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Article

The Hermitian Solution to a New System of Commutative Quaternion Matrix Equations

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Abstract: This paper considers the Hermitian solutions of a new system of commutative quaternion matrix equations, we establish both necessary and sufficient conditions for the existence of solutions. Furthermore, we derive an explicit general expression when it is solvable. To illustrate our main findings, we present two numerical algorithms and examples in this paper.

Keywords: commutative quaternion algebra; matrix equations; Hermitian matrix; complex representation

MSC: 15A09; 15A24; 15B33; 15B57

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1. Introduction

In 1843, Hamilton introduced the concept of real quaternions which are defined by [1]

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = j^2 = k^2 = -1, ijk = -1, q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

which is a four-dimensional noncommutative associative algebra over real number field. Quaternions have been used in many areas such as statistic of quaternion random signals [2], color image processing [3] and face recognition [4]. Real quaternions are an extension of the complex numbers. However, the multiplication of the real quaternions is non-commutative which causes many difficulties.

A commutative quaternion, which was introduced by Segre [5] in 1892, is in the form of $q = q_0 + q_1 i + q_2 j + q_3 k$, where q_0, q_1, q_2, q_3 belong to the real number field, and the imaginary identities i, j, k satisfy $i^2 = k^2 = -1, j^2 = 1, ijk = -1, ij = ji = k, jk = kj = i, ki = ik = -j$. The most prominent feature of a commutative quaternion is the satisfaction of the multiplication commutative rule. The collection of commutative quaternions comprises a four-dimensional Clifford algebra, forming a ring. Within this set, we can find noteworthy attributes such as nontrivial idempotents, zero divisors, and nilpotent elements. There are many applications of the commutative quaternion algebra in Hopfield neural netwoks, digital signal, image processing [6–10], and so on. Commutative quaternions are also extensively researched. Kösal et al. [11] gave complex representations of commutative quaternion matrices and discussed several related properties. In [12], Kösal et al. proposed the real representation of a commutative quaternion matrix, and derived some explicit expression of the solutions of the commutative quaternion matrix equations $X - A\bar{\bar{\bar{X}}}B = C, X - A\bar{\tilde{\bar{X}}}B = C$ and $X - A\tilde{\bar{X}}B = C$, which are called the Kalman-Yakubovich-conjugate matrix equations, by means of real representation of a commutative quaternion matrix. Based on this, Kösal et al. [13] gave an expression of the general solution to the matrix equation $AX = B$ over the commutative quaternion ring.

Hermitian matrix has drawn a lot of attentions due to its great importance. In [14], Yu et al. studied Hermitian solutions to the generalizaed quaternion matrix equation $AXB + CX^*D = E$ through the real representation method. Yuan et al. [15] discussed Hermitian solutions to the split quaternion matrix equation $AXB + CXD = E$ by using the complex representation method. In [24], Kyrchei obtained the determinantal representation formulas of η-(η-skew)-Hermitian solutions to

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the quaternion matrix equations $AX = B$ and $AXA^{\eta*} = B$. As far as we know that the Sylvester matrix equations have a large number of applications in different fields. For example, the Sylvester matrix equation $A_1X + XB_1 = C_1$ and the Sylvester-like matrix equation $A_1X + YB_1 = C_1$ have been applied in singular system control [16], perturbation theory [17], sensitivity analysis [18] and control theory [19]. Wang et al. [20,21] considered the system of coupled Sylvester-like quaternion matrix equations. In [22], Wang et al. derived solvability conditions and expressions of the general solution to the system of two-sides coupled Sylvester-like quaternion matrix equations. Kyrchei [23] gave the determinantal representation formulas of solutions to the generalized Sylvester quaternion matrix equation $A_1X_1B_1 + A_2X_2B_2 = C$.

Motivated by keeping interests in Hermitian solutions and applications of the system of commutative quaternion matrix equations, we in this paper intend to investigate the solvability conditions and the Hermitian solutions to the following system of commutative quaternion matrix equations

$$\begin{cases} A_1X = C_1, \\ YB_1 = D_1, \\ A_2Z = C_2, ZB_2 = D_2, \\ A_3W = C_3, WB_3 = D_3, A_4WB_4 = C_4, \\ A_5X + YB_5 + A_6ZB_6 + A_7WB_7 = C_5, \end{cases} \tag{1}$$

where X, Y, Z, W are unknown Hermitian commutative quaternion matrices.

This paper is organized as follows. In Section 2, we review some useful properties and the structures of $\mathrm{vec}(AXB)$ over the commutative quaternion algebra when X is a Hermitian commutative quaternion matrix. In Section 3, we derive some practical necessary and sufficient conditions for the existence of Hermitian solutions to the system (1) over $\mathbb{H}_c$, and the numerical examples are given in Section 4.

2. Preliminaries

Throughout this paper, let $\mathbb{R}^{m\times n}, \mathbb{SR}^{m\times n}, \mathbb{ASR}^{m\times n}, \mathbb{C}^{m\times n}, \mathbb{H}_c, {\mathbb{H}_c}^n, {\mathbb{H}_c}^{m\times n}$ be the set of all $m \times n$ real matrices, the set of all $n \times n$ real symmetric matrices, the set of all $n \times n$ real anti-symmetric matrices, the set of all $m \times n$ complex matrices, the set of commutative quaternions, the set of n dimensional commutative quaternion column vectors, and the set of all $m \times n$ commutative quaternion matrices, respectively. The symbol $r(A)$ denotes the rank of A. Let the symbols $I, O, A^T, A^\dagger$ stand for the identity matrix, the zero matrix with appropriate size, the transpose of A, and the Moore-Penrose inverse of a matrix A, respectively. $\bar{A}$ and A^H denote the conjugate matrix, the conjugate transpose matrix of A, respectively. We call $A \in {\mathbb{H}_c}^{n\times n}$ is a Hermitian matrix if $A^H = A$, and denote it by $A \in {\mathbb{HH}_c}^{n\times n}$, where ${\mathbb{HH}_c}^{n\times n}$ is the set of all Hermitian commutative quaternion matrices with the size of $n \times n$.

For any $A \in {\mathbb{H}_c}^{m\times n}$, A can be uniquely expressed as $A = A_0 + A_1i + A_2j + A_3k$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$. It can also be uniquely expressed as $A = C_1 + C_2j$, where $C_1 = A_0 + A_1i, C_2 = A_2 + A_3i \in \mathbb{C}^{m\times n}$.

Proposition 1. [11] *The complex representation matrix for commutative quaternion $q = d_1 + d_2j, d_1 = q_0 + q_1i, d_2 = q_2 + q_3i$ is denoted as*

$$g(q) = \begin{pmatrix} d_1 & d_2 \\ d_2 & d_1 \end{pmatrix} \in \mathbb{C}^{2\times 2}.$$

Similarly, for any given $A = A_1 + A_2j \in {\mathbb{H}_c}^{m\times n}$, the complex representation matrix of A is

$$G(A) = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \in \mathbb{C}^{2m\times 2n}. \tag{2}$$

Obviously, $G(A)$ is uniquely determined by A. It is straightforward to confirm that the following statements are valid.

Proposition 2. [11] *If $A, B \in \mathbb{H}_c{}^{n\times n}$, then*

(a) $A = B$ *if and only if* $G(A) = G(B)$,
(b) $G(A + B) = G(A) + G(B)$,
(c) $G(I_n) = I_{2n}$,
(d) $G(AB) = G(A)G(B)$.

Suppose $A = (a_{ij}) \in \mathbb{H}_c^{m\times n}$ and $B = (b_{ij}) \in \mathbb{H}_c^{s\times t}$, the Kronecker product of A and B is defined as $A \otimes B = (a_{ij}B) \in \mathbb{H}_c^{ms\times nt}$. Considering commutative quaternion matrices A, B, C, D, E with appropriate dimensions, along with the real number p, we establish

$$(pA) \otimes B = A \otimes (pB) = p(A \otimes B),$$
$$(A, B, C) \otimes D = (A \otimes D, B \otimes D, C \otimes D),$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes E = \begin{pmatrix} A\otimes E & B\otimes E \\ C\otimes E & D\otimes E \end{pmatrix}.$$

The vec-operator of $A = (a_{ij}) \in \mathbb{H}_c^{m\times n}$ is defined as

$$\mathrm{vec}(A) = (a_1, a_2, \ldots, a_n)^T, a_j = (a_{1j}, a_{2j}, \ldots, a_{mj}), j = 1, 2, \ldots, n.$$

To investigate the Hermitian solutions of a system of matrix equations (1) within the framework of the commutative quaternion algebra, we need to review some certain definitions and fundamental properties.

Assume that $A = A_1 + A_2 j \in \mathbb{H}_c{}^{m\times n}, A_1, A_2 \in \mathbb{C}^{m\times n}$, then we have

$$A_1 + A_2 j = A \cong \Phi_A = (A_1, A_2),$$

where the symbol $\cong$ represents an equivalence relation. For a given matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, the corresponding Frobenius norm is defined as follows:

$$||A|| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} \|a_{ij}\|^2}, \quad \|a_{ij}\|^2 = (\mathrm{Re}\, a_{ij})^2 + (\mathrm{Im}\, a_{ij})^2.$$

According to the previously mentioned definition of Frobenius norm for complex matrices, we can define the Frobenius norm for commutative quaternion matrix $A = A_1 + A_2 j \in \mathbb{H}_c{}^{m\times n}$ as follows:

$$\|\acute{A}\| = \sqrt{\|\mathrm{Re}\, A_1\|^2 + \|\mathrm{Im}\, A_1\|^2 + \|\mathrm{Re}\, A_2\|^2 + \|\mathrm{Im}\, A_2\|^2},$$

where $\acute{A} = \begin{pmatrix} \mathrm{Re}(A_1) & \mathrm{Im}(A_1) & \mathrm{Re}(A_2) & \mathrm{Im}(A_2) \end{pmatrix}$, then we have

$$||\Phi_A|| = ||\acute{A}|| = ||\,\mathrm{vec}_{\Phi_A}||.$$

Theorem 1. [28] *Let $p \in \mathbb{R}, A, B \in \mathbb{H}_c{}^{m\times n}$ and $C \in \mathbb{H}_c{}^{n\times s}$. Then*

(a) $A = B$ *if and only if* $\Phi_A = \Phi_B$,
(b) $\Phi_{A+B} = \Phi_A + \Phi_B, \Phi_{pA} = p\Phi_A$,
(c) $(AC)^T = C^T A^T$,
(d) $(AC)^{-1} = C^{-1}A^{-1}$, *if the matrices A, C and AC are invertible,*
(e) $\Phi_{AC} = \Phi_A G(C)$.

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For the purpose of deriving the Hermitian solutions of the system (1), we introduce some relevant definitions and conclusions.

Definition 1. [15] *For the matrix* $A = (a_{ij}) \in \mathbb{H}_c^{n\times n}$*, set* $a_1 = \left(a_{11}, \sqrt{2}a_{21}, \ldots, \sqrt{2}a_{n1}\right), a_2 = \left(a_{22}, \sqrt{2}a_{32}, \ldots, \sqrt{2}a_{n2}\right), \ldots, a_{n-1} = \left(a_{(n-1)(n-1)}, \sqrt{2}a_{n(n-1)}\right), a_n = a_{nn}$*, and denote by* $\mathrm{vec}_S(A)$ *the following vector:*

$$\mathrm{vec}_S(A) = (a_1, a_2, \ldots, a_{n-1}, a_n)^T \in \mathbb{H}_c^{(n(n+1))/2}. \tag{3}$$

Definition 2. [15] *For the matrix* $B = (b_{ij}) \in \mathbb{H}_c^{n\times n}$*, set* $b_1 = (b_{21}, b_{31}, \ldots, b_{n1}), b_2 = (b_{32}, b_{42}, \ldots, b_{n2}), \ldots, b_{n-2} = \left(b_{(n-1)(n-2)}, b_{n(n-2)}\right), b_{n-1} = b_{n(n-1)}$*, and denote by* $\mathrm{vec}_A(B)$ *the following vector:*

$$\mathrm{vec}_A(B) = \sqrt{2}\,(b_1, b_2, \ldots, b_{n-2}, b_{n-1})^T \in \mathbb{H}_c^{(n(n-1))/2}. \tag{4}$$

Proposition 3. [25] *Suppose that* $X \in \mathbb{R}^{n\times n}$*, then*
(1)

$$X \in \mathbb{SR}^{n\times n} \Longleftrightarrow \mathrm{vec}(X) = K_S\,\mathrm{vec}_S(X), \tag{5}$$

where the matrix $K_S \in \mathbb{R}^{n^2\times(n(n+1)/2)}$ *is of the following form:*

$$K_S = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0 \\ 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n \end{pmatrix},$$

and e_i *is the ith column of the identity matrix of order n.*
(2)

$$X \in \mathbb{ASR}^{n\times n} \Longleftrightarrow \mathrm{vec}(X) = K_A\,\mathrm{vec}_A(X), \tag{6}$$

$\mathrm{vec}_A(X)$ *is described as (4) and the matrix* $K_A \in \mathbb{R}^{n^2\times(n(n-1)/2)}$ *is of the following form:*

$$K_A = \frac{1}{\sqrt{2}}\begin{pmatrix} e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 \\ 0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & 0 & 0 & & 0 \\ 0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n \\ 0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e_2 & \cdots & -e_{n-1} \end{pmatrix},$$

where e_i *is the column of the identity matrix of order n. It is apparent that* $K_S^T K_S = I_{(n(n+1))/2}, K_A^T K_A = I_{(n(n-1))/2}$.

Next, we explore the relationships between the Hermitian commutative quaternion matrices and symmetric matrices, as well as anti-symmetric matrices.

If $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$, where $X_1, X_2 \in \mathbb{C}^{n\times n}$, we can get

$$X \in \mathbb{HH}_c^{n\times n} \Leftrightarrow \begin{cases} \mathrm{Re}\,(X_1)^T = \mathrm{Re}\,(X_1), & \mathrm{Im}\,(X_1)^T = -\mathrm{Im}\,(X_1), \\ \mathrm{Re}\,(X_2)^T = -\mathrm{Re}\,(X_2), & \mathrm{Im}\,(X_2)^T = -\mathrm{Im}\,(X_2). \end{cases}$$

Apparently, $\mathrm{Re}(X_1)$ is symmetric, and $\mathrm{Im}(X_1), \mathrm{Re}(X_2), \mathrm{Im}(X_2)$ are antisymmetric. By means of Proposition 3, we have the following:

Theorem 2. [29] *Assume that* $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$*, then we obtain*

$$\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} = M \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}, \tag{7}$$

in which

$$M = \begin{pmatrix} K_S & iK_A & 0 & 0 \\ 0 & 0 & K_A & iK_A \end{pmatrix}. \tag{8}$$

Theorem 3. [29] *Suppose that* $A = A_1 + A_2 j \in \mathbb{H}_c{}^{m\times n}$, $B = B_1 + B_2 j \in \mathbb{H}_c{}^{s\times t}$ *and* $X = X_1 + X_2 j \in \mathbb{H}_c{}^{n\times s}$*, where* $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1, B_2 \in \mathbb{C}^{s\times t}$ *and* $X_1, X_2 \in \mathbb{C}^{n\times s}$*. Then*

$$\mathrm{vec}(\Phi_{AXB}) = G\left[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2)j\right] \begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix}. \tag{9}$$

Note that the results of $\mathrm{vec}(\Phi_{AXB})$ is very important for figuring out the system of commutative quaternion matrix equations (1). Analogous methods and related conclusions can be found in [15]. By incorporating Theorem 3 with Theorem 2, we can gain the following outcome.

Theorem 4. [29] *If* $A = A_1 + A_2 j \in \mathbb{H}_c^{m\times n}$, $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$*, and* $B = B_1 + B_2 j \in \mathbb{H}_c^{n\times s}$*, where* $A_i \in \mathbb{C}^{m\times n}$, $X_i \in \mathbb{C}^{n\times n}$*, and* $B_i \in \mathbb{C}^{n\times s}(i = 1, 2)$*. Consequently,*

$$\mathrm{vec}(\Phi_{AXB}) = G\left[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2)j\right] M \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}. \tag{10}$$

Lemma 1. [26] *The matrix equation* $Ax = b$*, with* $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$*, has a solution* $x \in \mathbb{R}^n$ *if and only if*

$$AA^{\dagger}b = b, \tag{11}$$

In this case, it has the general solution

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y, \tag{12}$$

where $y \in \mathbb{R}^n$ *is an arbitrary vector, and it has the unique solution* $x = A^{\dagger}b$ *for the case when* $r(A) = n$*. The solution of the matrix equation* $Ax = b$ *with the least norm is* $x = A^{\dagger}b$*.*

3. The Hermitian solution to the system

In accordance with the above discussion, now we focus on solving the system (1), for ease of description, We firstly state the following notations.

Let $A_1 = A_{11} + A_{12}j, A_2 = A_{21} + A_{22}j, A_3 = A_{31} + A_{32}j \in \mathbb{H}_c^{m\times n}, C_1, C_2, C_3 \in \mathbb{H}_c^{m\times n}, B_1 = B_{11} + B_{12}j, B_2 = B_{21} + B_{22}j, B_3 = B_{31} + B_{32}j \in \mathbb{H}_c^{n\times k}, D_1, D_2, D_3 \in \mathbb{H}_c^{n\times k}, A_4 = A_{41} + A_{42}j \in \mathbb{H}_c^{s\times n}, B_4 =$

$B_{41}+B_{42}j\in\mathbb{H}_c^{n\times t}, C_4\in\mathbb{H}_c^{s\in\times t}, A_5=A_{51}+A_{52}j, A_6=A_{61}+A_{62}j, A_7=A_{71}+A_{72}j, B_5=B_{51}+B_{52}j, B_6=B_{61}+B_{62}j, B_7=B_{71}+B_{72}j\in\mathbb{H}_c^{n\times n}$ and $C_5\in\mathbb{H}_c^{n\times n}$. We set

$$E=\begin{pmatrix} G[(I\otimes A_{11})+(I\otimes A_{12})j] \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ G[(I\otimes A_{51})+(I\otimes A_{52})j] \end{pmatrix} M,\ F=\begin{pmatrix} 0 \\ 0 \\ 0 \\ G[(B_{11}^T\otimes I)+(B_{12}^T\otimes I)j] \\ 0 \\ 0 \\ 0 \\ G[(B_{51}^T\otimes I)+(B_{52}^T\otimes I)j] \end{pmatrix} M,$$

$$P=\begin{pmatrix} 0 \\ G[(I\otimes A_{21})+(I\otimes A_{22})j] \\ 0 \\ 0 \\ G[(B_{21}^T\otimes I)+(B_{22}^T\otimes I)j] \\ 0 \\ 0 \\ G[(B_{61}^T\otimes A_{61}+B_{62}^T\otimes A_{62})+(B_{62}^T\otimes A_{61}+B_{61}^T\otimes A_{62})j] \end{pmatrix} M,$$

$$Q=\begin{pmatrix} 0 \\ 0 \\ G[(I\otimes A_{31})+(I\otimes A_{32})j] \\ 0 \\ 0 \\ G[(B_{31}^T\otimes I)+(B_{32}^T\otimes I)j] \\ G[(B_{41}^T\otimes A_{41}+B_{42}^T\otimes A_{42})+(B_{42}^T\otimes A_{41}+B_{41}^T\otimes A_{42})j] \\ G[(B_{71}^T\otimes A_{71}+B_{72}^T\otimes A_{72})+(B_{72}^T\otimes A_{71}+B_{71}^T\otimes A_{72})j] \end{pmatrix} M,$$

$$T=\begin{pmatrix} \mathrm{vec}\,(\Phi_{C_1}) \\ \mathrm{vec}\,(\Phi_{C_2}) \\ \mathrm{vec}\,(\Phi_{C_3}) \\ \mathrm{vec}\,(\Phi_{D_1}) \\ \mathrm{vec}\,(\Phi_{D_2}) \\ \mathrm{vec}\,(\Phi_{D_3}) \\ \mathrm{vec}\,(\Phi_{C_4}) \\ \mathrm{vec}\,(\Phi_{C_5}) \end{pmatrix}, T_1=\begin{pmatrix} \mathrm{vec}\,(\mathrm{Re}\,\Phi_{C_1}) \\ \mathrm{vec}\,(\mathrm{Re}\,\Phi_{C_2}) \\ \mathrm{vec}\,(\mathrm{Re}\,\Phi_{C_3}) \\ \mathrm{vec}\,(\mathrm{Re}\,\Phi_{D_1}) \\ \mathrm{vec}\,(\mathrm{Re}\,\Phi_{D_2}) \\ \mathrm{vec}\,(\mathrm{Re}\,\Phi_{D_3}) \\ \mathrm{vec}\,(\mathrm{Re}\,\Phi_{C_4}) \\ \mathrm{vec}\,(\mathrm{Re}\,\Phi_{C_5}) \end{pmatrix}, T_2=\begin{pmatrix} \mathrm{vec}\,(\mathrm{Im}\,\Phi_{C_1}) \\ \mathrm{vec}\,(\mathrm{Im}\,\Phi_{C_2}) \\ \mathrm{vec}\,(\mathrm{Im}\,\Phi_{C_3}) \\ \mathrm{vec}\,(\mathrm{Im}\,\Phi_{D_1}) \\ \mathrm{vec}\,(\mathrm{Im}\,\Phi_{D_2}) \\ \mathrm{vec}\,(\mathrm{Im}\,\Phi_{D_3}) \\ \mathrm{vec}\,(\mathrm{Im}\,\Phi_{C_4}) \\ \mathrm{vec}\,(\mathrm{Im}\,\Phi_{C_5}) \end{pmatrix}, \epsilon=\begin{pmatrix} T_1 \\ T_2 \end{pmatrix},$$

$$\mathcal{W}=\begin{pmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{pmatrix}, \mathfrak{W}=\begin{pmatrix} \mathcal{W} & 0 & 0 & 0 \\ 0 & \mathcal{W} & 0 & 0 \\ 0 & 0 & \mathcal{W} & 0 \\ 0 & 0 & 0 & \mathcal{W} \end{pmatrix},$$

$$\mathrm{vec}(\vec{X})=\begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}, \mathrm{vec}(\vec{Y})=\begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_2)) \end{pmatrix},$$

$$\mathrm{vec}(\vec{Z})=\begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Z_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_2)) \end{pmatrix}, \mathrm{vec}(\vec{W})=\begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(W_1)) \\ \mathrm{vec}_A(\mathrm{Im}(W_1)) \\ \mathrm{vec}_A(\mathrm{Re}(W_2)) \\ \mathrm{vec}_A(\mathrm{Im}(W_2)) \end{pmatrix}, \tag{13}$$

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and

$$\begin{aligned}
&E_1 = \mathrm{Re}\, E, \quad E_2 = \mathrm{Im}\, E, \quad F_1 = \mathrm{Re}\, F, \quad F_2 = \mathrm{Im}\, F, \\
&P_1 = \mathrm{Re}\, P, \quad P_2 = \mathrm{Im}\, P, \quad Q_1 = \mathrm{Re}\, Q, \quad Q_2 = \mathrm{Im}\, Q, \\
&U_1 = (E_1, F_1, P_1, Q_1), \\
&U_2 = (E_2, F_2, P_2, Q_2).
\end{aligned} \tag{14}$$

For further study the structure of Hermitian solution of the system of matrix equations (1), it is necessary to study the generalized inverse of matrix in the form of column blocks. The following notations are required. Let

$$\begin{aligned}
&d = 6mn + 6kn + 2st + 2n^2, \\
&H = \left(I_{8n^2-4n} - U_1^{\dagger} U_1\right) U_2^T, \\
&K = \left(I_d + \left(I_d - H^{\dagger} H\right) U_2 U_1^{\dagger} U_1^{\dagger T} U_2^T \left(I_d - H^{\dagger} H\right)\right)^{-1}, \\
&J = H^{\dagger} + \left(I_d - H^{\dagger} H\right) K U_2 U_1^{\dagger} U_1^{\dagger T} \left(I_{8n^2-4n} - U_2^T H^{\dagger}\right), \\
&R_{11} = I_d - U_1 U_1^{\dagger} + U_1^{\dagger T} U_2^T K \left(I_d - H^{\dagger} H\right) U_2 U_1^{\dagger}, \\
&R_{12} = -U_1^{\dagger T} U_2^T \left(I_d - H^{\dagger} H\right) K, \\
&R_{22} = \left(I_d - H^{\dagger} H\right) K.
\end{aligned}$$

From the findings [27] presented above, it can be inferred that

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} = \left(U_1^{\dagger} - J^T U_2 U_1^{\dagger}, J^T\right), \quad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = U_1^{\dagger} U_1 + H H^{\dagger}, \tag{15}$$

and

$$I_{2d} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} = \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix}. \tag{16}$$

Taking into account the aforementioned results, we then turn our attention to the Hermitian solution of the system (1).

Theorem 5. *Let $A_1, A_2, A_3, C_1, C_2, C_3 \in \mathbb{H}_c^{m\times n}, B_1, B_2, B_3, D_1, D_2, D_3 \in \mathbb{H}_c^{n\times k}, A_4 \in \mathbb{H}_c^{s\times n}, B_4 \in \mathbb{H}_c^{n\times t}, C_4 \in \mathbb{H}_c^{s\times t}, A_5, A_6, A_7, B_5, B_6, B_7 \in \mathbb{H}_c^{n\times n}$ and $C_5 \in \mathbb{H}_c^{n\times n}$. U_1, U_2 and ϵ are in the form of (3) and (14), respectively. Then the system of commutative quaternion matrix equations (1) has a solution $X, Y, Z, W \in \mathbb{H}\mathbb{H}_c{}^{n\times n}$ if and only if*

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon = \epsilon. \tag{17}$$

In this case, the set of Hermitian solutions is as follows:

$$\Lambda = \left\{ (X, Y, Z, W) | \begin{pmatrix} \mathrm{vec}(\acute{X}) \\ \mathrm{vec}(\acute{Y}) \\ \mathrm{vec}(\acute{Z}) \\ \mathrm{vec}(\acute{W}) \end{pmatrix} = \mathfrak{W} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon + \mathfrak{W}(I_{8n^2-4n} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}) y \right\}, \tag{18}$$

where y is an arbitrary vector of appropriate order. Then the system (1) has a unique solution $(X, Y, Z, W) \in \Lambda$ if and only if

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$$r\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 8n^2 - 4n. \tag{19}$$

If this condition satisfies, then

$$\Lambda = \left\{ (X,Y,Z,W) | \begin{pmatrix} \mathrm{vec}(\acute{X}) \\ \mathrm{vec}(\acute{Y}) \\ \mathrm{vec}(\acute{Z}) \\ \mathrm{vec}(\acute{W}) \end{pmatrix} = \mathfrak{W} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon \right\}. \tag{20}$$

Proof. By virtue of Theorem 1 and Theorem 4 , we obtain

$$\begin{cases} A_1X = C_1, \\ YB_1 = D_1, \\ A_2Z = C_2, ZB_2 = D_2, \\ A_3W = C_3, WB_3 = D_3, A_4WB_4 = C_4, \\ A_5X + YB_5 + A_6ZB_6 + A_7WB_7 = C_5, \end{cases}$$

$$\Longleftrightarrow \begin{cases} \Phi_{A_1X} = \Phi_{C_1}, \\ \Phi_{YB_1} = \Phi_{D_1}, \\ \Phi_{A_2Z} = \Phi_{C_2}, \Phi_{ZB_2} = \Phi_{D_2}, \\ \Phi_{A_3W} = \Phi_{C_3}, \Phi_{WB_3} = \Phi_{D_3}, \Phi_{A_4WB_4} = \Phi_{C_4}, \\ \Phi_{A_5X} + \Phi_{YB_5} + \Phi_{A_6ZB_6} + \Phi_{A_7WB_7} = \Phi_{C_5}, \end{cases}$$

$$\Longleftrightarrow E \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + F \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_2)) \end{pmatrix} + P \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Z_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_2)) \end{pmatrix} + Q \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(W_1)) \\ \mathrm{vec}_A(\mathrm{Im}(W_1)) \\ \mathrm{vec}_A(\mathrm{Re}(W_2)) \\ \mathrm{vec}_A(\mathrm{Im}(W_2)) \end{pmatrix} = T,$$

$$\Longleftrightarrow (\mathrm{Re}\,E + i\,\mathrm{Im}\,E) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + (\mathrm{Re}\,F + i\,\mathrm{Im}\,F) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_2)) \end{pmatrix}$$

$$+ (\mathrm{Re}\,P + i\,\mathrm{Im}\,P) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Z_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_2)) \end{pmatrix} + (\mathrm{Re}\,Q + i\,\mathrm{Im}\,Q) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(W_1)) \\ \mathrm{vec}_A(\mathrm{Im}(W_1)) \\ \mathrm{vec}_A(\mathrm{Re}(W_2)) \\ \mathrm{vec}_A(\mathrm{Im}(W_2)) \end{pmatrix} = T_1 + iT_2,$$

$$\Longleftrightarrow \begin{pmatrix} \mathrm{Re}\,E & \mathrm{Re}\,F & \mathrm{Re}\,P & \mathrm{Re}\,Q \\ \mathrm{Im}\,E & \mathrm{Im}\,F & \mathrm{Im}\,P & \mathrm{Im}\,Q \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\vec{X}) \\ \mathrm{vec}(\vec{Y}) \\ \mathrm{vec}(\vec{Z}) \\ \mathrm{vec}(\vec{W}) \end{pmatrix} = \epsilon,$$

$$\Longleftrightarrow \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\vec{X}) \\ \mathrm{vec}(\vec{Y}) \\ \mathrm{vec}(\vec{Z}) \\ \mathrm{vec}(\vec{W}) \end{pmatrix} = \epsilon.$$

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By Lemma 2, we conclude that the system (1) has a Hermitian solution $(X, Y, Z, W) \in \Lambda$ if and only if (17) satisfy, thus we have

$$\begin{pmatrix} \text{vec}(\vec{X}) \\ \text{vec}(\vec{Y}) \\ \text{vec}(\vec{Z}) \\ \text{vec}(\vec{W}) \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon + \left(I_{8n^2-4n} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \right) y.$$

On account of

$$\text{vec}(\acute{X}) = \begin{pmatrix} \text{vec}(\text{Re}(X_1)) \\ \text{vec}(\text{Im}(X_1)) \\ \text{vec}(\text{Re}(X_2)) \\ \text{vec}(\text{Im}(X_2)) \end{pmatrix} = \begin{pmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{pmatrix} \begin{pmatrix} \text{vec}_S(\text{Re}(X_1)) \\ \text{vec}_A(\text{Im}(X_1)) \\ \text{vec}_A(\text{Re}(X_2)) \\ \text{vec}_A(\text{Im}(X_2)) \end{pmatrix} = \mathcal{W}\,\text{vec}(\vec{X}),$$

similarly, we can derive $\text{vec}(\acute{Y}) = \mathcal{W}\,\text{vec}(\vec{Y}), \text{vec}(\acute{Z}) = \mathcal{W}\,\text{vec}(\vec{Z})$ and $\text{vec}(\acute{W}) = \mathcal{W}\,\text{vec}(\vec{W})$, then we have

$$\begin{aligned} \begin{pmatrix} \text{vec}(\acute{X}) \\ \text{vec}(\acute{Y}) \\ \text{vec}(\acute{Z}) \\ \text{vec}(\acute{W}) \end{pmatrix} &= \begin{pmatrix} \mathcal{W} & 0 & 0 & 0 \\ 0 & \mathcal{W} & 0 & 0 \\ 0 & 0 & \mathcal{W} & 0 \\ 0 & 0 & 0 & \mathcal{W} \end{pmatrix} \begin{pmatrix} \text{vec}(\vec{X}) \\ \text{vec}(\vec{Y}) \\ \text{vec}(\vec{Z}) \\ \text{vec}(\vec{W}) \end{pmatrix} \\ &= \mathfrak{W} \begin{pmatrix} \text{vec}(\vec{X}) \\ \text{vec}(\vec{Y}) \\ \text{vec}(\vec{Z}) \\ \text{vec}(\vec{W}) \end{pmatrix} \\ &= \mathfrak{W} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon + \mathfrak{W}(I_{8n^2-4n} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix})y. \end{aligned}$$

It means that (18) is true, if (17) holds, the system (1) has a unique solution $(X, Y, Z, W) \in \Lambda$ if and only if

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = I_{8n^2-4n}.$$

Thus by (19), we can obtain (20). □

Corollary 1. *The system* (1) *has a solution* $X, Y, Z, W \in \mathbb{HH}_c$ *if and only if*

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} \epsilon = 0. \tag{21}$$

Under this circumstances, the set of Hermitian solution of the system (1) *can be represented as follows:*

$$\Lambda = \left\{ (X, Y, Z, W) | \begin{pmatrix} \text{vec}(\acute{X}) \\ \text{vec}(\acute{Y}) \\ \text{vec}(\acute{Z}) \\ \text{vec}(\acute{W}) \end{pmatrix} = \mathfrak{W} \left(U_1^{\dagger} - J^T U_2 U_1^{\dagger}, J^T \right) \epsilon + \mathfrak{W}(I_{8n^2-4n} - U_1^{\dagger} U_1 - HH^{\dagger}) y \right\}, \tag{22}$$

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in which y is an arbitrary vector of appropriate size. Then the system (1) *has an unique solution* $(X, Y, Z, W) \in \Lambda$ *when* (21) *and* (19) *are obeyed. In this case,*

$$\Lambda = \left\{ (X, Y, Z, W) | \begin{pmatrix} \mathrm{vec}(\acute{X}) \\ \mathrm{vec}(\acute{Y}) \\ \mathrm{vec}(\acute{Z}) \\ \mathrm{vec}(\acute{W}) \end{pmatrix} = \mathfrak{W} \left(U_1^\dagger - J^T U_2 U_1^\dagger, J^T \right) \epsilon \right\}. \tag{23}$$

4. Numerical exemplification

In this section, on the basis of discussions in Section 2 and Section 3, we provide two algorithms for solving the system (1), and present two numerical examples to verify the feasibility of algorithms.

Algorithm 1. For the system (1)

(1) **Input:** the matrix:$A_1, A_2, A_3 \in \mathbb{H}_c^{m\times n}, C_1, C_2, C_3 \in \mathbb{H}_c^{m\times n}, B_1, B_2, B_3 \in \mathbb{H}_c^{n\times k}, D_1, D_2, D_3 \in \mathbb{H}_c^{n\times k}, A_4 \in \mathbb{H}_c^{s\times n}, B_4 \in \mathbb{H}_c^{n\times t}, C_4 \in \mathbb{H}_c^{s\in\times t}, A_5, A_6, A_7, B_5, B_6, B_7 \in \mathbb{H}_c^{n\times n}, C_5 \in \mathbb{H}_c^{n\times n}, K_S$ and K_A.
(2) Compute U_1, U_2 and ϵ.
(3) If both (17) and (19) hold, then calculate the unique solution $(X, Y, Z, W) \in \Lambda$ by (20).
(4) If (17) holds, then calculate $(X, Y, Z, W) \in \Lambda$ according to (18).
(5) **Output:**(X, Y, Z, W).

Algorithm 2. For the system (1)

(1) **Input:** the matrix:$A_1, A_2, A_3 \in \mathbb{H}_c^{m\times n}, C_1, C_2, C_3 \in \mathbb{H}_c^{m\times n}, B_1, B_2, B_3 \in \mathbb{H}_c^{n\times k}, D_1, D_2, D_3 \in \mathbb{H}_c^{n\times k}, A_4 \in \mathbb{H}_c^{s\times n}, B_4 \in \mathbb{H}_c^{n\times t}, C_4 \in \mathbb{H}_c^{s\in\times t}, A_5, A_6, A_7, B_5, B_6, B_7 \in \mathbb{H}_c^{n\times n}, C_5 \in \mathbb{H}_c^{n\times n}, K_S$ and K_A.
(2) Compute $U_1, U_2, H, K, J, R_{11}, R_{12}, R_{22}$ and ϵ.
(3) If both (19) and (21) hold, then calculate the unique solution $(X, Y, Z, W) \in \Lambda$ by (23).
(4) If (21) holds, then calculate $(X, Y, Z, W) \in \Lambda$ according to (22).
Otherwise stop.
(5) **Output:** (X, Y, Z, W).

From the previous theoretical analysis, if the system (1) is solvable, then

$$\theta_1 = \left\| \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^\dagger \epsilon - \epsilon \right\|, \quad \theta_2 = \left\| \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} \epsilon \right\|$$

and

$$\theta_3 = \left\| I_{2d} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^\dagger - \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} \right\|$$

are small enough.

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Example 1. *Let* $m = 6, n = 4, k = 5, s = 2, t = 3$, *and*

$$
\begin{aligned}
&A_1 = A_{11} + A_{12}j, \quad A_2 = A_{21} + A_{22}j, \quad A_3 = A_{31} + A_{32}j,\\
&B_1 = B_{11} + B_{12}j, \quad B_2 = B_{21} + B_{22}j, \quad B_3 = B_{31} + B_{32}j,\\
&A_4 = A_{41} + A_{42}j, \quad B_4 = B_{41} + B_{42}j, \quad A_5 = A_{51} + A_{52}j,\\
&A_6 = A_{61} + A_{62}j, \quad A_7 = A_{71} + A_{72}j, \quad B_5 = B_{51} + B_{52}j,\\
&B_6 = B_{61} + B_{62}j, \quad B_7 = B_{71} + B_{72}j,\\
&\dot{X} = \dot{X}_1 + \dot{X}_2 j, \quad \dot{Y} = \dot{Y}_1 + \dot{Y}_2 j,\\
&\dot{Z} = \dot{Z}_1 + \dot{Z}_2 j, \quad \dot{W} = \dot{W}_1 + \dot{W}_2 j,\\
&C_1 = A_1\dot{X}, \quad C_2 = A_2\dot{Z}, \quad C_3 = A_3\dot{W},\\
&D_1 = \dot{Y}B_1, \quad D_2 = \dot{Z}B_2, \quad D_3 = \dot{W}B_3,\\
&C_4 = A_4\dot{W}B_4, \quad C_5 = A_5\dot{X} + \dot{Y}B_5 + A_6\dot{Z}B_6 + A_7\dot{W}B_7.
\end{aligned}
$$

where

$$
\begin{aligned}
&A_{11} = \begin{pmatrix} I_4 \\ O_{2\times4} \end{pmatrix}, \quad A_{12} = O_{6\times4}, \quad A_{21} = O_{6\times4}, \quad A_{22} = \begin{pmatrix} I_4 \\ O_{2\times4} \end{pmatrix},\\
&A_{31} = O_{6\times4}, \quad A_{32} = \begin{pmatrix} O_{2\times4} \\ I_4 i \end{pmatrix}, \quad A_{41} = \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \end{pmatrix}, \quad A_{42} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\\
&A_{51} = I_4 i, \quad A_{52} = O_{4\times4}, \quad A_{61} = \begin{pmatrix} I_2 & O_{2\times2} \\ O_{2\times2} & O_{2\times2} \end{pmatrix}, \quad A_{62} = O_{4\times4},\\
&A_{71} = \begin{pmatrix} O_{2\times2} & I_2 \\ I_2 & O_{2\times2} \end{pmatrix}, \quad A_{72} = \begin{pmatrix} O_{2\times2} & -I_2 i \\ O_{2\times2} & O_{2\times2} \end{pmatrix}, \quad B_{11} = O_{4\times5}, \quad B_{12} = \begin{pmatrix} I_4 & O_{4\times1} \end{pmatrix},\\
&B_{21} = \begin{pmatrix} -I_4 i & O_{4\times1} \end{pmatrix}, \quad B_{22} = O_{4\times5}, \quad B_{31} = O_{4\times5}, \quad B_{32} = \begin{pmatrix} I_4 & O_{4\times1} \end{pmatrix},\\
&B_{41} = \begin{pmatrix} I_3 \\ O_{1\times3} \end{pmatrix}, \quad B_{42} = O_{4\times3}, \quad B_{51} = O_{4\times4}, \quad B_{52} = -I_4 i,\\
&B_{61} = I_4, \quad B_{62} = O_{4\times4}, \quad B_{71} = \begin{pmatrix} O_{2\times2} & I_2 \\ I_2 & O_{2\times2} \end{pmatrix}, \quad B_{72} = O_{4\times4},\\
&C_1 = \begin{pmatrix} 1 & -1+i & j & 0 \\ -1-i & 2 & i & k \\ -j & -i & 0 & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} j & 0 & 0 & k \\ 0 & j & 0 & -1 \\ 0 & 0 & j & 0 \\ -k & 1 & 0 & j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

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$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -j & i & 0 \\ j & 0 & k & 0 \\ -i & k & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & -k & j & 0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} -i & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & k & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & k & j & 0 & 0 \\ -k & 0 & -1+k & 0 & 0 \\ j & 1-k & 0 & k & 0 \\ -1 & -k & 0 & -i & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & k & 1 & 0 & 0 \\ -k & 0 & j & 0 & 0 \\ -1 & j & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -1-k & 2i+j & j \\ -i & 0 & 1-i+j \end{pmatrix},$$

$$C_5 = \begin{pmatrix} 1+i & -i+j+k & i-j & 1+i-k \\ -i-j-k & 1+2i & -1+i+j & -2j \\ j-2k & 1-i-j & 0 & i+j \\ 1 & j & -i-j & i-k \end{pmatrix}.$$

We take

$$\dot{X}_1 = \begin{pmatrix} 1 & -1+i & 0 & 0 \\ -1-i & 2 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \dot{X}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\dot{Y}_1 = \begin{pmatrix} 0 & i & 1 & 0 \\ -i & 0 & i & 0 \\ 1 & -i & 0 & i \\ 0 & 0 & -i & 1 \end{pmatrix}, \quad \dot{Y}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\dot{Z}_1 = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}, \quad \dot{Z}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\dot{W}_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \dot{W}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

Let

$$\Phi_{C_1} = \Phi_{A_1} G(\dot{X}),$$
$$\Phi_{D_1} = \Phi_{\dot{Y}} G(B_1),$$
$$\Phi_{C_2} = \Phi_{A_2} G(\dot{Z}), \quad \Phi_{D_2} = \Phi_{\dot{Z}} G(B_2),$$
$$\Phi_{C_3} = \Phi_{A_3} G(\dot{W}), \quad \Phi_{D_3} = \Phi_{\dot{W}} G(B_3), \quad \Phi_{C_4} = \Phi_{A_4} G(\dot{W}) G(B_4),$$
$$\Phi_{C_5} = \Phi_{A_5} G(\dot{X}) + \Phi_{\dot{Y}} G(B_5) + \Phi_{A_6} G(\dot{Z}) G(B_6) + \Phi_{A_7} G(\dot{W}) G(B_7).$$

From MATLAB and Algorithm 2, we can obtain

$$r\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 112 = 8n^2 - 4n, \quad \theta_2 = 7.0360 \times 10^{-15}.$$

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According to Algorithm 2, the system of matrix equations (1) *has an unique solution* $(X, Y, Z, W) \in \Lambda$, *and we derive* $\theta_1 = 1.7493 \times 10^{-14}, \theta_3 = 2.3111 \times 10^{-14}$ *and* $\|\Phi_{(X,Y,Z,W)} - \Phi_{(\dot{X},\dot{Y},\dot{Z},\dot{W})}\| = 1.1628 \times 10^{-14}$.

Example 2. *Let* $m = 2, n = 2, k = 2, s = 2, t = 2$, *and*

$$A_1 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$A_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_7 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad B_1 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix},$$
$$B_6 = O_2, \quad B_7 = \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad C_1 = \begin{pmatrix} i & -1+i \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix},$$
$$C_3 = \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 \\ 0 & j+2k \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1 & 1+i+3j+k \\ 1-i & -k \end{pmatrix},$$
$$D_1 = \begin{pmatrix} i & 0 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 \\ 2-i & 0 \end{pmatrix},$$

taking

$$X = \begin{pmatrix} 1 & 1+i \\ 1-i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix}.$$

From MATLAB and Algorithm 2, we obtain

$$r\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 23, \quad \theta_2 = 2.1412 \times 10^{-15}.$$

According to Algorithm 2, the system (1) *has infinite solutions* $(X, Y, Z, W) \in \Lambda$. *We can also obtain* $\theta_1 = 3.3532 \times 10^{-15}, \theta_3 = 4.4730 \times 10^{-15}$. *Then the optimization problem*

$$\min_{(X,Y,Z,W)\in\Lambda} \left(\left\| \Phi_{(X,Y,Z,W)} \right\| \right)$$

has an unique minimizer $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W})$, *it can also be expressed as*

$$\begin{pmatrix} \mathrm{vec}(\acute{\tilde{X}}) \\ \mathrm{vec}(\acute{\tilde{Y}}) \\ \mathrm{vec}(\acute{\tilde{Z}}) \\ \mathrm{vec}(\acute{\tilde{W}}) \end{pmatrix} = \mathfrak{W}\left(U_1^{\dagger} - J^T U_2 U_1^{\dagger}, J^T \right) \epsilon.$$

Therefore, we can get $\|\Phi_{(X,Y,Z,W)} - \Phi_{(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W})}\| = 1$, *and*

$$\tilde{X} = \begin{pmatrix} 1 & 1+i \\ 1-i & 0 \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix}.$$

5. Conclusion

In this paper, we have given the necessary and sufficient conditions for the existence of the Hermitian solutions to the system of commutative quaternion matrix equations (1), and also have established an expression of the Hermitian solutions to the system (1) when it is consistent. Some numerical algorithms and examples are provided to illustrate our results.

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