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Symmetry Transformations in Cosmological and Black Hole Analytical Solutions

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Abstract: We analyze the transformation of a very broad class of metrics that can be put in terms of static coordinates. Starting from a general ansatz, we obtain a relation for the parameters in which one can impose further symmetries, or restrictions. One of the simplest restrictions leads to FLRW cases, while transforming from the initial static to other static-type coordinates can lead to near horizon coordinates, Wheeler-Regge, isotropic coordinates, among others. As less restrictive cases, we show an indirect route for obtaining Kruskal-Szekeres within this approach, as well as Lemaître coordinates. We use Schwarzschild spacetime as prototype for testing the procedure in individual cases. However, application to other spacetimes, such as de-Sitter, Reissner-Nordström or Schwarzschild de Sitter, can be readily generalized.

Keywords: analytical solutions; black holes; cosmology

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1. Introduction

The role of coordinates in general relativity is a subtle issue which must be taken with caution for each one of the individual solutions to Einstein's field equations. There is a class of coordinate systems that are useful in a specific patch of the spacetime. Usually, these type of transformations are suitable for very specific physical interpretations, such as when one can attach them to a type of observers. This occurs, for instance, in Painlevé-Gullstrand (or rain) coordinates for black hole static solutions [1]. Isotropic coordinates for static spacetimes can also be included in this class, which allow comparing asymptotic behavior at spatial infinity with a Newtonian metric [2].

A second type of transformations includes those that allow the continuity of spacetime to be considered. In particular, the new coordinates can clarify the coordinate character of singularities associated with cosmological and event horizons. For example, Lemaître system provided one of the first descriptions, for the Schwarzschild case, that allowed a continuous crossing of the event horizon. Eddington-Finkelstein coordinates, adapted to null coordinates, also satisfy this property [3,4].

Other coordinates are useful to visualize relevant regions of the spacetime in the same diagram, or even expand it to include possible copies of it. In fact, we can interpret Schwarzschild and other spherically symmetric spacetimes as a section of the broader Kruskal-Szekeres spacetime [5–8]. Finally, other representations allow us to visualize effects such as how space is curved (e.g. with embedding diagrams) or to visualize the behavior of the complete spacetime in a compact diagram, as is the case with conformal diagrams [9,10]. It is worth emphasizing that some coordinates may possess properties that make them ubiquitous in this classification.

In this article we analyze transformations for spherically symmetric spacetimes that can be described with static coordinates. Some of the most paradigmatic solutions in cosmology and black hole theory can be formulated in these terms. Some examples include: Schwarzschild, Reissner-Nördstrom, Schwarzschild-de-Sitter, de-Sitter and Anti-de-Sitter spacetimes, just to name a few. We start precisely with the static version, where $g_{11}g_{00} = -1$ [11]. From there, we consider the transformation to a generic form that maintains emphasis on the isotropy with respect to $r = 0$. We obtain a crucial equation, from which we can impose restrictions that allow us to solve for specific cases. Although we maintain the spirit of showing general relationships, we mostly use Schwarzschild solution's as a specific example of the procedures.

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The structure for the rest of this article is as follows. We present in Sect. 2 the initial form of the metric and also derive useful relations that lead to the master equation that dictates the way to obtain specific solutions. Section 3 is dedicated to briefly discussing how and which FLRW solutions emerge. Furthermore, starting from the restricted Friedmann equations that appear, we summarize the cosmological solutions that can be put into static form. Section 4 has a broader scope, as it includes many possibilities: those leading from static coordinates to other static coordinates. Specifically, we obtain near-horizon coordinates, Regge-Wheeler, isotropic coordinates and a transformation that resembles Lemaître coordinates. In Sect. 5 we obtain, Kruskal-Szekeres as well as the correct Lemaître coordinates for Schwarzschild spacetime,. Both are examples of specific coordinates with the characteristics analyzed throughout the article, but without the restrictions of the last previous two sections. Finally, in Sect. 6 we make some remarks about the method and the results.

2. The General Transformations

It is well known that Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = 8\pi G T_{\mu\nu}, \tag{1}$$

admit several analytic solutions that can be put in the form

$$ds^2_{(1)} = g_{\mu\nu} dx^{\mu\nu} = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2. \tag{2}$$

Here, we have $f = f(r)$ and $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. This encompass a broad class of spherically symmetric metrics that can be put in a static form [12–14]. There is a subclass of FLRW-metrics can be put in the form (2), such as de Sitter (dS) and AdS space, as well as Milne and Lanczos universes. Other class of possibilities include black hole solutions, such as Schwarzschild, Reissner-Nordström and Schwarzschild-de-Sitter, to name a few.

From the coordinates that define the metric $ds^2_{(1)}$, namely $x^\alpha = (t, r, \theta\, \phi)$, we consider a very general transformation to a metric defined in the new coordinates $x^{\alpha'} = (\tau, \rho, \theta\, \phi)$:

$$ds^2_{(2)} = \gamma_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} = -N^2 d\tau^2 + b^2 \left(g^2 d\rho^2 + \rho^2 d\Omega^2 \right). \tag{3}$$

The bases for the new coordinate are orthogonal again, and the metric components have dependence $N = N(\tau, \rho)$, $b = b(\tau, \rho)$ and $g = g(\rho)$. This allows plenty of possibilities. For instance, as we shall see, in the gauge $N = const.$ the solutions can be associated with a restricted class of FLRW metrics. For instance, $N = 1$ with $bg = \exp(\sqrt{\Lambda/3}\tau)$ corresponds to de-Sitter space, of main importance as the asymptotic limit for the standard Λ-CDM cosmology. In general $N = 1$ with $b = b(\tau)$ is the adequate ansatz for FLRW metrics. Other restrictions can lead to Kruskal or other types of coordinates representing the Schwarzschild spacetime, and so on [15].

Now, the general metric transformation that interest us is

$$g_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} g_{\alpha\beta}, \tag{4}$$

where the indices run from 0 to 3 in the order mentioned for both set of coordinates. From here on, an overdot denotes partial derivative respet to τ, such as in $\dot{t} = \partial t / \partial \tau$. In a similar way, a prime shall denote a partial derivative respect to ρ, such as in $r' = \partial r / \partial \rho$. The angular components of (4), $g_{2'2'}$ and $g_{3'3'}$, are tantamount to $r = b\rho$. This implies that $\dot{r} = \dot{b}\rho$ and $r' = b + b'\rho$. On the other hand, the expansion for $g_{0'0'}$, $g_{1'1'}$ and $g_{0'1'}$ in (4) leads, after rearrangements, to

$$\dot{t}^2 = f^{-2}(f N^2 + \dot{b}^2 \rho^2), \tag{5}$$

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$$t'^2 = f^{-2}\left[(b + b'\rho)^2 - fb^2g^2\right] \tag{6}$$

and

$$f^2\dot{t}t' = \dot{b}\rho(b + b'\rho), \tag{7}$$

respectively. By inserting (5) and (6) in (7) and simplificating, we obtain the useful relation

$$f = b^{-2}g^{-2}(b + b'\rho)^2 - N^{-2}\dot{b}^2\rho^2. \tag{8}$$

In fact, by inserting it back in (5) and (6), they simplify to

$$\dot{t} = Nf^{-1}b^{-1}g^{-1}(b + b'\rho) \tag{9}$$

and

$$t' = N^{-1}f^{-1}b\dot{b}g\rho. \tag{10}$$

Partial derivation of (9) respect to ρ leads to

$$\begin{gathered} \tfrac{\partial^2 t}{\partial\rho\partial\tau} = f^{-2}b^{-2}g^{-2}\{Nfbg(2b' + b''\rho) \\ +[N'fbg - N(bgf' + fgb' + fbg')](b + b'\rho)\}. \end{gathered} \tag{11}$$

In a similar way, from (10) we have

$$\frac{\partial^2 t}{\partial\tau\partial\rho} = N^{-2}f^{-2}g\rho[Nf(\dot{b}^2 + b\ddot{b}) - (N\dot{f} + f\dot{N})b\dot{b}]. \tag{12}$$

We shall equate these two relations. It is useful to note that from (8) one obtains explicitly

$$\dot{f} = 2b^{-3}g^{-2}\rho(b + b'\rho)(b\dot{b}' - \dot{b}b') + 2N^{-3}(\dot{N}\dot{b}^2 - N\dot{b}\ddot{b})\rho^2. \tag{13}$$

as well as

$$\begin{gathered} f' = 2b^{-3}g^{-3}(b + b'\rho)\left[bg(b' + b''\rho) - b'^2g\rho - bg'(b + b'\rho)\right] \\ +2N^{-3}\dot{b}\rho\left[N'\dot{b}\rho - N(\dot{b}'\rho + \dot{b})\right]. \end{gathered} \tag{14}$$

Using these two relations in the equality between (11) and (12), after some algebra, the following relation appears:

$$\begin{gathered} N^4\left(1 + \tfrac{b'}{b}\rho\right)^2\left[\left(\tfrac{g'}{g} + \tfrac{N'}{N}\right)\left(1 + \tfrac{b'}{b}\rho\right) + \tfrac{b'^2}{b^2}\rho - \left(\tfrac{b'}{b} + \tfrac{b''}{b}\rho\right)\right] \\ +N^2g^2\rho^2\left\{\left[\left(\tfrac{g'}{g} + \tfrac{N'}{N} - \tfrac{b'}{b}\right) + 4\tfrac{\dot{b}'}{\dot{b}}\right]\left(1 + \tfrac{b'}{b}\rho\right) - \tfrac{b''}{b}\rho\right\}\dot{b}^2 \\ -N^2b^2g^2\rho\left(1 + \tfrac{b'}{b}\rho\right)^2\left(\tfrac{\ddot{b}}{\dot{b}} + \tfrac{\dot{b}}{b} - \tfrac{\dot{N}}{N}\right)\tfrac{\dot{b}}{b} + 2N^2b^2g^2\rho\tfrac{\dot{b}^2}{b^2} \\ = b^4g^4\rho^3\left(\tfrac{\ddot{b}}{\dot{b}} - \tfrac{\dot{b}}{b} - \tfrac{\dot{N}}{N}\right)\tfrac{\dot{b}^3}{b^3}. \end{gathered} \tag{15}$$

This is the crucial equation for the main part of the rest of the article, where we restrict to several cases of interest, remembering that the generic form of the metric is

$$ds^2_{(2)} = -N^2d\tau^2 + b^2\left(g^2d\rho^2 + \rho^2d\Omega^2\right). \tag{16}$$

From here on, we shall assume some restrictions in the variables N, b and g, to see which type of space is obtained, and at the same time to visualize if the identification turns out to be unique.

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3. FLRW Cases

When $N = 1$ and $b = b(\tau)$ in this metric, one has precisely the ansatz for the comoving coordinates (ρ, θ, ϕ) of an isotropic and homogeneous expanding universe. Substitution of $N = 1$ and $b' = 0$ in (15) yields massive simplifications:

$$\frac{g'}{g^3 \rho} = b\ddot{b} - \dot{b}^2. \tag{17}$$

Since the dependences of the left and right parts of the equality are on ρ and τ respectively, this relation is equal to a constant κ. This allows to integrate $g' g^{-3} = \kappa\rho$, yielding the important relation

$$g^2 = \frac{1}{B - \kappa\rho^2}, \tag{18}$$

where B is an integration constant and κ can have any sign. Now, since locally -where the term $\kappa\rho^2$ can be neglected- the homogeneous space is flat, $B = 1$. That is, we have recovered the usual FLRW solution.

The right hand of Eq. (17) is also equal to κ, allowing to writte the relation as

$$\frac{\ddot{b}}{b} = \frac{\dot{b}^2 + \kappa}{b^2}. \tag{19}$$

This equation can be integrated, for instance by making the reduction of order $w(\tau) = \dot{b}^2 + \kappa$, and then $\dot{w} = 2\dot{b}\ddot{b}$, turning (19) into

$$\frac{\dot{w}}{w} = 2\frac{\dot{b}}{b}, \tag{20}$$

with solution $w = \Gamma b^2$, where Γ is another integration constant. Then we can reinterpret (19) as two relations:

$$\frac{\dot{b}^2 + \kappa}{b^2} = \Gamma, \tag{21}$$

and

$$\frac{\ddot{b}}{b} = \Gamma. \tag{22}$$

These two relations can be recognized as the two Friedmann equations, with cosmological constant $\Lambda = 3\Gamma$ as the only source to energy momentum tensor [16].

Even more, we have argued before that (18) should be written as $g^{-2} = 1 - \kappa\rho^2$. The substitution of this, as well as $r = b\rho$ and the relation $\dot{b}^2 = \Gamma b^2 - \kappa$ (see Eq. (21)) in (8) leads to $f = 1 - \Gamma r^2$.

Depending on the value of Γ, the only FLRW solutions that can be put in the metric static form (2) are de Sitter and Lanczos universes for $\Gamma > 0$, Minkowski and Milne for $\Gamma = 0$, and Anti-de-Sitter for $\Gamma < 0$ [17,18]. All the solutions to the scale factor can be obtained from (21). The resulting metrics, given by (3), are summarized at the following table.

Density Γ	Curvature	Metric
$\Gamma = 0$	$k = 0$	$ds^2 = -d\tau^2 + d\rho^2 + \rho^2 d\Omega^2$ Minkowski
$\Gamma = 0$	$k = -1$	$ds^2 = -d\tau^2 + \tau^2\left(\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega^2\right)$ Milne universe
$\Gamma > 0$	$k = 0$	$ds^2 = -d\tau^2 + e^{2\sqrt{\Gamma}T}\left(d\rho^2 + \rho^2 d\Omega^2\right)$ de Sitter
$\Gamma > 0$	$k = 1$	$ds^2 = -d\tau^2 + \frac{\cosh^2\left(\sqrt{\Gamma}T\right)}{\Gamma}\left(\frac{d\rho^2}{1-\rho^2} + \rho^2 d\Omega^2\right)$ Lanczos (1)
$\Gamma > 0$	$k = -1$	$ds^2 = -d\tau^2 + \frac{\sinh^2\left(\sqrt{\Gamma}T\right)}{\Gamma}\left(\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega^2\right)$ Lanczos (2)
$\Gamma < 0$	$k = -1$	$ds^2 = -d\tau^2 + \frac{\sin^2\left(\sqrt{\vert\Gamma\vert}T\right)}{\vert\Gamma\vert}\left(\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega^2\right)$ Anti de Sitter

Only FLRW solutions expressible in static form.

4. From Static Coordinates (t, r) to Static Coordinates (τ, ρ).

This assumption includes many subcases, of which Minkowski, Schwarzschild and de Sitter spacetimes can be seen as archetypical. Static coordinates is the same as having $\dot{N} = \dot{b} = 0$. Then only the first row suvives in (15), which leads to

$$b' + b''\rho - \frac{b'^2}{b}\rho = \left(\frac{N'}{N} + \frac{g'}{g}\right)(b + b'\rho). \tag{23}$$

This can be rewritten as

$$\frac{d(b + b'\rho)}{d\rho} - (b + b'\rho)\frac{b'}{b} = \left(\frac{N'}{N} + \frac{g'}{g}\right)(b + b'\rho). \tag{24}$$

Dividing by the factor $b + b'\rho = dr/d\rho$, this is equivalent to $d\ln(b + b'\rho) = d\ln Ngb$. Integration yields

$$b + b'\rho = \frac{dr}{d\rho} = 2\alpha Nbg, \tag{25}$$

where α is a constant. Remember that in this section bg only depends on ρ, which means that one could obtain the explicit form for $r = r(\rho)$ directly by integration. Also, from (8) the function f is given by

$$f = 4\alpha^2 N^2. \tag{26}$$

Since r is a function of ρ and vice-versa, the timelike variables τ and t are the same except a multiplicative constant. As a check of consistency, take the trivial possibility $N = bg = 1$. Then, by selecting $\alpha = 1/2$ one obtains $f = 1$. Therefore both (16) and (2) would be the same, Minkowski spacetime.

The last two relations are of main importance for the remainder of this section.

4.1. Near Horizon Coordinates

We take a further step beyond Minkowski, and take as a simple possibility $N = \rho$, preserving $bg = 1$. Now (25) is the same as $d(b\rho)/d\rho = 2\alpha\rho$, with solution $b = \alpha\rho + \beta\rho^{-1}$. That is, from $r = b\rho$ we obtain the relation

$$r = \alpha\rho^2 + \beta. \tag{27}$$

From it, $\alpha N^2 = r - \beta$, and (26) is

$$f = 4\alpha(r - \beta). \tag{28}$$

What is the meaning of these results? Clearly, the chosen relations correspond to Rindler (1+1) metric $ds^2 = -\rho^2 d\tau^2 + d\rho^2$ in (16) when considering constant angles. However, even this subdimensional identification, it cannot be representing Rindler space globally. In fact we have been using $r = b\rho$ all along, and it comes from the angular terms. Also contemplate the substitution (28) in (2): it prohibits transformation to Minkowski, contrary to the Rindler case.

The coordinate system induced by $N = \rho$ can be obtained by making the identification $\alpha = (8M)^{-1}$ and $\beta = 2M$. It turns out that for the Schwarzschild case, ρ corresponds to a radial coordinate that measures the proper distance at certain r very near the event horizon $r = 2M$ [19]. A simple derivation of near horizon coordinates is performed in Appendix A. In particular, compare (27) with Eq. (A.5).

4.2. Tortoise Coordinates

Assume conformal flatness at constant angles. That is, we impose $N = bg$, and then (16), -at $d\Omega^2 = 0$- reduces to

$$ds^2_{(2)} = N^2(-d\tau^2 + d\rho^2). \tag{29}$$

while (25) now is

$$dr = 2\alpha N^2 d\rho. \tag{30}$$

For a moment, consider what occurs for a non trivial case, v.g. let us take $N^2 = \rho$. Then the solution to (29) is $r = b\rho = \alpha\rho^2 + \beta$. That is, one obtains the same dependence for r seen in (27). However, in that case we impossed $bg = 1$ with $N^2 = \rho^2$, while in the actual case we have $b^2g^2 = N^2 = \rho$. Then (26) leads to $f = 4\alpha^2\rho = 4\sqrt{\alpha^3(r-\beta)}$, with a very different behaviour than the local Rindler radial form. Less simple forms appear by considering other values for k in $N = \rho^k$.

However, another point of view is more useful here, in order to obtain more realistic metrics. As we stressed out before, for $\dot{N} = \dot{b} = 0$ we have that r is a function of ρ. We assume invertibility in a suitable patch, such as the exterior to the event horizon in black holes. This allows to consider $N = N(r)$, and by using (26) in (30) we have

$$d\rho = 2\alpha f^{-1} dr. \tag{31}$$

We are allowed to select $2\alpha = 1$, and the result is the known Regge-Wheeler coordinate, for any spacetime characterized in the metric form given by (2). For instance, for the Schwarzschild case $f = 1 - 2M/r$, where $2M$ is the Schwarzschild radius. Integration leads to $\rho^* = r + 2M\ln[r/(2M) - 1]$ for the exterior solution. This is the known Tortoise coordinate, which pushes the horizon event $r = 2M$ to $-\infty$ in the radial coordinate ρ^*. Although it does not allow continuous crossing of the event horizon, it is useful to obtain the Eddington-Finkelstein and Kruskal-Szekeres coordinates, via some exponentiations and rotations [7,8].

Also take into account that one needs $r = r(\rho)$ in order to obtain $N = N(\rho)$ in (29). For Schwarzschild spacetime, the mentioned transformation to Tortoise yields, after exponentiation, $\exp(\rho^*/(2M) - 1) = (r/(2M) - 1)\exp(r/(2M) - 1)$. Inversion involves Lambert W function, in the form

$$r = 2M\left\{1 + W\left[\exp\left(\frac{\rho^*}{2M} - 1\right)\right]\right\}, \tag{32}$$

from which is clear that the turning point from positive to negative ρ^* is approximately 1.278 times the Schwarzschild radius. Also, $\rho^* \to -\infty$ when $r \to 2M$.

In Ref. [20] this line of thinking -about (1+1)-conformal property- was generalized. The result was the analysis of several possibilities, that included the Kruskal-Szekeres transformations among another proposals not considered before. However, to obtain them, one must abandon the assumption $N = N(\rho)$: it is more natural to consider an inverse route to the one explored in this article, considering the inverse transformation of (4).

4.3. Isotropic Coordinates

This system maintains explicit spherical symmetry for the spacetimes considered, while putting the metric in a conformal flat form when time is constant in (2). That is achieved transforming (2) into $-fdt^2 + \lambda^2(d\rho^2 + \rho^2 d\Omega^2)$. Here f is obtained as a function of ρ, and λ^2 is the conformal factor to flat 3-dimensional space, at constant t. For Schwarzschild spacetime, for instance, isotropic coordinates allow to match directly the spacetime with the weak Newtonian metric.

Note that in (16) we just need to make the association $f \propto N^2$ -consistent with relation (26)- as well as $\lambda = b$, while setting $g = 1$. Notice the distinction with the case (a): now we have $N \neq 1$ in general.

The two relations (25) and (26), imply for this case, that

$$\frac{d\rho}{\rho} = \frac{dr}{r\sqrt{f}}. \tag{33}$$

Here, we have also used the fact that $r = b\rho$ and chosen adequate signs. Compare with the general Tortoise case (31). Here is also convenient having the explicit form $f = f(r)$ in order to obtain

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the relation between ρ and r. And again, we take the Schwarzschild case in order to test that the developments yield the known solution. As before, select $f = 1 - 2M/r$, which allows to express (33) in the form

$$\frac{d\rho}{\rho} = \frac{dr}{M\sqrt{\left(\frac{r}{M} - 1\right)^2 - 1}}. \tag{34}$$

The solution is $\ln(\beta\rho) = \cosh^{-1}(r/M - 1)$, where β is a positive constant. Performing little algebra, we obtain

$$\frac{\beta^2\rho^2 + 1}{\beta\rho} = 2\left(\frac{r}{M} - 1\right). \tag{35}$$

Here we ask that the asymptotic behavior of the radial coordinates ρ and r are the same as they approach infinity, that is we impose that $\rho \to r$ for $r \gg 2M$. This leads to $\beta = 2/M$ in (35). Solving for r, we have

$$r = \rho\left(1 + \frac{M}{2\rho}\right)^2. \tag{36}$$

Substituting this in both $f = 1 - 2M/r$ and $b = r/\rho$, and also by recalling that $N^2 = f$ and $g = 1$, we have that (16) can be rewritten as

$$ds^2 = -\left(\frac{1 - \frac{M}{2\rho}}{1 + \frac{M}{2\rho}}\right)^2 d\tau^2 + \left(1 + \frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right). \tag{37}$$

This is the usual isotropic form for Schwarzschild spacetime [4].

4.4. Lemaître-Type for Schwarzschild

Lemaître coordinates (τ, ρ) for Schwarzschild spacetime are such that the metric takes the form

$$ds^2 = -N^2 d\tau^2 + \frac{\kappa}{r} d\rho^2 + r^2 d\Omega^2, \tag{38}$$

with $N = 1$ and $\kappa = 2M$ [3,7]. By considering (26), this form implies that $2\alpha = 1$ if one takes $t = \tau$, and the fact that $f = 1$ in (2) indicates that we are constrained to Minkowski spacetime from the start. We can just take into account that $r dr^2 = \kappa d\rho^2$ to show that $4r^3 = 9\kappa\rho^2$ is an appropiate transformation for this case.

All the conditions imposed in this section, in particular that N and b depend only on ρ, need to be relaxed in order to obtain the Lemaître form for general spacetimes. However, for the conditions in this very section, we can obtain a form that resembles it for Schwarzschild spacetime, for constant t.

Consistence with (38) demands that $b^2 g^2 = 2M/r$, and from (26) we have $4\alpha^2 N^2 = 1 - 2M/r$. Then (25) can be expressed as $r dr/\sqrt{(r/2M) - 1} = 2M d\rho$, which is solved by

$$\rho = \frac{4M}{3}\left(\frac{r}{2M} + 2\right)\sqrt{\frac{r}{2M} - 1}. \tag{39}$$

We have set to zero an integration constant. It mantains the range of ρ from 0 to ∞ when r goes from $2M$ to ∞, the exterior patch of the black hole. As a matter of contrast, remember that the range is pushed from $(2M, \infty)$ in r to $(-\infty, \infty)$ in ρ^*, the Tortoise coordinate appearing in (32). Notice also that for $r \gg 2M$, the behavior is $4r^3 = 9(2M)\rho^2$, the same for Minkowski case mentioned above. That is, we have explicit asymptotic flatness for Schwarzschild spacetime. However, as interesting as it may sound, the range of coordinates induced by the transformation (39) limits its validity to the exterior of the event horizon in Schwarzschild spacetime. By contrast, the most attractive feature of Lemaître coordinates is precisely that they allow a continuous crossing of the event horizon [3,7].

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5. Other Relevant Cases

There are several cases of interest in which it is necessary to abandon the assumption $N = N(\rho)$ and $b = b(\rho)$. In particular, in the Kruskal-Szekeres coordinates, N depends on both τ and ρ. The same is true for the factor bg in Lemaître coordinates for Schwarzschild. As we mentioned before, the first case readily appears by taking the inverse transformation, instead of the one used in this article. That is, going from the metric form (3) to that of (2), as in [**?**][**?**]. One can obtain those coordinates within our scheme, albeit indirectly.

5.1. Kruskal-Szekeres (Indirect Route)

s Maintain $\dot{N} = \dot{b} = 0$, in such a way that the relations in the first part of Sect. 4 are still valid. Take the case $N = bg\rho$, that converts (16) into

$$ds^2_{(2)} = n^2(-\rho^2 d\tau^2 + d\rho^2) + r^2 d\Omega^2, \tag{40}$$

where we have defined $n = N/\rho$. Now we again have the conformal Rindler form, at hypersurfaces where $d\Omega^2 = 0$.

Notice that now (25) and (26) imply the relation

$$\frac{d\rho}{\rho} = 2\alpha \frac{dr}{f}. \tag{41}$$

Its solution is $\rho = ke^{\rho^*}$, where ρ^* is the generic solution to (31). Actually, this exponentiation constitutes the starting point of the textbook route for obtaining Kruskal-Szekeres -via intermediate definition of Eddington-Finkelstein coordinates [2,7,8].

Let us define $U = \rho e^{\tau}$, $V = -\rho e^{-\tau}$, that allows to express the metric (40) as $-n^2 dU dV + r^2 d\Omega^2$. Now we obtain a suggestive representation, by performing the custom rotation $V = T - X$ and $U = T + X$, in such a way that we have

$$ds^2_{(2)} = n^2(-dT^2 + dX^2) + r^2 d\Omega^2. \tag{42}$$

It is useful to compare this relation with (29). Here, $n = n(\rho)$ in the $(\tau, \rho, \theta, \phi)$ system, while $n = n(T, X)$ in (42). Even more, in the Kruskal-Szekeres case it is best to keep the conformal factor in terms of the original radial coordinate r.

More precisely, for Schwarzschild spacetime, the association $n^2\rho^2 = 8M(1 - 2M/r)$ can be made. Furthermore, the solution to (41) is $\rho = ke^{\rho^*}$, where we have the (rescaled-) Tortoise coordinate, given by

$$4M\rho^* = r + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{43}$$

Here we have chosen $2\alpha = 1/(4M)$ in (41). Also, with $k\sqrt{2M} = 1$ in $\rho = ke^{\rho^*}$, we have

$$\rho^2 = \frac{1}{2M}\left(\frac{r}{2M} - 1\right)e^{\frac{r}{2M}}. \tag{44}$$

Substituing in $n^2\rho^2 = 4r_s(1 - r_s/r)$, (42) can be written as

$$ds^2_{(2)} = \frac{32M^3}{r} e^{-\frac{r}{2M}}(-dT^2 + dX^2) + r^2 d\Omega^2, \tag{45}$$

that is the usual Kruskal-Szekeres form for Schwarzschild spacetime.

5.2. Lemaître Coordinates

The Lemaître form for Schwarzschild spacetime appears by imposing $N = 1$ and $b^2 g^2 = 2M/r$ in (16) [3,7]. Compare with the last example of Sect. 4: here we consider $r = r(\tau, \rho)$.

Remembering that $r = b\rho$, we have $b + b'\rho = \partial r/\partial\rho$ and $\dot{b}\rho = \partial r/\partial\tau$. Then Eq. (8) can be written as

$$1 - \frac{2M}{r} = \frac{r}{2M}\left(\frac{\partial r}{\partial\rho}\right)^2 - \left(\frac{\partial r}{\partial\tau}\right)^2. \tag{46}$$

This form suggests to propose $r = A(\rho-\tau)^n$, as it allows to factorize. Substitution and simplification yields

$$\left[nA(\rho-\tau)^{n-1}\right]^2 = \frac{2M}{r}. \tag{47}$$

We use again $r = A(\rho-\tau)^n$, leading to $n^2A^3(\rho-\tau)^{3n-2} = 2M$. This alone implies $n = 2/3$ and also that $(2/3)^2A^3 = 2M$. That is, $r = r(\tau,\rho)$ is solved as

$$r = \left[\frac{9}{2}M(\rho-\tau)^2\right]^{\frac{1}{3}}. \tag{48}$$

Now consider, that for this case, (9) and (10) can be recasted as

$$\frac{\partial t}{\partial\tau} = \frac{1}{1-\frac{2M}{r}}\sqrt{\frac{r}{2M}}\left(\frac{\partial r}{\partial\rho}\right) \tag{49}$$

and

$$\frac{\partial t}{\partial\rho} = \frac{1}{1-\frac{2M}{r}}\sqrt{\frac{2M}{r}}\left(\frac{\partial r}{\partial\tau}\right), \tag{50}$$

respectively. Also, the differential of (48) is

$$dr = \sqrt{\frac{2M}{r}}(d\rho - d\tau), \tag{51}$$

where we have used (48) in the form $(\rho-\tau)^{-\frac{1}{3}} = \left(\sqrt{9M/2}\right)^{\frac{1}{3}} r^{-1/2}$. This indicates that $\partial r/\partial\rho = -\partial r/\partial\tau = \sqrt{2M/r}$, which can be substituted into (49) and (50). In turn, this allows us to obtain $dt = (\partial t/\partial\tau)d\tau + (\partial t/\partial\rho)d\rho$. The result is

$$dt = \frac{1}{1-\frac{2M}{r}}\left(d\tau - \frac{2M}{r}d\rho\right). \tag{52}$$

The inversion of (51) and (52) is

$$d\tau = dt + \sqrt{\frac{2M}{r}}\frac{dr}{1-\frac{2M}{r}} \tag{53}$$

and

$$d\rho = dt + \sqrt{\frac{r}{2M}}\frac{dr}{1-\frac{2M}{r}}. \tag{54}$$

These two relations, together with (48), constitute the usual Lemaître transformation [3,7], with metric

$$ds^2 = -d\tau^2 + \frac{2M}{r}d\rho^2 + r^2 d\Omega^2. \tag{55}$$

6. Discussion

In this article we have presented a consistent and selfcontained method to obtain distinct representations of spherically symmetric spacetimes that share a metric form

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

in static coordinates. We transformed it to the general form

$$ds^2_{(2)} = -N^2 d\tau^2 + b^2 \left(g^2 d\rho^2 + \rho^2 d\Omega^2 \right),$$

where N and b in general are functions of (τ, ρ) and $g = g(\rho)$. This allows for a very general class of representations. This alone suggested including FLRW solutions and, in fact, this was the topic of the first part of the article. As summarized at the end of Sect. 3, there are six different FLRW solutions that satisfy the aforementioned transformation.

Section 4 was dedicated to analyzing metrics that transform from the initial static version to another static form, in the sense that N and b are independent of τ in the new system. Many of the known analytical solutions to Einstein's equations satisfy this requeriment. In particular, making small assumptions we obtained Near Horizon Coordinates, as well as Tortoise and Isotropic coordinates, for the Schwarzschild solution. At the end of Sect. 3 we also briefly obtained a Lemaître-like case, on hypersurfaces with t constant. However, it is more of an illustrative example, since it does not share desirable properties of the correct Lemaître system, such as allowing a continuous crossing of the event horizon, or the interpretation of the time coordinate with a free falling observer -same time τ of Painleve-Gullstrand.

At the final part we have included two important examples, the Kruskal-Szekeres and Lemaître coordinates. Although both were obtained with relations from the previous sections, these two coordinate systems needed some additional assumptions: This is the result of allowing $b = b(\tau, \rho)$ in the transformed system.

Throughout developments, we have emphasized the Schwarzschild spacetime. However, the method may be readily applied to other spacetimes, such as Reissner-Nordström, de Sitter, Anti de Sitter, among others [16,20,21].

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Appendix A

Start by noticing that a (shell) observer fixed at a Schwarzschild coordinate r measures a radial spacelike distance ($dt = d\theta = d\phi$) given by

$$ds = \frac{dr}{\sqrt{1 - \frac{2M}{r}}}, \tag{A.1}$$

that yields directly the known radial length contraction for the Schwarzschild observer as one approaches $r_s = 2M$. This induces a radial coordinate that can be rewritten as

$$d\varrho = \frac{rdr}{\sqrt{r^2 - 2Mr}}. \tag{A.2}$$

Defining $w = r^2 - 2Mr$ we have $dw = 2(r - M)dr$, the last expression is the same as

$$d\varrho = \frac{dw}{2\sqrt{w}} + \frac{Mdr}{\sqrt{(r - M)^2 - M^2}}, \tag{A.3}$$

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that integrated is

$$\varrho = \sqrt{r(r-2M)} + M\cosh^{-1}\left(\frac{r}{M} - 1\right). \tag{A.4}$$

Note that the integration constant is zero, in such a way that $\varrho = 0$ when $r = 2M$.

Now, since $\cosh^{-1}(2x^2+1) = 2\sinh x$ for $x \geq 0$, the second term is $2M\sinh\sqrt{r/2M-1}$. For r very near to $r = 2M$, it can be approximated to $2M\sqrt{r/2M-1}$, while the first term is $\sqrt{2M(r-2M)} = 2M\sqrt{r/2M-1}$. That is, when r is very near to $2M$, (A.4) is $\varrho \approx 2\sqrt{2M(r-2M)}$.In terms of the new coordinate, then

$$r = \frac{\varrho^2}{8M} + 2M, \tag{A.5}$$

and by substituting in the Schwarschild metric (at constant angles), we obtain

$$ds^2 = -\left(\frac{\varrho^2}{\varrho^2 + 16M^2}\right)dt^2 + d\varrho^2. \tag{A.6}$$

Near the horizon, $\varrho^2 + 16M^2 \approx 16M^2$, and scaling the time coordinate as we have scaled the time coordinate $T = t/4M$, we obtain

$$ds^2 \approx -\varrho^2 dT^2 + d\varrho^2, \tag{A.7}$$

that is, the (1+1) Rindler space that can be useful to make an association between accelerated frames in Minkowski space with Schwarschild space, remarkably the Unruh effect and Hawking radiation.

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