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Carolin Hannusch *

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The Feit-Thompson Conjecture: An Example for Mathematical Research Based on Contradictory Sources

Carolin Hannusch

Faculty of Informatics, University of Debrecen, Debrecen, Hungary; hannusch.carolin@inf.unideb.hu

Abstract: The present study aims to show on the one hand how easily one can get confused when doing research on the internet. Different platforms, fora and homepages supply the visitor with different - and sometimes contradictory - information. On the other hand, we show how to overcome these contradictions and produce valuable research based on almost all sources. For our study, we use the example of the Feit-Thompson conjecture, which is - in opposite to the well-known Feit-Thompson theorem - not widely known. We try to clarify the scientific literature on this topic and we embed our own result into the present state of art. We prove that $\frac{p^q-1}{p-1}$ does not divide $\frac{q^p-1}{q-1}$ if p and q are twin primes, i.e. $p = q + 2$. Thus we prove that the Feit-Thompson conjecture is true for twin prime numbers.

Keywords: Feit-Thompson conjecture; prime numbers; mathematical research; research education; twin primes; generalized twin prime conjecture

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1. Introduction

This paper has two main goals. On the one hand, we prove that the Feit-Thompson conjecture on prime numbers is also true for twin primes. On the other hand, we want to use the Feit-Thompson conjecture as an example for doing research with / despite contradictory sources, which can be easily accessed. As a result, we give a detailed overview of the scientific and more or less scientific literature on the state of art of the conjecture. The Feit-Thompson conjecture is - in contrast to the Feit-Thompson theorem - not widely known. Therefore the literature which can be found about it is not too much, but as it turns out, the little existing literature on the topic contains contradictions, which we want to clarify in the current paper. The present paper is organized as follows. We start with a short survey about prime numbers and pairs of twin prime numbers in Section 2. After that, we introduce the available contradictory sources to the reader in Section 3. In Section 4 we prove our own results on Feit-Thompson conjecture for twin primes. Finally, we draw our conclusions in Section 5.

2. Twin Primes

A pair of twin prime numbers is a pair of two prime numbers whose difference is 2, i.e. they are neighbouring or consecutive odd numbers. The smallest pair of twin primes is $(3, 5)$. The prime number 5 is the only prime number which is part of two distinct pairs of twin primes, since $(5, 7)$ is also a pair of twin primes. A list of twin primes can be found at OEIS [1] under the entry A077800. Furthermore, for larger pairs of twin primes we know that they have the form $(6n - 1, 6n + 1)$ for some natural number n, since all prime numbers greater than 2 and 3 lie next to a number which is divisible by 6 [2]. We can prove this statement easily in the following way. Every natural number can be expressed as $6k + r$ for some integers k and r. The number r is called the remainder and $r \in \{0, 1, 2, 3, 4, 5\}$. If p is prime ($p > 2$) and $p = 6k + r$, then r cannot be even, thus $r \notin \{0, 2, 4\}$. If $r = 3$, then $p = 6k + 3 = 3(2k + 1)$, which implies that p is divisible by 3, which is a contradiction to p prime. Therefore we have $r \in \{1, 5\}$. Since the difference of twin primes is 2 by definition, we have that $(p, p + 2)$ is a pair of twin primes if and only if $p = 6k + 5$, which implies $p = 6n - 1$ for $n = k + 1$. Actually, in 1957 the largest known pair of twin primes was $(1000000009649, 1000000009651)$ [3], whereas today the largest known pair of

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twin primes is $(2996863034895 \cdot 21290000 + 1, 2996863034895 \cdot 21290000 - 1)$ [4]. This fact shows how the development of computational ability on computers could speed up the state of art of prime search within the last century.

The famous twin prime conjecture says that there exist infinitely many pairs of twin primes, i.e. there exist infinitely many prime numbers p, such that $p + 2$ is also a prime number. For some more background and elementary number theoretical knowdledge we refer the reader for example to [5] and [3]. For more details and recent developments on the twin prime conjecture, we refer the reader to [6] and for some history on the origin of the twin prime conjecture to [7].

A more general conjecture goes back to de Polignac [8] who conjectured that there is an infinite number of primes p, such that $p + 2k$ is also prime for some natural number k. The generalized form of the twin prime conjecture is connected to the (strong) Goldbach conjecture [9] which states that every integer greater than 4 can be expressed as the sum of two prime numbers [10].

For general k, several partial results were achieved during the last years. Zhang [11] showed that there exists a number $N < 10^7$, such that there exist infinitely many prime numbers p and q, such that $p - q = N$. Maynard could improve this bound to $N < 600$ [12]. Recently, Wright improved the bound of Maynard-Tao, Baker-Irving, and the participants of the project Polymath 8b in [13].

It is clear that for $k = 1$ the conjecture of de Polignac coincides with the twin prime conjecture. Nevertheless, it is not sure when exactly was the first time the twin prime conjecture was stated, it may go back to Euclid's time [14].

A prime number is called isolated if it is not part of a pair of twin primes. The sequence of the first isolated primes can be seen under A007510 in [1].

According to Brun's theorem [15], [16], the sum of the reciprocals of twin primes converges to a finite value (called Brun's constant). As a consequence, almost all prime numbers are isolated, i.e. almost all prime numbers are not contained in a pair of twin primes.

Partial results make us believe that the twin prime conjecture is true [17], [18], [19]. Nevertheless, the proof of the twin prime conjecture remains unsolved as per today.

3. Contradictory Sources

Feit and Thompson published an introductory paper in 1962 [20] preceding their famous proof of their theorem on solvable groups [21]. In this introductory paper the authors state that "The validity of ... $\frac{p^q-1}{p-1}$ never divides $\frac{q^p-1}{q-1}$ if p, q are distinct primes would also simplify the proof..." Based on this remark, we can state the Feit-Thompson conjecture:

Conjecture 1. *[Feit-Thompson] Let p and q be distinct prime numbers. Then $\frac{p^q-1}{p-1}$ does not divide $\frac{q^p-1}{q-1}$.*

The first who referred to the remark of Feit and Thompson was Stephens in 1971 [22]. In his paper, the author gives a counterexample to the conjecture that for distinct primes p and q the numbers $\frac{p^q-1}{p-1}$ and $\frac{q^p-1}{q-1}$ are always coprime, i.e. their greatest common divisor is 1. This conjecture was later called the Stephens conjecture:

Conjecture 2. *[Stephens] Let p and q be distinct prime numbers. Then $\frac{p^q-1}{p-1}$ and $\frac{q^p-1}{q-1}$ are coprime.*

If this conjecture would have been true, then the Feit-Thompson conjecture would have been true as well. But the Stephens conjecture is not true [22] and therefore we still do not know if the Feit-Thompson conjecture is true or not, as Stephens concludes this in the last sentence of his paper. So, instead of one conjecture we have two conjectures which have a big difference in mathematical sense. This difference is important for any research on this topic. In the present paper, we investigate Conjecture 1, which is regarded as the *real* Feit-Thompson conjecture by the mathematical community.

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Motose [23], [24] was the first who differed between Feit-Thompson conjecture and Stephens conjecture. Motose shows partial solutions to the Feit-Thompson conjecture. Le shows in [25] that the Feit-Thompson conjecture is true if $q = 3$.

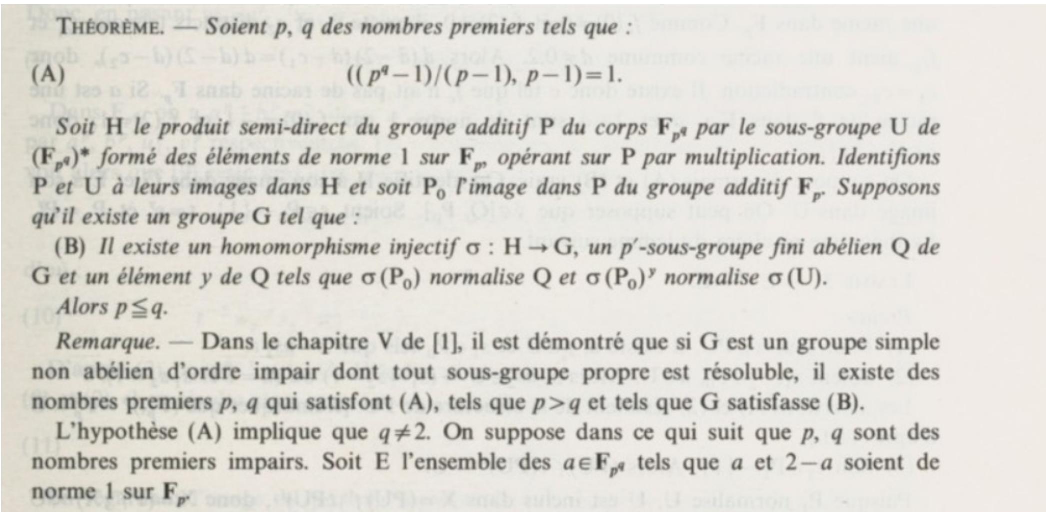
In contrast to the previously mentioned sources, on Wolfram Math World [26] the following is stated: "The Feit-Thompson conjecture asserts that there are no primes p and q for which $(p^q - 1)/(p - 1)$ and $(q^p - 1)/(q - 1)$ have a common factor." Further, Le writes in [25] that "E.T.Parker observed that the very long proof by W.Feit and J.Thompson ... that every group of odd order is solvable would be shortened if it could be proved that $\frac{p^q-1}{p-1}$ never divides $\frac{q^p-1}{q-1}$." This statement is remarkable and it is contradictory to the fact that Feit and Thompson themselves stated this in their paper [20]. Parkers name also occurs on the Wolfram Math website, again without reference.

The article on Wikipedia [27] remarks in "External links" the confusion of the Feit-Thompson conjecture and Stephens conjecture, but on Groups Wiki [28] this remark is missing.

Things become even more interesting when reading the math overflow forum on this topic [29] where one user draws the attention to a paper of Peterfalvi [30] which simplifies the proof of the Feit-Thompson theorem by two pages. Peterfalvi's proof is group theoretical and working on generators of groups. Indeed, if we go back to Feit's and Thompson's paper [20] then we can read in the last paragraph:

"The proof would be simplified considerably if it is true that nonabelian simple groups never contain self-normalizing cyclic subgroups. The validity of the con- jecture that $\frac{p^q-1}{p-1}$ never divides $\frac{q^p-1}{q-1}$ if p, q are distinct primes would also simplify the proof, rendering unnecessary the detailed use of generators and relations. If it is true that nonidentity Sylow subgroups of simple groups always contain nonidentity abelian weakly closed subgroups, short proofs of the necessary group-theoretic lemmas could be given."

Indeed, the (positive) proof of the number theoretical Feit-Thompson conjecture would make the original proof of the Feit-Thompson theorem simpler, but this simplification was already achieved by the group theoretical approach of Peterfalvi [30].



THÉORÈME. — *Soient p, q des nombres premiers tels que :*

(A) $((p^q-1)/(p-1),\ p-1)=1.$

Soit H *le produit semi-direct du groupe additif* P *du corps* $\mathbf{F}_{p^q}$ *par le sous-groupe* U *de* $(\mathbf{F}_{p^q})^*$ *formé des éléments de norme* 1 *sur* $\mathbf{F}_p$, *opérant sur* P *par multiplication. Identifions* P *et* U *à leurs images dans* H *et soit* P_0 *l'image dans* P *du groupe additif* $\mathbf{F}_p$. *Supposons qu'il existe un groupe* G *tel que :*

(B) *Il existe un homomorphisme injectif* $\sigma : H \to G$, *un* p'*-sous-groupe fini abélien* Q *de* G *et un élément* y *de* Q *tels que* $\sigma(P_0)$ *normalise* Q *et* $\sigma(P_0)^y$ *normalise* $\sigma(U)$.

Alors $p \leq q$.

Remarque. — Dans le chapitre V de [1], il est démontré que si G est un groupe simple non abélien d'ordre impair dont tout sous-groupe propre est résoluble, il existe des nombres premiers p, q qui satisfont (A), tels que $p>q$ et tels que G satisfasse (B).

L'hypothèse (A) implique que $q \neq 2$. On suppose dans ce qui suit que p, q sont des nombres premiers impairs. Soit E l'ensemble des $a \in \mathbf{F}_{p^q}$ tels que a et $2-a$ soient de norme 1 sur $\mathbf{F}_p$.

Figure 1. Extract from Peterfalvi's paper [30]

Summarizing, we can say that the Feit-Thompson conjecture is not of much group theoretical interest any longer, but nevertheless it remains an unsolved number theoretical problem.

Both, the solution of the twin prime conjecture and the solution of the Feit-Thompson conjecture would be a great contribution to the field of Number Theory.

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4. Embedding our Result into the Scientific Literature

4.1. Auxiliary Result

In the proof of our main result in the next section we will need the following Lemma, which we recall here. We use the notation of linear congruence and the notation $a \mid b$ for a divides b:

$$x \equiv y \mod z \Leftrightarrow z \mid x - y.$$

Lemma 1. *Let m be a positive integer. Then*

$$(4m+5)^{4m+3} - 1 \equiv 0 \mod 4$$

and

$$(4m+3)^{4m+5} - 1 \equiv 2 \mod 4.$$

Proof.

$$(4m+5)^{4m+3} - 1 \equiv (4m+4+1)^{4m+3} - 1 \equiv 1^{4m+3} - 1 \equiv 0 \mod 4$$

and

$$(4m+3)^{4m+5} - 1 \equiv 3^{4m+5} - 1 \equiv 3 - 1 \equiv 2 \mod 4$$

since $4m+5$ is odd.

4.2. Main Result and Proof

Theorem 1. *There do not exist twin primes p and $q = p+2$, such that $\frac{q^p-1}{q-1}$ divides $\frac{p^q-1}{p-1}$.*

Proof. First, we consider the case $p = 4m+3$ for some integer m and we show that in this case $\frac{(p+2)^p-1}{p+1}$ does not divide $\frac{p^{p+2}-1}{p-1}$. We assume indirectly that there exist p and $p+2$ prime numbers such that $\frac{(p+2)^p-1}{p+1} \mid \frac{p^{p+2}-1}{p-1}$. Then there exists $k \in \mathbb{N}$ such that

$$\frac{k((p+2)^p - 1)}{p+1} = \frac{p^{p+2}-1}{p-1}.$$

This implies that

$$k = \frac{p+1}{p-1} \cdot \frac{p^{p+2}-1}{(p+2)^p - 1} \in \mathbb{N}.$$

Therefore we have either

$$\frac{p^{p+2}-1}{(p+2)^p-1} \in \mathbb{N} \text{ and } \frac{2}{p-1} \cdot \frac{p^{p+2}-1}{(p+2)^p-1} \in \mathbb{N}$$

or

$$\left\{ \frac{p^{p+2}-1}{(p+2)^p-1} \right\} + \left\{ \frac{2}{p-1} \cdot \frac{p^{p+2}-1}{(p+2)^p-1} \right\} = 1,$$

where $\{x\}$ denotes the fractional part of x.

Considering the first case

$$\frac{p^{p+2}-1}{(p+2)^p-1} \in \mathbb{N} \text{ and } \frac{2}{p-1} \cdot \frac{p^{p+2}-1}{(p+2)^p-1} \in \mathbb{N},$$

if $p \equiv 3 \mod 4$, then by Lemma 1 $p^{p+2} - 1$ cannot be divisible by $(p+2)^p - 1$. Therefore we have a contradiction in this case. If $p \equiv 1 \mod 4$, then $q \equiv 3 \mod 4$, and therefore by Motose's proposition [23] $\frac{(p+2)^p-1}{p+1}$ does not divide $\frac{p^{p+2}-1}{p-1}$.

Considering the second case, we denote $x := \frac{p^{p+2}-1}{(p+2)^p-1}$. Then we need to consider for which x we have

$$\{x\} + \left\{\frac{2}{p-1} \cdot x\right\} = 1.$$

That means we need to solve

$$x \cdot \frac{p+1}{p-1} = y \text{ for some } y \in \mathbb{N}.$$

Since y can be even or odd, we need to solve the following congruences for some prime number p

$$x \cdot \frac{p+1}{p-1} \equiv 0 \mod 2 \tag{1}$$

and

$$x \cdot \frac{p+1}{p-1} \equiv 1 \mod 2. \tag{2}$$

Congruence 1 is equivalent to $2 \mid \frac{(p^{p+2}-1)(p+1)}{((p+2)^p-1)(p-1)}$ and congruence 2 is equivalent to $2 \mid \frac{(p^{p+2}-1)(p+1)}{((p+2)^p-1)(p-1)} - 1$. Both are only possibly if

$$\frac{(p^{p+2}-1)(p+1)}{((p+2)^p-1)(p-1)} \in \mathbb{N}.$$

We know that $\frac{(p^{p+2}-1)}{(p-1)} \in \mathbb{N}$, therefore $\frac{(p^{p+2}-1)(p+1)}{(p-1)} \in \mathbb{N}$. Thus, we need to answer the question if $(p+2)^p - 1 \mid \frac{(p^{p+2}-1)(p+1)}{(p-1)}$ for some p. Solving the following congruence

$$\frac{(p^{p+2}-1)(p+1)}{(p-1)} \equiv 0 \mod (p+2)^p - 1,$$

we get $p = 2$ as the only possible solution. In the next section we will show that $p = 2$ does not fulfill the Feit-Thompson conjecture. By the way, $p = 2$, is not part of a twin prime pair.

4.3. *The Cases $p = 2, p = 3, q = 2$*

It is easy to see that if $p = 2$, then $2^q - 1$ needs to divide $q + 1$. This is only possible for $q = 2$. Similarly, if $p = 3$, then $\frac{3^q-1}{2}$ needs to be a divisor of $q^2 + q + 1$, which is impossible for $q \geq 4$. Furthermore, if $q = 2$, then $p + 1$ is a divisor of $2^p - 1$. Since by assumption p is a prime number distinct from q, we have that p is odd. Therefore $p + 1$ is even, whereas $2^p - 1$ is odd, thus $p + 1$ cannot be a divisor of $2^p - 1$. Together with Le's result [25] we have $p, q \notin \{2, 3\}$.

5. Conclusion

In this paper, we had two goals (see Section 1). First, we proved in Section 4 that the Feit-Thompson conjecture is true for pairs of twin prime numbers by using elementary mathematical methods. According to Section 2, we may believe that this means for an infinite set of prime numbers p, we have that $\frac{(p+2)^p-1}{p+1}$ never divides $\frac{p^{p+2}-1}{p-1}$.

The Feit-Thompson conjecture remains unsolved in general.

For our second goal, we showed through an example how research questions can be found and answered in the times of digitalization. Since nowadays a lot of information and sources are available

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to a broad part of society through encyclopediae, fora, books, journals, etc. in the internet [31], [32], it is necessary to sensitize the next generation of researchers for the advantages and disadvantages of research in the internet. People today have access to much more information than people ever had before, but not all information are complete and/or reliable. We need to learn to select useful and correct information from the mass of all information in order to build up our research on the right facts and correct information. On the one hand, it is good that many people have access to information today. On the other hand, information is not equal to education. The role of universities is changing, since universities have to deal with the fact that they are not the only place where knowledge is available [33], [34], [35], [36]. Next to the Civic University movement, where universities try to change their education into a market-driven way of educating young employees for factories and companies [37], we should not forget about the research component. Universities need to remain also a place for research [38], [39]. In order to make the improvement and development of science possible, we also need young researchers to be educated to do research among the wideness of information today. This education also needs to include the investigation of incomplete and even unreliable sources in order to educate researchers. We want to draw the attention of university leaders and educators to this - probably growing - problem of half-reliable, incomplete and confusing sources in both, scientific and more-or-less scientific literature, all easily accessible. The author believes that only if researchers are aware of the fact how easily they could do research based on non-reliable or only half-reliable sources will prevent science from developing into self-destructive or non-trustworthy directions in the future.

Data Availability Statement: The author declares that no data was used for the current research. All mentioned sources are freely accessible on the given platform.

Conflicts of Interest: The author declares that there is no potential conflict of interests, neither of personal origin nor of financial one.

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