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Hawking Radiation as a Manifestation of Spontaneous Symmetry Breaking

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Abstract: We demonstrate that the black hole evaporation can be modelled as a process where one symmetry of the system is spontaneously broken continuously. We then identify three free-parameters of the system. The sign of one of the free-parameters, governs whether the particles emitted by the black-hole are fermions or bosons. The present model explains why the Black Hole evaporation process is so universal. Interestingly, this universality emerges naturally inside certain modifications of gravity.

Keywords: hawking radiation; spontaneous symmetry breaking; black holes

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0. Introduction

Black holes are highly compact objects, generating very strong gravitational fields. They concentrate all their mass inside the gravitational radius $2GM$, which becomes to be an event horizon for the case of spherical symmetry [1]. For Kerr-black holes (rotating), as well as charged black-holes, some modifications for this expression are done [1,2]. In any case, once an object crosses the event horizon of a black-hole, it can never escape from the huge gravitational attraction experienced at such scales. This is a classical perspective of a black hole. Quantum mechanically however, it is known that the black holes emit particles in the form of a spectrum of radiation. This was the seminal discover of Hawking [3]. Although Hawking's original formulation focused on the use of Bogoliubov transformations, several other methods were developed, including path integrals, etc [4]. The Hawking's mechanism (radiation), brought with itself a huge theoretical problem, namely, the black-hole information paradox [5]. The paradox suggests that since the black holes emit the particles in the form of radiation, then they come out without any information from the past. This is called lost of unitarity. Unitarity is one of the fundamental conditions satisfied in Quantum Mechanics [6]. By the date there is no a universally accepted solution for the black-hole information paradox, although it is strongly suspected that the solution must be connected with the concepts related to the holographic principle [7]. Considering this important information problem, it becomes highly priority to analyze the Hawking radiation from different perspectives. This possibility is open now with the outcome of analogue models, some of them including holographic principles [8]. In previous papers, the black-hole evaporation by using analogue models has been done, including an analysis from the perspective of neural networks [9,10]. In this paper we demonstrate that the Hawking radiation is equivalent to the spontaneous breaking of some symmetry of the system. The general idea is that the Black Hole starts with a state with some specific mass, angular momentum and charge (M_1, L_1 , Q_1). Although there are many possible internal Black Hole configurations able to reproduce these combinations (degenerate vacuum states), at the moment when the radiation is emitted, only one of those configurations is allowed. This is equivalent to a system selecting one specific vacuum state. Subsequently, after the emission of radiation, the system will be in a different state with M_2, L_2 and Q_2. Since there are many different states with internal configurations able to satisfy this condition, the system again selects one specific vacuum configuration. Each time that the system selects a particular vacuum state, then the symmetry is spontaneously broken and then radiation is emitted. In this paper we construct a model where a scalar field represents the particle number for the particles emitted by the Black-Hole. The scalar field Lagrangian then contains a potential term with a scalar field expansion of the order of quadratic, cubic and quartic order. The system then has three free-parameters. The relation between the different

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parameters, changes the vacuum configuration and then the behavior of the particles. In particular, the sign of the parameter related to the cubic interaction term for the particle-number field, defines whether the particles evaporating are fermions or bosons. In the limit, when this parameter vanishes, there is no distinction between fermions and bosons. We interpret this result in the language of Black-Holes.

1. Standard Formulation of the Black Hole Evaporation

The black hole evaporation process is a quantum effect, derived originally inside a semiclassical approach [3]. It is semiclassical because the derivation is not carried inside a full formulation of quantum gravity. Instead, the calculation works inside the classical background of General Relativity with a quantum field moving around. The emission of particles coming from a Black Hole can be explained when we observe a Penrose diagram as it is shown in the Figure 1. We can then imagine an observer located at the infinity with respect to the event horizon and observing the dynamic of a quantum scalar field, which represents the matter content perceived at that point in spacetime. The quantum field at that point can be expanded as

$$\phi(x,t) = \sum_{\mathbf{p}} \left(f_{\mathbf{p}} \hat{b}_{\mathbf{p}} + \bar{f}_{\mathbf{p}} \hat{b}^{+}_{\mathbf{p}} \right). \tag{1}$$

This expansion contains all the information of the quantum field. The original argument of Hawking, suggests that the radiation process is a natural consequence of a comparison between the vacuum located at the future null infinity, namely, the vacuum defined after the formation of the Black-Hole (but defined by an observer located at large scales with respect to the event horizon of the Black-Hole); with respect the vacuum located in the past null-infinity, namely, the one defined before the Black-Hole formation and devoid of particles. Before the Black-Hole formation (past-null infinity) there are no particles; then we can settle the vacuum in such a case as

$$\hat{b}_{\mathbf{p}} |\bar{0}>= 0. \tag{2}$$

Then the expansion (1) corresponds to the field expansion at the past-null infinity (before the formation of the black hole). If the radiation effect is true, then we should expect the observers located at the future null-infinity to detect particles, even if these are not supposed to escape from the event horizon of the Black-Hole. This occurs because after the formation of the black-hole, the vacuum state changes completely and then it is defined in a different way. Actually, after the creation of the black-hole, the same field defined in eq. (1), can be expanded now in terms of two different modes as follows

$$\phi(x,t) = \sum_{\mathbf{p}} \left(p_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \bar{p}_{\mathbf{p}} \hat{a}^{+}_{\mathbf{p}} + q_{\mathbf{p}} \hat{c}_{\mathbf{p}} + \bar{q}_{\mathbf{p}} \hat{c}^{+}_{\mathbf{p}} \right). \tag{3}$$

Both equations, (1) and (3) carry the same information. The vacuum $\hat{a}_{\mathbf{p}}|0>= 0$ is defined at the future null-infinity. On the other hand, the modes $q_{\mathbf{p}}$, together with the operators $\hat{c}_{\mathbf{p}}$, are defined at the event horizon. These are the modes which the observer located at the asymptotic future cannot see because they are hidden. Then here we focus only on the modes which are perceived by the observers located at large scales from the event horizon. In order to relate the modes defined by $f_{\mathbf{p}}$ and those defined by $p_{\mathbf{p}}$, we employ the Bogoliubov transformations. Following the arguments of Hawking in [3], we can find that the relation between the modes under discussion, respect the following relation

$$\hat{a}_{\mathbf{p}} = u_{\mathbf{p},\mathbf{p}'} \hat{b}_{\mathbf{p}'} - v_{\mathbf{p},\mathbf{p}'} \hat{b}^{+}_{\mathbf{p}'}. \tag{4}$$

Then we can see why the vacuum at the future null infinity is not empty even if it is devoid of particles in the past null infinity. The density of particles emitted by the black hole, is defined by the following expression

$$<\bar{0}|\hat{n}^{a}_{\mathbf{p}}|\bar{0}>= |v_{\mathbf{p},\mathbf{p}'}|^2. \tag{5}$$

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The arguments of Hawking, based on the behavior of a quantum field moving around the black-hole, demonstrated that the fraction of particles escaping the Black-Hole, follow the statistical distribution

$$< \bar{0}|\hat{n}^a_{\mathbf{p}}|\bar{0} >= \frac{\Gamma_{\mathbf{p},\mathbf{p}'}}{e^{\frac{2\pi\omega}{\kappa}} \pm 1}. \tag{6}$$

The negative sign is taken if the particles escaping the Black-Hole are bosons, while the positive sign is taken if the emitted particles are fermions. In the previous expression, $\Gamma_{\mathbf{p},\mathbf{p}'}$ represents the fraction of particles entering the collapsing body (Black-Hole) [3]. In the Hawking calculation, the relation between the Bogoliubov coefficients is given as

$$|u_{\mathbf{p},\mathbf{p}'}| = e^{\frac{\pi\omega}{\kappa}}|v_{\mathbf{p},\mathbf{p}'}|. \tag{7}$$

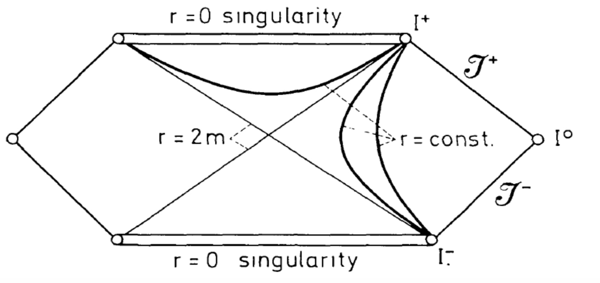

Figure 1. The Penrose diagram for the Schwarzschild geometry in GR as it is showed in [3]. The past-null infinity (J^-) of the diagram, corresponds to the events where the black hole has not yet formed. The future null infinity on the other hand, corresponds to the cases where the black-hole is already formed.

2. Black Hole Evaporation as a Consequence of Spontaneous Symmetry Breaking

The black hole evaporation effect can be also obtained if we take the particle number operator $\hat{n}^a_{\mathbf{p}}$ as quantum scalar field. Then we define its Lagrangian as the one containing a kinetic term and and potential term with the field expansion done up to fourth-order, without omitting the cubic term in the field expansion. Working in this way, we have a Lagrangian of the form

$$\pounds = \frac{1}{2}\partial^\mu \hat{n}^a_{\mathbf{p}}(\omega)\partial_\mu \hat{n}^a_{\mathbf{p}}(\omega) - V(\hat{n}^a(\omega)), \tag{8}$$

with the partial derivative taken with respect to the frequency ω, namely, $\partial_\mu = \partial_\omega = \frac{\partial}{\partial\omega}$. The potential term in eq. (8) is

$$V(\hat{n}_{\mathbf{p}}) = \frac{1}{2}m^2\hat{n}^2_{\mathbf{p}} + \frac{\beta}{3}\hat{n}^3_{\mathbf{p}} + \frac{\lambda}{4}\hat{n}^4_{\mathbf{p}}. \tag{9}$$

In eq. (9) we have omitted the index a for simplicity. Although in principle, the ground state for the potential is defined by solving the equation $\partial V/\partial\hat{n} = 0$; we cannot assume this to be the correct result in this case because the spacetime curvature forces the field $\hat{n}_{\mathbf{p}}$ to have a kinetic term at its most stable configuration. Still, it can be proved that the symmetry of the system under exchange of internal configurations consistent with the Black-Hole entropy (exchange of particles), is spontaneously broken when $m^2 < 0$. In addition, the signature of the parameter β tells us whether the particles evaporating are bosons or fermions. In other words, we have two solutions for this case. The first solution corresponds to the boson statistics and the second one corresponds to the fermion statistics. Both statistics are identical to the ones defined in eq. (6) and equally obtained inside the standard formalism due to Hawking. The black hole naturally emits both types of particles. When the surface gravity of the black-hole tends to zero, the statistics of fermions and bosons emitted by the black-hole, have a similar behavior. However, when the surface gravity of the Black Hole increases, then more bosons than fermions are emitted [3]. From these explanations, it is clear that the parameter β, must be related to the surface gravity as we will demonstrate in a moment. Now we can now proceed to analyze the behavior of the solution of the Euler-Lagrange equations obtained from the Lagrangian defined in eq. (8).

2.1. Emission of Particles

Here we will explore the solutions for the field equations, corresponding to the Lagrangian in eq. (8). The system in general, has the tendency for looking for some equilibrium configuration, but it is unable to reach such condition and instead it emits particles at each instant of selection of some configuration. The solution to the field equations, obtained from the Lagrangian (8), is given by

$$\hat{n}^a = \frac{A}{e^{-\gamma\omega} \pm 1}, \tag{10}$$

with the corresponding parameters $m^2 = -\gamma^2$, $\beta = 3\gamma^2/A$ and $\lambda = -2\gamma^2/A$, as they are defined inside the potential (9). The sign of β defines whether the emitted particles are bosons or fermions. If we compare eq. (10) with eq. (6), then γ takes negative values in this case. Note that the parameters m, β and λ do not depend on the sign taken by γ. After doing the corresponding comparisons, then γ is related to the surface gravity as

$$\gamma = -\frac{2\pi}{\kappa}. \tag{11}$$

Then the surface gravity is equivalent to $\kappa = -2\pi/\gamma$. It is a trivial task to notice that $A = \Gamma_{\mathbf{p},\mathbf{p}'}$ inside the present comparison. Note that since m^2 is negative in this case, the symmetry under exchange of particles (keeping the same total mass, angular momentum and charge) is spontaneously broken. Then a black-hole is just an unstable system, permanently trying to find its ground state configuration but it never reaches it until it evaporates completely.

2.2. The Connection of β with the Particle Statistic

For seeing the connection between the sign of β and the statistic followed by the particles emitted, we analyze the Euler Lagrange equations obtained from eq. (8), but considering dimensionless coupling constants. In such a case, we get

$$(\partial^\mu\partial_\mu + \bar{m}^2)n_R^a + \bar{\beta}(n_R^a)^2 + \bar{\lambda}(n_R^a)^3 = 0. \tag{12}$$

Here the sub-index R makes reference to the fact that we are dealing with a dimensionless equation which will give a dimensionless solution. We can convert easily the dimensionless solution toward the full solution (10). For getting the fermionic statistic, we need to satisfy the set $\bar{\lambda} = -2$, $\bar{\beta} = 3$ and $\bar{m}^2 = -1$. On the other hand, for getting the bosonic statistic, the same set of parameters are valid, except the value of $\bar{\beta}$, which for the bosonic case takes the value $\bar{\beta} = -3$. It is a trivial task to demonstrate the following expressions

$$m^2 = \bar{m}^2\gamma^2, \qquad \beta = \frac{\bar{\beta}\gamma^2}{A}, \qquad \lambda = \frac{\bar{\lambda}\gamma^2}{A}. \tag{13}$$

Then the change in sign of the dimensionless parameter $\bar{\beta}$, affects the sign of β, and then the statistic following by the emitted particles depends on the signature taken by β. Combining eq. (13), with (11), we get

$$\beta = \frac{4\pi^2\bar{\beta}}{A\kappa^2}. \tag{14}$$

From this expression, it is clear that when $\bar{\beta} \to 0$, then $\kappa \to 0$. Then more properly, when $\bar{\beta} = 0$, equal amount of fermions and bosons are emitted by the black-hole. However, if instead we keep $\bar{\beta} = \pm 3$

in agreement with the paragraphs following eq. (12), then the condition for getting equal amount of fermions and bosons is $\beta \to \pm\infty$, with the positive sign corresponding to the fermionic statistic.

3. Curvature Effects Appearing from the Particle Lagrangian

The first impression coming from the Lagrangian (8), is that gravity is apparently absent and then all the calculations would represent a simple coincidence between the results obtained by Hawking in eq. (6) and the one obtained in this paper in eq. (10). However, these types of coincidences do not exist and here we will prove that in fact, gravity appears implicit inside the Lagrangian defined in eq. (8). The Lagrangian of a standard scalar field moving along a flat spacetime (without gravity), is defined as

$$£ = \frac{1}{2}\partial^{\mu}\phi(x)\partial_{\mu}\phi(x) - \bar{m}^2\phi^2(x). \tag{15}$$

Here $\bar{m}$ is the mass of the Quantum field moving along the flat spacetime. The vacuum state of this Quantum field is simply $\phi(x) = 0$, if we ignore the residual vacuum energy coming from the ground state of the Quantum harmonic oscillator [11]. Now let's introduce gravity over this system such that the Quantum field moves now along a curved spacetime with minimal coupling. In such a case, the Lagrangian takes the form

$$£ = \frac{1}{2}\sqrt{-g}\Big(g^{\mu\nu}\phi_{,\mu}(x)\phi_{,\nu}(x) - m^2\phi^2(x)\Big) \tag{16}$$

Here the gravity effects emerge from the deviations of the metric with respect to the Minkowski spacetime. Although the spacetime curvature generated by a Black-Hole is very large, for an initial explanation, we can apply perturbation theory over the spacetime metric, in order to analyze how the terms appearing on the potential (9) emerge. Perturbative theory takes the small deviations of the metric $g_{\mu\nu}$ with respect to Minkowski as $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$. Here $\eta_{\mu\nu}$ is the Minkowski spacetime, while $h_{\mu\nu}$ is the perturbation around Minkowski. In this way, $\sqrt{-g} \approx 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}^2 + ...$ up to second order, with $h = 0$ in vacuum and $h \propto T$ when there is a source term $T \propto \phi^2(x)$ at the ground state. We can also make similar statements for the case $h_{\mu\nu} \propto T_{\mu\nu}$ since at this point we are only concerned about proportionality relations. Then the Lagrangian near the ground state (ignoring kinetic terms) now becomes

$$£ \approx -\frac{1}{2}\left(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{2}h_{\mu\nu}^2\right)\bar{m}^2\phi^2. \tag{17}$$

If we expand the Lagrangian (16) by considering the previous comments and the result (17), then it is evident that terms of different orders on the field $\phi(x)$ will emerge if we take into account that $h \propto T \propto \phi^2(x)$. This also means that terms of different orders in the particle number operator $\hat{n}_{\mathbf{p}}$ will emerge, considering that naively $\hat{n}_{\mathbf{p}} \propto \phi^2(x)$, given the fact that the scalar fields are linear functions of the annihilation and creation operators. With these arguments, the Lagrangian (17), generate terms of the form

$$£ = a\hat{n}_{\mathbf{p}} + b\hat{n}_{\mathbf{p}}^2 + c\hat{n}_{\mathbf{p}}^3 + ... \tag{18}$$

The expansion include higher order terms at the non-linear level, which increase in relevance. If we compare eq. (18) with eq. (17), then obviously certain terms in the expansion in eq. (18), would correspond to the terms in the potential (9) in a direct way. There will be other terms in the expansion difficult to compare, unless a re-summation between terms emerge at the event horizon level. Yet still, we can see that each term in eq. (9) can be reproduced from the Einstein-Hilbert expansion no matter what. These type of re-summation methods appear in massive gravity in order to eliminate an undesirable ghost at the non-linear level [12–14]. Since massive gravity converges to General Relativity when the gravitational field is strong, then the amount of particles emitted from the event horizon of a black-hole in General Relativity is the same amount of particles emitted from the event horizon of a

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black-hole inside the non-linear theory of massive gravity, as it was demonstrated in [15,16]. Based on this interesting aspect for the black-holes, it is important to realize that although the metric expansions developed in [12–14] were done thinking on a massive theory of gravity (non-linear), still the same formalism is general in the sense that we can use it for analyzing certain aspects of gravity. In [12] is illustrated how the deviations with respect to Minkowski can be represented in a non-linear theory of gravity as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = H_{\mu\nu} + Special\ \ terms$. Here the special terms refer to those terms carrying out gravitational degrees of freedom by using the Stückelberg trick [17]. In this way, at the end of the calculations, it was demonstrated that if we want to find the source term of the Einstein equations, it can be calculated from a potential term containing quadratic, cubic and fourth order terms in the metric (the same order corresponds to the particle number operator) [14]

$$U_{source} = U_2 + \alpha_3 U_3 + \alpha_4 U_4. \tag{19}$$

This potential contains three free-parameters which can be paired with three free-parameters of the Einstein-Hilbert action after considering the field equations [14]

$$G_{\mu\nu} = -m^2 X_{\mu\nu}, \tag{20}$$

with $X_{\mu\nu} = \frac{\delta U}{\delta g^{\mu\nu}} - \frac{1}{2} U g_{\mu\nu}$. What is important to remark here is that there is a direct connection between the series expansion of the standard Einstein-Hilbert action and the potential expansion defined in eq. (19) through the Euler-Lagrange equations, which give us the field equations in (20). In eq. (19), since each term $U_n \propto g^n_{\mu\nu} \propto \phi^{2n} \propto n^n_{\mathbf{p}}$, then we have a direct correspondence between the potential defined in eq. (19) and the potential proposed in eq. (9). Then the Hawking radiation effect is so general, that the form of the Lagrangian reproducing it from eq. (9) appears in theories intending to generate source terms with degrees of freedom being able to move through an event horizon. The result is generic and it explains why the Hawking radiation obeys the bosonic/fermionic statistics of a black-body. In other words, if the Lagrangian (8) had a different potential term instead of (9), then the statistics followed by the spectrum of Black-hole would change dramatically. This can be seen if we evaluate the Euler-Lagrange equations over eq. (8).

4. Conclusions

In this paper we have proved that it is possible to model the black-hole evaporation process as a mechanism of continuous symmetry breaking, where the black hole is permanently selecting some specific vacuum configuration, forcing then the system to emit particles. This means that a black-hole, having a degenerate vacuum state in agreement with its entropy, will select a ground state among all the possibilities consistent with its mass, angular momentum and charge. Once this occurs, the system emits radiation, decaying towards a new configuration with different values of charge, angular momentum and mass. The process then continues until the black-hole evaporates completely. This means that the black-hole is permanently looking for one stable configuration but it never reaches it and that's why the Hawking radiation process emerges. Finally, we have also identified and analyzed some free parameters for the field theory which is able to model the Black-Hole evaporation process. Inside the free-parameters, the signature of β or the signature of its dimensionless counterpart $\bar{\beta}$, determine whether the particles evaporating are fermions or bosons. The Lagrangian proposed for modelling the scalar field moving around a Black-Hole, is naturally coupled to gravity because the higher order terms or self-interaction terms are precisely consequence of this coupling. Finally, we have demonstrated the generic character of the Hawking radiation by showing that similar potential forms emerge from theories reproducing the propagation of gravitational degrees of freedom through black-hole event horizons [14]. In fact, by using the standard and generic method where a non-linear formulation of gravity can include a source term (depending on the scalar field), a potential of the form (9) emerges. This should not be a surprise because in massive gravity the degrees of freedom

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propagating in addition to the tensorial (spin-2) component is a scalar (spin-0) component. Then in gravity in general, the propagation of a scalar field in the presence of curvature effects is generic.

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