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Article

Finding All Possibly Efficient Solutions of an Interval Multiple Objective Linear Programming Problem

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Abstract

Finding all possibly efficient solutions of an interval multiple objective linear programming (IMOLP) problem with interval coefficients in the objective functions, the constraint matrix and the right-hand side vector is dealt with. Up to now, there are no known methods that can find all possibly efficient solutions of an IMOLP problem with interval coefficients in the objective functions and the right-hand side vector. In this paper, we propose a method to find all possibly efficient solutions of an IMOLP problem with interval coefficients in the objective functions and the right-hand side vector. Some sufficient conditions to obtain all possibly efficient solutions of an IMOLP problem in a general case are also given.

Keywords: interval multiple objective linear programming; the efficient basic set; the possibly efficient set

MSC: 90C29; 90C50; 90C90

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1. Introduction

Multiple objective linear programming (MOLP) models play an important role in solving and investigating real-life practical problems. There is a practical fact that the exact values of the data of practical problems are very difficult to determine but intervals containing them can be easily determined. Thus, interval multiple objective linear programming (IMOLP) models can describe practical problems more correctly and more easily than MOLP models do. For brevity of presentation, we shall use the following notation: For two matrices *A* and *B* of the same size, $A \le B$ if and only if $a_{ij} \le b_{ij}$, where a_{ij} and b_{ij} are elements of *A* and *B*, respectively.

An interval multiple objective linear programming (IMOLP) problem can be stated as follows:

$$\text{“maximize”}\ Cx\,, \quad (1)$$

$$Ax \ \le \ b\,, \quad (2)$$

$$C \in IC\,, \quad A \in IA\,, \quad b \in ib\,, \quad (3)$$

where $IA = \left\{ A \middle| \underline{A} \le A \le \bar{A} \right\}$ is an $m \times n$ interval matrix, $IC = \left\{ C \middle| \underline{C} \le C \le \bar{C} \right\}$ is a $k \times n$ interval matrix, $ib = \left\{ b \middle| \underline{b} \le b \le \bar{b} \right\}$ is an m interval vector, $\underline{A}$, $\bar{A}$, $\underline{C}$, $\bar{C}$, $\underline{b}$ and $\bar{b}$ are determined. Problem (1)-(3) is denoted by $P(IA, IC, ib)$. For every $A \in IA$, $C \in IC$ and $b \in ib$ a multiple objective linear programming (MOLP) problem, denoted by $P(A, C, b)$, is obtained from problem (1)-(3). Let $L(A, b)$ and $L(IA, ib)$ be the feasible sets of $P(A, C, b)$ and problem (1)-(3), respectively. A point (solution) $x \in L(A, b)$ is called *efficient* for $P(A, C, b)$ if there is no $y \in L(A, b)$ such that $Cx \le Cy$

and $Cx \neq Cy$. A feasible point (solution) of problem (1)-(3) is called *possibly efficient* for it if there are $A \in IA$, $C \in IC$ and $b \in ib$ such that it is efficient for $P(A,C,b)$. The set of all efficient solutions of a problem $P(A,C,b)$, denoted by $E(A,C,b)$, is called an *efficient set* of it. The set of all possibly efficient solutions of a problem $P(IA,IC,ib)$), denoted by $E_p(IA,IC,ib)$, is called a *possibly efficient set* of it. It is easily seen that

$$L(IA,ib) = \left\{ x \in R^m \middle| \begin{array}{l} Ax \leq b, \\ A \in IA, b \in ib \end{array} \right\}, (4)$$

$$L(IA,ib) = \cup \{ L(A,b) | A \in IA, b \in ib \}, (5)$$

$$E_p(IA,IC,ib) = \cup \{ E(A,C,b) | A \in IA, C \in IC, b \in ib \}. (6)$$

The notion of possibly efficient solutions of IMOLP problem (1)-(3) can be found in, for example, Allahdadi and Batamiz [1], Bitran [6], Inuiguchi and Kume [17], Oliveira and Antunes [24], Tu [34]. There are many known methods for finding the efficient set of an MOLP problem, see, for example, Yu and Zeleny [42], Isermann [19], Ecker et al. [10], Dauer and Liu [9], Armand and Malivert [3], Armand [2], Rudloff et at. [31], Sayin [32], Dauer and Gallagher [8], Benson [5], Tu ([34]-[38]), Yan et al. [41], Foroughi and Jafari [12], Pourkarimi et al. [25], Krichen et al. [21], Tohidi and Hassasi [33]. It can be easily seen that all the known methods for finding the efficient set of an MOLP problem, in general, must recompute the efficient set when the data of the MOLP problem is changed. Therefore, known methods for finding the efficient set of an MOLP problem cannot find the possibly efficient set of an IMOLP problem in a general case.

Solving an IMOLP problem is much more difficult than solving an MOLP problem. Almost known methods investigate an IMOLP problem with interval coefficients only in objective functions, for example, Chanas and Kuchta [7], Ishibuchi and Tanaka [20], Wang and Wang [44] . Some possibly efficient solutions can be found by methods given by Urli and Nadeau [43] based on an interactive method, by Inuiguchi and Kume [18] based on a goal programming method, by Rivaz and Yaghoobi [27] based on a weighted sum of maximum regrets, Hajiagha et al. [14], Rivaza and Saeidib [26] based on fuzzy programming methods, etc.. Possibly efficient extreme points of an IMOLP problem are dealt with in Inuiguchi and Kume [17]. If the coefficients of an IMOLP problem are satisfied probability distributions, then stochastic programming methods can be used to study this problem, see, for example, Batamiz and Allahdadi [4]. All possibly efficient solutions can be found by the method given in [34] for an IMOLP problem with interval coefficients only in the right-hand side vector. The method given in [44] can find the possibly efficient set of an IMOLP problem with interval coefficients only in the objective functions, but this method requires a lot of computations and is difficult to implement. However, there are no known methods that can find all possibly efficient solutions of an IMOLP problem with interval coefficients in the objective functions and the right-hand side vector. Theoretically, an IMOLP problem can be stated based on fuzzy numbers and can be solved by fuzzy programming methods. However, huge difficulties in this way lie in constructing adequate membership functions and finding all possibly efficient solutions of an IMOLP problem.

In this paper, we propose a method to find all possibly efficient solutions of an IMOLP problem with interval coefficients in the objective functions and the right-hand side vector. We also give some sufficient conditions to obtain all possibly efficient solutions of an IMOLP problem in a general case (with interval coefficients in the objective functions, the constraint matrix and the right-hand side vector). This method is developed based on the methods given in [34,35] and can be easily implemented.

2. Finding the Possibly Efficient Set of IMOLP Problem (1)-(3)

2.1. A General Case

Corresponding to IMOLP problem (1)-(3), we consider the following set:

$$\bar{L}\left(IA,IC\right)=\left\{p=\left(y,z\right)^T\in R^{m+k}\left|\begin{matrix}y^TA-z^TC=e^TC,y\geq 0,z\geq 0,\\ A\in IA,\ C\in IC\end{matrix}\right.\right\},\ (7)$$

where $e=(1,...,1)^T\in R^k$. For every $A\in IA$ and $C\in IC$, the following convex polyhedron, denoted by $\bar{L}\left(A,C\right)$, is directly obtained from (7):

$$\bar{L}\left(A,C\right)=\left\{p=\left(y,z\right)^T\in R^{m+k}\left|y^TA-z^TC=e^TC,y\geq 0,z\geq 0\right.\right\}.$$

Thus, it is clear that

$$\bar{L}\left(IA,IC\right)=\cup\left\{\bar{L}\left(A,C\right)\middle|A\in IA,C\in IC\right\}.\ (8)$$

A formula presented in the following property to describe the solution set of a system of interval linear equations is given by Oettli-Prager [23]:

Property 2.1. $X=\left\{x\middle|\left|A_cx-b_c\right|\leq\Delta|x|+\delta\right\}$, *where* $X=\left\{x\in R^n\left|Ax=b,A\in IA,b\in ib\right.\right\}$, $A_c=\left(\underline{A}+\bar{A}\right)/2$, $b_c=\left(\underline{b}+\bar{b}\right)/2$, $\Delta=\left(\bar{A}-\underline{A}\right)/2$, $\delta=\left(\bar{b}-\underline{b}\right)/2$, $|x|=\left(|x_1|,...,|x_n|\right)^T$, IA *and* ib *are defined in problem* (1)-(3).

This property is proven by Rohn ([29], Theorem 2.1, p.43) in a special case when $n=m$ but his proof can be easily modified to prove Property 2.1. In the case when $x\geq 0$, we get the following property:

Property 2.2. *If* $x\geq 0$, *then* $X=\left\{x\in R^n\left|\begin{matrix}\underline{A}x\leq\bar{b},\\ -\bar{A}x\leq-\underline{b},x\geq 0\end{matrix}\right.\right\}$.

Proof. Since $x\geq 0$ and $\left|A_cx-b_c\right|\leq\Delta|x|+\delta$ $\Leftrightarrow ?x\geq$ and $\left|(\bar{A}x-\underline{b})-(\bar{b}-\underline{A}x)\right|\leq(\bar{A}x-\underline{b})+(\bar{b}-\underline{A}x)$ $\Leftrightarrow\begin{cases}\bar{A}x\geq\underline{b}\\ \bar{b}\geq\underline{A}x,x\geq 0\end{cases}$. Therefore, based on Property 2.1 we have $X=\left\{x\in R^n\left|\begin{matrix}\underline{A}x\leq\bar{b},\\ -\bar{A}x\leq-\underline{b},x\geq 0\end{matrix}\right.\right\}$. The proof is complete. □

The solution set of a system of interval linear inequalities with non-negative variables is given in the following property:

Property 2.3. $IS\left(IA,ib\right)=\left\{x\in R^n\left|\begin{matrix}\underline{A}x+I_my\leq\bar{b},\\ -\bar{A}x-I_my\leq-\underline{b},\\ x\geq 0,y\geq 0,y\in R^m\end{matrix}\right.\right\}$,
where

$$IS\left(IA,ib\right)=\left\{x\in R^n\left|\begin{matrix}Ax\leq b,x\geq 0,\\ A\in IA,b\in ib\end{matrix}\right.\right\}\ \textit{and}\ I_m\ \textit{is the unit matrix in}\ R^m.$$

Proof. $IS\left(IA,ib\right)=\left\{x\in R^n\left|\begin{matrix}Ax\leq b,x\geq 0,\\ A\in IA,b\in ib\end{matrix}\right.\right\}=\left\{x\in R^n\left|\begin{matrix}Ax+I_my=b,\\ A\in IA,b\in ib,\\ x\geq 0,y\geq 0,y\in R^m\end{matrix}\right.\right\}=$ $\left\{x\in R^n\left|\begin{matrix}Dz=b,z=\left(x,y\right)^T\geq 0,\\ D\in ID,b\in ib\end{matrix}\right.\right\}$

where $D=(A,I_m)$, $\bar{D}=(\bar{A},I_m)$, $\underline{D}=(\underline{A},I_m)$ and $ID=\{D \mid \underline{D}\le D\le \bar{D}\}$ is an $m\times(m+n)$ interval matrix. Based on Property 2.2 we have $IS(IA,ib)=\left\{x\in R^n \,\middle|\, \begin{array}{l} \underline{A}x+I_m y\le \bar{b}, \\ -\bar{A}x-I_m y\le -\underline{b}, \\ x\ge 0, y\ge 0, y\in R^m \end{array}\right\}$. The proof is complete.

□

Remark 2.1. The solution set of a system of interval linear equations or inequalities can be not convex, see, for example, Hensen [15], Fiedler et al. [11], Rohn [28,30]. From Properties 2.2 and 2.3 it can be easily seen that the solution set of a system of interval linear equations or inequalities is a convex polyhedron if all the variables of it are bounded.

Noting that the variables in (7) are non-negative, based on Property 2.2 the following property can be easily obtained:

Property 2.4. $\bar{L}(IA,IC)=\left\{p=(y,z)^T\in R^{m+k} \,\middle|\, \begin{array}{l} y^T\underline{A}-z^T\bar{C}\le e^T\bar{C}, \\ -y^T\bar{A}+z^T\underline{C}\le -e^T\underline{C} \\ y\ge 0, z\ge 0 \end{array}\right\}$.

Remark 2.2. Since $\bar{L}(IA,IC)$ is a convex polyhedron described by a system of linear inequalities with non-negative variables, $\bar{L}(IA,IC)$ has an extreme point if and only if it is not empty.

In order to find extreme points of $\bar{L}(IA,IC)$ based on the simplex method, $\bar{L}(IA,IC)$ is stated in the following form:

$$\bar{\bar{L}}=\left\{p=(y,y^1,y^2,z)^T\in R^{m+2n+k} \,\middle|\, \begin{array}{l} y^T\underline{A}+I_n y^1-z^T\bar{C}=e^T\bar{C}, \\ -y^T\bar{A}+I_n y^2+z^T\underline{C}=-e^T\underline{C}, \\ y\ge 0, y^1\ge 0, y^2\ge 0, z\ge 0 \end{array}\right\}.$$

Let

$I_+(p)=\{i\in\{1,\dots,m\} \mid p_i>0\}$ for every point p of $\bar{L}(IA,IC)$ or $\bar{\bar{L}}$, where p_i is the i-th component of p,

$T_0(IA,IC)=\{p \mid p=(y,z)^T \text{ is an extreme point of } \bar{L}(IA,IC)\}$,

$T_1(IA,IC)=\{I_+(p) \mid p\in T_0(IA,IC)\}$,

$T_2(IA,IC)$ is a set consisting of all minimal elements of $T_1(IA,IC)$ by inclusion.

Let $T_2(A,C)$ be a set established based on $\bar{L}(A,C)$ by a way similar to that establishing $T_2(IA,IC)$ based on $\bar{L}(IA,IC)$.

It can be easily seen that the set of all extreme points of $\bar{L}(IA,IC)$ can be found by finding extreme points of $\bar{\bar{L}}$. Based on this property, the set $T_2(IA,IC)$ can be found by the method given in Tu [35] based on $\bar{\bar{L}}$ without determining all extreme points of this polyhedron.

A relation between $T_2(IA,IC)$ and $T_2(A,C)$ is considered in the following property:

Property 2.5. *If* $T_2(A,C)\ne\varnothing$, *then for every* $I\in T_2(A,C)$ *there is* $J\in T_2(IA,IC)$ *such that* $J\subseteq I$.

Proof. From the definition of $T_2(A,C)$ and $T_2(A,C)\ne\varnothing$ it follows that there is an extreme point of p^0 of $\bar{L}(A,C)$ such that $I_+(p^0)=I$. From (8) it follows that $p^0\in\bar{L}(IA,IC)$. Based on a proof similar to that of Property 2.4 in [34], it can be easily seen that there is an extreme point p^1 of $\bar{L}(IA,IC)$ such that $I_+\left(p^1\right) \subseteq I_+\left(p^0\right)$. From the definition of $T_2\left(IA, IC\right)$ it follows that there is $J \in T_2\left(IA, IC\right)$ such that $J \subseteq I_+\left(p^1\right)$. Therefore, $J \subseteq I$. The proof is complete. □

Let

$$S\left(I, A, b\right) = \left\{x \in L\left(A, b\right) \middle| a_i x = b_i, i \in I\right\},$$

$$S\left(I, IA, ib\right) = \cup\left\{S\left(I, A, b\right) \middle| A \in IA, b \in ib\right\},$$

where $\left(a_i, b_i\right)$ is the *i*-th row of a matrix $\left(A, b\right)$ defined in (4).

A property of the possibly efficient set of IMOLP problem (1)-(3) is shown in the following property:

Property 2.6. *If* $E_p\left(IA, IC, ib\right) \neq \varnothing$*, then* $T_2\left(IA, IC\right) \neq \varnothing$ *and* $E_p\left(IA, IC, ib\right) \subseteq \bigcup_{I \in T_2\left(IA, IC\right)} S\left(I, IA, ib\right)$.

Proof. For every element $x^0 \in E_p\left(IA, IC, ib\right)$ there are $A \in IA$, $C \in IC$ and $b \in ib$ such that $x^0 \in E\left(A, C, b\right)$. From Property 2.2 in [34] it follows that $\bar{L}\left(A, C\right) \neq \varnothing$. Thus, $\bar{L}\left(IA, IC\right) \neq \varnothing$. From Remark 2.2 and the definition of $T_2\left(IA, IC\right)$ it follows that $T_2\left(IA, IC\right) \neq \varnothing$. Based on Property 2.4 in [34], there is $J \in T_2\left(A, C\right)$ such that $J \subseteq ID\left(x^0, A, b\right)$, where

$$ID\left(x^0, A, b\right) = \left\{i \in \{1, \ldots, m\} \middle| a_i x^0 = b_i\right\}.$$

Since $x^0 \in S\left(ID\left(x^0, A, b\right), A, b\right) \subseteq S\left(J, A, b\right)$, $x^0 \in S\left(J, A, b\right)$. From Property 2.5 it follows that there is $I \in T_2\left(IA, IC\right)$ such that $I \subseteq J$. Thus, $x^0 \in S\left(J, A, b\right) \subseteq S\left(I, A, b\right) \subseteq S\left(I, IA, ib\right)$. Therefore, $E_p\left(IA, IC, ib\right) \subseteq \bigcup_{I \in T_2\left(IA, IC\right)} S\left(I, IA, ib\right)$. □

Property 2.7. *If* $E_p\left(IA, IC, ib\right) \neq \varnothing$ *and* $\bar{A} = \underline{A}$*, then* $E_p\left(IA, IC, ib\right) = \bigcup_{I \in T_2\left(IA, IC\right)} S\left(I, IA, ib\right)$.

Proof. From Property 2.6 it follows that $T_2\left(IA, IC\right) \neq \varnothing$ and $E_p\left(IA, IC, ib\right) \subseteq \bigcup_{I \in T_2\left(IA, IC\right)} S\left(I, IA, ib\right)$. Conversely, for every element $x^0 \in \bigcup_{I \in T_2\left(IA, IC\right)} S\left(I, IA, ib\right)$ there is $I \in T_2\left(IA, IC\right)$ such that $x^0 \in S\left(I, IA, ib\right)$. From the definition of $T_2\left(IA, IC\right)$ it follows that there is an extreme point $p^0 = \left(y^0, z^0\right)^T \in \bar{L}\left(IA, IC\right)$ such that $I_+\left(p^0\right) = I$. From (8) it follows that there are $A \in IA$, $C \in IC$ such that $p^0 \in \bar{L}\left(A, C\right)$. From $x^0 \in S\left(I, IA, ib\right)$ and $\bar{A} = \underline{A}$ it follows that there is $b^0 \in ib$ such that $x^0 \in S\left(I, A, b^0\right)$. Hence, it follows that $I \subseteq ID\left(x^0, A, b^0\right)$. Noting that $I_+\left(p^0\right) = I$, $I_+\left(p^0\right) \subseteq ID\left(x^0, A, b^0\right)$ and $p^0 = \left(y^0, z^0\right)^T \in \bar{L}\left(A, C\right)$, it can be easily seen that $y^0 \in \bar{F}\left(ID\left(x^0, A, b^0\right), x^0\right)$, where

$$\bar{F}\left(ID\left(x^0,A,b^0\right),x^0\right)=\left\{y\in R^m \,\middle|\, \begin{array}{l} y^TA=\left(e+z^0\right)^T C,\\ y_i=0 \text{ for all } i\in\{1,\dots,m\}\setminus ID\left(x^0,A,b^0\right),\\ y_i\ge 0 \text{ for all } i\in ID\left(x^0,A,b^0\right)\end{array}\right\}.$$

Thus, based on the complementary theorem of linear programming, x^0 is an optimal solution of the linear programming problem max $\left\{\left(e+z^0\right)^T Cx \,\middle|\, x\in L\left(A,b^0\right)\right\}$. Thus, $x^0\in E\left(A,C,b^0\right)$. Therefore, $x^0\in E_p\left(IA,IC,ib\right)$. The proof is complete. ☐

Now we deal with finding the possibly efficient set of IMOLP problem (1)-(3) in a general case.

Property 2.8. *If* $J\in T_2\left(IA,IC\right)$ *and* $\bar{L}\left(A,C,J\right)\neq\varnothing$ *for every* $A\in IA$ *and some* $C\in IC$ *, then* $S\left(J,IA,ib\right)\subseteq E_p\left(IA,IC,ib\right)$*,*

where

$$\bar{L}\left(A,C,J\right)=\left\{p=\left(y,z\right)^T\in\bar{L}\left(A,C\right)\,\middle|\, y_i=0 \textit{ for all } i\in\bar{J}\right\}.$$

Proof. If $S\left(J,IA,ib\right)=\varnothing$, then the proof is obvious. In the other case, for every element $x^0\in S\left(J,IA,ib\right)$, there are $A^0\in IA$ and $b^0\in ib$ such that $x^0\in S\left(J,A^0,b^0\right)$. It is easily seen that $J\subseteq ID\left(x^0,A^0,b^0\right)$. Let C^0 be an element of IC such that $\bar{L}\left(A^0,C^0,J\right)\neq\varnothing$. Since $\bar{L}\left(A^0,C^0,J\right)$ is a non-empty convex polyhedron with non-negative variables, there is an extreme point $p^0=\left(y^0,z^0\right)^T$ of $\bar{L}\left(A^0,C^0,J\right)$. It is easily seen that p^0 also is an extreme point of $\bar{L}\left(A^0,C^0\right)$ and $I_+\left(p^0\right)\subseteq J$. From the definition of $T_2\left(IA,IC\right)$ and $J\in T_2\left(IA,IC\right)$ it follows that $I_+\left(p^0\right)=J$. By a proof similar to that presented in Property 2.7, we have $y^0\in\bar{F}\left(ID\left(x^0,A^0,b^0\right),x^0\right)$. Thus, $x^0\in E\left(A^0,C^0,b^0\right)$. Therefore, $x^0\in E_p\left(IA,IC,ib\right)$. ☐

An interval linear equation $y^TA-z^TC=e^TC$, $A\in IA$, $C\in IC$, denoted by $EQ\left(IA,IC\right)$, is called an (IA)-*strongly feasible for* J if for each $A\in IA$ there is $C\in IC$ such that $\bar{L}\left(A,C,J\right)\neq\varnothing$. This notion when $J=\varnothing$ is introduced and investigated by Li et al. [22].

Corollary 2.9. *If* $E_p\left(IA,IC,ib\right)\neq\varnothing$ *and* $EQ\left(IA,IC\right)$ *is an* (IA)-*strongly feasible for* J *for every* $J\in\bar{T}_2\left(IA,IC\right)$*, then* $E_p\left(IA,IC,ib\right)=\bigcup_{I\in T_2\left(IA,IC\right)} S\left(I,IA,ib\right)$*,*

where

$$\bar{T}_2\left(IA,IC\right)=\left\{J\in T_2\left(IA,IC\right)\,\middle|\, S\left(J,IA,ib\right)\neq\varnothing\right\}.$$

Proof. From Property 2.6 it follows that $T_2\left(IA,IC\right)\neq\varnothing$ and $E_p\left(IA,IC,ib\right)\subseteq\bigcup_{I\in T_2\left(IA,IC\right)} S\left(I,IA,ib\right)$. From Property 2.8 it follows that $E_p\left(IA,IC,ib\right)\supseteq\bigcup_{I\in T_2\left(IA,IC\right)} S\left(I,IA,ib\right)$. Therefore, we have $E_p\left(IA,IC,ib\right)=\bigcup_{I\in T_2\left(IA,IC\right)} S\left(I,IA,ib\right)$. The proof is complete. ☐

Since the solution set of a system of interval linear equations, in general, is not convex, see, for example, Hensen [15], Fiedler et al. [11], Rohn [28,30]. Therefore, the sets $S\left(I,IA,ib\right)$ defined in

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Property 2.6 can be not convex polyhedrons. This can cause difficulties in finding most preferred solutions from the possibly efficient set of IMOLP problem (1)-(3).

Now we consider an IMOLP problem of which the possibly efficient set can be computed by a union of a finite number of convex polyhedrons.

2.2. A Special Case

We consider the following IMOLP problem:

$$\text{“maximize” } Cx \ (9)$$

$$A^1 x \ \le \ b^1, \ \ x \ge 0, \ (10)$$

$$C \in IC, \ \ b^1 \in ib^1, \ (11)$$

where A^1 is an $m_2 \times n$ matrix, $IC = \left\{ \underline{C} \le C \le \overline{C} \right\}$ is defined in proplem (1)-(3), $ib^1 = \left\{ \underline{b}^1 \le b^1 \le \overline{b}^1 \right\}$ is an m_2 interval vector. Problem (9)-(11) is a special case of IMOLP problem (1)-(3) because its variables are restricted in sign. It is clear that IMOLP problem (9)-(11) can be easily solved by the above presented method for solving problem (1)-(3). To do this, we restate problem (9)-(11) in the form of problem (1)-(3) by defining $A = \begin{pmatrix} A^1 \\ -I_n \end{pmatrix} \in R^{(m_2+n)\times n}$, $b = \begin{pmatrix} b^1 \\ O_{n\times 1} \end{pmatrix} \in R^{(m_2+n)\times 1}$, $\underline{b} = \begin{pmatrix} \underline{b}^1 \\ O_{n\times 1} \end{pmatrix} \in R^{(m_2+n)\times 1}$, $\overline{b} = \begin{pmatrix} \overline{b}^1 \\ O_{n\times 1} \end{pmatrix} \in R^{(m_2+n)\times 1}$ and $m = m_2 + n$, where I_n is the identity matrix in R^n and $O_{n\times 1}$ is the n column vector with components being 0. Thus, the possibly efficient set of problem (9)-(11), denoted by $E_p^+\left(A^1, IC, ib^1\right)$, can be computed by the formula given in Property 2.7. Now we represent this formula with using the data of problem (9)-(11).

Property 2.10. $E_p^+\left(A^1, IC, ib^1\right) = \bigcup_{I \in T_2(A,IC)} S_+(I)$,

where

$$S_+(I) = \left\{ x \in R^n \;\middle|\; \begin{array}{c} \underline{b}_i^1 \le a_i^1 x \le \overline{b}_i^1, i \in I \cap \{1,\dots,m_2\}, \\ x_i = 0, \ \ (i+m_2) \in I \ \textit{and} \ i \in \{1,\dots,n\}, \\ \underline{b}_i^1 \le a_i^1 x + y_i \le \overline{b}_i^1, i \in \overline{I} \cap \{1,\dots,m_2\}, \\ x_i = 0, (i+m_2) \in \overline{I} \cap \{m_2+1,\dots,m_2+n\}, \\ y_i = 0, i \in \overline{I} \cap \{m_2+1,\dots,m_2+n\}, \\ x \ge 0, y_i \ge 0, i \in \overline{I} \end{array} \right\},$$

$\overline{I} = \{1,\dots,m\} \setminus I$, $\left(a_i^1, \underline{b}_i^1\right)$ and $\left(a_i^1, \overline{b}_i^1\right)$ are the i-th row of the matrix $\left(A^1, \underline{b}^1\right)$ and $\left(A^1, \overline{b}^1\right)$, respectively.

Proof. Let $A(I), IA(I), b(I)$ and $ib(I)$ be the matrices obtained from the matrices A, IA, b and ib by dropping rows whose indices are not in I, respectively. Based on Properties 2.2 and 2.3, it can be easily seen that $S(I, A, ib) = \left\{ x \in L(A, ib) \middle| a_i x = b_i, i \in I \right\}$ = $\left\{ x \in R^n \;\middle|\; \begin{array}{l} a_i x = b_i, i \in I, \\ Ax \le b, x \ge 0, \\ b \in ib \end{array} \right\}$ =

$\left\{ x \in R^n \,\middle|\, \begin{array}{l} A(I)x = b(I), x \geq 0, \\ b(I) \in ib(I), \\ A(\overline{I})x \leq b(\overline{I}), x \geq 0, \\ b(\overline{I}) \in ib(\overline{I}) \end{array} \right\} = \left\{ x \in R^n \,\middle|\, \begin{array}{c} -a_i x \leq -\underline{b}_i, i \in I, \\ a_i x \leq \overline{b}_i, i \in I, \\ a_i x + y_i \leq \overline{b}_i, i \in \overline{I}, \\ -a_i x - y_i \leq -\underline{b}_i, i \in \overline{I}, \\ x \geq 0, y_i \geq 0, i \in \overline{I} \end{array} \right\}$ ($(a_i, \overline{b}_i)$ and $(a_i, \underline{b}_i)$ are the *i*-th rows of the matrices $(A, \overline{b})$ and $(A, \underline{b})$, respectively) $= \left\{ x \in R^n \,\middle|\, \begin{array}{c} \underline{b}_i \leq a_i x \leq \overline{b}_i, i \in I, \\ \underline{b}_i \leq a_i x + y_i \leq \overline{b}_i, i \in \overline{I}, \\ x \geq 0, y_i \geq 0, i \in \overline{I} \end{array} \right\} =$

$$\left\{ x \in R^n \,\middle|\, \begin{array}{c} \underline{b}_i^1 \leq a_i^1 x \leq \overline{b}_i^1, i \in I \cap \{1,\ldots,m_2\}, \\ x_i = 0, \ \ (i + m_2) \in I \text{ and } i \in \{1,\ldots,n\}, \\ \underline{b}_i^1 \leq a_i^1 x + y_i \leq \overline{b}_i^1, i \in \overline{I} \cap \{1,\ldots,m_2\}, \\ x_{i-m_2} + y_i = 0, i \in \overline{I} \cap \{m_2+1,\ldots,m_2+n\}, \\ x \geq 0, y_i \geq 0, i \in \overline{I} \end{array} \right\} = \left\{ x \in R^n \,\middle|\, \begin{array}{c} \underline{b}_i^1 \leq a_i^1 x \leq \overline{b}_i^1, i \in I \cap \{1,\ldots,m_2\}, \\ x_i = 0, \ \ (i + m_2) \in I \ and \ i \in \{1,\ldots,n\}, \\ \underline{b}_i^1 \leq a_i^1 x + y_i \leq \overline{b}_i^1, i \in \overline{I} \cap \{1,\ldots,m_2\}, \\ x_i = 0, (i + m_2) \in \overline{I} \cap \{m_2+1,\ldots,m_2+n\}, \\ y_i = 0, i \in \overline{I} \cap \{m_2+1,\ldots,m_2+n\}, \\ x \geq 0, y_i \geq 0, i \in \overline{I} \end{array} \right\}.$$

Based on Property 2.7, the proof is complete. □

Remark 2.3. IMOLP problem (9)-(11) is a popular problem used in investigating practical problems because the condition of the variables is almost satisfied for the practical problems. Since its possibly efficient set can be computed by the union of convex polyhedrons, finding most preferred solutions based on IMOLP problem (9)-(11) has many advantages.

Remark 2.4. Interval linear programming (ILP) problems are extensively investigated by many researchers, for example, Garajova and Hladik [13], Hladik [16]. Since ILP problems are a special case of IMOLP problems, the above presented results for IMOLP problem (1)-(3) are also valid for ILP problems.

3. Conclusions

We propose a method to find all possibly efficient solutions of an IMOLP problem with interval coefficients in objective functions and the right-hand side vector. The set of all possibly efficient solutions of an IMOLP problem can be computed by a union of a finite number of systems of interval linear equations or inequalities, and can be computed by a union of a finite number of convex polyhedrons when the variables of the problem are non-nagative. The proposed method is simple, is easy to implement and illustrated by a numerical example. Some sufficient conditions to obtain all possibly efficient solutions of an IMOLP problem in a general case (with interval coefficients in the objective functions, the constraint matrix and the right-hand side vector) are also given. Other sufficient conditions can be found based on investigating a system of interval linear equations and inequalities. This will be dealt with in another opportunity.

We would like to introduce some new notions about the efficiencies of an IMOLP problem. For a feasible point (solution) $x \in L(IA, ib)$, x is said to be *efficient* for $P(IA, IC, ib)$, if there is $C \in IC$ such that there is no $y \in L(IA, ib)$ such that $Cx \leq Cy$ and $Cx \neq Cy$, x is called *necessarily efficient* for $P(IA, IC, ib)$ if there are $A \in IA$ and $b \in ib$ such that it is efficient of $P(A, C, b)$ for all $C \in IC$, and is called *strongly efficient* for $P(IA, IC, ib)$ if it is efficient for $P(IA, IC, ib)$ for all $C \in IC$. The set of all efficient solutions of a problem $P(IA, IC, ib)$) is called an *efficient set* of $P(IA, IC, ib)$ and denoted by $E(IA, IC, ib)$. The set of all necessarily efficient solutions of a problem $P(IA, IC, ib)$), denoted by $E_n(IA, IC, ib)$, is called a *necessarily efficient set* and the set of all strongly efficient solutions of a problem $P(IA,IC,ib)$), denoted by $E_s(IA,IC,ib)$, is called a *strongly efficient set* of $P(IA,IC,ib)$. It is easily seen that

$$E_s(IA,IC,ib) \subseteq E(IA,IC,ib) \subseteq E_p(IA,IC,ib) \text{ and}$$

$$E_s(IA,IC,ib) \subseteq E_n(IA,IC,ib) \subseteq E_p(IA,IC,ib).$$

Based on these, if strongly efficient solutions exist, based on the set $E_s(IA,IC,ib)$, the decision maker can find most preferred solutions with more advantages than based on $E_p(IA,IC,ib)$ or $E_n(IA,IC,ib)$. It can be seen that $E(IA,IC,ib) \neq \emptyset$ if and only if $E_p(IA,IC,ib) \neq \emptyset$, and most preferred solutions chosen from $E_p(IA,IC,ib)$ are efficient. Therefore, determining the efficient and strongly efficient sets plays an important role in finding most preferred solutions. Methods to do these are dealt with in detail in Tu ([39,40]).

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