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Not peer-reviewed version

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Posted Date: 12 March 2024

doi: 10.20944/preprints202403.0732.v1

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Article

The Generating Functions and Goldbach Conjecture

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Abstract: It is proved that there exist two odd prime numbers, p_i and p_j, such that $p_i + p_j = 2r + 2(r \geqslant 2)$, where p_n denote the n-th odd prime number, $1 \leqslant i, j \leqslant n$. By the theory of generating functions, we prove that there exist two positive integers h_i and h_j such that $h_i + h_j = r(r \geqslant 2)$, where $h_n = (p_n - 1)/2(p_n \geqslant 3)$.Then we complete the proof.

Keywords: generating functions; integer partitions; prime number; the Goldbach conjecture

MSC: 05A15; 11A41; 11P32

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1. Introduction

In 1742, Goldbach proposed two conjectures: the binary Goldbach conjecture, which states that every even integer greater than 2 can be expressed as the sum of two primes, and the ternary Goldbach conjecture, which states that every odd integer greater than 5 can be expressed as the sum of three primes [7].

In 2013, Helfgott successfully proved the ternary Goldbach conjecture [6]. The binary Goldbach conjecture is still unresolved.

For the binary Goldbach conjecture, by the sieve method, since Brun first proved $9 + 9$ in 1920 [3]. Currently the best result is due to Jingrun Chen [4], who proved $1 + 2$ in 1973, which prompts that every sufficiently large even integer can be written as the sum of a prime and product of at most two primes. Obviously, it is difficult to solve Goldbach conjecture by studying prime numbers directly.

The Goldbach conjecture is actually a special type of integer partitions. G.E. Andrews and K.Eriksson use generating functions as products to find the number of partitions of n into multiple parts [2]. Therefore, in this paper we use the theory of generating functions to prove that there exist two positive integers h_i and h_j, such that $h_i + h_j = r(r \geqslant 2)$, where $h_n = (p_n - 1)/2(p_n \geqslant 3)$. Then we complete the proof of Goldbach conjecture.

2. Notation

p_n, p_i, p_j: odd prime numbers.
h_n, h_i, h_j: the generator of odd prime numbers, positive integers.
c_n: the generator of odd composite numbers, positive integers.
o_n: the generator of odd numbers, positive integers.
n, r, s, t: positive integers.
a_r, a_t, a'_r: nonnegative integers.
$\{p_n\}_1^\infty, \{h_n\}_1^\infty, \{c_n\}_1^\infty$: infinite sequence.
P, H, C, D, O:set.
$C^{h_{n+1}-1}$:a set containing only $c_i(c_i \leqslant h_{n+1} - 1)$.
$Y(x), H(x), D(x), C(x), O(x)$: formal power series.
$Y^{h_n}(x)$: a formal power series containing only $a_r x^r(1 \leqslant r \leqslant h_n)$; others are similar.

3. Lemma

Lemma 1. *If p_n denote the n-th odd prime, then there exist two positive integers h_i and h_j, such that $h_i + h_j = r(r \geqslant 2)$, where $h_n = (p_n - 1)/2$.*

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Proof. a . Let $\{p_n\}_1^\infty$ denote the odd prime sequence. Let

$$h_n = (p_n - 1)/2, \tag{1}$$

and we call h_n as the generator of odd prime numbers. So we have

$$H = \{h_n\}_1^\infty = \{1, 2, 3, 5, 6, 8, 9, 11, 14 \cdots\}.$$

Similarly, let c_n be the generator of odd composite numbers. We have

$$C = \{c_n\}_1^\infty = \{4, 7, 10, 12, 13, 16, 17 \cdots\}.$$

Let o_n be the generator of odd numbers, where the odd numbers do not contain 1. We have

$$O = H + C = \{1, 2, 3, 4, 5, 6, 7 \cdots\}.$$

b. Let us construct a generating function for a_r , the number of partitions of the positive integer $r(r \geqslant 2)$ which into two generator of odd prime numbers.
Let

$$H(x) = x^1 + x^2 + x^3 + x^5 + x^6 + \cdots + x^{h_n} + x^{h_{n+1}} + \cdots . \tag{2}$$

So, the generating function $Y(x)$ as follows:

$$\begin{aligned} Y(x) = H(x)^2 = &(x^1 + x^2 + x^3 + x^5 + x^6 + \cdots + x^{h_n} + x^{h_{n+1}} + \cdots) \\ &\cdot (x^1 + x^2 + x^3 + x^5 + x^6 + \cdots + x^{h_n} + x^{h_{n+1}} + \cdots). \end{aligned} \tag{3}$$

c. Obviously, it is only necessary to prove that $a_r \geqslant 1 (r \geqslant 2)$, and the lemma is proved.

Since the value of a_r cannot be obtained by the theory of generating functions, and Euclid's proof demonstrates that the set of prime numbers is countable infinite [5].

So, let us prove it by mathematical induction.

STEP 1: Easy to know

$$a_1 = 0, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 2, \ldots .$$

Thus, $a_t \geqslant 1 (2 \leqslant t \leqslant 5)$, When $r = h_4 = 5$.

STEP 2: Let us assume $a_t \geqslant 1 (2 \leqslant t \leqslant h_n)$ is true for $r = h_n (n \geqslant 5)$.

STEP 3: Let's consider the statement with $r = h_{n+1}$.

For every $n \geqslant 1$, there is some prime number p with $n < p \leqslant 2n$. It was first proved for all n by Pafnuty Chebyshev in 1850. [1]. In other words, it has

$$13 \leqslant p_n < p_{n+1} \leqslant 2p_n (n \geqslant 5). \tag{4}$$

Thus, we have

$$6 \leqslant h_n < h_{n+1} \leqslant 2h_n (n \geqslant 5). \tag{5}$$

So let

$$h_{n+1} = h_n + k (1 \leqslant k \leqslant h_n). \tag{6}$$

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Obviously, $h_n+1, h_n+2, \cdots, h_n+k-1$ are the generator of odd composite numbers. Let

$$D = \{h_n+1, h_n+2, \cdots, h_n+k-1\}. \tag{7}$$

Thus

$$C^{h_{n+1}-1} = \{4, 7, 10, \cdots, h_n-1, h_n+1, \cdots, h_n+k-1\}. \tag{8}$$

Consider the formal power series as follows:

$$\begin{aligned} H^{h_{n+1}}(x) =& x^1 + x^2 + x^3 + x^5 + \cdots + x_n^h + x^{h_{n+1}} \\ =& H^{h_n}(x) + x^{h_{n+1}} \end{aligned} \tag{9}$$

$$D(x) = x^{h_n+1} + x^{h_n+2} + x^{h_n+3} + \cdots + x^{h_n+k-1} \tag{10}$$

$$\begin{aligned} C^{h_{n+1}-1}(x) =& x^4 + x^7 + x^{10} + \cdots + x^{h_n-1} + x^{h_n+1} + \cdots + x^{h_n+k-1} \\ =& C^{h_n-1}(x) + D(x) \end{aligned} \tag{11}$$

$$\begin{aligned} O^{h_{n+1}}(x) =& x^1 + x^2 + x^3 + \cdots + x^{h_n} + \cdots + x^{h_{n+1}} \\ =& O^{h_n}(x) + D(x) + x^{h_{n+1}}. \end{aligned} \tag{12}$$

Thus, we have

$$\begin{aligned} & Y^{h_{n+1}}(x) \\ &= H^{h_{n+1}}(x)^2 \\ &= O^{h_{n+1}}(x)^2 - C^{h_{n+1}-1}(x)^2 - 2C^{h_{n+1}-1}(x) \cdot H^{h_{n+1}}(x) \\ &= O^{h_{n+1}}(x)^2 - (C^{h_n-1}(x) + D(x))^2 - 2(C^{h_n-1}(x) + D(x)) \cdot (H^{h_n}(x) + x^{h_{n+1}}) \\ &= O^{h_{n+1}}(x)^2 - C^{h_n-1}(x)^2 - 2C^{h_n-1}(x) \cdot D(x) - D(x)^2 - 2C^{h_n-1}(x) \cdot H^{h_n}(x) \\ &\quad - 2D(x) \cdot H^{h_n}(x) - 2C^{h_n-1}(x) \cdot x^{h_{n+1}} - 2D(x) \cdot x^{h_{n+1}} \\ &= O^{h_{n+1}}(x)^2 - O^{h_n}(x)^2 + H^{h_n}(x)^2 - 2D(x) \cdot O^{h_n}(x) - 2C^{h_{n+1}-1}(x) \cdot x^{h_{n+1}} - D(x)^2. \end{aligned}$$

Obviously,

$$O^{h_{n+1}}(x)^2 - O^{h_n}(x)^2 = 2x^{h_n+1} + 4x^{h_n+2} + \cdots + 2kx^{h_n+k} + \cdots + x^{2(h_n+k)}. \tag{13}$$

$$H^{h_n}(x)^2 = a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_{h_n}x^{h_n} + \cdots + x^{2h_n}. \tag{14}$$

$$D(x) \cdot O^{h_n}(x) = x^{h_n+2} + 2x^{h_n+3} + \cdots + (k-1)x^{h_n+k} + \cdots + x^{2h_n+k-1}. \tag{15}$$

$$C^{h_{n+1}-1}(x) \cdot x^{h_{n+1}} = x^{h_n+k+4} + x^{h_n+k+7} + \cdots + x^{2h_n+2k-1}. \tag{16}$$

$$D(x)^2 = x^{2h_n+2} + 2x^{2h_n+3} + 3x^{2h_n+4} + \cdots + x^{2h_n+2k-2}. \tag{17}$$

The above data are sorted out in the following table:

Table 1. The values of $a_t(h_n+1 \leqslant t \leqslant h_n+k)$.

	a_t	x^{h_n+1}	x^{h_n+2}	x^{h_n+3}	$\cdots$	x^{h_n+k}
a	$O^{h_{n+1}}(x)^2 - O^{h_n}(x)^2$	2	4	6	$\cdots$	$2k$
b	$H^{h_n}(x)^2$	$\geqslant 0$	$\geqslant 0$	$\geqslant 0$	$\cdots$	$\geqslant 0$
c	$D(x) \cdot O^{h_n}(x)$	0	1	2	$\cdots$	$k-1$
d	$C^{h_{n+1}-1}(x) \cdot x^{h_{n+1}}$	0	0	0	$\cdots$	0
e	$D(x)^2$	0	0	0	$\cdots$	0
$a+b-2c-2d-e$	$Y^{h_{n+1}}(x)$	$\geqslant 2$	$\geqslant 2$	$\geqslant 2$	$\cdots$	$\geqslant 2$

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As can be seen from the table, we have

$$a_t \geqslant 2(h_n < t \leqslant h_{n+1}). \tag{18}$$

So $a_t \geqslant 1(h_n < t \leqslant h_{n+1})$ is true. Therefore, by principle of mathematical induction, $a_r \geqslant 1(r \geqslant 2)$.

Thus,this indicates that there exist two positive integers h_i and h_j such that

$$h_i + h_j = r \quad (h_n = (p_n - 1)/2, r \geqslant 2). \tag{19}$$

In addition, considering symmetry, let a'_r be the number of non repeating partitions of the integer r, so

$$a'_r = \begin{cases} (a_r + 1)/2, & a_r = 2s - 1(s \geqslant 1) \\ a_r/2, & a_r = 2s(s \geqslant 1), \end{cases} \tag{20}$$

which proves the lemma. □

4. Demonstration of Goldbach Conjecture

Theorem 1. *There exist two odd primes, p_i and p_j, such that $p_i + p_j = 2r + 2(r \geqslant 2)$.*

Proof. Consider the sequence

$$\{2r + 2, r \geqslant 2\} = \{6, 8, 10, 12, \ldots, 2r + 2, \ldots\}.$$

With help of Lemma 1, we have

$$2r + 2 = 2(h_i + h_j) + 2 = (2h_i + 1) + (2h_j + 1). \tag{21}$$

Since $p_n = 2h_n + 1$, so we have

$$2r + 2 = p_i + p_j. \tag{22}$$

Hence there exist two odd primes p_i and p_j such that

$$p_i + p_j = 2r + 2(r \geqslant 2), \tag{23}$$

which proves the theorem. □

Thus, With help of Theorem 1, for the binary Goldbach conjecture, We have

$$2 + 2 = 4, \quad p_i + p_j = 2r + 2(r \geqslant 2). \tag{24}$$

For the ternary Goldbach conjecture, We have

$$2 + 2 + 3 = 7, \quad p_i + p_j + 3 = 2r + 2 + 3 = 2r + 5(r \geqslant 2). \tag{25}$$

This proves the Goldbach conjecture.

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