Article

Not peer-reviewed version

Novel Approach to Study Soft α-Open Sets and Its Applications via Fuzzy Soft Topologies

Islam Taha * and Wafa Alqurashi

Posted Date: 13 February 2024

doi: 10.20944/preprints202402.0734.v1

Keywords: Fuzzy soft set; fuzzy soft topological space; r-fuzzy soft α-closed (α-open) set; fuzzy soft α-closure (α-interior) operator; fuzzy soft α-continuity; fuzzy soft almost (weakly) α-continuity; connectedness; compactness.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Article

Novel Approach to Study Soft α-Open Sets and Its Applications via Fuzzy Soft Topologies

Islam M. Taha [1,*] and Wafa Alqurashi [2]

[1] Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt
[2] Department of Mathematics, Faculty of Science, Umm Al-Qura University, Makkah, Saudi Arabia
[*] Correspondence: itaha@ut.edu.sa or imtaha2010@yahoo.com

Abstract: In this manuscript, we first introduce some properties of r-fuzzy soft α-open sets in fuzzy soft topological spaces based on the manuscript Aygünoğlu et al. (Hacet. J. Math. Stat. 2014, 43, 193-208). In addition, we define the concepts of fuzzy soft α-closure (α-interior) operators, and investigate some properties of them. Furthermore, the concept of r-fuzzy soft α-connected sets is defined and characterized with help of fuzzy soft α-closure operators. Thereafter, we introduce and study the concepts of fuzzy soft almost (weakly) α-continuous mappings, which are weaker forms of fuzzy soft α-continuous mappings, and we discuss some properties of fuzzy soft α-continuity. Moreover, we establish that fuzzy soft α-continuity $\Rightarrow$ fuzzy soft almost α-continuity $\Rightarrow$ fuzzy soft weakly α-continuity, but the converse may not be true. It is also we show that the composition $\varphi^*_{\psi^*} \circ \varphi_\psi$ is fuzzy soft almost α-continuous mapping if, φ_ψ is fuzzy soft α-continuous mapping and $\varphi^*_{\psi^*}$ is fuzzy soft almost continuous mapping. Finally, some new types of compactness via r-fuzzy soft α-open sets are defined, and the relationships between them are examined.

Keywords: Fuzzy soft set; fuzzy soft topological space; r-fuzzy soft α-closed (α-open) set; fuzzy soft α-closure (α-interior) operator; fuzzy soft α-continuity; fuzzy soft almost (weakly) α-continuity; connectedness; compactness

MSC: 54A05; 54A40; 54C05; 54C10; 54D30

1. Introduction

Theory of soft sets was first introduced by Molodtsov [1], which is a completely new approach for modeling uncertainty and vagueness. He demonstrated many applications of these theory in solving several practical problems in mathematics, engineering, economics, and social science etc. Akdag and Ozkan [2] defined the concept of soft α-open sets on soft topological spaces, and some properties are specified. The concept of soft β-open sets was defined and studied by the authors of [3,4]. Also, the concepts of soft semi-open, somewhere dense and Q-sets were studied by the authors of [5,6]. Al-shami et al. [7] introduced the concept of weakly soft semi-open sets and obtained its main properties. Moreover, Al-shami et al. [8] initiated the concept of weakly soft β-open sets and examined weakly soft β-continuity. Kaur et al. [9] introduced a new approach to studying soft continuous mappings using an induced mapping based on soft sets. Al Ghour and Al-Mufarrij [10] defined two new concepts of mappings over soft topological spaces: soft somewhat-r-continuity and soft somewhat-r-openness. In 2024, Ameen et al. [11] explored more properties of soft somewhere dense continuous mappings.

The concept of fuzzy soft sets was introduced by Maji et al. [12], which combines soft sets [1] and fuzzy sets [13]. The concept of fuzzy soft topology is defined and some characterized such as fuzzy soft interior (closure) set, fuzzy soft continuity and fuzzy soft subspace is studied in [14,15] based on fuzzy topologies in the sense of Šostak [16]. A new approach to studying separation and regularity axioms via fuzzy soft sets was introduced by the author of [17,18] based on the paper Aygünoğlu et al. [14]. The concept of r-fuzzy soft regularly open sets was introduced by Çetkin and Aygün [19]. Also, the concepts of r-fuzzy soft pre-open (resp. β-open) sets were defined by Taha [20].

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

In our study, the layout is designed as follows.

• Firstly, we introduce the concepts of fuzzy soft α-closure (α-interior) operators in fuzzy soft topological space (W, τ_N) based on the paper Aygünoğlu et al. [14], and examine some of its properties. Also, the concept of r-fuzzy soft α-connected sets is introduced and studied.

• Secondly, we are going to investigate some properties of fuzzy soft α-continuous mappings between two fuzzy soft topological spaces (W, τ_N) and (V, η_F). Moreover, we define and study the concepts of fuzzy soft weakly (almost) α-continuous mappings, which are weaker forms of fuzzy soft α-continuous mappings. Also, the relationships between these classes of mappings are investigated with the help of some examples.

• Finally, several types of fuzzy soft compactness via r-fuzzy soft α-open sets are defined, and the relationships between them are specified.

• In the end, we close this manuscript with some conclusions and proposed some future works in Section 5.

In this work, nonempty sets will be denoted by W, V etc. N is the set of all parameters for W and $C \subseteq N$. The family of all fuzzy sets on W is denoted by I^W (where $I_\circ = (0,1],\ I = [0,1]$), and for $s \in I$, $\underline{s}(w) = s$, for all $w \in W$.

The following concepts and results will be used in the next sections.

Definition 1.1. [14,21,22] A fuzzy soft set h_C on W is a mapping from N to I^W such that $h_C(n)$ is a fuzzy set on W, for each $n \in C$ and $h_C(n) = \underline{0}$, if $n \notin C$. The family of all fuzzy soft sets on W is denoted by $\widetilde{(W, N)}$.

Definition 1.2. [23] The difference between two fuzzy soft sets h_C and g_B is a fuzzy soft set defined as follows, for each $n \in N$: $(h_C \overline{\sqcap} g_B)(n) = \begin{cases} \underline{0}, & \text{if } h_C(n) \leq g_B(n), \\ h_C(n) \wedge (g_B(n))^c, & \text{otherwise.} \end{cases}$

Definition 1.3. [24] A fuzzy soft point n_{w_s} on W is a fuzzy soft set defined as follows:

$$n_{w_s}(k) = \begin{cases} w_s, & \text{if} \quad k = n, \\ \underline{0}, & \text{if} \quad k \in N - \{n\}, \end{cases}$$

where w_s is a fuzzy point on W. n_{w_s} is called belong to a fuzzy soft set f_A, denoted by $n_{w_s} \tilde{\in} f_A$, if $s \leq f_A(n)(w)$. The family of all fuzzy soft points on W is denoted by $\widetilde{P_s(W)}$.

Definition 1.4. [25] A fuzzy soft point $n_{w_s} \in \widetilde{P_s(W)}$ is called a soft quasi-coincident with $h_C \in \widetilde{(W, N)}$ and denoted by $n_{w_s} \tilde{q} h_C$, if $s + h_C(n)(w) > 1$. A fuzzy soft set $h_C \in \widetilde{(W, N)}$ is called a soft quasi-coincident with $g_B \in \widetilde{(W, N)}$ and denoted by $h_C \tilde{q} g_B$, if there is $n \in N$ and $w \in W$ such that $h_C(n)(w) + g_B(n)(w) > 1$. If h_C is not soft quasi-coincident with g_B, $h_C \tilde{\not{q}} g_B$.

Definition 1.5. [14] A mapping $\tau : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ is called a fuzzy soft topology on W if it satisfies the following, for each $n \in N$:

(1) $\tau_n(\Phi) = \tau_n(\widetilde{N}) = 1$,

(2) $\tau_n(h_C \sqcap g_B) \geq \tau_n(h_C) \wedge \tau_n(g_B)$, for each $h_C, g_B \in \widetilde{(W, N)}$,

(3) $\tau_n(\sqcup_{\delta \in \Delta}(h_C)_\delta) \geq \wedge_{\delta \in \Delta} \tau_n((h_C)_\delta)$, for each $(h_C)_\delta \in \widetilde{(W, N)}, \delta \in \Delta$.

Then, (W, τ_N) is called a fuzzy soft topological space (briefly, FSTS) in the sense of Šostak.

Definition 1.6. [14] Let (W, τ_N) and (V, η_F) be a FSTSs. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W, N)} \longrightarrow \widetilde{(V, F)}$ is called a fuzzy soft continuous if, $\tau_n(\varphi_\psi^{-1}(h_C)) \geq \eta_k(h_C)$ for each $h_C \in \widetilde{(V, F)}$, $n \in N$ and $(k = \psi(n)) \in F$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Definition 1.7. [15,19] In a FSTS (W, τ_N), for each $h_C \in \widetilde{(W, N)}$, $n \in N$ and $r \in I_0$, we define the fuzzy soft operators C_τ and $I_\tau : N \times \widetilde{(W, N)} \times I_\circ \to \widetilde{(W, N)}$ as follows:

$C_\tau(n, h_C, r) = \sqcap \{g_B \in \widetilde{(W, N)} : h_C \sqsubseteq g_B,\ \tau_n(g_B^c) \geq r\}$,

$I_\tau(n, h_C, r) = \sqcup \{g_B \in \widetilde{(W, N)} : g_B \sqsubseteq h_C,\ \tau_n(g_B) \geq r\}$.

Definition 1.8. Let (W, τ_N) be a FSTS and $r \in I_0$. A fuzzy soft set $h_C \in \widetilde{(W, N)}$ is called an r-fuzzy soft regularly open [19] (resp. β-open [20], pre-open [20], α-open [26] and semi-open [26]) if, $h_C = I_\tau(n, C_\tau(n, h_C, r), r)$ (resp. $h_C \sqsubseteq C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r)$, $h_C \sqsubseteq I_\tau(n, C_\tau(n, h_C, r), r)$, $h_C \sqsubseteq I_\tau(n, C_\tau(n, I_\tau(n, h_C, r), r), r)$ and $h_C \sqsubseteq C_\tau(n, I_\tau(n, h_C, r), r)$) for each $n \in N$.

Definition 1.9. [19] Let (W, τ_N) be a FSTS and $r \in I_0$. A fuzzy soft set $h_C \in \widetilde{(W, N)}$ is called an r-fuzzy soft regularly closed if, $h_C = C_\tau(n, I_\tau(n, h_C, r), r)$ for each $n \in N$.

Definition 1.10. [26] Let (W, τ_N) and (V, η_F) be a FSTSs and $r \in I_0$. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W, N)} \longrightarrow \widetilde{(V, F)}$ is called a fuzzy soft almost (resp. weakly) continuous if, for any $n_{w_s} \in \widetilde{P_s(W)}$ and any $f_A \in \widetilde{(V, F)}$ with $\eta_k(f_A) \geq r$ containing $\varphi_\psi(n_{w_s})$, there is $h_C \in \widetilde{(W, N)}$ with $\tau_n(h_C) \geq r$ containing n_{w_s} such that $\varphi_\psi(h_C) \sqsubseteq I_\eta(k, C_\eta(k, f_A, r), r)$ (resp. $\varphi_\psi(h_C) \sqsubseteq C_\eta(k, f_A, r)$).

Remark 1.1. [26] From Definitions 1.6 and 1.10, we have: Fuzzy soft continuity $\Rightarrow$ fuzzy soft almost continuity $\Rightarrow$ fuzzy soft weakly continuity, but the converse may not be true.

Lemma 1.1. Let (W, τ_N) and (V, η_F) be a FSTSs and $r \in I_0$. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W, N)} \longrightarrow \widetilde{(V, F)}$ is fuzzy soft almost continuous if, $\tau_n(\varphi_\psi^{-1}(h_C)) \geq r$ for each $h_C \in \widetilde{(V, F)}$ is r-fuzzy soft regularly open, $n \in N$ and $(k = \psi(n)) \in F$.

Proof. Easily proved from Definition 1.10.
□

Definition 1.11. Let (W, τ_N) and (V, η_F) be a FSTSs. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W, N)} \longrightarrow \widetilde{(V, F)}$ is called a fuzzy soft open if, $\eta_k(\varphi_\psi(h_C)) \geq \tau_n(h_C)$ for each $h_C \in \widetilde{(W, N)}$, $n \in N$ and $(k = \psi(n)) \in F$.

The basic concepts and results are found in [14,15], which we need in the next sections.

2. On r-Fuzzy Soft α-Open Sets

Here, we introduce and discuss the notions of fuzzy soft α-closure (α-interior) operators in fuzzy soft topological spaces based on the paper Aygünoğlu et al. [14]. Also, the notion of r-fuzzy soft α-connected sets is defined and studied with help of fuzzy soft α-closure operators.

Definition 2.1. Let (W, τ_N) be a FSTS and $r \in I_0$. A fuzzy soft set $h_C \in \widetilde{(W, N)}$ is called an r-fuzzy soft α-closed (resp. semi-closed, β-closed and pre-closed) if, $C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r) \sqsubseteq h_C$ (resp. $I_\tau(n, C_\tau(n, h_C, r), r) \sqsubseteq h_C$, $I_\tau(n, C_\tau(n, I_\tau(n, h_C, r), r), r) \sqsubseteq h_C$ and $C_\tau(n, I_\tau(n, h_C, r), r) \sqsubseteq h_C$) for each $n \in N$.

Remark 2.1. The complement of an r-fuzzy soft α-open [26] (resp. semi-open [26], β-open [20] and pre-open [20]) set is an r-fuzzy soft α-closed (resp. semi-closed, β-closed and pre-closed) set.

Lemma 2.1. Let (W, τ_N) be a FSTS and $r \in I_0$. Then, any intersection (resp. union) of r-fuzzy soft α-closed (resp. α-open) sets is an r-fuzzy soft α-closed (resp. α-open) set.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Proof. Easily proved from Definitions 1.8 and 2.1. □

Proposition 2.1. Let (W, τ_N) be a FSTS, $h_C \in \widetilde{(W, N)}$, $n \in N$ and $r \in I_0$. Then, the following statements are equivalent.

(1) h_C is r-fuzzy soft α-closed.

(2) h_C is r-fuzzy soft semi-closed and r-fuzzy soft pre-closed.

Proof. (1) ⇒ (2) Let h_C be an r-fuzzy soft α-closed, $h_C \sqsupseteq C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r) \sqsupseteq I_\tau(n, C_\tau(n, h_C, r), r)$. This shows that h_C is r-fuzzy soft semi-closed.

Since $h_C \sqsupseteq C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r)$ and $C_\tau(n, h_C, r) \sqsupseteq h_C$, then $h_C \sqsupseteq C_\tau(n, I_\tau(n, h_C, r), r)$. Therefore, h_C is r-fuzzy soft pre-closed

(2) ⇒ (1) Let h_C be an r-fuzzy soft semi-closed and r-fuzzy soft pre-closed, then $h_C \sqsupseteq C_\tau(n, I_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r), r) = C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r)$. This shows that h_C is r-fuzzy soft α-closed.

□

Proposition 2.2. Let (W, τ_N) be a FSTS, $g_B, h_C \in \widetilde{(W, N)}$, $n \in N$ and $r \in I_0$. If g_B is r-fuzzy soft semi-closed set such that $g_B \sqsupseteq h_C \sqsupseteq C_\tau(n, I_\tau(n, g_B, r), r)$, h_C is r-fuzzy soft α-closed.

Proof. Let g_B be an r-fuzzy soft semi-closed and $g_B \sqsupseteq h_C$, then $g_B \sqsupseteq I_\tau(n, C_\tau(n, g_B, r), r) \sqsupseteq I_\tau(n, C_\tau(n, h_C, r), r)$. Let $h_C \sqsupseteq C_\tau(n, I_\tau(n, g_B, r), r)$, then $h_C \sqsupseteq C_\tau(n, I_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r), r) = C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r)$. Therefore, h_C is r-fuzzy soft α-closed.

□

Lemma 2.2. Let (W, τ_N) be a FSTS, $g_B, h_C \in \widetilde{(W, N)}$, $n \in N$ and $r \in I_0$. If g_B is r-fuzzy soft α-closed set such that $g_B \sqsupseteq h_C \sqsupseteq C_\tau(n, I_\tau(n, g_B, r), r)$, h_C is r-fuzzy soft α-closed.

Proof. It is easily proved from every r-fuzzy soft α-closed set is r-fuzzy soft semi-closed set.

□

Remark 2.2. From the previous definition, we can summarize the relationships among different types of fuzzy soft sets as in the next diagram.

$$\alpha - closed\ \ set$$

$$\Downarrow \qquad \Downarrow$$

$$semi - closed\ \ set \qquad pre - closed\ \ set$$

$$\Downarrow \qquad \Downarrow$$

$$\beta - closed\ \ set$$

Remark 2.3. The converses of the above relationships may not be true, as shown by Examples 2.1 and 2.2.

Example 2.1. Let $W = \{w_1, w_2\}$, $N = \{n_1, n_2\}$ and define $f_N, g_N, h_N \in \widetilde{(W, N)}$ as follows: $f_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}\})\}$, $g_N = \{(n_1, \{\frac{w_1}{0.6}, \frac{w_2}{0.2}\}), (n_2, \{\frac{w_1}{0.6}, \frac{w_2}{0.2}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.5}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.5}\})\}$. Define fuzzy soft topology $\tau_N : N \longrightarrow [0, 1]^{\widetilde{(W,N)}}$ as follows:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

$$\tau_{n_1}(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{1}{2}, & \text{if} \quad t_N=f_N,\\ \frac{2}{3}, & \text{if} \quad t_N=g_N,\\ \frac{2}{3}, & \text{if} \quad t_N=f_N\sqcap g_N,\\ \frac{1}{2}, & \text{if} \quad t_N=f_N\sqcup g_N,\\ 0, & \text{otherwise,}\end{cases}$$

$$\tau_{n_2}(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{1}{4}, & \text{if} \quad t_N=f_N,\\ \frac{1}{2}, & \text{if} \quad t_N=g_N,\\ \frac{1}{2}, & \text{if} \quad t_N=f_N\sqcap g_N,\\ \frac{1}{4}, & \text{if} \quad t_N=f_N\sqcup g_N,\\ 0, & \text{otherwise.}\end{cases}$$

Then, h_N is $\frac{1}{4}$-fuzzy soft semi-closed and $\frac{1}{4}$-fuzzy soft β-closed, but it is neither $\frac{1}{4}$-fuzzy soft α-closed nor $\frac{1}{4}$-fuzzy soft pre-closed.

Example 2.2. Let $W=\{w_1,w_2\}$, $N=\{n_1,n_2\}$ and define $f_N,h_N\in\widetilde{(W,N)}$ as follows: $f_N=\{(n_1,\{\frac{w_1}{0.4},\frac{w_2}{0.5}\}),(n_2,\{\frac{w_1}{0.4},\frac{w_2}{0.5}\})\}$, $h_N=\{(n_1,\{\frac{w_1}{0.7},\frac{w_2}{0.6}\}),(n_2,\{\frac{w_1}{0.7},\frac{w_2}{0.6}\})\}$. Define fuzzy soft topology $\tau_N:N\longrightarrow[0,1]^{\widetilde{(W,N)}}$ as follows:

$$\tau_{n_1}(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{1}{3}, & \text{if} \quad t_N=f_N,\\ 0, & \text{otherwise,}\end{cases}$$

$$\tau_{n_2}(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{1}{2}, & \text{if} \quad t_N=f_N,\\ 0, & \text{otherwise.}\end{cases}$$

Then, h_N is $\frac{1}{3}$-fuzzy soft pre-closed and $\frac{1}{3}$-fuzzy soft β-closed, but it is neither $\frac{1}{3}$-fuzzy soft semi-closed nor $\frac{1}{3}$-fuzzy soft α-closed.

Definition 2.2. In a FSTS (W,τ_N), for each $h_C\in\widetilde{(W,N)}$, $n\in N$ and $r\in I_0$, we define a fuzzy soft α-closure operator $\alpha C_\tau: N\times\widetilde{(W,N)}\times I_\circ\to\widetilde{(W,N)}$ as follows: $\alpha C_\tau(n,h_C,r)=\sqcap\{f_A\in\widetilde{(W,N)}: h_C\sqsubseteq f_A,\ f_A \text{ is } r\text{-fuzzy soft } \alpha\text{-closed}\}$.

Theorem 2.1. In a FSTS (W,τ_N), for each $g_B,h_C\in\widetilde{(W,N)}$, $n\in N$ and $r\in I_0$, the operator $\alpha C_\tau : N\times\widetilde{(W,N)}\times I_\circ\to\widetilde{(W,N)}$ satisfies the following properties.

(1) $\alpha C_\tau(n,\Phi,r)=\Phi$.
(2) $h_C\sqsubseteq\alpha C_\tau(n,h_C,r)\sqsubseteq C_\tau(n,h_C,r)$.
(3) $\alpha C_\tau(n,h_C,r)\sqsubseteq\alpha C_\tau(n,g_B,r)$ if, $h_C\sqsubseteq g_B$.
(4) $\alpha C_\tau(n,\alpha C_\tau(n,h_C,r),r)=\alpha C_\tau(n,h_C,r)$.
(5) $\alpha C_\tau(n,h_C\sqcup g_B,r)\sqsupseteq\alpha C_\tau(n,h_C,r)\sqcup\alpha C_\tau(n,g_B,r)$.
(6) $\alpha C_\tau(n,h_C,r)=h_C$ iff h_C is r-fuzzy soft α-closed.
(7) $\alpha C_\tau(n,C_\tau(n,h_C,r),r)=C_\tau(n,h_C,r)$.

Proof. (1), (2), (3) and (6) are easily proved from Definition 2.2.

(4) From (2) and (3), $\alpha C_\tau(n,h_C,r)\sqsubseteq\alpha C_\tau(n,\alpha C_\tau(n,h_C,r),r)$. Now we show that $\alpha C_\tau(n,h_C,r)\sqsupseteq\alpha C_\tau(n,\alpha C_\tau(n,h_C,r),r)$. Suppose that $\alpha C_\tau(n,h_C,r)$ is not contain $\alpha C_\tau(n,\alpha C_\tau(n,h_C,r),r)$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Then, there is $w \in W$ and $s \in (0,1)$ such that $\alpha C_\tau(n, h_C, r)(n)(w) < s < \alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)(n)(w)$. (A)

Since $\alpha C_\tau(n, h_C, r)(n)(w) < s$, by the definition of αC_τ, there is g_B is r-fuzzy soft α-closed with $h_C \sqsubseteq g_B$ such that $\alpha C_\tau(n, h_C, r)(n)(w) \leq g_B(n)(w) < s$. Since $h_C \sqsubseteq g_B$, we have $\alpha C_\tau(n, h_C, r) \sqsubseteq g_B$. Again, by the definition of αC_τ, we have $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r) \sqsubseteq g_B$. Hence $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)(n)(w) \leq g_B(n)(w) < s$, it is a contradiction for (A). Thus, $\alpha C_\tau(n, h_C, r) \sqsupseteq \alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)$. Then, $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r) = \alpha C_\tau(n, h_C, r)$.

(5) Since h_C and $g_B \sqsubseteq h_C \sqcup g_B$, hence by (3), $\alpha C_\tau(n, h_C, r) \sqsubseteq \alpha C_\tau(n, h_C \sqcup g_B, r)$ and $\alpha C_\tau(n, g_B, r) \sqsubseteq \alpha C_\tau(n, h_C \sqcup g_B, r)$. Thus, $\alpha C_\tau(n, h_C \sqcup g_B, r) \sqsupseteq \alpha C_\tau(n, h_C, r) \sqcup \alpha C_\tau(n, g_B, r)$.

(7) From (6) and $C_\tau(n, h_C, r)$ is r-fuzzy soft α-closed set, hence $\alpha C_\tau(n, C_\tau(n, h_C, r), r) = C_\tau(n, h_C, r)$.

□

Theorem 2.2. In a FSTS (W, τ_N), for each $h_C \in \widetilde{(W,N)}$, $n \in N$ and $r \in I_0$, we define a fuzzy soft α-interior operator $\alpha I_\tau : N \times \widetilde{(W,N)} \times I_\circ \to \widetilde{(W,N)}$ as follows: $\alpha I_\tau(n, h_C, r) = \sqcup \{f_A \in \widetilde{(W,N)} : f_A \sqsubseteq h_C,\ f_A$ is r-fuzzy soft α-open$\}$. Then, for each g_B and $h_C \in \widetilde{(W,N)}$, the operator αI_τ satisfies the following properties.

(1) $\alpha I_\tau(n, \widetilde{N}, r) = \widetilde{N}$.

(2) $I_\tau(n, h_C, r) \sqsubseteq \alpha I_\tau(n, h_C, r) \sqsubseteq h_C$.

(3) $\alpha I_\tau(n, h_C, r) \sqsubseteq \alpha I_\tau(n, g_B, r)$ if, $h_C \sqsubseteq g_B$.

(4) $\alpha I_\tau(n, \alpha I_\tau(n, h_C, r), r) = \alpha I_\tau(n, h_C, r)$.

(5) $\alpha I_\tau(n, h_C, r) \sqcap \alpha I_\tau(n, g_B, r) \sqsupseteq \alpha I_\tau(n, h_C \sqcap g_B, r)$.

(6) $\alpha I_\tau(n, h_C, r) = h_C$ iff h_C is r-fuzzy soft α-open.

(7) $\alpha I_\tau(n, h^c_C, r) = (\alpha C_\tau(n, h_C, r))^c$.

Proof. (1), (2), (3) and (6) are easily proved from the definition of αI_τ.

(4) and (5) are easily proved by a similar way in Theorem 2.1.

(7) For each $h_C \in \widetilde{(W,N)}$, $n \in N$ and $r \in I_0$, we have $\alpha I_\tau(n, h^c_C, r) = \sqcup\{f_A \in \widetilde{(W,N)} : f_A \sqsubseteq h^c_C,\ f_A$ is r-fuzzy soft α-open$\} = [\sqcap\{f^c_A \in \widetilde{(W,N)} : h_C \sqsubseteq f^c_A,\ f^c_A$ is r-fuzzy soft α-closed$\}]^c = (\alpha C_\tau(n, h_C, r))^c$.

□

Definition 2.3. Let (W, τ_N) be a FSTS, $r \in I_0$ and $g_B, h_C \in \widetilde{(W,N)}$. Then, we have:

(1) Two fuzzy soft sets g_B and h_C are called r-fuzzy soft α-separated iff $g_B \bar{\tilde{q}}\ \alpha C_\tau(n, h_C, r)$ and $h_C \bar{\tilde{q}}\ \alpha C_\tau(n, g_B, r)$ for each $n \in N$.

(2) Any fuzzy soft set which cannot be expressed as the union of two r-fuzzy soft α-separated sets is called an r-fuzzy soft α-connected.

Theorem 2.3. In a FSTS (W, τ_N), we have:

(1) If f_A and $g_B \in \widetilde{(W,N)}$ are r-fuzzy soft α-separated and $h_C, t_D \in \widetilde{(W,N)}$ such that $h_C \sqsubseteq f_A$ and $t_D \sqsubseteq g_B$, then h_C and t_D are r-fuzzy soft α-separated.

(2) If $f_A \bar{\tilde{q}}\ g_B$ and either both are r-fuzzy soft α-open or both r-fuzzy soft α-closed, then f_A and g_B are r-fuzzy soft α-separated.

(3) If f_A and g_B are either both r-fuzzy soft α-open or both r-fuzzy soft α-closed, then $f_A \sqcap g^c_B$ and $g_B \sqcap f^c_A$ are r-fuzzy soft α-separated.

Proof. (1) and (2) are obvious.

(3) Let f_A and g_B be an r-fuzzy soft α-open. Since $f_A \sqcap g^c_B \sqsubseteq g^c_B$, $\alpha C_\tau(n, f_A \sqcap g^c_B, r) \sqsubseteq g^c_B$ and hence $\alpha C_\tau(n, f_A \sqcap g^c_B, r)\ \bar{\tilde{q}}\ g_B$. Then, $\alpha C_\tau(n, f_A \sqcap g^c_B, r)\ \bar{\tilde{q}}\ (g_B \sqcap f^c_A)$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Again, since $g_B \sqcap f^c_A \sqsubseteq f^c_A$, $\alpha C_\tau(n, g_B \sqcap f^c_A, r) \sqsubseteq f^c_A$ and hence $\alpha C_\tau(n, g_B \sqcap f^c_A, r) \tilde{\not q} f_A$. Then, $\alpha C_\tau(n, g_B \sqcap f^c_A, r) \tilde{\not q} (f_A \sqcap g^c_B)$. Thus, $f_A \sqcap g^c_B$ and $g_B \sqcap f^c_A$ are r-fuzzy soft α-separated. The other case follows similar lines.

□

Theorem 2.4. In a FSTS (W, τ_N), then $f_A, g_B \in \widetilde{(W, N)}$ are r-fuzzy soft α-separated iff there exist two r-fuzzy soft α-open sets h_C and t_D such that $f_A \sqsubseteq h_C$, $g_B \sqsubseteq t_D$, $f_A \tilde{\not q}\, t_D$ and $g_B \tilde{\not q}\, h_C$.

Proof. $(\Rightarrow)$ Let f_A and $g_B \in \widetilde{(W, N)}$ be an r-fuzzy soft α-separated, $f_A \sqsubseteq (\alpha C_\tau(n, g_B, r))^c = h_C$ and $g_B \sqsubseteq (\alpha C_\tau(n, f_A, r))^c = t_D$, where t_D and h_C are r-fuzzy soft α-open, then $t_D \tilde{\not q} \alpha C_\tau(n, f_A, r)$ and $h_C \tilde{\not q} \alpha C_\tau(n, g_B, r)$. Thus, $g_B \tilde{\not q}\, h_C$ and $f_A \tilde{\not q}\, t_D$. Hence, we obtain the required result.

$(\Leftarrow)$ Let h_C and t_D be an r-fuzzy soft α-open such that $g_B \sqsubseteq t_D$, $f_A \sqsubseteq h_C$, $g_B \tilde{\not q}\, h_C$ and $f_A \tilde{\not q}\, t_D$. Then, $g_B \sqsubseteq h^c_C$ and $f_A \sqsubseteq t^c_D$. Hence, $\alpha C_\tau(n, g_B, r) \sqsubseteq h^c_C$ and $\alpha C_\tau(n, f_A, r) \sqsubseteq t^c_D$. Then, $\alpha C_\tau(n, g_B, r) \tilde{\not q}\, f_A$ and $\alpha C_\tau(n, f_A, r) \tilde{\not q}\, g_B$. Thus, g_B and f_A are r- fuzzy soft α-separated. Hence, we obtain the required result.

□

Theorem 2.5. In a FSTS (W, τ_N), if $g_B \in \widetilde{(W, N)}$ is r-fuzzy soft α-connected such that $g_B \sqsubseteq f_A \sqsubseteq \alpha C_\tau(n, g_B, r)$, then f_A is r-fuzzy soft α-connected.

Proof. Suppose that f_A is not r-fuzzy soft α-connected, then there is r-fuzzy soft α-separated sets h^*_C and $t^*_D \in \widetilde{(W, N)}$ such that $f_A = h^*_C \sqcup t^*_D$. Let $h_C = g_B \sqcap h^*_C$ and $t_D = g_B \sqcap t^*_D$, then $g_B = t_D \sqcup h_C$. Since $h_C \sqsubseteq h^*_C$ and $t_D \sqsubseteq t^*_D$, hence by Theorem 2.3(1), h_C and t_D are r-fuzzy soft α-separated, it is a contradiction. Thus, f_A is r-fuzzy soft α-connected, as required.

□

3. Continuity via r-Fuzzy Soft α-Open Sets

Here, we investigate some properties of fuzzy soft α-continuous mappings. Additionally, we introduce and study the notions of fuzzy soft almost (weakly) α-continuous mappings, which are weaker forms of fuzzy soft α-continuous mappings. Also, we show that fuzzy soft α-continuity $\Rightarrow$ fuzzy soft almost α-continuity $\Rightarrow$ fuzzy soft weakly α-continuity.

Definition 3.1. [26] Let (W, τ_N) and (V, η_F) be a FSTSs and $r \in I_\circ$. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W, N)} \longrightarrow \widetilde{(V, F)}$ is called a fuzzy soft α-continuous if, $\varphi^{-1}_\psi(h_C)$ is r-fuzzy soft α-open set for each $h_C \in \widetilde{(V, F)}$ with $\eta_k(h_C) \geq r$, $n \in N$, $(k = \psi(n)) \in F$.

Theorem 3.1. Let (W, τ_N) and (V, η_F) be a FSTSs, and $\varphi_\psi : \widetilde{(W, N)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft mapping. The following statements are equivalent for each $f_A \in \widetilde{(V, F)}$, $n \in N$, $(k = \psi(n)) \in F$ and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft α-continuous.
(2) For each f_A with $\eta_k(f^c_A) \geq r$, $\varphi^{-1}_\psi(f_A)$ is r-fuzzy soft α-closed.
(3) $\alpha C_\tau(n, \varphi^{-1}_\psi(f_A), r) \sqsubseteq \varphi^{-1}_\psi(C_\eta(k, f_A, r))$.
(4) $\varphi^{-1}_\psi(I_\eta(k, f_A, r)) \sqsubseteq \alpha I_\tau(n, \varphi^{-1}_\psi(f_A), r)$.
(5) $C_\tau(n, I_\tau(n, C_\tau(n, \varphi^{-1}_\psi(f_A), r), r), r) \sqsubseteq \varphi^{-1}_\psi(C_\eta(k, f_A, r))$.

Proof. (1) $\Leftrightarrow$ (2) Follows from Remark 2.1 and $\varphi^{-1}_\psi(f^c_A) = (\varphi^{-1}_\psi(f_A))^c$.

(2) $\Rightarrow$ (3) Let $f_A \in \widetilde{(V, F)}$, hence by (2), $\varphi^{-1}_\psi(C_\eta(k, f_A, r))$ is r-fuzzy soft α-closed. Then, we obtain $\alpha C_\tau(n, \varphi^{-1}_\psi(f_A), r) \sqsubseteq \varphi^{-1}_\psi(C_\eta(k, f_A, r))$.

(3) $\Leftrightarrow$ (4) Follows from Theorem 2.2(7).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

(3) $\Rightarrow$ (5) Let $f_A \in \widetilde{(V,F)}$, hence by (3), we obtain $C_\tau(n, I_\tau(n, C_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r) \sqsubseteq \alpha C_\tau(n, \varphi_\psi^{-1}(f_A), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$.

(5) $\Rightarrow$ (1) Let $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$, hence by (3), we obtain $(\varphi_\psi^{-1}(f_A))^c = \varphi_\psi^{-1}(f_A^c) \sqsupseteq C_\tau(n, I_\tau(n, C_\tau(n, \varphi_\psi^{-1}(f_A^c), r), r), r) = (I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r))^c$. Then, $\varphi_\psi^{-1}(f_A) \sqsubseteq I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r)$, so $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-open. Hence, φ_ψ is fuzzy soft α-continuous.
□

Lemma 3.1. Every fuzzy soft continuous mapping [14] is fuzzy soft α-continuous.

Proof. Follows from Definitions 1.6 and 3.1.
□

Remark 3.1. The converse of Lemma 3.1 is not true, as shown by Example 3.1.

Example 3.1. Let $W = \{w_1, w_2, w_3\}$, $N = \{n_1, n_2\}$ and define $f_N, g_N, h_N \in \widetilde{(W,N)}$ as: $f_N = \{(n_1, \{\frac{w_1}{0.4}, \frac{w_2}{0.5}, \frac{w_3}{0.5}\}), (n_2, \{\frac{w_1}{0.4}, \frac{w_2}{0.5}, \frac{w_3}{0.5}\})\}$, $g_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.4}\})\}$. Define fuzzy soft topologies $\tau_N, \eta_N : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ as follows: $\forall n \in N$,

$$\tau_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N, \\ \frac{2}{3}, & \text{if} \quad t_N = g_N, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N, \\ \frac{1}{3}, & \text{if} \quad t_N = h_N, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft mapping $\varphi_\psi : (W, \tau_N) \longrightarrow (W, \eta_N)$ is fuzzy soft α-continuous, but it is not fuzzy soft continuous.

Definition 3.2. Let (W, τ_N) and (V, η_F) be a FSTSs. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is called fuzzy soft almost (resp. weakly) α-continuous if, for each $n_{w_s} \in \widetilde{P_s(W)}$ and each $g_B \in \widetilde{(V,F)}$ with $\eta_k(g_B) \geq r$ containing $\varphi_\psi(n_{w_s})$, there is $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-open set containing n_{w_s} such that $\varphi_\psi(h_C) \sqsubseteq I_\eta(k, C_\eta(k, g_B, r), r)$ (resp. $\varphi_\psi(h_C) \sqsubseteq C_\eta(k, g_B, r)$), $n \in N$, $(k = \psi(n)) \in F$ and $r \in I_\circ$.

Lemma 3.2. (1) Every fuzzy soft α-continuous mapping is fuzzy soft almost α-continuous.
(2) Every fuzzy soft almost α-continuous mapping is fuzzy soft weakly α-continuous.

Proof. Follows from Definitions 3.1 and 3.2. □

Remark 3.2. The converse of Lemma 3.2 is not true, as shown by Examples 3.2 and 3.3.

Example 3.2. Let $W = \{w_1, w_2, w_3\}$, $N = \{n_1, n_2\}$ and define $g_N, h_N \in \widetilde{(W,N)}$ as follows: $g_N = \{(n_1, \{\frac{w_1}{0.5}, \frac{w_2}{0.5}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.5}, \frac{w_2}{0.5}, \frac{w_3}{0.4}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\})\}$. Define fuzzy soft topologies $\tau_N, \eta_N : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ as follows: $\forall n \in N$,

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

$$\tau_n(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{1}{2}, & \text{if} \quad t_N=g_N,\\ 0, & \text{otherwise},\end{cases}$$

$$\eta_n(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{1}{2}, & \text{if} \quad t_N=g_N,\\ \frac{1}{3}, & \text{if} \quad t_N=h_N,\\ 0, & \text{otherwise}.\end{cases}$$

Then, the identity fuzzy soft mapping $\varphi_\psi:(W,\tau_N)\longrightarrow(W,\eta_N)$ is fuzzy soft almost α-continuous, but it is not fuzzy soft α-continuous.

Example 3.3. Let $W=\{w_1,w_2,w_3\}$, $N=\{n_1,n_2\}$ and define $g_N,h_N\in\widetilde{(W,N)}$ as follows: $g_N=\{(n_1,\{\frac{w_1}{0.5},\frac{w_2}{0.5},\frac{w_3}{0.5}\}),(n_2,\{\frac{w_1}{0.5},\frac{w_2}{0.5},\frac{w_3}{0.5}\})\}$, $h_N=\{(n_1,\{\frac{w_1}{0.3},\frac{w_2}{0},\frac{w_3}{0.5}\}),(n_2,\{\frac{w_1}{0.3},\frac{w_2}{0},\frac{w_3}{0.5}\})\}$. Define fuzzy soft topologies $\tau_N,\eta_N:N\longrightarrow[0,1]^{\widetilde{(W,N)}}$ as follows: $\forall n\in N$,

$$\tau_n(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{2}{3}, & \text{if} \quad t_N=g_N,\\ 0, & \text{otherwise},\end{cases}$$

$$\eta_n(t_N)=\begin{cases}1, & \text{if} \quad t_N\in\{\Phi,\widetilde{N}\},\\ \frac{1}{3}, & \text{if} \quad t_N=h_N,\\ 0, & \text{otherwise}.\end{cases}$$

Then, the identity fuzzy soft mapping $\varphi_\psi:(W,\tau_N)\longrightarrow(W,\eta_N)$ is fuzzy soft weakly α-continuous, but it is not fuzzy soft almost α-continuous.

Theorem 3.2. Let (W,τ_N) and (V,η_F) be a FSTSs, and $\varphi_\psi:\widetilde{(W,N)}\longrightarrow\widetilde{(V,F)}$ be a fuzzy soft mapping. The following statements are equivalent for each $f_A\in\widetilde{(V,F)}$, $n\in N$, $(k=\psi(n))\in F$ and $r\in I_\circ$:

(1) φ_ψ is fuzzy soft almost α-continuous.

(2) $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-open, for each f_A is r-fuzzy soft regularly open.

(3) $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-closed, for each f_A is r-fuzzy soft regularly closed.

(4) $\alpha C_\tau(n,\varphi_\psi^{-1}(f_A),r)\sqsubseteq\varphi_\psi^{-1}(C_\eta(k,f_A,r))$, for each f_A is r-fuzzy soft β-open.

(5) $\alpha C_\tau(n,\varphi_\psi^{-1}(f_A),r)\sqsubseteq\varphi_\psi^{-1}(C_\eta(k,f_A,r))$, for each f_A is r-fuzzy soft semi-open.

(6) $\alpha I_\tau(n,\varphi_\psi^{-1}(I_\eta(k,C_\eta(k,f_A,r),r)),r)\sqsupseteq\varphi_\psi^{-1}(f_A)$, for each f_A with $\eta_k(f_A)\geq r$.

Proof. (1) $\Rightarrow$ (2) Let $n_{w_s}\in\widetilde{P_s(W)}$ and $f_A\in\widetilde{(V,F)}$ be an r-fuzzy soft regularly open set containing $\varphi_\psi(n_{w_s})$, hence by (1), there is $h_C\in\widetilde{(W,N)}$ is r-fuzzy soft α-open set containing n_{w_s} such that $\varphi_\psi(h_C)\sqsubseteq I_\eta(k,C_\eta(k,f_A,r),r)$.

Thus, $h_C\sqsubseteq\varphi_\psi^{-1}(I_\eta(k,C_\eta(k,f_A,r),r))=\varphi_\psi^{-1}(f_A)$ and $n_{w_s}\tilde{\in}h_C\sqsubseteq\varphi_\psi^{-1}(f_A)$. Then, $n_{w_s}\tilde{\in}I_\tau(n,C_\tau(n,I_\tau(n,\varphi_\psi^{-1}(f_A),r),r),r)$ and $\varphi_\psi^{-1}(f_A)\sqsubseteq I_\tau(n,C_\tau(n,I_\tau(n,\varphi_\psi^{-1}(f_A),r),r),r)$. Therefore, $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-open set.

(2) $\Rightarrow$ (3) Let f_A be an r-fuzzy soft regularly closed set, hence by (2), $\varphi_\psi^{-1}(f_A^c)=(\varphi_\psi^{-1}(f_A))^c$ is r-fuzzy soft α-open set. Then, $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-closed set.

(3) $\Rightarrow$ (4) Let f_A be an r-fuzzy soft β-open set. Since $C_\eta(k,f_A,r)$ is r-fuzzy soft regularly closed set, hence by (3), $\varphi_\psi^{-1}(C_\eta(k,f_A,r))$ is r-fuzzy soft α-closed set. Since $\varphi_\psi^{-1}(f_A)\sqsubseteq\varphi_\psi^{-1}(C_\eta(k,f_A,r))$, then we have $\alpha C_\tau(n,\varphi_\psi^{-1}(f_A),r)\sqsubseteq\varphi_\psi^{-1}(C_\eta(k,f_A,r))$.

(4) $\Rightarrow$ (5) This is obvious from each r-fuzzy soft semi-open set is r-fuzzy soft β-open.

(5) $\Rightarrow$ (3) Let f_A be an r-fuzzy soft regularly closed set, hence f_A is r-fuzzy soft semi-open. Then by (5), $\alpha C_\tau(n, \varphi_\psi^{-1}(f_A), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r)) = \varphi_\psi^{-1}(f_A)$. Therefore, $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-closed set.

(3) $\Rightarrow$ (6) Let $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$ and $n_{w_s} \tilde{\in} \varphi_\psi^{-1}(f_A)$, then we have $n_{w_s} \tilde{\in} \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r))$. Since $[I_\eta(k, C_\eta(k, f_A, r), r)]^c$ is r-fuzzy soft regularly closed set, hence by (3), $\varphi_\psi^{-1}([I_\eta(k, C_\eta(k, f_A, r), r)]^c)$ is r-fuzzy soft α-closed set. Thus, $\varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r))$ is r-fuzzy soft α-open set and $n_{w_s} \tilde{\in} \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r)$. Then, $\varphi_\psi^{-1}(f_A) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r)$.

(6) $\Rightarrow$ (1) Let $n_{w_s} \in \widetilde{P_s(W)}$ and $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$ containing $\varphi_\psi(n_{w_s})$, hence by (6), $\varphi_\psi^{-1}(f_A) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r)$.

Since $n_{w_s} \tilde{\in} \varphi_\psi^{-1}(f_A)$, then we obtain $n_{w_s} \tilde{\in} \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r) = h_C$ (say). Hence, there is $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-open set containing n_{w_s} such that $\varphi_\psi(h_C) \sqsubseteq I_\eta(k, C_\eta(k, f_A, r), r)$. Therefore, φ_ψ is fuzzy soft almost α-continuous.

□

In a similar way, we can prove the following theorem.

Theorem 3.3. Let (W, τ_N) and (V, η_F) be a FSTSs, and $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ be a fuzzy soft mapping. The following statements are equivalent for each $f_A \in \widetilde{(V,F)}$, $n \in N$, $(k = \psi(n)) \in F$ and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft weakly α-continuous.

(2) $I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, f_A, r)), r), r), r) \sqsupseteq \varphi_\psi^{-1}(f_A)$, if $\eta_k(f_A) \geq r$.

(3) $C_\tau(n, I_\tau(n, C_\tau(n, \varphi_\psi^{-1}(I_\eta(k, f_A, r)), r), r), r) \sqsubseteq \varphi_\psi^{-1}(f_A)$, if $\eta_k(f_A^c) \geq r$.

(4) $\alpha C_\tau(n, \varphi_\psi^{-1}(I_\eta(k, f_A, r)), r) \sqsubseteq \varphi_\psi^{-1}(f_A)$, if $\eta_k(f_A^c) \geq r$.

(5) $\alpha C_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$.

(6) $\alpha I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, I_\eta(k, f_A, r), r)), r) \sqsupseteq \varphi_\psi^{-1}(I_\eta(k, f_A, r))$.

(7) $\varphi_\psi^{-1}(f_A) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, f_A, r)), r)$, if $\eta_k(f_A) \geq r$.

Remark 3.3. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft continuity as in the next diagram.

continuity	$\Rightarrow$	α-continuity
$\Downarrow$		$\Downarrow$
almost continuity	$\Rightarrow$	almost α-continuity
$\Downarrow$		$\Downarrow$
weakly continuity	$\Rightarrow$	weakly α-continuity

Proposition 3.1. Let (W, τ_N), (V, η_F) and (U, γ_E) be a FSTSs, and $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$, $\varphi^*_{\psi^*} : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ be two fuzzy soft functions. Then, the composition $\varphi^*_{\psi^*} \circ \varphi_\psi$ is fuzzy soft almost α-continuous if, φ_ψ is fuzzy soft α-continuous and $\varphi^*_{\psi^*}$ is fuzzy soft almost continuous (resp. continuous).

Proof. The proof is obvious. □

Let $\mathcal{H}$ and $\mathcal{I} : N \times \widetilde{(W,N)} \times I_\circ \to \widetilde{(W,N)}$ be operators on $\widetilde{(W,N)}$, and $\mathcal{J}$ and $\mathcal{K} : F \times \widetilde{(V,F)} \times I_\circ \to \widetilde{(V,F)}$ be operators on $\widetilde{(V,F)}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Definition 3.3. [26] Let (W, τ_N) and (V, η_F) be a FSTSs. $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is said to be a fuzzy soft $(\mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K})$-continuous mapping if, $\mathcal{H}[n, \varphi_\psi^{-1}(\mathcal{K}(k, h_C, r)), r] \sqcap \mathcal{I}[n, \varphi_\psi^{-1}(\mathcal{J}(k, h_C, r)), r] = \Phi$ for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$, $n \in N$ and $(k = \psi(n)) \in F$.

In (2023), Alshammari et al. [26] defined the notion of fuzzy soft α-continuous mappings: $\varphi_\psi^{-1}(h_C) \sqsubseteq I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(h_C), r), r), r)$, for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$. We can see that Definition 3.3 generalizes the concept of fuzzy soft continuous functions, when we choose $\mathcal{H}$ = identity operator, $\mathcal{I}$ = interior closure interior operator, $\mathcal{J}$ = identity operator and $\mathcal{K}$ = identity operator.

A historical justification of Definition 3.3:

(1) In Section 3, we obtained the notion of fuzzy soft almost α-continuous mappings: $\varphi_\psi^{-1}(h_C) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, h_C, r), r)), r)$, for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$. Here, $\mathcal{H}$ = identity operator, $\mathcal{I}$ = α-interior operator, $\mathcal{J}$ = interior closure operator and $\mathcal{K}$ = identity operator.

(2) In Section 3, we obtained the notion of fuzzy soft weakly α-continuous mappings: $\varphi_\psi^{-1}(h_C) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, h_C, r)), r)$, for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$. Here, $\mathcal{H}$ = identity operator, $\mathcal{I}$ = α-interior operator, $\mathcal{J}$ = closure operator and $\mathcal{K}$ = identity operator.

4. New Types of Fuzzy Soft Compactness

Here, several types of fuzzy soft compactness via r-fuzzy soft α-open sets are given, and the relationships between them are studied.

Definition 4.1. Let (W, τ_N) be a FSTS and $r \in I_\circ$. Then, $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid \tau_n((g_B)_\delta) \geq r$ for each $n \in N\}_{\delta \in \Delta}$ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta_\circ}(g_B)_\delta$.

Definition 4.2. Let (W, τ_N) be a FSTS and $r \in I_\circ$. Then, $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft α-compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid (g_B)_\delta$ is r-fuzzy soft α-open$\}_{\delta \in \Delta}$ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta_\circ}(g_B)_\delta$.

Lemma 4.1. Let (W, τ_N) be a FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-compact, then h_C is r-fuzzy soft compact.

Proof. Follows from Definitions 4.1 and 4.2.
□

Theorem 4.1. Let $\varphi_\psi : (W, \tau_N) \longrightarrow (V, \eta_F)$ be a fuzzy soft α-continuous mapping. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-compact, then $\varphi_\psi(h_C)$ is r-fuzzy soft compact.

Proof. Let $\{(g_B)_\delta \in \widetilde{(V,F)} \mid \eta_k((g_B)_\delta) \geq r\}_{\delta \in \Delta}$ with $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta \in \Delta}(g_B)_\delta$ for each $k \in F$. Then, $\{\varphi_\psi^{-1}((g_B)_\delta) \in \widetilde{(W,N)} \mid \varphi_\psi^{-1}((g_B)_\delta)$ is r-fuzzy soft α-open$\}_{\delta \in \Delta}$ (by φ_ψ is fuzzy soft α-continuous) such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta}\varphi_\psi^{-1}((g_B)_\delta)$. Since h_C is r-fuzzy soft α-compact, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta_\circ}\varphi_\psi^{-1}((g_B)_\delta)$. Then, $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta \in \Delta_\circ}(g_B)_\delta$. Hence, the proof is completed. □

Definition 4.3. Let (W, τ_N) be a FSTS and $r \in I_\circ$. Then, $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft almost compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid \tau_n((g_B)_\delta) \geq r\}_{\delta \in \Delta}$ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta_\circ}C_\tau(n, (g_B)_\delta, r)$ for each $n \in N$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Definition 4.4. Let (W, τ_N) be a FSTS and $r \in I_\circ$. Then, $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft almost α-compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid (g_B)_\delta$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ} C_\tau(n, (g_B)_\delta, r)$ for each $n \in N$.

Lemma 4.2. Let (W, τ_N) be a FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft almost α-compact, then h_C is r-fuzzy soft almost compact.

Proof. Follows from Definitions 4.3 and 4.4.
□

Lemma 4.3. Let (W, τ_N) be a FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-compact (resp. compact), then h_C is r-fuzzy soft almost α-compact (resp. almost compact).

Proof. Follows from Definitions 4.1–4.4.
□

Theorem 4.2. Let $\varphi_\psi : (W, \tau_N) \longrightarrow (V, \eta_F)$ be a fuzzy soft continuous mapping. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft almost α-compact, then $\varphi_\psi(h_C)$ is r-fuzzy soft almost compact.

Proof. Let $\{(g_B)_\delta \in \widetilde{(V,F)} \mid \eta_k((g_B)_\delta) \geq r\}_{\delta\in\Delta}$ with $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$ for each $k \in F$. Then, $\{\varphi_\psi^{-1}((g_B)_\delta) \in \widetilde{(W,N)} \mid \varphi_\psi^{-1}((g_B)_\delta)$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$ (by φ_ψ is fuzzy soft α-continuous) such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}\varphi_\psi^{-1}((g_B)_\delta)$. Since h_C is r-fuzzy soft almost α-compact, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ} C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r)$. Since φ_ψ is fuzzy soft continuous mapping, it follows

$$\sqcup_{\delta\in\Delta_\circ} C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r) \sqsubseteq$$

$$\sqcup_{\delta\in\Delta_\circ} \varphi_\psi^{-1}(C_\eta(k, (g_B)_\delta, r)) =$$

$$\varphi_\psi^{-1}(\sqcup_{\delta\in\Delta_\circ} C_\eta(k, (g_B)_\delta, r)).$$

Then, $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta_\circ} C_\eta(k, (g_B)_\delta, r)$. Hence, the proof is completed. □

Definition 4.5. Let (W, τ_N) be a FSTS and $r \in I_\circ$. Then, $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft nearly compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid \tau_n((g_B)_\delta) \geq r\}_{\delta\in\Delta}$ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ} I_\tau(n, C_\tau(n, (g_B)_\delta, r), r)$ for each $n \in N$.

Definition 4.6. Let (W, τ_N) be a FSTS and $r \in I_\circ$. Then, $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft nearly α-compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid (g_B)_\delta$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ} I_\tau(n, C_\tau(n, (g_B)_\delta, r), r)$ for each $n \in N$.

Lemma 4.4. Let (W, τ_N) be a FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft nearly α-compact, then h_C is r-fuzzy soft nearly compact.

Proof. Follows from Definitions 4.5 and 4.6.
□

Lemma 4.5. Let (W, τ_N) be a FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-compact (resp. compact), then h_C is r-fuzzy soft nearly α-compact (resp. nearly compact).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 February 2024 doi:10.20944/preprints202402.0734.v1

Proof. Follows from Definitions 4.1, 4.2, 4.5 and 4.6.
□

Theorem 4.3. Let $\varphi_\psi : (W, \tau_N) \longrightarrow (V, \eta_F)$ be a fuzzy soft continuous and fuzzy soft open mapping. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft nearly α-compact, then $\varphi_\psi(h_C)$ is r-fuzzy soft nearly compact.

Proof. Let $\{(g_B)_\delta \in \widetilde{(V,F)} \mid \eta_k((g_B)_\delta) \geq r\}_{\delta\in\Delta}$ with $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$ for each $k \in F$. Then, $\{\varphi_\psi^{-1}((g_B)_\delta) \in \widetilde{(W,N)} \mid \varphi_\psi^{-1}((g_B)_\delta)$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$ (by φ_ψ is fuzzy soft α-continuous) such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}\varphi_\psi^{-1}((g_B)_\delta)$. Since h_C is r-fuzzy soft nearly α-compact, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ} I_\tau(n, C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r), r)$. Since φ_ψ is fuzzy soft continuous and fuzzy soft open mapping, it follows

$$\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta_\circ}\varphi_\psi(I_\tau(n, C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r), r))$$

$$\sqsubseteq \sqcup_{\delta\in\Delta_\circ} I_\eta(k, \varphi_\psi(C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r)), r)$$

$$\sqsubseteq \sqcup_{\delta\in\Delta_\circ} I_\eta(k, \varphi_\psi(\varphi_\psi^{-1}(C_\eta(k, (g_B)_\delta, r))), r)$$

$$\sqsubseteq \sqcup_{\delta\in\Delta_\circ} I_\eta(k, C_\eta(k, (g_B)_\delta, r), r).$$

Hence, the proof is completed. □

Lemma 4.6. Let (W, τ_N) be a FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft nearly α-compact (resp. nearly compact), then h_C is r-fuzzy soft almost α-compact (resp. almost compact).

Proof. Follows from Definitions 4.3, 4.4, 4.5 and 4.6.
□

Remark 4.1. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft compactness as in the next diagram.

$$\begin{array}{ccc}
\alpha\text{-compactness} & \Rightarrow & \text{compactness} \\
\Downarrow & & \Downarrow \\
\text{nearly } \alpha\text{-compactness} & \Rightarrow & \text{nearly compactness} \\
\Downarrow & & \Downarrow \\
\text{almost } \alpha\text{-compactness} & \Rightarrow & \text{almost compactness}
\end{array}$$

5. Conclusions and Future Work

The main achievements of this study are:

(**1**) In Section 2, the concepts of fuzzy soft α-closure (α-interior) operators are introduced in fuzzy soft topological spaces based on the paper Aygünoğlu et al. [14], and some of their basic properties have been investigated. Furthermore, the notion of r-fuzzy soft α-connected sets is defined and studied with help of fuzzy soft α-closure operators.

(**2**) In Section 3, some properties of fuzzy soft α-continuous mappings are obtained between two fuzzy soft topological spaces (W, τ_N) and (V, η_F). Moreover, as a weaker forms of the notion of fuzzy soft α-continuous mappings, the notions of fuzzy soft almost (weakly) α-continuous mappings are introduced, and some of their basic properties and characterizations have been investigated. Also, we show that fuzzy soft α-continuity $\Rightarrow$ fuzzy soft almost α-continuity $\Rightarrow$ fuzzy soft weakly α-continuity, and we have the following results:

 doi:10.20944/preprints202402.0734.v1

- Fuzzy soft $(id_W, I_\tau(C_\tau(I_\tau)), id_V, id_V)$-continuous mapping is fuzzy soft α-continuous.
- Fuzzy soft $(id_W, \alpha I_\tau, I_\eta(C_\eta), id_V)$-continuous mapping is fuzzy soft almost α-continuous.
- Fuzzy soft $(id_W, \alpha I_\tau, C_\eta, id_V)$-continuous mapping is fuzzy soft weakly α-continuous.

(**3**) In Section 4, several types of soft compactness via r-fuzzy soft α-open sets are explored, and the relationships between them are studied.

In upcoming manuscripts, we will use the fuzzy soft α-closure operator to define some new separation axioms on fuzzy soft topological space based on the paper Aygünoğlu et al. [14]. Additionally, we shall discuss some of the notions given here in the frames of fuzzy soft r-minimal structures [20].

Author Contributions: Methodology, Islam M. Taha; Formal Analysis, Islam M. Taha and Wafa Alqurashi; Investigation, Islam M. Taha and Wafa Alqurashi; Writing Original Draft Preparation, Islam M. Taha and Wafa Alqurashi; Writing Review and Editing, Islam M. Taha and Wafa Alqurashi; Funding Acquisition, Islam M. Taha and Wafa Alqurashi.

Data Availability Statement: No data were used to support this study.

Acknowledgments: In this section you can acknowledge any support given which is not covered by the author contribution or funding sections. This may include administrative and technical support, or donations in kind (e.g., materials used for experiments).

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Use of AI tools declaration: The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

References

1. Molodtsov, D. Soft set theory-first results. *Comput. Math. Appl.* 1999, 37, 19–31.
2. Akdag, M.; Ozkan, A. Soft α-open sets and soft α-continuous functions. *Abst. Appl. Anal.* 2014, 2014, 891341.
3. Akdag, M.; Ozkan, A. On soft β-open sets and soft β-continuous functions. *Sci. World J.* 2014, 2014, 843456.
4. El-Sheikh, S.A.; Hosny, R.A.; Abd El-latif, A.M. Characterizations of β-soft separation axioms in soft topological spaces. *Inf. Sci. Lett.* 2015, 4, 125–133.
5. Al-shami, T.M. Soft somewhere dense sets on soft topological spaces. *Commun. Korean Math. Soc.* 2018, 33, 1341–1356.
6. Al-Ghour, S. Boolean algebra of soft Q-Sets in soft topological spaces. *Appl. Comput. Intell. Soft Comput.* 2022, 2022, 5200590.
7. Al-shami, T.M.; Mhemdi, A.; Abu-Gdairid, R. A Novel framework for generalizations of soft open sets and its applications via soft topologies. *Mathematics* 2023, 11, 840.
8. Al-shami, T.M.; Arar, M.; Abu-Gdairi, R.; Ameen, Z.A. On weakly soft β-open sets and weakly soft β-continuity. *J. Inte. Fuzzy Syst.* 2023, 45, 6351–6363.
9. Kaur, S.; Al-shami, T.M.; Ozkan, A.; Hosny, M. A new approach to soft continuity. *Mathematics*, 11, 3164, 2023.
10. Al-Ghour S.; Al-Mufarrij, J. Between soft complete continuity and soft somewhat-continuity. *Symmetry* 2023, 15, 1–14.
11. Ameen, Z.A.; Abu-Gdairi, R.; Al-shami, T.M.; Asaad, B.A.; Arar, M. Further properties of soft somewhere dense continuous functions and soft Baire spaces. *J. Math. Computer Sci.* 2024, 32, 54–63.
12. Maji, P.K.; Biswas, R.; Roy, A.R. Fuzzy soft sets. *J. Fuzzy Math.* 2001, 9, 589–602.
13. Zadeh, L.A. Fuzzy Sets. *Inform. Control* 1965, 8, 338–353.
14. Aygünoğlu, A.; Çetkin, V.; Aygün, H. An introduction to fuzzy soft topological spaces. *Hacet. J. Math. Stat.* 2014, 43, 193–208.
15. Çetkin, V.; Aygünoğlu, A.; Aygün, H. On soft fuzzy closure and interior operators. *Util. Math.* 2016, 99, 341–367.
16. Šostak, A.P. On a fuzzy topological structure. *In: Proceedings of the 13th winter school on abstract analysis, Section of topology, Palermo: Circolo Matematico di Palermo* 1985, 89–103.
17. Taha, I.M. A new approach to separation and regularity axioms via fuzzy soft sets. *Ann. Fuzzy Math. Inform.* 2020, 20, 115–123.

doi:10.20944/preprints202402.0734.v1

18. Taha, I.M. Some new separation axioms in fuzzy soft topological spaces. *Filomat* 2021, 35, 1775–1783.
19. Çetkin, V.; Aygün, H. Fuzzy soft semiregularization spaces. *Ann. Fuzzy Math. Inform.* 2014, 7, 687–697.
20. Taha, I.M. Compactness on fuzzy soft r-minimal spaces. *Int. J. Fuzzy Logic Intell. Syst.* 2021, 21, 251–258.
21. Ahmad, B.; Kharal, A. On fuzzy soft sets. *Adv. Fuzzy Syst.* 2009, 2009, 586507.
22. Çağman, N.; Enginoğlu, S.; Çitak, F. Fuzzy soft set theory and its applications. *Iran. J. Fuzzy Syst.* 2011, 8, 137–147.
23. Taha, I.M. Some new results on fuzzy soft r-minimal spaces. *AIMS Math.* 2022, 7, 12458–12470.
24. Mishra, S.; Srivastava, R. Hausdorff fuzzy soft topological spaces. *Ann. Fuzzy Math. Inform.* 2015, 9, 247–260.
25. Atmaca, S.; Zorlutuna, I. On fuzzy soft topological spaces. *Ann. Fuzzy Math. Inform.* 2013, 5, 377–386.
26. Alshammari, I.; Alqahtani, M.H.; Taha, I.M. On r-fuzzy soft δ-open sets and applications via fuzzy soft topologies. *Preprints* 2023, 2023121240.