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Fiber Structure and Stability Theorems for Symmetry Breaking Transitions

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Abstract: We discuss three key points: defining mathematically the notions of *phase*, *symmetry breaking*, *topological defects* and their *stability* in a formal language of homotopy theory and fiber bundle formalism; postulating and proving two theorems for successive symmetry breaking scenarios in condensed matter physics and cosmology [1], and demonstrating application of theorems to superfluid Helium 3 and Kibble-Lazarides-Shafi (KLS) walls, supporting and mathematically justifying the results of [2].

Keywords: Fiber bundle; phase transitions; Lie groups; homotopy; fundamental group; Stability Theorem; Stability Lemma; Fiber Structure Theorem; superfluid Helium

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1. Motivation

In physics, fiber bundles and homotopy theory are fundamental for understanding fields and forces, offering a geometric interpretation of various physical phenomena, including gauge theories and aspects of general relativity. [3–15]. Employing the language of fiber bundles offers significant advantages in studying the phase transitioning systems. Fiber bundles are able to reveal the contrast between local simplicity and global complexity making these differences the key points in understanding topological properties of the symmetry breaking systems. Additionally, the homotopy lifting property on a fiber bundle enables conversion from *relative homotopy groups* to *absolute homotopy group*: $\pi_n(X, x_0) \cong \pi_n(G, H, e_0)$, naturally extending the results of [2]. In the fiber bundle framework, every path in a topological space X based at x_0 will be lifted to a unique path on fiber bundle G (with a fiber H and a base $B \equiv G/H$) starting at e_0 and ending in fiber H up to homotopy. In first section, we show how fibers and bases of fiber bundles together with homotopy can be used to define the notions of an *order parameter space*, *topological defect*, *symmetry breaking*, *phases* and *phase transitions*. The Defect and Fiber structure theorems we present in the second section of this work allow us to expose explicitly the mathematical conditions at which long exact homotopy sequence of the absolute homotopy group can be used to track the stability of a topological defect in successive phase transitions. Finally, we provide the examples of how these theorems may be used on the most well-studied condensed matter phase transitioning system- superfluid Helium [16–20].

2. Preliminary Knowledge

Suppose that B and H are Hausdorff topological spaces. A **fiber bundle G** with base B and fiber H is a Hausdorff topological space X together with a continuous map $p : X \to B$ (called **projection map**) with the following properties:

- For each $b \in B$, the preimage of $p^{-1}(b)$ of $b \in X$ is homeomorphic to H
- For each $b \in B$, $\exists$ neighborhood $U(b)$: **fibers** $p^{-1}(U)$ are homeomorphic to $X = U \times H$, such that $p(x) = p_1(\Phi(x)), p_1 : U \times H \to U$ is the map $p_1(u, h) = u$ and $x \in X$

Equivalence relation is a binary relation which is reflexive ($a \sim a$), symmetric ($a \sim b, b \sim a$), and transitive ($a \sim b, b \sim c, a \sim c$).

Equivalence class is a subset $Y \subset X$, such that $n \sim m, \forall n, m \in Y$ and never a case for $n \in Y$, but $m \notin Y$. The equivalence class to which n belongs is denoted as $[n] \equiv \{x \in X : n \sim x\}$. All elements of X equivalent to each other are also elements of the same equivalence class. The equivalence class of a path f under the equivalence relation of homotopy called the homotopy class of f.

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The n-th homotopy group $\pi_n(X, x_0)$**.** Let I^n be n-cube consisting of points $t = (t_1, t_2, ..., t_n)$ in Euclidean n-space, such that $0 \leq t_i \leq 1$, $i = 1, ..., n$. Setting $t_i = 0$ or $t_i = 1$ gives us the $(n-1)$ -face , and union of such $(n-1)$ -faces forms the boundary ∂I^n of I^n. Consider maps of I^n into X, such that ∂I^n maps into x_0 , then the elements of $\pi_n(X, x_0)$ are homotopy classes of such maps. Alternatively, an element of $\pi_n(X, x_0)$ can be defined as a homotopy class of maps of S^n into X, with $y_0 \rightarrow x_0$, where y_0 is the reference point on S^n to which the boundary ∂I^n is squished.

Normal subgroup. The subgroup $H \subseteq G$ is called normal if and only if $gHg^{-1} = H$ for $\forall g \in G$. Equally, a subgroup $H \subseteq G$ is normal if and only if $gH = Hg, \forall g \in G$.

Quotient space. The left coset of $H, \forall g \in G$ is a subset of $G : \{gh | h \in H\}$. Collection of all cosets of H is denoted by $G/H = \{gh | g \in G\}$, so elements of G/H are **cosets** , not elements of G. If H is a normal subgroup of G, then G/H can be turned into a group and we call G/H **the quotient group**.

Covering theorem. Let $\tilde{X}$ be the covering space for X, such that there exists a projection map $p : \tilde{X} \rightarrow X$, such that $p(\tilde{x_0}) = x_0$. Then exists induced homomorphism p^*, such that $p^* : \pi_n(\tilde{X}, \tilde{x_0}) \rightarrow \pi_n(X, x_0)$ is the isomorphism onto for $n \geq 2$.

Universal property: Let X be a space and $\sim$ equivalence relation on X with corresponding quotient space $Y = X/\sim$. The projection map $q : X \rightarrow Y$ has the universal property: for a given map $f : X \rightarrow Z$, such that $f(x) = f(y)$ if $x \sim y$, there $\exists$ a unique map $g : Y \rightarrow Z$, such that $f = g \cdot q$. Every map $g : Y \rightarrow Z$ lifts to a map on X that is constant on equivalence classes.

Homotopy Lifting Property. A fiber bundle G with the base B and projection map $p : G \rightarrow B$ has the homotopy lifting property with respect to all CW pairs (X, A) [21]

Although some of the order parameter spaces can be directly observed by physicists in condensed matter laboratories, the topological defects defined as the elements of the $n-$ the homotopy group are not easily related to the physical realms. The term "defect" is quite ambiguous and used in various disciplines implying different meanings [22–25]. In this work, a topological defect is referred to a mathematical structure uniquely associated to a phenomena observed by physicists in the laboratory. The observed phenomena by experimentalists will be referred as **physical defects**. For example, if two physical defects, such as vortices or phonons, are observed in some condensed matter experiment, we associate each observed physical defect to a unique element of the fundamental group of the parameter space and call it a topological defect.

Order parameter space and topological defects. Let G be having a structure of fiber bundle of dimension m with fiber H of dimension k, and base G/H of dimension $m-k$. The projection map $p : G \rightarrow G/H$ is given by $p(g) = [x]$, where $g \in G$ and $[x]$ denotes the coset $xH \in (G/H)^{m-k}$. The base $B \equiv G/H$ is called the **order parameter space** of dimension $m-k$. The elements of the n-th homotopy group $\pi_n(B)$ are called **topological defects** (or simply defects).

Symmetry breaking. Let G be a group corresponding to the maximal symmetries of some physical system named K. We say that system K undergoes *symmetry breaking* if exists a normal subgroup H in G and continuous map $p : G \rightarrow G/H$. Then H represent the set of symmetries the system X is invariant after the symmetry breaking, and the quotient group G/H represents the symmetries under which X is no longer invariant.

Symmetry breaking may also be discussed in the language of fiber bundles. For some arbitrary system K with its maximal symmetries described by the fiber bundle G , single spontaneous symmetry breaking is given by the short fiber bundle sequence $N \longrightarrow G \xrightarrow{p} (G/N)$, where N is the fiber of G, which represents the unbroken or remaining symmetries in the system G and the base G/N represents the group of broken symmetries.

Phases and transitions. Let G be a group of the maximal symmetries of the considered system and have a structure of the fiber bundle. Let N_1 be a closed subgroup of G and N_2 be a closed subgroup of N_1, so that $N_2 \subseteq N_1 \subseteq G$. The number of such closed "nested" subgroups in G is called the number of **phases** which the system can transit to, and our case is given for the two phases. The system is said to experience **two-phase transitions**, if there exists a continuous map p from $G/N_2 \xrightarrow{p} G/N_1$.

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3. The Defect and Fiber Structure Theorems

Local cross section f. Let N be a closed subgroup of G, so that N is a point $n_0 \in G/N$. A local cross section of N in G is a function f that continuously mapping a neighborhood $N_\epsilon(n_0)$ of (n_0) into G such that $pf(n) = n$ for $\forall n \in N_\epsilon$. Such f exists for every fiber bundle G over G/N [26].

Fiber structure theorem. Let N_1 be a closed subgroup of G such that it admits a local cross section f , and N_2 be a closed subgroup of N_1: $N_2 \subseteq N_1 \subseteq G$, and let $p : G/N_2 \to G/N_1$ be the the map induced by the inclusion of cosets. Two statements can be made:

Statement 1. G/N_2 has a fiber bundle structure relative to p with the fiber N_1/N_2, and the group of the bundle is N_1/N_{2_0} with N_{2_0} being the largest subgroup of N_2 invariant in N_1 acting in N_1/N_2 as left translations. There exists a short homotopy sequence group, $N_1/N_2 \to G/N_2 \to^p G/N_1$ (In other notation, $F \to E \to^p B$).

Statement 2. For path-connected base G/N_1, there exists a long homotopy sequence: $\pi_i(N_1/N_2) \to \pi_i(G/N_2) \to \pi_i(G/N_1) \to \pi_{i-1}(N_1/N_2) \to \pi_{i-1}(G/N_2) \to \pi_{i-1}(G/N_1) \to ....$

Proof:

- *Proof of statement 1.* By the bundle structure theorem in [26] , one can assign a bundle structure to G/N_2 relative to p with the fiber N_1/N_2.
- *Proof of statement 2:* By Proposition 4.48 in [21], a fiber bundle $p : G/N_2 \to G/N_1$, (base is G/N_1 and fiber N_1/N_2), has the homotopy lifting property with respect to all CW pairs (X, A). By Theorem 4.41 and Proposition 4.1 in [21], there exists a long exact homotopy sequence $\pi_i(N_1/N_2) \to \pi_i(G/N_2) \to \pi_i(G/N_1) \to \pi_{i-1}(N_1/N_2) \to \pi_{i-1}(G/N_2) \to \pi_{i-1}(G/N_1) \to ....$

Corollary. For two phase-transitions, the long homotopy sequence,
$\pi_i(N_1/N_2) \to \pi_i(G/N_2) \to \pi_i(G/N_1) \to \pi_{i-1}(N_1/N_2) \to ...$, where the fiber is $F \equiv N_1/N_2$, the fiber bundle is $E \equiv G/N_2$, and the base is $B \equiv G/N_1$ and the short sequence is $N_1/N_2 \to G/N_2 \to G/N_1$. However, if N_2 is trivial, then the original fiber bundle is reduced to $E_r \equiv G$, with the reduced fiber $F_r \equiv N_1$, and the base $B_r \equiv G/N_1$, so the short sequence for the reduced fiber bundle becomes $N_1 \to G \to G/N_1$, and the long homotopy sequence becomes $\pi_i(N_1) \to \pi_i(G) \to \pi_i(G/N_1) \to \pi_{i-1}(N_1) \to \pi_{i-1}(G) \to \pi_{i-1}(G/N_1) \to ...$

Stability lemma. For a topological space X, these conditions are equivalent:

1. Any map $S^n \to X$ is homotopic without regard to basepoints to a constant map, with image a point
2. Any map $S^n \to X$ extends to a map $D^{n+1} \to X$
3. $\pi_n(X, x_0) = 0$ for all $x_0 \in X$

Proof:

- 1. $\to$ 2. Consider $S^n \times I$ with the defined equivalence relation $\sim$ on it and I being $n-$dimensional version of the interval $I = [0, 1]$. The map $q : S^n \times I \to S^n \times I/\sim (S^n \times (0, ..., 0)$ collapsing the top of the mapping cylinder $S^n \times (0, 0, ..., 0)$ to a single point gives us a quotient cone $Y \equiv S^n \times I/\sim (S^n \times (0, ..., 0)$. Consider a given map $g : S^n \to X$ that is homotopic without regard to a basepoints to a constant map with image a point, so that $g(x) = g(y)$ for $x \sim y, x \in S^n \times I, y \in Y$. Then by the universal property, there exists a unique map $l : Y \to X$, such that $q \cdot l = g$, and every map $l : Y \to X$ lifts to a map from S^n that is constant on equivalence classes. Define a map $d : Y \to D^{n+1}$, such that $d(y) = d(x)$ for $x \sim y$. Then , again, by the universal property, exists a unique map $h : D^{n+1} \to X$ such that $d \cdot h = l$ and every map $h : D^{n+1} \to X$ lifts to a map from Y that is constant on equivalence classes. Denote map $d \cdot q \equiv q'$, so now $q' : S^n \times I \to D^{n+1}$. Then $h : D^{n+1} \to X$ and $q' \cdot h = g$, so any map $S^n \to X$ extends to a map $D^{n+1} \to X$.

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$$\begin{array}{ccc} S^n \times I & \xrightarrow{g} & X \\ \downarrow q & \nearrow l & \uparrow h \\ Y & \xrightarrow{d} & D^{n+1} \end{array}$$

$$\begin{array}{ccc} S^n \times I & \xrightarrow{g} & X \\ q'=d\circ q \uparrow & \nearrow h & \\ D^{n+1} & & \end{array}$$

- 2. $\rightarrow$ 3. Adopt the second diagram from the part above and consider that any map $S^n \rightarrow X$ extends to a map $D^{n+1} \rightarrow X$. Moreover, consider the map $g : S^n \times (1, ..., 1) \rightarrow (X, x_0)$ to be restriction of the map $h : (D^{n+1}, (1, ..., 1) \rightarrow (X, x_0)$.

$$\begin{array}{ccc} S^n \times (1,..,1) & \xrightarrow{g} & (X, x_0) \\ q' \uparrow & \nearrow h & \\ D^{n+1}, \times (1,..,1) & & \end{array}$$

$$\begin{array}{ccc} \pi_n(S^n \times (1,..,1)) & \xrightarrow{g^*} & \pi_n(X, x_0) \\ q'^* \uparrow & \nearrow h^* & \\ \pi_n(D^{n+1}, \times (1,..,1)) & & \end{array}$$

Consider the based homotopy class $[g'] \in \pi_n(X, x_0)$. Since D^{n+1} is deformation contractible to a point, any image of the loop of $\pi_n(S^n, 1)$ can also be deformation-contracted to a point. So the images of loops of $\pi_n(S^n, 1)$ and $\pi_n(D^{n+1}, 1)$ are homotopy equivalent to a point, hence $\pi_n(X, x_0)$ is trivial $\forall x_0 \in X$.

- 3. $\rightarrow$ 1. Adopting a hypothesis that $\pi_n(X, x_0)$ is trivial $\forall x_0 \in X$, consider any map $g : S^n \rightarrow X$ and let $[g] \in \pi_n(X, x_0)$, with $g(s) = x_0$ for some $s \in S^n$. Then $[g]$ is nullhomotopic.

Physical stability theorem. Let r be an arbitrary dimension of the physical defect bounded to $m-$dimensional space. An $r-$ dimensional physical defect cannot be physically stable if its associated homotopy class is homotopic to a constant map (or, equivalently, the fundamental group $\pi_{m-r-1}(G/N)$ is trivial).

Proof: If the $\pi_{m-r-1}(G/N)$ is trivial, then by Stability lemma, any map from $S^{m-r-1} \rightarrow G/N$ is homotopic without regard to basepoints to a constant map, having a point as its image. So, any topological defects formed in G/H is homotopic to a point, and hence associated physical defects cannot be stable.

4. Examples in Condensed Matter Physics

4.1. Superfluid Helium 4

From physics experiments, superfluid Helium 4 is known to have only one superfuid phase and the complex order parameter described by $U(1) \cong S^1$ [27] .

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To connect mathematical language with the physics language, let G be having a structure of a fiber bundle represent all the symmetries of Helium 4 before the transition to its superfluid phase, and N_1 be s closed subgroup of G representing the remaining symmetries after the system underwent the phase transition from conventional fluid to the superfluid. Then the base of the fiber bundle is G/N_1 representing the order parameter of the superfluid, so that $G/N_1 = U(1) \cong S^1$.

Unlike the order parameter space, it is challenging to observe G and N_1 in the laboratory, however one can deduce G topological structure from the long homotopy sequence and assuming that N_1 is trivial. Given relatively simple nature of the order parameter, one may only consider lower order fundamental groups in the long homotopy sequence:

$$\pi_2(N_1) \xrightarrow{f} \pi_2(G) \xrightarrow{g} \pi_2(G/N_1) \xrightarrow{j} \pi_1(N_1) \rightarrow ... \tag{1}$$

Setting $\pi_2(N_1) = 0, \pi_1(N_1) = 0$, one arrives to a part in a long homotopy sequence $0 \xrightarrow{f} \pi_2(G) \xrightarrow{g} \pi_2(G/N_1) \xrightarrow{j} 0$. Because the long homotopy sequence is exact, each map is a homomorphism such that the image of each map is equal to the kernel of the following map. For example, the image of the map f is trivial, which means that the kernel of g is also trivial, and hence the image of g must be trivial as well. Given that $\pi_1(N_1)$ is trivial, the kernel of j must be $\pi_2(G/N_1)$, so the image of g is $\pi_2(G/N_1)$. As result, g is isomorphism of $\pi_2(G)$ with $\pi_2(G/N_1)$. From these results, we can deduce the Trivial fiber theorem for the transitioning systems with maximal symmetries described by fiber bundle G and single trivial fiber N_1:

***The Trivial Fiber Theorem**:* Suppose that G is a fiber bundle with the fiber N_1 and base G/N_1. If $\pi_1(N_1)$ and $\pi_2(N_1)$ are both trivial, then $\pi_2(G)$ is isomorphic $\pi_2(G/N_1)$.

Although the fiber bundle G for superfluid Helium 4 is experimentally not accessible, from the the fact that $\pi_2(G)$ is isomorphic to $\pi_2(G/N_1)$ and that $\pi_2(G/N_1) = \pi_2(S^1)$ is trivial, the second fundamental group $\pi_2(G)$ is also trivial.

Similarly, consider another part of the long homotopy sequence $\pi_1(N_1) \xrightarrow{f'} \pi_1(G) \xrightarrow{g'} \pi_1(G/N_1) \xrightarrow{j'} \pi_0(N_1)$ Since $\pi_1(N_1)$ is trivial and since $\pi_0(N_1)$ is also trivial, given that there is only one element and hence the number of connected components, we deduce that $\pi_1(G)$ isomorphic to Z , since $\pi_1(G/N_1) = \pi_1(S^1) \cong Z$. This result means that the fiber bundle G of superfluid Helium 4 contains non-contractible loops whose behavior is very much like the behavior of loops S^1.

The Physical Stability Theorem can be applied to discuss the stability of physical defects which can be observed in superfluid Helium 4.

For given dimension $m = 3$ of physical space, such as a bulk of superfluid Helium 4, and r being the physical dimension of a defect,

- The 0-dimensional defects cannot be stable since $\pi_2(S^1)$ is trivial,
- There exist 1-dimensional stable defects since $\pi_1(S^1) \cong Z$,
- The 2-dimensional defects cannot be stable since $\pi_0(S^1)$ is trivial.

One does not need to consider $r = 3$ scenario representing 3D topological defect in the system, because it is physically impossible. In fact, the $m-$ dimensional physical space M^m may be considered isomorphic to $m-$dimensional Euclidean space, $M^m \cong R^m$,and the defect space K^r of dimension r is isomorphic to $k-$ dimensional Euclidean space R^k, $K^r \cong R^k$. So, space-wise, $M^m/K^k \cong R^m - R^k$. To illustrate this idea in an explicit example, consider the topological space $R^3 - R^1$, which can be envisioned as observing the cross section of a conventional three-dimensional space formed by Cartesian coordinates with removed axis, let it be z -axis for convenience. Removing the z axis leads to a cross-section is missing a point, resulting in the space $R^3 - R^1$ behavior equivalent to R^2 with a missing point. Three dimensional topological defect would be equivalent to an object embedded in R^3, and hence having it in 3D topological medium would lead to R^3 minus R^3 behavior, leading to breaking the continuous space apart, which might cost infinitely amount of energy.

Bundle structure theorem: Suppose G is a matrix Lie group and H is a closed subgroup of G: Then G has the structure of a fibre bundle with base G/H and fiber H, where projection map

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$p : G \to G/H$ is given by $p(x) = [x]$, with $[x]$ denoting the coset $xH \in G/H$ Prove is given for Proposition 13.8 in [28].

4.2. Stability of the Defects in Superfluid Helium 3

The order parameter space for superfluid Helium 3 is given as $SO(3) \times SO(3) \times U(1)$, which we refer as the *initial base space* [29] . Given that the base space is a matrix Lie group $SO(3) \times SO(3) \times U(1)$, it is having a structure of a fiber bundle G with its own fibers N_i representing remaining symmetries, and bases $B_i = G/N_i$, corresponding to the order parameters of various phases by the Bundle structure theorem[19]. We convert the results of [30] for continuous symmetries to a fiber-bundle language (Table 1), and apply Stability lemma and Physical Stability theorem in Table 2 for some of the phases of superfluid Helium 3, where each fundamental group's multiplication rule dictates the behavior of the zero-dimensional and 1-dimensional physical defects.

Table 1. Fibers N_i and Base spaces B_i for phases of superfluid Helium 3.

i	N_i	B_i
(1)	$SO(3)_{L+S}$	$SO(3)_{L,S} \times U(1)_\phi$
(2)	$U(1)_{S_z} \times U(1)_{L_z}$	$S^2_S \times S^2_L \times U(1)_\phi$
(3)	$U(1)_{L_z} \times U(1)_\phi$	$SO(3)_S \times S^2_L \times U(1)_\phi$
(4)	$U(1)_{S_z}$	$S^2_S \times SO(3)^2_L \times U(1)_\phi$
(5)	$U(1)_{S_z+L_z}$	$S^2_S \times S^2_L \times U(1)_{L_z+S_z} \times U(1)_\phi$
(6)	$U(1)_{L_z-\phi} \times U(1)_{S_z}$	$S^2_S \times SO(3)_{L,\phi}$
(7)	$U(1)_{L_z-\phi}$	$SO(3)_S \times SO(3)_{L,\Phi}$
(8)	$U(1)_{S_z-\phi}$	$SO(3)_L \times SO(3)_{S,\Phi}$
(10)	$U(1)_{L_z+\phi} \times U(1)_{S_z-\phi}$	$S^2_S \times S^2_L \times U(1)_{L_z,S_z,\phi}$
(11)	$U(1)_{L_z+S_z-\phi}$	$S^2_S \times S^2_L \times U(1)_{L_z,\phi} \times U(1)_\phi$

Table 2. 0-dimensional and 1-dimensional defects represented by $\pi_2(B_i)$ and $\pi_1(B_i)$ correspondingly for a base space B_i for each of the $i-$phase of superfluid Helium 3.

i	B_i	$\pi_2(B_i)$ for 0d-defects	$\pi_1(B_i)$ for 1d-defects
(1)	$SO(3)_{L,S} \times U(1)_\phi$	0×0	$Z_2 \times Z$
(2)	$S^2_S \times S^2_L \times U(1)_\phi$	$Z \times Z \times 0$	$0 \times 0 \times Z$
(3)	$SO(3)_S \times S^2_L \times U(1)_\phi$	$0 \times Z \times 0$	$Z_2 \times 0 \times Z$
(4)	$S^2_S \times SO(3)^2_L \times U(1)_\phi$	$Z \times 0 \times 0$	$0 \times Z_2 \times Z$
(5)	$S^2_S \times S^2_L \times U(1)_{L_z+S_z} \times U(1)_\phi$	$Z \times Z \times 0$	$0 \times 0 \times Z$
(6)	$S^2_S \times SO(3)_{L,\phi}$	$Z \times 0$	$0 \times Z_2$
(7)	$SO(3)_S \times SO(3)_{L,\Phi}$	0×0	$Z_2 \times Z_2$
(8)	$SO(3)_L \times SO(3)_{S,\Phi}$	0×0	$Z_2 \times Z_2$
(10)	$S^2_S \times S^2_L \times U(1)_{L_z,S_z,\phi}$	$Z \times Z \times 0$	$0 \times 0 \times Z$
(11)	$S^2_S \times S^2_L \times U(1)_{L_z,\phi} \times U(1)_\phi$	$Z \times Z \times 0 \times 0$	$0 \times 0 \times Z \times Z$

Although the phase diagram of superfluid Helium 3 is very complex [31], for simplicity of the argument, consider only two phases of superfluid Helium 3 called *phase-A* and *phase-B* to illustrate usage of Fiber structure theorem. Additionally, consider a part of a phase diagram where phase-B follows from phase-A. In a physical laboratory, such scenario may happen if pressure remains fixed during the experiment, but the temperature gradually lowered, leading to encountering phase-A and followed by phase B [19]. The defects formed during phase-A may transit to the phase-B, and stability of those defects may be studied using the long homotopy sequence of the Fiber structure theorem. For example, in 3-dimensional case ($m = 3$), by statements of Physics stability theorem, the 0-dimensional defects (often referred as *monopoles* by physicists) represented by the second homotopy group of the order parameter space B_i, where i corresponds to the number of phases. So, for phase-A, the order parameter space would be given by $B_A = G/N_1$, where N_1 is the group of symmetries under which the system remains invariant even at phase-A after the transition. Similarly, $B_B = G/N_2$, where N_2

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is the remaining symmetries for G during phase-B, and N_2 is a closed subgroup of N_1, while N_1 is a closed subgroup of G admitting a local cross section f. Then, for the 0-dimensional objects, according to the second statement of the Fiber structure theorem, a part of the long homotopy sequence:

$$\pi_2(N_A/N_B) \to \pi_2(G/N_B) \to \pi_2(G/N_A) \to \pi_1(N_A/N_B) \to \pi_1(G/N_B) \to \pi_1(G/N_A) \tag{2}$$

To illustrate an example, for the simplicity of the argument consider dipole-locked phase-A of superfluid Helium 3 transitioning to the phase-B. The order parameter for dipole-locked phase-A is given by $G/N_A = SO(3)$ and the group of remaining symmetries N_A consists of the identity alone. Similarly, for the phase-B, $G/N_B = SO(3) \times U(1)$ with $N_B = SO(3)$. So, the fiber N_A/N_B reduces to N_B given the triviality of N_A, $N_A/N_B \to N_B \cong SO(3)$ [4,11,27].

The long homotopy sequence may be used to evaluate stability of physical defects formed from one phase to another. For example, by Physical stability theorem, $\pi_1(SO(3)) \cong Z_2$ and hence dipole-locked phase-A supports stable 1d-defects. To evaluate the question whether these 1d-defects remain stable as superfluid Helium 3 enters the phase-B, consider the relevant part of the long homotopy sequence

$$\pi_1(SO(3)) \to \pi_1(SO(3) \times U(1)) \to \pi_1(SO(3)) \tag{3}$$

Using Seifert-van Kampen Theorem, $\pi_1(SO(3) \times U(1)) \cong \pi_1(SO(3)) \times \pi_1(U(1)) \cong Z_2 \times Z$, which infinite abelian group consisting of pairs (a, b) where first element a belongs to the cyclic group of order two $Z_2 = \{0, 1\}$ and second element b belongs to Z, the group of integers under addition. So the partial sequence becomes

$$Z_2 \to Z_2 \times Z \to Z_2 \tag{4}$$

The stability of 1d-defects of dipole-locked phase-A in phase-B depends on whether elements of $\pi_1(SO(3))$ remain non-trivial when mapped to $\pi_1(SO(3) \times U(1))$. If non-trivial element of $\pi_1(SO(3))$ maps to a trivial element of $\pi_1(SO(3) \times U(1))$, then 1d-defects corresponding to these elements are considered unstable in the order parameter space of phase-B.

For the 1d defects to be stable, the element of $\pi_1(SO(3))$ (which is Z_2) must map to an element of $\pi_1(SO(3) \times U(1)) \cong Z_2 \times Z$ that is not in the kernel of the subsequent map to $\pi_1(SO(3))$. If the map from $\pi_1(SO(3) \times U(1))$ to $\pi_1(SO(3))$ sends this element to a non-trivial element, then the 1d-defects are stable in the total space $SO(3) \times U(1)$, because they are not "killed" when projecting down to $SO(3)$.

Given the homotopy sequence $Z_2 \to Z_2 \times Z \to Z_2$, where the map is the inclusion of cosets from $Z_2 \times Z$ to Z_2, we analyze the stability of strings. The first map in the sequence, from $\pi_1(SO(3))$ to $\pi_1(SO(3) \times U(1))$, likely maps Z_2 into $Z_2 \times Z$ as $\{(0,0), (1,0)\}$ or a similar structure. The second map, the inclusion of cosets, maps elements (a, b) from $Z_2 \times Z$ to Z_2, where typically the Z component maps to the identity. If the inclusion map $p : G/N_B \to G/N_A$ preserves the Z_2 part, then non-trivial strings in $SO(3) \times U(1)$ remain non-trivial in $SO(3)$, indicating stability.

Similarly, the long homotopy sequence may be used to determine 0-dimensional defects, such as monopoles as superfluid Helium 3 undergoes transition from dipole-locked phase-A to phase-B [2]. Considering the homotopy sequence for monopoles, which are classified by the second homotopy group π_2:

$$\pi_2(N_A/N_B) \to \pi_2(G/N_B) \to \pi_2(G/N_A)$$

A monopole represented by an element in $\pi_2(G/N_B)$ is stable if it maps to the trivial element in $\pi_2(G/N_A)$, which suggests the absence of an attachment to higher-dimensional defects. Conversely, if the element maps to a non-trivial element in $\pi_2(G/N_A)$, the monopole is considered unstable as it indicates an attachment to a higher-dimensional defect or transformation into a 1d-defect in the space G/N_A. For a monopole to be stable, its corresponding element must not be in the kernel of the map from $\pi_2(G/N_B)$ to $\pi_2(G/N_A)$. So, in the given fibration with $G/N_B = SO(3) \times U(1)$ and $G/N_A = SO(3)$, the relevant part of the long exact sequence is:

$$0 \to 0 \to 0$$

This sequence indicates that the second homotopy group is trivial for both the total space and the base space. Consequently, there are no non-trivial monopole defects to consider, and thus, by definition, any monopoles would be stable [32,33]. However, this stability is somewhat vacuous since the triviality of the groups suggests the absence of monopoles in the spaces under consideration due to Physical stability theorem.

4.3. Combined Effects in KLS Cosmic Walls

Various sequential symmetry breaking scenarios described in [2] and corresponding references in detail. For example, given the context of polar distorted B phase (PdB) phase in the KLS walls, the combined topological objects are described by the relative homotopy group $\pi_n(R_1, R_2)$, where $R_1 = G/H_{PdB}$ and $R_2 = H_P/H_{PdB}$ corresponding the vacuum spaces. Using the homotopy lifting properties, such relative homotopy group can be converted to the absolute homotopy group $\pi_n(G, N)$ with the fiber bundle G and corresponding fiber N and base B. So in such formalism, the fiber $F \equiv N_1/N_2$ is H_P/H_{PdB} , the fiber bundle $E \equiv G/H_{PdB}$ and base $B \equiv G/H_P$, and using Statement 1 in the Fiber structure theorem, the short homotopy sequence group $F \to E \to^p B$

$$H_P/H_{PdB} \to G/H_{PdB} \to^p G/H_P \tag{5}$$

By applying the Physical stability theorem and Statement 2 of the Fiber structure theorem, one may recover results of [2,16] on exact sequence of homomorphism.

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