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Abraham A. Ungar *

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When Four Cyclic Antipodal Points Are Ordered Counterclockwise in Euclidean and Hyperbolic Geometry

Abraham A. Ungar

Department of Mathematics, North Dakota State University; abraham.ungar@ndsu.edu

Abstract: A cyclic antipodal points of a circle is a pair of points that are the intersection of the circle with a diameter of the circle. A recent proof of Ptolemy's Theorem, simultaneously in both (i) Euclidean geometry; and (ii) the relativistic model of hyperbolic geometry (which is identified with the Klein model of hyperbolic geometry), motivates in this article the study of four cyclic antipodal points of a circle, ordered arbitrarily counterclockwise. The translation of results from Euclidean geometry into hyperbolic geometry is obtained by means of hyperbolic trigonometry, called gyrotrigonometry, to which Einstein addition gives rise. Formulas that extend the Pythagorean formula in both Euclidean and hyperbolic geometry are obtained as byproducts.

Keywords: cyclic antipodal points; relativistic model of hyperbolic geometry; gyrovector space; gyrotrigonometry

1. Introduction

Trigonometric identities, viewed in analytic Euclidean geometry can be viewed in analytic hyperbolic geometry as well, as demonstrated in [1]. A trigonometric identity made up of sines of half angles is said to be *half-angled*, if viewed in Euclidean geometry, or *half-gyroangled*, if viewed in hyperbolic geometry. Thus, for instance, the trigonometric identities (1), (4) and (47) are half-angled when viewed in Euclidean geometry, and are half-gyroangled when viewed in hyperbolic geometry. For a fruitfull reading of this article, familiarity with [1] is required.

The study of Ptolemy's Theorem (i) in Euclidean geometry and (ii) in the relativistic model of hyperbolic geometry in [1] and (iii) in the Poincaré ball model of hyperbolic geometry in [2] is solely based on the half-angled trigonometric identity

$$\begin{aligned} \sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\delta}{2} &= \sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2} \\ \alpha+\beta+\gamma+\delta &= 2\pi \end{aligned} \tag{1}$$

for any $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that $\alpha+\beta+\gamma+\delta=2\pi$. Identity (1) is viewed both trigonometrically (that is, in Euclidean geometry) and gyrotrigonometrically (that is, in hyperbolic geometry).

Remarkably, thus, (1) gives rise simultaneously to the Euclidean and to the hyperbolic Ptolemy's Theorem, as shown in [1] where the novel hyperbolic Ptolemy's Theorem is established.

Trigonometric identities are commonly viewed in analytic Euclidean geometry. However, they can be viewed in analytic hyperbolic geometry as well, where they are said to be *gyrotrigonometric identities*, as explained in [1]. Accordingly, if viewed in hyperbolic geometry, (1) is said to be a *half-gyroangled gyrotrigonometric identity*. Furthermore, we note the following observations in [1] and [2]:

1. It is shown in [1] that if viewed in Euclidean geometry, the half-angled trigonometric identity (1) gives rise to Ptolemy's Theorem in Euclidean geometry.
2. It is shown in [1] that if viewed in the relativistic model of hyperbolic geometry, the half-gyroangled gyrotrigonometric identity (1) gives rise to the novel hyperbolic Ptolemy's Theorem in the relativistic model of hyperbolic geometry. Remarkably, the relativistic model of hyperbolic geometry, in turn, is identical with the well-known Klein model of hyperbolic geometry [3].

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3. It is shown in [2] that if viewed in the Poincaré ball model of hyperbolic geometry, the half-gyroangled gyrotrigonometric identity (1) gives rise to the novel hyperbolic Ptolemy's Theorem in the Poincaré ball model of hyperbolic geometry.

The study of Ptolemy's Theorem in [1] and [2] is based on the half-angled trigonometric identity (1). Similarly, the study of cyclic antipodal points in this article is based on the half-angled trigonometric identity (4). Identity (4) is viewed both (i) *trigonometrically*, as a half-angled trigonometric identity in Euclidean geometry, and (ii) *gyrotrigonometrically*, as a half-gyroangled gyrotrigonometric identity in hyperbolic geometry.

Any half-angled trigonometric identity admits a geometric realization in both Euclidean geometry and hyperbolic geometry. In this article, identity (4) is realized geometrically (i) in Euclidean geometry, giving rise to the Four Cyclic Antipodal Points Theorem 1 in the Euclidean plane; and (ii) in hyperbolic geometry, giving rise to the Four Gyrocyclic Antipodal Points Theorem 3 in the relativistic model of the hyperbolic plane.

Finally, (i) in Section 8 we realize geometrically the half-angled trigonometric identity (47) in Euclidean geometry, obtaining the Pythagorean identity; and (ii) in Section 9 we realize (47) geometrically in the relativistic model of hyperbolic geometry, obtaining a novel hyperbolic Pythagorean identity.

2. Four Cyclic Antipodal Points

A cyclic antipodal points of a circle in a Euclidean plane $\mathbb{R}^2$ is a pair (A, A') of points $A, A' \in \mathbb{R}^2$ that are the intersection of the circle with a diameter of the circle. Illustrating Theorem 1, a circle with four cyclic antipodal points is depicted in Figure 1. The Euclidean distance $|AB|$ between two points $A, B \in \mathbb{R}^2$ is given by

$$|AB| := \| -A + B\| \,. \tag{2}$$

Equivalently, $|AB|$ is the Euclidean length of the segment AB that joins the points A and B. Contrastingly, in the context of hyperbolic geometry, $|AB|$ is a corresponding hyperbolic length, called *gyrolength*, as in (31).

Theorem 1. (A Four Cyclic Antipodal Points Theorem). *Let $\Sigma(O, r)$ be a circle in the Euclidean plane $\mathbb{R}^2$ with radius r, centered at $O \in \mathbb{R}^2$, with four cyclic antipodal points (A, A'), (B, B'), (C, C') and (D, D'), such that the eight points $A, B, C, D, A', B', C', D' \in \mathbb{R}^2$ are arbitrarily ordered counterclockwise (or clockwise), as shown in Figure 1.*

Then, the four cyclic antipodal points satisfy the identity

$$\begin{aligned} |AB'||BC'||CD'| - |AB'||BC||CD| - |AB||BC'||CD| \\ - |AB||BC||CD'| = 4r^2|A'D| \,. \end{aligned} \tag{3}$$

Proof. Clearly, $r = \frac{1}{2}|AA'| = \frac{1}{2}|BB'| = \frac{1}{2}|CC'| = \frac{1}{2}|DD'|$.

The proof of the theorem is based on the elegant half-angled trigonometric identity

$$\begin{aligned} \sin\frac{\alpha+\pi}{2}\sin\frac{\beta+\pi}{2}\sin\frac{\gamma+\pi}{2} - \sin\frac{\alpha+\pi}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \\ - \sin\frac{\alpha}{2}\sin\frac{\beta+\pi}{2}\sin\frac{\gamma}{2} \\ - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma+\pi}{2} = \sin\frac{\delta}{2} \end{aligned} \tag{4}$$

which holds for all $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$ that satisfy the condition

$$\alpha + \beta + \gamma + \delta = \pi \,. \tag{5}$$

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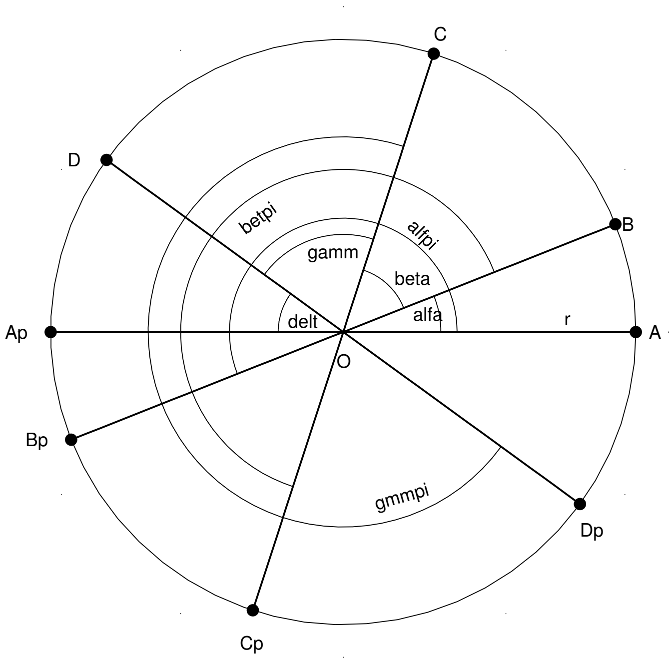

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Figure 1. Four cyclic antipodal points, (A, A'), (B, B'), (C, C'), (D, D'), on a circle $\Sigma(O, r)$ centered at O with radius r, and their corresponding O-vertex angles α, β, γ, δ. The points $A, B, C, D, A', B', C', D'$ are arbitrarily ordered counterclockwise, implying $\alpha + \beta + \gamma + \delta = \pi$. The identity of Theorem 1 is shown, where $|AB| = \| - A + B\|$, *etc.*

Noting that $\sin \frac{\alpha+\pi}{2} = \cos \frac{\alpha}{2}$, the trigonometric identity in (4)–(5) can readily be verified. When realized geometrically in Figure 1, this trigonometric identity reveals its unexpected grace and elegance.

In order to realize (4)–(5) geometrically by a circle with four cyclic antipodal points, shown in Figure 1, we define the four O-vertex angles in Figure 1 as follows.

$$\begin{aligned} \alpha &= \angle AOB \\ \beta &= \angle BOC \\ \gamma &= \angle COD \\ \delta &= \angle DOA' \,. \end{aligned} \tag{6}$$

We note that $\alpha + \beta + \gamma + \delta = \pi$, as required by Condition (5), since the points $A, B, C, D, A', B', C', D'$ are ordered counterclockwise.

Then, consequently, the remaining three O-vertex angles in Figure 1 are

$$\begin{aligned} \angle AOB' &= \alpha + \pi \\ \angle BOC' &= \beta + \pi \\ \angle COD' &= \gamma + \pi \,. \end{aligned} \tag{7}$$

Applying the law of cosines to triangle AOB yields

$$\begin{aligned} |AB|^2 &= 2r^2 - 2r^2 \cos \alpha \\ &= 2r^2(1 - \cos \alpha) \\ &= 4r^2 \sin^2 \frac{\alpha}{2} \end{aligned} \tag{8}$$

so that

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$$|AB| = 2r\sin\frac{\alpha}{2}\,. \tag{9}$$

Similarly to (9), by means of (6)–(7), we obtain the following seven results:

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \frac{1}{2r}|AB| \\
\sin\frac{\beta}{2} &= \frac{1}{2r}|BC| \\
\sin\frac{\gamma}{2} &= \frac{1}{2r}|CD| \\
\sin\frac{\delta}{2} &= \frac{1}{2r}|A'D| \\
\sin\frac{\alpha+\pi}{2} &= \frac{1}{2r}|AB'| \\
\sin\frac{\beta+\pi}{2} &= \frac{1}{2r}|BC'| \\
\sin\frac{\gamma+\pi}{2} &= \frac{1}{2r}|CD'|\,.
\end{aligned} \tag{10}$$

Substituting the sines in (10) into the half-angled trigonometric identity (4) yields (3), as desired. □

We say that (3) realizes the half-angled trigonometric identity (4) geometrically in the Euclidean plane.

The proof of Theorem 1 is motivated by the proof of Ptolemy's Theorem in [1]. In [1] and [2] Ptolemy's Theorem is extended to hyperbolic geometry. Similarly, Theorem 1 can be extended to hyperbolic geometry as well, as we will see in Section 6.

3. Special Cases

In the special case when $A = B$ and, hence, $A' = B'$, the result (3) of Theorem 1 descends to

$$\begin{aligned}
&|BB'|BC'|CD'| - |BB'||BC||CD| - |BB||BC'||CD| \\
&\quad - |BB||BC||CD'| = 4r^2|B'D|\,.
\end{aligned} \tag{11}$$

Noting that $|BB'| = 2r$ and $|BB| = 0$, (11) yields

$$|BC'||CD'| - |BC||CD| = 2r|B'D|\,. \tag{12}$$

Formalizing the result in (12) we obtain the result (17) of Corollary 2.

In a second special case, when $D = A'$ and, hence, $D' = A$, the result (3) of Theorem 1 descends to

$$\begin{aligned}
&|AB'||BC'||AC| - |AB'||BC||A'C| - |AB||BC'||A'C| \\
&\quad - |AB||BC||AC| = 4r^2|A'A'| = 0\,.
\end{aligned} \tag{13}$$

Equation (13) gives rise to the elegant equation

$$\frac{|AB'|}{|AB|}\frac{|BC'|}{|BC|} - \frac{|AB'|}{|AB|}\frac{|A'C|}{|AC|} - \frac{|BC'|}{|BC|}\frac{|A'C|}{|AC|} = 1\,. \tag{14}$$

Equation (14) gives rise to the inequalities

$$\frac{|AB'|}{|AB|} > \frac{|A'C|}{|AC|} \tag{15}$$

and

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$$\frac{|BC'|}{|BC|} > \frac{|A'C|}{|AC|}\,. \tag{16}$$

Formalizing, we have the following Corollary.

Corollary 2. (A Three Cyclic Antipodal Points Theorem). *Let $\Sigma(O,r)$ be a circle in the Euclidean plane $\mathbb{R}^2$ with radius r, centered at $O \in \mathbb{R}^2$, with three cyclic antipodal points (A,A'), (B,B') and (C,C'). The six points $A,B,C,A',B',C' \in \mathbb{R}^2$ are arbitrarily ordered counterclockwise (or clockwise), as shown in Figure 2.*

Then,

$$|AB'||BC'| - |AB||BC| = 2r|A'C| \tag{17}$$

and

$$\frac{|AB'|}{|AB|}\frac{|BC'|}{|BC|} - \frac{|AB'|}{|AB|}\frac{|A'C|}{|AC|} - \frac{|BC'|}{|BC|}\frac{|A'C|}{|AC|} = 1 \tag{18}$$

and

$$|AB||BC'| + |AB'||BC| = 2r|AC|\,. \tag{19}$$

Proof. Equations (17) and (18) are proved, respectively, in (12) and (14). Equation (19) follows from (17) and (18) by eliminating the term

$$(|AB'|/|AB|)(|BC'|/|BC|)$$

between these two equations, □

In order to recover the Pythagorean theorem we consider (17) and (19) in the special case when $C = A'$ and, hence, $C' = A$. In this special case (17) descends to

$$|AB'||AB| - |AB||A'B| = 0 \tag{20}$$

implying

$$|AB'| = |A'B| \tag{21}$$

and (19) descends to

$$|AB|^2 + |AB'||A'B| = 2r|AA'| = 4r^2\,. \tag{22}$$

Equations (21) and (22) recover the Pythagorean theorem

$$|AB|^2 + |AB'|^2 = 4r^2 = |BB'|^2 \tag{23}$$

for right angled triangles ABB', where (A,A') and (B,B') are any distinct pairs of cyclic antipodal points.

4. Einstein Addition, Gyrogroups, Gyrovector Spaces and the Relativistic Model of Hyperbolic Geometry

In this section we set the stage for the extension of Theorem 1 from Euclidean geometry to the relativistic model of hyperbolic geometry. The relativistic model of hyperbolic geometry, in turn, stems from Einstein addition, and is identical with the well-known Klein model of hyperbolic geometry [3,4].

Einstein addition $\oplus : \mathbb{R}^n_s \times \mathbb{R}^n_s \to \mathbb{R}^n_s$ is a binary operation in the s-ball $\mathbb{R}^n_s$ of the Euclidean n-space $\mathbb{R}^n$,

$$\mathbb{R}^n_s = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\}\,, \tag{24}$$

$n \in \mathbb{N}$ ($n = 3$ in physical applications), where $s > 0$ is an arbitrarily fixed constant that in physical applications represents the vacuum speed of light $s = c$. Einstein addition $\oplus$ is given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{25}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}} \,. \tag{26}$$

For later reference we note that it follows from (26) that

$$\frac{||\mathbf{v}||^2}{s^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \tag{27}$$

so that

$$\gamma_{\mathbf{v}}^2 - 1 = \frac{1}{s^2}\gamma_{\mathbf{v}}^2||\mathbf{v}||^2 \tag{28}$$

and, hence,

$$\sqrt{\gamma_{\mathbf{v}} - 1} = \frac{1}{s}\frac{\gamma_{\mathbf{v}}}{\sqrt{\gamma_{\mathbf{v}} + 1}}||\mathbf{v}|| \,. \tag{29}$$

Einstein addition (25) of relativistically admissible velocities was introduced by Einstein in his 1905 paper [5, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (25) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [6] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (25).

Seemingly structureless, Einstein addition is neither commutative nor associative. Therefore, as F. Chatelin noted [7], Einstein addition of 3-velocities (representing relativistically admissible velocities) was fully inconceivable: Einstein's vision was much ahead of its time. A more understandable version of Special Relativity was conceived by Minkowski in 1907 by dressing up as a physical concept the Lorentz transformation group. This is the version adopted until today in physics textbooks. Einstein's intuition was left dormant for 83 years until it was brought back to a new mathematical life in the seminal papers [8,9].

Since 1988 the author has crafted an algebraic language for hyperbolic geometry which sheds a natural light on the physical theory of Special Relativity. In the resulting language one prefixes a gyro to a classical term to mean the analogous term in hyperbolic geometry. As an example, the relativistic gyrotrigonometry of Einstein's Special Relativity is developed in [4] and employed to the study of the stellar aberration phenomenon in astronomy.

Being both gyrocommutative and gyroassociative, Einstein addition encodes a novel group-like object called a gyrogroup. Gyrogroup theory shares important analogies with group theory as shown, for instance, in [10,11]. As such, Einstein addition provides a velocity-symmetry approach to special relativity theory.

Some gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces. Einstein (Möbius) gyrovector spaces form the algebraic setting for Klein (Poincaré) ball model of hyperbolic geometry with spectacular gain in clarity and simplicity, just as vector spaces form the algebraic setting for Euclidean geometry. In our work, the Poincaré ball model of hyperbolic geometry stems from *Möbius addition* [2] and the Klein model of hyperbolic geometry stems from Einstein addition [1]. Hence, when studied analytically by means of Einstein addition, we refer the Klein model to as the relativistic model.

Both Einstein gyrovector plane and Möbius gyrovector plane admit their own gyrotrigonometry, which is fully analogous to the familiar trigonometry in the Euclidean plane.

An attractive review of gyrogroups, gyrovector spaces and gyrotrigonometry that Einstein addition encodes is available in [1, Sections 2-12]. Familiarity with this review is necessary for a fruitful reading of the sequel.

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5. The Law of Gyrocosines in the Relativistic Model of Hyperbolic Geometry

The proof of Theorem 1 is based on the law of cosines of trigonometry. Accordingly, for translating Theorem 1 from Euclidean geometry into the relativistic model of hyperbolic geometry, we consider the law of gyrocosines of gyrotrigonometry in the relativistic model of hyperbolic geometry, which is reviewed in [1, Section 12].

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ along with its standard notation shown in [1, Figure 4]. According to [12, Sect. 7.3], the gyrotriangle ABC obeys the following three identities, each of which represents its *law of gyrocosines*,

$$\begin{aligned} \gamma_a &= \gamma_b \gamma_c (1 - b_s c_s \cos\alpha) \\ \gamma_b &= \gamma_a \gamma_c (1 - a_s c_s \cos\beta) \\ \gamma_c &= \gamma_a \gamma_b (1 - a_s b_s \cos\gamma)\,. \end{aligned} \tag{30}$$

The elements a, b, c and $a_s = a/s, b_s = b/s, c_s = c/s$ and the gyroangles α, β, γ in the law of gyrocosines (30) are defined in [1, Figure 4]. The gamma factors in (30) are defined in (26), $\gamma_a = (1 - a_s^2)^{-\frac{1}{2}}$.

Note that *cos* in (30) is the *gyrocosine* function of gyrotrigonometry, defined in [1, Eq. 35] and illustrated in [1, Figure 3]. The use of the symbol *cos* to represent both (i) the cosine function of trigonometry; and (ii) the gyrocosine function of gyrotrigonometry is justified by means of the *trigonometry – gyrotrigonometry duality* explained in [1, Section 11].

In Section 2, Figure 1 depicts a circle in a Euclidean plane $\mathbb{R}^2$, along with its eight points $A, B, C, D, A', B', C', D' \in \mathbb{R}^2$. Accordingly, Figure 1 is viewed in a Euclidean plane.

In contrast, in this section the same Figure 1 is viewed in the relativistic model of a hyperbolic plane. As such, it depicts a gyrocircle in a hyperbolic plane $\mathbb{R}^2_s$, along with its eight points $A, B, C, D, A', B', C', D' \in \mathbb{R}^2_s$.

In full analogy with (2), in the context of the relativistic model of hyperbolic geometry we use the notation

$$|AB| = \|\ominus A \oplus B\| \tag{31}$$

for any $A, B \in \mathbb{R}^2_s$, where $\oplus$ denotes Einstein addition (25) in $\mathbb{R}^2_s$ (reviewed in [1]), and where $\ominus A = -A$ is the inverse of A. Here $|AB|$ represents the gyrodistance between A and B, that is, the hyperbolic distance between A and B in the relativistic model. Equivalently, $|AB|$ is the gyrolength of the gyrosegment AB that joins the points A and B.

It should be noted that in the Euclidean limit, $s \to \infty$, the hyperbolic $|AB|$, given by (31), descends to the Euclidean $|AB|$, given by (2), since

$$\lim_{s\to\infty} \|\ominus A \oplus B\| = \| - A + B\|\,. \tag{32}$$

Let $A, B \in \mathbb{R}^2_s$ be two points of a gyrocircle with gyroradius r centered at $O \in \mathbb{R}^2_s$, as shown in Figure 1. The gyroradius of the gyrocircle in Figure 1 is given by $r = |OA| = |OB|$ in the hyperbolic plane regulated by the Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$.

Then, by the law of gyrocosines (30) applied to gyrotriangle ABO in Figure 1, we have

$$\gamma_{|AB|} = \gamma_r^2 (1 - r_s^2 \cos\alpha) \tag{33}$$

where we use the usual notation $\gamma_{|AB|} = (1 - |AB|^2/s^2)^{-1/2}$ and $r_s = r/s$, $\gamma_r = (1 - r_s^2)^{-1/2}$.

Solving (33) for $\cos\alpha$, noting (27), yields

$$\cos\alpha = \frac{\gamma_r^2 - \gamma_{|AB|}}{\gamma_r^2 - 1} \tag{34}$$

so that

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$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{1}{2}\frac{\gamma_{|AB|}-1}{\gamma_r^2 r_s^2} \tag{35}$$

and, hence,

$$\sin\frac{\alpha}{2} = \frac{1}{\sqrt{2}}\frac{\sqrt{\gamma_{|AB|}-1}}{\gamma_r\, r_s}\,. \tag{36}$$

Hence, finally, by means of (29) we have

$$\sin\frac{\alpha}{2} = \frac{1}{\sqrt{2}\gamma_r\, r}|AB|_h \tag{37}$$

where we define

$$|AB|_h := \frac{\gamma_{|AB|}}{\sqrt{1+\gamma_{|AB|}}}|AB| \tag{38}$$

for any $A, B \in \mathbb{R}^2_s$.

Note that here *sin* and *cos* represent the *gyrosine* and the *gyrocosine* functions in the gyrotrigonometry of the relativistic model of hyperbolic geometry.

The result in (37) is obtained by applying the law of gyrocosines to the O-gyrovertex gyrotriangle OAB with the O-gyrovertex gyroangle α. Expressing (37) to all the O-gyrovertex gyroangles in Figure 1, which are $\alpha, \beta, \gamma, \delta, \alpha+\pi, \beta+\pi$ and $\gamma+\pi$, where $\alpha+\beta+\gamma+\delta=\pi$, yields the following seven equations:

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|AB|_h \\
\sin\frac{\beta}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|BC|_h \\
\sin\frac{\gamma}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|CD|_h \\
\sin\frac{\delta}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|A'D|_h \\
\sin\frac{\alpha+\pi}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|AB'|_h \\
\sin\frac{\beta+\pi}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|BC'|_h \\
\sin\frac{\gamma+\pi}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|CD'|_h\,.
\end{aligned} \tag{39}$$

Substituting the sines in (39) into the half-gyroangled gyrotrigonometric identity (4) yields the equation

$$\begin{aligned}
|AB'|_h|BC'|_h|CD'|_h - |AB'|_h|BC|_h|CD|_h - |AB|_h|BC'|_h|CD|_h \\
- |AB|_h|BC|_h|CD'|_h = 2\gamma_r^2 r^2 |A'D|_h\,.
\end{aligned} \tag{40}$$

We say that (40) realizes the half-gyroangled gyrotrigonometric identity (4) geometrically in the relativistic model of the hyperbolic plane.

6. Four Gyrocyclic Antipodal Points

Formalizing the result in (40), we obtain the following theorem.

Theorem 3. (A Four Gyrocyclic Antipodal Points Theorem). *Let $\Sigma(O, r)$ be a gyrocircle in the hyperbolic plane $(\mathbb{R}^2_s, \oplus, \otimes)$ with gyroradius r, centered at $O \in \mathbb{R}^2_s$, with four gyrocyclic antipodal points (A, A'),*

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(B,B'), (C,C') and (D,D'), such that the eight points $A,B,C,D,A',B',C',D' \in \mathbb{R}^2_s$ are arbitrarily ordered counterclockwise (or clockwise), as shown in Figure 1 (viewed in the hyperbolic plane).

Then, the four gyrocyclic antipodal points satisfy the identity

$$\begin{aligned} |AB'|_h|BC'|_h|CD'|_h - |AB'|_h|BC|_h|CD|_h - |AB|_h|BC'|_h|CD|_h \\ - |AB|_h|BC|_h|CD'|_h = 2\gamma_r^2 r^2 |A'D|_h \end{aligned} \tag{41}$$

where

$$|AB|_h := \frac{\gamma_{|AB|}}{\sqrt{1+\gamma_{|AB|}}}|AB|\,. \tag{42}$$

for any $A,B \in \mathbb{R}^2_s$, where $|AB|$ is the hyperbolic length, gyrolength, given by (31).

7. Three Gyrocyclic Antipodal Points

In the special case when $D = A'$ and, hence, $D' = A$ the result (41) of Theorem 3 descends to

$$\begin{aligned} |AB'|_h|BC'|_h|AC|_h - |AB'|_h|BC|_h|A'C|_h - |AB|_h|BC'|_h|A'C|_h \\ - |AB|_h|BC|_h|AC|_h = 2\gamma_r^2 r^2 |A'A'| = 0\,. \end{aligned} \tag{43}$$

Equation (43) gives rise to the elegant equation (44) in Corollary 4
Formalizing, we have the following Corollary.

Corollary 4. (A Three Gyrocyclic Antipodal Points Theorem). *Let $\Sigma(O,r)$ be a gyrocircle in the relativistic model of the hyperbolic plane $\mathbb{R}^2_s$ with gyroradius r, centered at $O \in \mathbb{R}^2_s$, with three gyrocyclic antipodal points (A,A'), (B,B') and (C,C'). The six points $A,B,C,A',B',C' \in \mathbb{R}^2_s$ are arbitrarily ordered counterclockwise (or clockwise), as shown in Figure 2 (viewed in the hyperbolic plane).*

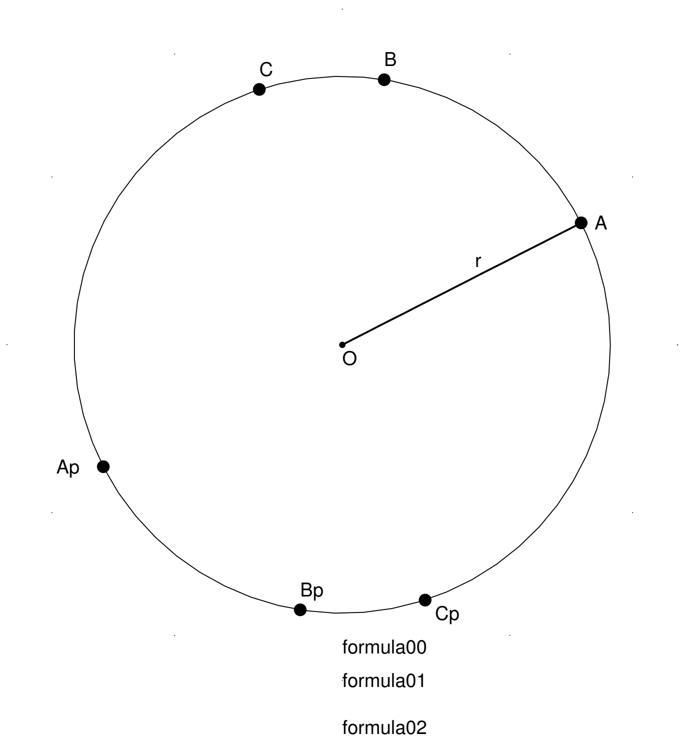

Figure 2. Three cyclic antipodal points, (A,A'), (B,B'), (C,C'), on a circle $\Sigma(O,r)$ centered at O with radius r. The points A,B,C,A',B',C' are arbitrarily ordered counterclockwise. The identities of Corrolary 2 are shown, where $|AB| = \|-A+B\|$, *etc.*

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Then,

$$\frac{|AB'|_h}{|AB|_h}\frac{|BC'|_h}{|BC|_h} - \frac{|AB'|_h}{|AB|_h}\frac{|A'C|_h}{|AC|_h} - \frac{|BC'|_h}{|BC|_h}\frac{|A'C|_h}{|AC|_h} = 1\,. \tag{44}$$

Proof. Equation (44) follows immediately from (43). □

Equation (44) is the hyperbolic counterpart of (14) in Euclidean geometry.

8. The Pythagorean Theorem is Recovered

The study of half-angled trigonometric identities is rewarding since they admit geometric realizations associated with *cyclic points*, that is, points on a circle, in both Euclidean and hyperbolic geometry. Accordingly, in this section we explore a simple half-angled trigonometric identity, which we realize in both Euclidean geometry (in this section) and hyperbolic geometry (in Section 9). Let us consider the trigonometric identity

$$\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} = 1\,. \tag{45}$$

Noting that

$$\cos\frac{\alpha}{2} = \sin\frac{\alpha+\pi}{2} \tag{46}$$

equation (45) can be written as a half-angled trigonometric identity,

$$\sin^2\frac{\alpha+\pi}{2} + \sin^2\frac{\alpha}{2} = 1\,. \tag{47}$$

In order to realize (47) geometrically in the Euclidean plane we consider Figure 1 in a Euclidean plane with only the three points $A, B, B' \in \mathbb{R}^2$ on the circle $\Sigma(O, r)$, and the associated O-vertex angles α and $\alpha + \pi$. Then, by (10),

$$\begin{aligned} \sin\frac{\alpha}{2} &= \frac{1}{2r}|AB| \\ \sin\frac{\alpha+\pi}{2} &= \frac{1}{2r}|AB'|\,. \end{aligned} \tag{48}$$

Substituting the sines in (48) into the half-angled trigonometric identity (47) yields

$$|AB|^2 + |AB'|^2 = 4r^2 = |BB'|^2 \tag{49}$$

thus recovering the Pythagorean theorem for the right-angled triangle ABB' in Figure 1 when viewed in a Euclidean plane.

It seems that exploring the half-angled trigonometric identity (47) is pointless since it yields an expected, well-known result. But, we are not finished. In Section 9 we will find that identity (47), when viewed gyrotrigonometrically as a half-gyroangled gyrotrigonometric identity, does yield an interesting novel result in hyperbolic geometry.

9. A Hyperbolic Pythagorean Theorem

In order to realize (47) geometrically in the relativistic model of the hyperbolic plane we consider Figure 1 in the relativistic model of the hyperbolic plane, with only the three points $A, B, B' \in \mathbb{R}^2$ on the gyrocircle $\Sigma(O, r)$, and the associated O-vertex gyroangles α and $\alpha + \pi$. Then, by (39),

$$\begin{aligned} \sin\frac{\alpha}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|AB|_h \\ \sin\frac{\alpha+\pi}{2} &= \frac{1}{\sqrt{2}\gamma_r\, r}|AB'|_h \end{aligned} \tag{50}$$

where

$$|AB|_h := \frac{\gamma_{|AB|}}{\sqrt{1+\gamma_{|AB|}}}|AB| \tag{51}$$

for any $A, B \in \mathbb{R}^2_s$.

Substituting the sines in (50) into the half-gyroangled gyrotrigonometric identity (47) yields the equation

$$|AB|^2_h + |AB'|^2_h = 2\gamma^2_r r^2 \,. \tag{52}$$

It can be shown by means of (51) and $BB' = r \oplus r = 2r/(1+r^2_s)$ that $|BB'|^2_h = 2\gamma^2_r r^2$. Hence, (52) can be written as

$$|AB|^2_h + |AB'|^2_h = |BB'|^2_h \tag{53}$$

for the gyrotriangle ABB', shown in Figure 1 when viewed in a hyperbolic plane.

Identity (53) is a novel hyperbolic Pythagorean-like theorem in the relativistic model. A different way to obtain (53) is presented in [1, Eq. 63]. Note that the gyrotriangle ABB', shown in Figure 1, associated with (53), is not right-gyroangled. However, as in the Euclidean case, the gyrotriangle ABB' is *gyrodiametric*, that is, one of its sides coincides with a gyrodiameter of its circumgyrocircle (see [1, Figure 7]).

Formalizing the result in (53), we have the following theorem.

Theorem 5. (A Hyperbolic Pythagorean Theorem). *Let ABB' be a gyrotriangle of which the side BB' coincides with a gyrodianeter of its circumgyrocircle in the relativistic model of the hyperbolic plane (Figure 1). Then*

$$|AB|^2_h + |AB'|^2_h = |BB'|^2_h \,. \tag{54}$$

The variation of Theorem 5 in the Poincaré ball model of hyperbolic geometry appears in [2, Theorem 3].

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