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The Dependence Structure of Hamiltonians of the SK-Model

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Abstract: The SK model proposed by Sherrington and Kirkpatrick is an important mean field model for spin glasses. The dependence structure of Hamiltonians of the SK model is considered in this paper. We have investigated the covariance matrix of Hamiltonians of the SK model, and establish a relationship between the rank of the corresponding covariance matrix and the dimension of the space spanned by the Hamiltonians. Finally, we also give an upper bound for the rank of the covariance matrix, which might be the best.

Keywords: SK model; Hamiltonians; rank; covariance matrix

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1. Introduction

Spin Glasses models originated in statistical physics, as a typical example of highly complex disordered systems, are very interesting and have an extremely wide range of applications. Therefore, they have always been favored by people for a long time. In the 1970s, in order to study the spin glasses, two very important mathematical models were proposed successively, namely EA model and SK model. The spin glasses question is so interesting to a large extent because neither model is easy to navigate. The EA model is a stochastic version of the classical Ising model, which is a lattice spin system. The SK model was proposed as a mean field model for spin glasses by [1] in 1975. Compared to the SK model, the EA model is more difficult to deal with, and we need to master the method based on numerical simulation for this model. Therefore, we consider the SK model simplification of the EA model in which all spins interact and, thus, the lattice is replaced by a complete graph on N sites in this paper.

Over the past several decades, the SK model has attracted much attention from physics and mathematics, including [2–6]. Many excellent results have also been achieved. For example, the Replica Symmetry Breaking method proposed by [3] can give the exact solution of the SK model according to this theory, the SK model has been solved in a certain sense. In the process of solving the problem, some unconventional material concepts are proposed, moreover, the thermodynamic limit of the SK model also requires a very complex structure to describe.

In the late 1980s, mathematicians began to use strict mathematical methods to explore the SK model. In particular, many mathematical conclusions about the SK model have been proved continuously. The work of [7] and the systematic programmes invented by Guerra and Talagrand all started at that time. In the following decades, scholars continued to pay attention to the SK model. [8] used Brownian motion instead of Gaussian orbit for the first time to study the mean-field SK model of Spin Glasses by means of stochastic analysis, which simplified the complex calculation and obtained the relevant asymptotic results. [4] gave variational upper bounds on the free energy. Under some conditions on the overlap function, [9] contributed to the entirely rigorous account of the original formulae proposed by [3], and in the special case of the inverse temperature parameter $\beta < 1$, one can usually prove that the formula of the limiting free energy. [10,11] utilized the ideas and methods of [9] to study the local behavior of the free energy. [12] considered the results of the limiting free energy of the SK model, proved the existence and uniqueness of the free energy, and gave an upper bound of the free energy of the SK model. This is consistent with the conclusion of [3], and it is the first time that this important result has been linked to a rigorous mathematical result. Later, [13] extended this method

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and obtained some results on the free energy of the SK model. [14,15] discussed the exponential inequality and symmetry breaking theory of SK model. Less than a year later, [9] announced that he could prove a corresponding lower bound on the free energy of the SK model. Up to this point, Parisi's solution to the SK model has been mathematically proved perfectly and rigorously. [16] considered central limit theorems for macroscopic observables of the model.

Recently, [17] has studied the fluctuations of the free energy of the two-spin spherical SK model at critical temperature $\beta = 1$. For more details on free energy in this field, see [18–20]. However, in the existing literature, most of the above studies on the SK model are based on the results of free energy. In this note, instead of considering the free energy of the SK model, we focus on the dependence structure of Hamiltonians of the SK model. In other words, the algebraic properties of Hamiltonians function of the SK model are studied. We examine the covariance matrix of Hamiltonians of the SK model, the relationship between the rank of the corresponding covariance matrix and the dimension of the space spanned by Hamiltonians is also established.

The rest of this paper is organized as follows. In Section 2, we describe the definition of the SK model and some notations. Section 3 is our main results. We examine the covariance matrix of Hamiltonians of the SK model, and build a relationship between its rank and the dimension of the space spanned by the Hamiltonians. We also obtain an upper bound for the rank of the covariance matrix, which might be the best.

2. Definition of the SK model and notations

In this section, we describe some details about the SK model, including its Hamiltonian fuction and some basic conclusions of the SK model.

Throughout the paper, we assume that $(\Omega, \mathcal{F}, P)$ is a given probability space with E being the expectation with respect to P, $\{\xi_{ij}\}_{1\leq i<j\leq N}$ is a sequence of independent standard Gaussian random variables on probability space $(\Omega, \mathcal{F}, P)$, where N is a given positive integer.

Consider the space of configurations $\Sigma_N = \{-1, +1\}^N$. A configuration $\vec{\sigma} \in \Sigma_N$ is a vector $(\sigma_1, \sigma_2, \ldots, \sigma_N)$ of spins σ_i that values ± 1. The SK model without an "external field" is defined on spin configurations, and its Hamiltonians are given by

$$H(\vec{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{i<j} \xi_{ij}\sigma_i\sigma_j, \quad \vec{\sigma} \in \Sigma_N, \tag{1}$$

where the normalization by $\sqrt{N}$ is introduced to keep the typical energy per spin of order 1, and can be viewed as a tuning of the interaction strength.

We denote by $H = (H(\vec{\sigma}))_{\vec{\sigma}\in\Sigma_N}$ the set of Hamiltonians. Obviously, $|\Sigma_N| = 2^N$, where $|\Sigma_N|$ denotes the cardinaliy of Σ_N. To simplify notation, we write $m = 2^N$ in the following.

Accordingly, the partition function of the SK model is defined as

$$Z_N = Z_N(\beta) = \sum_{\vec{\sigma}\in\Sigma^N} \exp\left(-\beta H_N(\vec{\sigma})\right),$$

where β is inversion temperature parameter.

It is obviously that

$$\mathrm{E}Z_N = \mathrm{E} \sum_{\vec{\sigma}\in\Sigma^N} \exp\left(-\beta H_N(\vec{\sigma})\right) = 2^N \exp\left(\frac{\beta^2}{4}(N-1)\right).$$

For any configuration $\vec{\sigma}$, $H(\vec{\sigma})$ is a random variable, and indeed a centered Gaussian one. Thus, given two configurations $\vec{\sigma}$ and $\vec{\tau}$, their covariance is given by

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$$\mathrm{E}(H(\vec{\sigma})H(\vec{\tau})) = \frac{N}{2}(\frac{1}{N}\sum_{i\leq N}\sigma_i\tau_i)^2 - \frac{1}{2}, \tag{2}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ and $\vec{\tau} = (\tau_1, \tau_2, \ldots, \tau_N)$.

In particular, we have

$$\mathrm{E}(H^2(\vec{\sigma})) = \frac{N-1}{2}. \tag{3}$$

3. Main results

For convenience, we endow these configurations with the natural order. For instance, when $N = 2$, the configurations can be listed in the natural order as:

$$\vec{\sigma}^{(1)} = (-1,-1), \quad \vec{\sigma}^{(2)} = (-1,+1), \quad \vec{\sigma}^{(3)} = (+1,-1), \quad \vec{\sigma}^{(4)} = (+1,+1).$$

Similarly, when $N = 3$, the configurations can be listed in the natural order as:

$$\begin{aligned} \vec{\sigma}^{(1)} &= (-1,-1,-1), & \vec{\sigma}^{(2)} &= (-1,-1,+1), \\ \vec{\sigma}^{(3)} &= (-1,+1,-1), & \vec{\sigma}^{(4)} &= (-1,+1,+1), \\ \vec{\sigma}^{(5)} &= (+1,-1,-1), & \vec{\sigma}^{(6)} &= (+1,-1,+1), \\ \vec{\sigma}^{(7)} &= (+1,+1,-1), & \vec{\sigma}^{(8)} &= (+1,+1,+1). \end{aligned}$$

When $N = 4$, the configurations can be listed in the natural order as:

$$\begin{aligned} \vec{\sigma}^{(1)} &= (-1,-1,-1,-1), & \vec{\sigma}^{(2)} &= (-1,-1,-1,+1), \\ \vec{\sigma}^{(3)} &= (-1,-1,+1,-1), & \vec{\sigma}^{(4)} &= (-1,+1,-1,-1), \\ \vec{\sigma}^{(5)} &= (-1,-1,+1,+1), & \vec{\sigma}^{(6)} &= (-1,+1,-1,+1), \\ \vec{\sigma}^{(7)} &= (-1,+1,+1,+1), & \vec{\sigma}^{(8)} &= (-1,+1,+1,-1), \\ \vec{\sigma}^{(9)} &= (+1,-1,-1,-1), & \vec{\sigma}^{(10)} &= (+1,-1,+1,-1) \\ \vec{\sigma}^{(11)} &= (+1,-1,-1,+1), & \vec{\sigma}^{(12)} &= (+1,+1,-1,+1) \\ \vec{\sigma}^{(13)} &= (+1,+1,+1,-1), & \vec{\sigma}^{(14)} &= (+1,-1,+1,+1) \\ \vec{\sigma}^{(15)} &= (+1,+1,-1,-1), & \vec{\sigma}^{(16)} &= (+1,+1,+1,+1) \end{aligned}$$

Therefore, under the natural order, the m configurations are listed uniquely and we write $(\vec{\sigma}^{(k)})_{k=1}^m$. Accordingly, the set of these Hamiltonians is denoted by $H = (H_k)_{k=1}^m$, where H_k is the kth Hamiltonian.

We can compute their covariances by using equations 2 and 3, and then we get a covariance matrix denoted by

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ C_{21} & C_{22} & \cdots & C_{2m} \\ & \cdots\cdots\cdots & & \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{pmatrix},$$

where $C_{kl} = \mathrm{Cov}(H_k H_l) = \mathrm{E}(H_k H_l)$.

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3.1. Algebraic properties of the covariance matrix

In the following, we write $r(C)$ to denote the rank of C as a matrix. The next proposition shows that the rank $r(C)$ has nothing to do with the choice of order for configurations $(\vec{\sigma}^{(k)})_{k=1}^{m}$.

Proposition 1 Let $H = (H_k)_{k=1}^{m}$ be the set of Hamiltonians and $C = (C_{kl})_{m\times m}$ be the covariance matrix. Then the rank $r(C)$ of C does not depend on the choice of order for configurations $(\vec{\sigma}^{(k)})_{k=1}^{m}$.

Proof. Give an order, accordingly, we obtain a covariance matrix. Thus, for two different orders, we can get two covariance matrices denoted by C_1 and C_2, respectively. However, we can obtain C_2 by exchanging rows and columns of C_1, which is an elementary transformation of matrix that doesn't change its rank. Therefore, the rank of the covariance matrix does not depend on the order of configurations. So it proves that $r(C_1) = r(C_2)$. □

In the following, we always use the natural order for configurations $(\vec{\sigma}^{(k)})_{k=1}^{m}$.

Definition 1 A collection of random variables $H_1, H_2, \ldots, H_n$ are called linearly dependent if there exist some real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_k \neq 0$ for some $k(1 \leq k \leq n)$ such that

$$\sum_{k=1}^{n} \lambda_k H_k = 0 \quad \text{almost surely.} \tag{4}$$

.

The next proposition gives a relationship between the rank of the covariance matrix C and the dimension of the space spanned by the Hamiltonians.

Proposition 2 Let span H be the space spanned by the set $H = (H_k)_{k=1}^{m}$ of the Hamiltonians of the SK model and $C = (C_{kl})_{m\times m}$ be the covariance matrix. Then the rank $r(C) = \dim \operatorname{span} H$.

Proof. For Hamiltonians $H_1, H_2, \ldots, H_n (n \leq m)$, we write

$$\begin{aligned} \vec{r_1} &= (C_{11}, C_{12}, \ldots, C_{1n}), \\ \vec{r_2} &= (C_{21}, C_{22}, \ldots, C_{2n}), \\ &\cdots \\ \vec{r_n} &= (C_{n1}, C_{n2}, \ldots, C_{nn}). \end{aligned}$$

Here, as mentioned above, $C_{kl} = \mathrm{Cov}(H_k H_l) = \mathrm{E}(H_k H_l)$. We can show that $H_1, H_2, \ldots, H_n$ are linearly dependent if and only if $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}$ are linearly dependent in R^n.

In fact, if $H_1, H_2, \ldots, H_n$ are linearly dependent, then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_k \neq 0$ for some $k(1 \leq k \leq n)$ such that $\sum\limits_{k=1}^{n} \lambda_k H_k = 0$ almost surely.

Thus

$$\begin{aligned} \sum_{k=1}^{n} \lambda_k \vec{r}_k &= (\sum_{k=1}^{n} \lambda_k C_{k1}, \sum_{k=1}^{n} \lambda_k C_{k2}, \ldots, \sum_{k=1}^{n} \lambda_k C_{kn}) \\ &= (\sum_{k=1}^{n} \lambda_k \mathrm{E}(H_k H_1), \sum_{k=1}^{n} \lambda_k \mathrm{E}(H_k H_2), \ldots, \sum_{k=1}^{n} \lambda_k \mathrm{E}(H_k H_n)) \\ &= (\mathrm{E}(\sum_{k=1}^{n} \lambda_k H_k H_1), \mathrm{E}(\sum_{k=1}^{n} \lambda_k H_k H_2), \ldots, \mathrm{E}(\sum_{k=1}^{n} \lambda_k H_k H_n)) \\ &= (0, 0, \ldots, 0), \end{aligned}$$

which means $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}$ are linearly dependent.

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Conversely, if $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ are linearly dependent, then there exist a set of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_k \neq 0$ for some $k(1 \leq k \leq n)$ such that $\sum\limits_{k=1}^{n} \lambda_k \vec{r}_k = 0$, namely

$$\sum_{k=1}^{n} \lambda_k \mathrm{E}(H_k H_1) = 0, \sum_{k=1}^{n} \lambda_k \mathrm{E}(H_k H_2) = 0, \cdots, \sum_{k=1}^{n} \lambda_k \mathrm{E}(H_k H_n) = 0,$$

which implies $\mathrm{E}(\sum\limits_{k=1}^{n} \lambda_k H_k)^2 = 0$. Thus $\sum\limits_{k=1}^{n} \lambda_k H_k = 0$ almost surely.

Without loss of generality, we may assume that $H_1, H_2, \ldots, H_n$ form a maximally linearly independent set of $H_1, H_2, \ldots, H_m$, which implies that $\dim \mathrm{span}\, H = n$. By using the previous conclusion, we see that $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ also form a maximally linearly independent set of $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$, which means $r(C) = n$. Thus $\dim \mathrm{span}\, H = r(C)$. This completes the proof. □

Our next result concerns an upper bound for the rank of the covariance matrix C of the Hamiltonians in the SK model.

Theorem 1 Let $H = (H_k)_{k=1}^{m}$ be the set of Hamiltonians and $C = (C_{kl})_{m \times m}$ be the covariance matrix. Then $r(C) \leq \frac{N(N-1)}{2}$.

Proof. Here, as denoted above, span H denotes the space spanned by the set $H = (H_k)_{k=1}^{m}$. We denote by span$\{ _{\text{ij}}\}$ the space spanned by $\{\xi_{ij}\}_{1 \leq i < j \leq N}$. It is easy to see that

$$\mathrm{span}\, H \subset \mathrm{span}\, \{ _{\text{ij}}\} \subset L^2(\Omega).$$

Hence

$$\dim \mathrm{span}\, H \leq \dim \mathrm{span}\{ _{\text{ij}}\}. \tag{5}$$

On the other hand, the dimension of span$\{ _{\text{ij}}\}$ as a subspace of $L^2(\Omega)$ just equals to $\frac{N(N-1)}{2}$, namely

$$\dim \mathrm{span}\{ _{\text{ij}}\} = \frac{N(N-1)}{2}. \tag{6}$$

Combining 5 with 6, we get

$$\dim \mathrm{span}\, H \leq \frac{N(N-1)}{2}, \tag{7}$$

which, together with Proposition 2, gives

$$r(C) \leq \frac{N(N-1)}{2}. \tag{8}$$

This completes the proof. □

3.2. Some special cases

In this subsection, we give some further discussions about the rank of the covariance matrix of Hamiltonians of the SK model. To do this, let us consider the following special cases.

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Example 1 Let $N = 2$. Then the corresponding covariance matrix is

$$C = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

It is easy to see that $r(C) = 1$. Note that $\frac{N(N-1)}{2} = 1$ in this case.
So we have $r(C) = \frac{N(N-1)}{2}$ for $N = 2$.

Example 2 Let $N = 3$. Then the corresponding covariance matrix reads

$$C = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \\ -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}.$$

By computing, we can get $r(C) = 3$. Note also that $\frac{N(N-1)}{2} = 3$ in this case.
So we again have $r(C) = \frac{N(N-1)}{2}$ for $N = 3$.

Example 3 Let $N = 4$. Then we can work out the corresponding covariance matrix as follows.

$$C = \begin{pmatrix}
\frac{3}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{3}{2} \\
0 & \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \\
0 & -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{3}{2} & \frac{3}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{3}{2} & \frac{3}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\
0 & \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \\
\frac{3}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{3}{2}
\end{pmatrix}.$$

By calculating, we can get $r(C) = 6$.
Similarly we have $r(C) = \frac{N(N-1)}{2}$ in this case.

4. Conclusion remark

In the present paper, we have investigated the dependence structure of Hamiltonians in the SK model, and establish the relationship between the rank of the covariance matrix and the dimension of the space spanned by the Hamiltonians. As is seen in the above, for special cases when $N = 2, 3$ and 4,

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it holds that $r(C) = \frac{N(N-1)}{2}$. In view of this, our result that $r(C) \leq \frac{N(N-1)}{2}$ may give the best upper bound for the rank of the covariance matrix of the Hamiltonians of the SK model.

Conflicts of Interest: The authors declare no conflict of interest.

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