^2(\alpha+\beta)$

[From ii and iii]

v) $\alpha^6 + \beta^6 = \{(\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]\}^2 - 2(\alpha\beta)^3$

Prove:

$\alpha^6 + \beta^6$

$= \boldsymbol{\alpha^6 + \beta^6 + 2\alpha^3\beta^3} - 2\alpha^3\beta^3$

$= (\boldsymbol{\alpha^3 + \beta^3})^2 - 2\alpha^3\beta^3$

$= \{(\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]\}^2 - 2(\alpha\beta)^3$ [From iii]

vi) $\alpha^7 + \beta^7 = [\boldsymbol{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}]\{[(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - [2(\alpha\beta)^2]\} - \alpha^3\beta^3(\beta+\alpha)$

Prove:

$\alpha^7 + \beta^7$

$= (\alpha^3+\beta^3)(\alpha^4+\beta^4) - \alpha^3\beta^4 - \beta^3\alpha^4$

$= [\boldsymbol{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}]\{[(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - [2(\alpha\beta)^2]\} - \alpha^3\beta^3(\beta+\alpha)$

From (ii) and (iii)

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vii) $\alpha^8 + \beta^8 = \{[(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - 2(\alpha\beta)^2\}^2 - 2(\alpha\beta)^4$

Prove:

$\alpha^8 + \beta^8$

$= \alpha^8 + \beta^8 + 2\alpha^4\beta^4 - 2\alpha^4\beta^4$

$= (\boldsymbol{\alpha}^4 + \boldsymbol{\beta}^4)^2 - 2\alpha^4\beta^4$

$=\{[(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - 2(\alpha\beta)^2\}^2 - 2(\alpha\beta)^4$ [From iii]

1.2. Example Quadratic Root function Calculation [Algebra Formula]

Example**1**[1]: If α, β are the zeroes of the polynomial $x^2 - 2x + 2$. find the value of i) $\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2$, ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$,iii) $\boldsymbol{\alpha}^3 + \boldsymbol{\beta}^3$, iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ v) $\boldsymbol{\alpha}^4 + \boldsymbol{\beta}^4$

Solution

Sum of the roots = $\alpha + \beta = -(-\frac{2}{1}) = 2$, Product of the roots = $\alpha\beta = \frac{c}{a} = 2$

$i)\ \boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 = (\alpha+\beta)^2 - 2\alpha\beta = \mathbf{2}^2 - 2\times2 = 4 - 4 = 0$

ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{0}{2} = 0$

iii) $\boldsymbol{\alpha}^3 + \boldsymbol{\beta}^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= (\mathbf{2})^3 - 3(\mathbf{2})(\mathbf{2})$ **from (i)**

$= 8 - 12$

$= -4$

iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

$= \frac{\alpha^3 + \beta^3}{\alpha\beta}$

$= \frac{-4}{2}$ from (iii)

$= -2$

v) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$

$= 0 - 2(\alpha\beta)^2$ from (i)

$= -2(\mathbf{2})^2$

$= -8$

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Example 2^2: If α and β are the roots of the equation $x^2 - 6x + 6=0$, what is $\alpha^3 + \beta^3 + \alpha^2 + \beta^2 + \alpha + \beta$ equal to?

Solution

Given quadratic equation is $x^2 - 6x + 6=0$, $\alpha+\beta = 6$, $\alpha\beta = 6$

We know

$\alpha^3+\beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$=(6)^3 - 3\times6\times6$

$= 216 - 108$

$= 108$

Also $\alpha^2+\beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$=6^2 - 2 \times 6$

$= 36 - 12$

$= 24$

Thus the value of $\alpha^3 + \beta^3 + \alpha^2 + \beta^2 + \alpha + \beta$

is 108+24+6 = 138.

1.3. Newton Sum (Newton's Identities)[3]

Newton's identities, also known as **Newton-Girard formulae**, is an efficient way to find the power sum of roots of polynomials without actually finding the roots. If $x_1, x_2, \ldots x_n$ are the roots of a polynomial equation, then Newton's identities are used to find the summations like

$$\sum_{i=1}^{n} x_i^k = x_1^k + x_2^k + \cdots + x_n^k.$$

It is mainly used in conjunction with **Vieta's formula** while working with the (complex) roots (say $a_1, a_2, \ldots.. a_k$) of a k^{th} degree polynomial. The main idea is that the **elementary symmetric polynomials** form an algebraic basis to produce all symmetric polynomials. Newton's identity gives us the calculation via a recurrence relation with known coefficients.

Newton's Identities for a Quadratic Polynomial

Suppose that you have a quadratic polynomial $P(x)$ with (complex) roots α_1 and α_2 . Now, you are asked to find the value of $\alpha_1{}^2 + \alpha_2{}^2$.

This seems very easy since you can use **Vieta's formula** along with the identity $(a+b)^2 = a^2 + b^2 + 2ab$ to find the required result. But what if you need to find ($\alpha_1{}^{10} + \alpha_2{}^{10}$) ? This would take a while if you were to simply use algebraic manipulations. But there's a clever way, using Newton's sums.

Let P(x) = ax^2 + bx + c . Then using Vieta`s formula,

we can get $\alpha_1 + \alpha_2 = -\frac{b}{a}$ and $\alpha_1 \alpha_2 = \frac{c}{a}$.

Denote P_i *as the* i^{th} *power sum of the roots, namely*

$$P_i = \alpha_1{}^i + \alpha_2{}^i \ .$$

Then we can obtain P_i *recursively as follows.*

$$P_0 = \alpha_1{}^0 + \alpha_2{}^0$$

$$= 2$$

$$P_1 = \alpha_1{}^1 + \alpha_2{}^1 = -\frac{b}{a}$$

$$P_2 = \alpha_1{}^2 + \alpha_2{}^2$$

$$= (\alpha_1 + \alpha_2)(\alpha_1{}^1 + \alpha_2{}^1) - 2\,\alpha_1\alpha_2$$

$$= -\frac{b}{a} P_1 - \frac{c}{a} P_0$$

$$P_3 = \alpha_1{}^3 + \alpha_2{}^3$$

$$= (\alpha_1 + \alpha_2)(\alpha_1{}^2 + \alpha_2{}^2) - \alpha_1\alpha_2(\alpha_1 + \alpha_2)$$

$$= -\frac{b}{a} P_2 - \frac{c}{a} P_1$$

$$\vdots$$

$$P_i = \alpha_1{}^i + \alpha_2{}^i$$

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$$= (\alpha_1 + \alpha_2)({\alpha_1}^{i-1} + {\alpha_2}^{i-1}) - \alpha_1\alpha_2\,({\alpha_1}^{i-2} + {\alpha_2}^{i-2})$$

$$= -\frac{b}{a}\,\mathrm{P}_{i-1} - \frac{c}{a}\,\mathrm{P}_{i-2}$$

This is a linear recurrence relation that gives us the i^{th} power sum. Note that solving this recurrence to get a closed-form solution is equivalent to finding the roots of the quadratic polynomial.

THEOREM

For a quadratic polynomial $f(x) = ax^2 + bx + c$ with (complex) roots α_1, α_2, we denote the i^{th} power sum of the roots as $P_i = \sum_{k=1}^{2} \alpha_k^i$ and the sum and product of the roots as A and B, respectively. Then we have

$$\begin{aligned} P_1 &= A \\ P_2 &= AP_1 - 2B \\ &\vdots \\ P_i &= AP_{i-1} - BP_{i-2} \ \forall i \geq 2. \end{aligned}$$

EXAMPLE

Let a polynomial $P(x)$ be defined as $P(x) = x^2 - 2x + 6$ with its (complex) roots a and b. Then what is the value of $a^{10} + b^{10}$?

By Vieta's formula, the sum of the roots is $2 = A$ (say) and the product of the roots is $6 = B$ (say). Now, using the recurrence relation found above and Newton's identities, we have

$$P_i = AP_{i-1} - BP_{i-2} = 2P_{i-1} - 6P_{i-2} \ \forall i \geq 2, \quad P_0 = 2, \quad P_1 = A = 2.$$

Now, we can simply use this recurrence relation obtained repeatedly to find $P_2, P_3, \ldots, P_9$ and then finally P_{10} which is the required answer. The answer comes out to be $2(-3808) - 6(-2528)$.

Hence, our final answer is 7552. $\square$

Full Calculation;

P(x) = x^2 - 2x + 6 , A=2 , B= 6.

$$\boldsymbol{P_i = \mathrm{A}\ \mathrm{P}_{i-1} - B\,\mathrm{P}_{i-2}}$$

$$\boldsymbol{= 2\ \mathrm{P}_{i-1} - 6\,\mathrm{P}_{i-2}}$$

$$\boldsymbol{\mathrm{P}_0 = {\alpha_1}^0 + {\alpha_2}^0 = 2}$$

$$\boldsymbol{P_1 = \alpha_1 + \alpha_2 = A = 2}$$

$$\boldsymbol{P_2 = 2\ \mathrm{P}_{2-1} - 6\,\mathrm{P}_{2-2}}$$

$$\boldsymbol{= 2\ \mathrm{P}_1 - 6\,\mathrm{P}_0}$$

= 2(2) - 6(2)

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= - 8

$P_3 = 2\ P_{3-1} - 6\ P_{3-2}$

$= 2\ P_2 - 6\ P_1$

= 2(-8) - 6(2)

= - 28

$P_4 = 2\ P_{4-1} - 6\ P_{4-2}$

$= 2\ P_3 - 6\ P_2$

= 2(-28) - 6(-8)

= - 8

$P_5 = 2\ P_{5-1} - 6\ P_{5-2}$

$= 2\ P_4 - 6\ P_3$

= 2(-8) - 6(-28)

= 152

$P_6 = 2\ P_5 - 6\ P_4$

= 2(152) - 6(-8)

= 352

$P_7 = 2\ P_6 - 6\ P_5$

= 2(352) - 6(152)

= -208

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$P_8 = 2\ P_7 - 6\ P_6$

= 2(-208) - 6(352)

= -2528

$P_9 = 2\ P_8 - 6\ P_7$

= 2(-2528) - 6(-208)

= -3808

$P_{10} = 2\ P_9 - 6\ P_8$

= 2(-3808) - 6(-2528)

= 7 552

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2. The Pros Newton's Sum in Quadratic Root Function Computations

2.1. Real Root Quadratic Equation

Example : If the roots of the equation $x^2 - 4x + 3 = 0$ are α and β, find the value of i) $\alpha^2+\beta^2$, ii) $\alpha^3+\beta^3$, iii) $\alpha^4+\beta^4$ iv) $\alpha^5+\beta^5$, v) $\alpha^6+\beta^6$, vi) $\alpha^7+\beta^7$ and vii) $\alpha^8+\beta^8$,

Solution:

Sum of the roots = $\alpha + \beta$

$= -\ \frac{b}{a}$

$= -\left(\frac{-4}{1}\right)$

= 4

Product of the roots = $\alpha\beta$

$= \frac{c}{a}$

$= \frac{3}{1}$

$= \mathbf{3}$

Newton's Sum method,

Let A= $\alpha + \beta = 4$ and B = $\alpha\beta = 3$

$\boldsymbol{P_i} = \boldsymbol{\alpha^i + \beta^i}$, So, $\boldsymbol{P_0} = \boldsymbol{\alpha^0 + \beta^0} = 2$. $\boldsymbol{P_1} = \boldsymbol{\alpha^1 + \beta^1} = 4$

$\boldsymbol{P_i = A\ P_{i-1} - B\ P_{i-2}}$

$\boldsymbol{= 4\ P_{i-1} - 3\ P_{i-2}}$

i) $\boldsymbol{\alpha^2 + \beta^2}$,

a)Algebraic formulas Method:

$\boldsymbol{\alpha^2 + \beta^2} = (\alpha + \beta)^2 - 2\alpha\beta$ **from 1.2**

$= \mathbf{4^2} - 2\times(3)$

$= 16 - 6$

$= 10$

$\therefore\ \boldsymbol{\alpha^2 + \beta^2 = 10}$

b) Newton's Sum method

$\boldsymbol{P_2 = 4\ P_1 - 3\ P_0}$

$\mathbf{= 4\ (4) - 3(2)}$

$\mathbf{= 10}$

$\therefore\ \boldsymbol{\alpha^2 + \beta^2 = 10}$

ii) $\boldsymbol{\alpha^3 + \beta^3}$,

a)Algebraic formulas Method:

$\boldsymbol{\alpha^3 + \beta^3 = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]}$ **from 1.2**

$= (4)[(4)^2 - 3(3)]$

$= 4(16 - 9)$

$= 28$

$\therefore \alpha^3+\beta^3 = 28$

b) Newton's Sum method

$P_3 = 4 \ P_2 - 3 \ P_1$

$= 4\ (10) - 3(4)$

$= 28$

$\therefore \alpha^3+\beta^3 = 28$

iii) $\alpha^4+\beta^4$,

a)Algebraic formulas Method:

$\alpha^4+\beta^4 = [(\alpha + \beta)^2 - 2(\alpha\beta)]^2 - 2(\alpha\beta)^2$ **from 1.2**

$= [(4)^2 - 2(3)]^2 - 2(3)^2$

$= [10]^2 - 18$

$= 82$

$\therefore \alpha^4+\beta^4 = 82$

b) Newton's Sum method

$P_4 = 4 \ P_3 - 3 \ P_2$

$= 4\ (28) - 3(10)$

$= 82$

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$\therefore\ \alpha^4 + \beta^4 = 82$

iv) $\alpha^5+\beta^5$,

a)Algebraic formulas Method:

$\alpha^5 + \beta^5$

$= [(\alpha + \beta)^2 - 2\alpha\beta]\{(\alpha + \beta)[(\alpha + \beta\}^2 - 3\alpha\beta]\} - (\alpha\beta)^2(\alpha + \beta)$

[From 1.2]

$= [(4)^2 - 2(3)]\{(4)[(4)^2 - 3(3)]\} - (3)^2(4)$

$= [10][(4)7] - 36$

= 244

$\therefore\ \alpha^5+\beta^5 = 244$

b) Newton's Sum method

$P_5 = 4\ P_4 - 3\ P_3$

$= 4\ (82) - 3(28)$

$= 244$

$\therefore\ \alpha^5 + \beta^5 = 244$

v) $\alpha^6+\beta^6$,

a)Algebraic formulas Method:

$\alpha^6 + \beta^6$

$= \{ (\alpha + \beta) [(\alpha + \beta)^2 - 3\alpha\beta] \}^2 - 2 (\alpha\beta)^3$ **[From 1.2]**

$= \{ (4) [(4)^2 - 3(3)] \}^2 - 2 (3)^3$

$= \{ (4) [7] \}^2 - 54$

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= 730

∴ $\alpha^6+\beta^6 = 730$

b) Newton's Sum method

$P_6 = 4\ P_5 - 3\ P_4$

= 4 (244) – 3(82)

= 730

∴ $\alpha^6+\beta^6 = 730$

vi) $\alpha^7+\beta^7$,

a)Algebraic formulas Method:

$\alpha^7 + \beta^7 = [(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)]\{[(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - [2(\alpha\beta)^2]\} - \alpha^3\beta^3(\beta+\alpha)$

[From 1.2]

$= [(4)^3 - 3(3)(4)]\{[(4)^2 - 2(3)]^2 - [2(3)^2]\} - 3^3(4)$

$= [28]\{[10]^2 - 18\} - 108$

= 2 188

∴ $\alpha^7+\beta^7 = 2\,188$

b) Newton's Sum method

$P_7 = 4\ P_6 - 3\ P_5$

= 4 (730) – 3(244)

= 2 188

∴ $\alpha^7+\beta^7 = 2\,188$

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vii) $\alpha^8+\beta^8$,

a)Algebraic formulas Method:

$\alpha^8 + \beta^8 = \{ [(\alpha + \beta)^2\text{-}2(\alpha\,\beta)]^2\text{-}2\,(\alpha\beta)^2\}^2 - 2(\alpha\beta)^4$

[From 1.2]

$= \{ [(4)^2\text{-}2(3)]^2\text{-}2\,(3)^2\}^2 - 2(3)^4$

$= \{ [10]^2\text{-}\,18\}^2 - 162$

= 6 562

∴ $\alpha^8+\beta^8$= 6 562

b) Newton's Sum method

$P_8 = 4\ P_7 - 3\ P_6$

= 4 (2 188) – 3(730)

= 6 562

∴ $\alpha^8+\beta^8$= 6 562

Note: The significant advantages of Newton's Sum method are obvious, such as freeing students from the burden of memorizing a large number of formulas. As shown above, if the algebraic formula method is used to calculate the quadratic root function, students need to remember a large number of formulas.

2.2. Complex Root Quadratic Equation

Example : If the roots of the equation $x^2 + x + 2 = 0$ are α and β, find the value of i) $\alpha^2+\beta^2$, ii) $\alpha^3+\beta^3$, iii) $\alpha^4+\beta^4$ iv) $\alpha^5+\beta^5$, v) $\alpha^6+\beta^6$, vi) $\alpha^7+\beta^7$ and vii) $\alpha^8+\beta^8$,

Solution:

Sum of the roots = $\alpha + \beta$ $= -\frac{b}{a} = -(\frac{1}{1})$ = -1

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Product of the roots = $\alpha\beta = \frac{c}{a} = \frac{2}{1} = 2$

Newton's Sum method,

Let A= $\alpha + \beta$ = -1 and B = $\alpha\beta$ = 2

$P_i = \alpha^i + \beta^i$, So, $P_0 = \alpha^0 + \beta^0$=2. $P_1 = \alpha^1 + \beta^1$= -1

$P_i = A\ P_{i-1} - B\ P_{i-2}$

$= -\ P_{i-1} - 2\ P_{i-2}$

i) $\alpha^2 + \beta^2$,

a)Algebraic formulas Method:

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ **from 1.2**

$= (-1)^2 - 2\times(2)$

$= 1 - 4$

= -3

$\therefore\ \alpha^2 + \beta^2 = -3$

b) Newton's Sum method

$P_2 = -\ P_1 - 2\ P_0$

= - (-1) – 2(2)

= - 3

$\therefore\ \alpha^2 + \beta^2 = -3$

ii) $\alpha^3 + \beta^3$,

a)Algebraic formulas Method:

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$\alpha^3 + \beta^3 = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$ **from 1.2**

$= (-1)[(-1)^2 - 3(2)]$

$= -(1 - 6)$

$= 5$

$\therefore \alpha^3 + \beta^3 = 5$

b) Newton's Sum method

$P_3 = - P_2 - 2 P_1$

$= -(-3) - 2(-1)$

$= 5$

$\therefore \alpha^3 + \beta^3 = 5$

iii) $\alpha^4 + \beta^4$,

a)Algebraic formulas Method:

$\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2(\alpha\beta)]^2 - 2(\alpha\beta)^2$ **from 1.2**

$= [(-1)^2 - 2(2)]^2 - 2(2)^2$

$= [-3]^2 - 8$

$= 1$

$\therefore \alpha^4 + \beta^4 = 1$

b) Newton's Sum method

$P_4 = - P_3 - 2 P_2$

$= -(5) - 2(-3)$

$= 1$

$\therefore \alpha^4 + \beta^4 = 1$

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iv) α^5+β^5,

a)Algebraic formulas Method:

$\alpha^5 + \beta^5$

$= [(\alpha + \beta)^2 - 2\alpha\,\beta]\{(\alpha + \beta)[(\alpha + \beta\}^2 - 3\alpha\,\beta]\} - (\alpha\beta)^2(\alpha + \beta)$

[From 1.2]

$= [(-1)^2 - 2(2)]\{(-1)[(-1)^2 - 3(2)]\} - (2)^2(-1)$

$= [-3][(-1)(-5)] + 4$

= - 11

∴ $\alpha^5+\beta^5= -11$

b) Newton's Sum method

$P_5 = - P_4 - 2\,P_3$

= - (1) – 2 (5)

= - 11

∴ $\alpha^5+\beta^5 = -11$

v) α^6+β^6,

a)Algebraic formulas Method:

$\alpha^6 + \beta^6$

$= \{ (\alpha + \beta) [(\alpha + \beta)^2 - 3\alpha\,\beta] \}^2 - 2\,(\alpha\beta)^3$ **[From 1.2]**

$= \{ (-1) [(-1)^2 - 3(2)] \}^2 - 2\,(2)^3$

$= \{ (-1) [-5] \}^2 - 16$

= 9

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$\therefore\ \alpha^6+\beta^6 = 9$

b) Newton's Sum method

$P_6 = - P_5 - 2\ P_4$

$= - (-11) - 2(1)$

$= 9$

$\therefore\ \alpha^6+\beta^6 = 9$

vi) $\alpha^7+\beta^7$,

a)Algebraic formulas Method:

$\alpha^7 + \beta^7 = [(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)]\{[(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - [2(\alpha\beta)^2]\} - \alpha^3\beta^3(\beta+\alpha)$

[From 1.2]

$= [(-1)^3 - 3(2)(-1)]\{[(-1)^2 - 2(2)]^2 - [2(2)^2]\} - 2^3(-1)$

$= [5]\{[-3]^2 - 8\} + 8$

$= 13$

$\therefore\ \alpha^7+\beta^7 = 13$

b) Newton's Sum method

$P_7 = - P_6 - 2\ P_5$

$= - (9) - 2(-11)$

$= 13$

$\therefore\ \alpha^7+\beta^7 = 13$

vii) $\alpha^8+\beta^8$,

a)Algebraic formulas Method:

$\alpha^8 + \beta^8 = \{[(\alpha+\beta)^2 - 2(\alpha\beta)]^2 - 2(\alpha\beta)^2\}^2 - 2(\alpha\beta)^4$

[From 1.2]

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$= \{ [(-1)^2 - 2(2)]^2 - 2\,(2)^2 \}^{\,2} - 2(2)^4$

$= \{ [-3]^2 - 8\}^{\,2} - 32$

= -31

$\therefore\ \alpha^8 + \beta^8 = -31$

b) Newton's Sum method

$P_8 = -\ P_7\ - 2\ P_6$

$= -\ (13) - 2(9)$

$= -\ 31$

$\therefore\ \alpha^8 + \beta^8 = 6\,562$

3. The Cons Newton's Sum in Quadratic Root Function Computations

Example : If the roots of the equation x^2 - 3x + 2 = 0 are α and β , find the value $\alpha^{25} + \beta^{25}$ ***by using*** **Newton's Sum method.**

Solution: Newton's Sum method. Let A= α + β = 3 and B = αβ = 2

$P_i = \alpha^i + \beta^i$, So, $P_0 = \alpha^0 + \beta^0 = 2$. $P_1 = \alpha^1 + \beta^1 = 3$

$P_i = A\ \ P_{i-1} - B\ P_{i-2}$

$= 3\ \ P_{i-1}\ - 2\ P_{i-2}$

$P_2 = 3\ P_1\ - 2\ P_0$

$= 3\ (3) - 2(2)$

$= 5$

$P_3 = 3\ P_2\ - 2\ P_1$

$= 3\ (5) - 2\ (3)$

$= 9$

$P_4 = 3\ \ P_3\ - 2\ P_2$

$= 3\ (9) - 2\ (5)$

$= 17$

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$P_5 = 3\ P_4 - 2\ P_3$

$= 3\ (17) - 2\ (9)$

$= 33$

$P_6 = 3\ P_5 - 2\ P_4$

$= 3\ (33) - 2\ (17)$

$= 65$

$P_7 = 3\ P_6 - 2\ P_5$

$= 3\ (65) - 2(33)$

$= 129$

$P_8 = 3\ P_7 - 2\ P_6$

$= 3\ (129) - 2\ (65)$

$= 257$

$P_9 = 3\ P_8 - 2\ P_7$

$= 3(257) - 2(129)$

$= 513$

$P_{10} = 3\ P_9 - 2\ P_8$

$= 3\ (513) - 2(257)$

$= 1\ 025$

$P_{11} = 3\ P_{10} - 2\ P_9$

$= 3\ (1025) - 2\ (513)$

$= 2049$

$P_{12} = 3\ P_{11} - 2\ P_{10}$

$= 3\ (2049) - 2\ (1025)$

$= 4097$

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$P_{13} = 3\ P_{12} - 2\ P_{11}$

$= 3\ (4097) - 2\ (2049)$

$= 8193$

$P_{14} = 3\ P_{13} - 2\ P_{12}$

$= 3\ (8193) - 2\ (4097)$

$= 16385$

$P_{15} = 3\ P_{14} - 2\ P_{13}$

$= 3\ (16385) - 2\ (8193)$

$= 32\ 769$

$P_{16} = 3\ P_{15} - 2\ P_{14}$

$= 3\ (32\ 769) - 2(16\ 385)$

$= 65\ 537$

$P_{17} = 3\ P_{16} - 2\ P_{15}$

$= 3\ (65\ 537) - 2\ (32\ 769)$

$= 131\ 073$

$P_{18} = 3\ P_{17} - 2\ P_{16}$

$= 3\ (131\ 073) - 2\ (65\ 537)$

$= 262\ 145$

$P_{19} = 3\ P_{18} - 2\ P_{17}$

$= 3\ (262\ 145) - 2(131\ 073)$

$= 524\ 289$

$P_{20} = 3\ P_{19} - 2\ P_{18}$

$= 3\ (524\ 289) - 2(262\ 145)$

$= 1\ 048\ 577$

$P_{21} = 3 \ P_{20} - 2 \ P_{19}$

= 3 (1048577) – 2(524 289)

= 2 097 153

$P_{22} = 3 \ P_{21} - 2 \ P_{20}$

= 3 (2 097 153) – 2(1 048 577)

= 4194305

$P_{23} = 3 \ P_{22} - 2 \ P_{21}$

= 3 (4 194 305) – 2(2 097 153)

= 8 388 609

$P_{24} = 3 \ P_{23} - 2 \ P_{22}$

= 3 (8 388 609) – 2(4194305)

= 16 777 217

$P_{25} = 3 \ P_{21} - 2 \ P_{20}$

= 3 (16 777 217) – 2 (8 388 609)

= 33 554 433

Note: As shown above, Newton's Sum is a historical calculation method that relies on previous results for subsequent calculations. This common historical basis reveals a limitation, particularly when dealing with higher-order quadratic root functions involving exponents such as $\alpha^{25} + \beta^{25}$. For higher-order quadratic root functions, using Newton's Sum involves many steps that can lead to calculation errors.

4. Asymmetric Function of a Quadratic's Roots

a) $\alpha - \beta = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$,

b) $\alpha^2 - \beta^2 = \pm(\alpha+\beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$,

c) $\alpha^3 - \beta^3 = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\ [\ (\alpha+\beta)^2 - \alpha\beta]$.

d) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)\ (\alpha^2 + \beta^2)$

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$$= \quad \pm[(\alpha+\beta)\sqrt{(\alpha+\beta)^{2} - 4\alpha\beta}\ ] \quad [(\alpha+\beta)^{2} - 2\alpha\beta]^{2} - 2(\alpha\beta)^{2}\ ]$$

Note: Newton Sum method not suitable for Asymmetric Quadratic Root Function Computations, adding an extra layer of complexity to its application.

5. Conclusion

In conclusion, "Mastering Newton's Sum" serves as a comprehensive and enlightening guide to navigating the intricate world of quadratic roots through the lens of Newton's Sum method.

Newton's Sum method notable advantages, such as freeing students from the burden of memorizing numerous formulas, are evident. However, as the narrative unfolds, the book draws parallels between Newton's Sum and the Pascal triangle method—both historical calculation methods reliant on prior results for subsequent calculations. This shared historical foundation reveals a limitation, particularly when addressing higher-order quadratic root functions involving exponents like $\alpha^{25} + \beta^{25}$.

Furthermore, the Newton Sum method proves not suitable for asymmetric quadratic root function computations, adding an extra layer of complexity to its application.

Beyond a mere instructional tool, the book encourages a profound understanding of when and where to apply Newton's Sum, adding a layer of insight to its practical use. As we reflect on the evolution from the Pascal triangle method to the binomial theorem, a clear imperative emerges to develop innovative approaches that overcome the limitations posed by Newton's Sum.

This necessity becomes a call to action, guiding readers toward future explorations and inspiring the creation of methods that transcend existing constraints. "Mastering Newton's Sum" is not just a guide; it is an invitation to embark on a journey of innovation, forging a dynamic intersection of history, creativity, and mathematical mastery.

As we chart new territories in the realm of quadratic roots, this book stands as a beacon, propelling readers towards unprecedented heights of understanding and application. It is a testament to the ongoing dialogue between tradition and innovation in the ever-evolving landscape of mathematical exploration.

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