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Article

Passive Aggressive Ensemble for Online Portfolio Selection

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Abstract: Developing effective trend estimators is a main method to solve the online portfolio selection problem. Although the existing portfolio strategies have demonstrated good performance through the development of various trend estimators, it is still challenging to determine in advance which estimator will yield the maximum final cumulative wealth in online portfolio selection tasks. This paper studies an online ensemble approach for online portfolio selection by leveraging the strengths of multiple trend estimators. Specifically, a return-based loss function and a cross-entropy-based loss function are first designed to evaluate the adaptiveness of different trend estimators in a financial environment. On this basis, a passive aggressive ensemble model is proposed to weigh these trend estimators within a unit simplex according to their adaptiveness. Extensive experiments are conducted on benchmark datasets from various real-world stock markets to evaluate the performance. The results show that the proposed strategy achieves the state-of-the-art performance including efficiency and cumulative return.

Keywords: Online Portfolio Selection; Online Ensemble Learning; Passive Aggressive Algorithm

1. Introduction

Online portfolio selection dynamically allocates wealth across real-world assets by analyzing stream data from financial markets [1,2]. To optimize the wealth allocation, portfolio selection leverages a range of machine learning techniques, including Newton gradient step [3] and nearest neighbor [4]. In addition to these methods, portfolio selection must also consider economic and financial theories and criteria, particularly investment behaviors [5,6]. Accordingly, many of the current leading portfolio selection systems employ historical price information to predict future prices to determine the optimal portfolio allocation.

Reversion and momentum are two significant characteristics in finance. Numerous the state-of-the-art portfolio selection strategies develop trend estimators based on these characteristics for decision-making. For example, trend estimators like the inverse price [7,8], the simple moving average [15], and the exponential moving average [9] assume that the asset price will reverse to some kind of its historical average. On the contrary, the peak price [10,11] assumes that the price of a good performing asset will continue to rise due to the irrational investing behaviors. Since each trend estimator is based on different prior assumptions, their effectiveness is contingent on the financial environment aligning with these assumptions. Consequently, it is challenging to predetermine which trend estimator will maximize the final cumulative wealth for online portfolio selection tasks.

To take the advantages of various trend estimators during the whole investment period, we pay attention to the online ensemble learning framework [12]. The online learning adeptly balances each trend estimator with evolving financial environments, while the ensemble learning combines these estimators to enhance robustness and precision. A simple but effective ensemble learning approach

is weighted averaging [13]. Consequently, it is a viable solution to develop a suitable online ensemble learning algorithm to amalgamate various trend estimators emerges.

In this paper, we address the problem of online portfolio selection with multiple trend estimators by developing a framework named Passive Aggressive Ensemble (PAE). The characteristics and contributions of the proposed PAE are summarized as follows:

1. The PAE framework is introduced to employ two distinct schemes to efficiently ensemble four different types of trend estimators.
2. The PAE framework augments the performance of original trend estimators through reasonable evaluation and weighting.
3. The hyperparameters are tuned based on the oldest dataset to avoid overfitting. Extensive experiments are conducted on six real-world datasets to demonstrate our algorithm not only outperforms competing strategies in terms of multiple evaluation criteria, but also has promising scalability of transaction costs.

The rest of the paper is organized as follows. Section 2 formally states portfolio selection in machine learning. Section 3 reviews some related work regarding trend estimator and ensemble learning in the context of online portfolio selection. Section 4 presents our PAE framework for portfolio selection. Section 5 presents experiments and results of PAE on real-world benchmark datasets. Finally, conclusions are made in Section 6.

2. Problem Setting

In this paper, we use a standard and universal setting of portfolio selection in machine learning [7–11]. Consider a financial market with m assets for n periods. At the end of the t^{th} period, a non-negative m -dimensional vector

$$\mathbf{p}_t \in \mathbb{R}_+^m, \quad t = 0, 1, \dots, n, \quad (1)$$

represents the close price of assets. A relative price vector is introduced to see the change of asset prices as

$$\mathbf{x}_t \triangleq \frac{\mathbf{p}_t}{\mathbf{p}_{t-1}}, \quad \mathbf{x}_t \in \mathbb{R}_+^m, \quad t = 1, 2, \dots, n, \quad (2)$$

where a division between two vectors represents element-wise division in this paper.

At the beginning of the t^{th} period, an investment in the market is specified by a portfolio vector in m dimensional unit simplex

$$\mathbf{b}_t \in \Delta_m := \left\{ \mathbf{b} \in \mathbb{R}_+^m : \sum_{i=1}^m \mathbf{b}^{(i)} = 1 \right\}, \quad (3)$$

where $\mathbf{b}_t^{(i)}$ denotes the proportion of total wealth invested in the i^{th} asset. The non-negative constrain means no short is allowed and the equality constraint means that the portfolio is self-financed.

For the t^{th} trading day, a portfolio \mathbf{b}_t generated by the portfolio selection algorithm results in a daily return $\mathbf{b}_t^\top \mathbf{x}_t$. Thus, the cumulative wealth can be calculated as $S_t = S_{t-1} \cdot (\mathbf{b}_t^\top \mathbf{x}_t)$, and the final cumulative wealth with a common initial wealth $S_0 = 1$ is given by

$$S_n = S_0 \prod_{t=1}^n (\mathbf{b}_t^\top \mathbf{x}_t) = \prod_{t=1}^n (\mathbf{b}_t^\top \mathbf{x}_t). \quad (4)$$

Finally, a portfolio learning algorithm sequential learns a set of portfolio vectors $\{\mathbf{b}_t\}_{t=1}^n$ to maximize the final cumulative wealth and satisfy some risk management metrics.

3. Related Works

In this section, we mainly review the related work of trend estimators and ensemble learning in online portfolio selection.

3.1. Trend Estimator

Following-the-winner strategies assume that the price of a good performing asset will continue to rise. Based on this assumption, researchers have designed different trend estimators. Exponential gradient [14] and its improved variants [15] estimate the asset price as it will continue to change as last day. The peak price tracking system [10] and short-term portfolio optimization with loss control [16] adopt the peak price (PP) as trend estimator. PP assumes the price continues moving in their max price of a recent time window but fails to capture continuous depreciation of asset price. Recently, there are many works to handle the shortcomings of peak price, *e.g.*, the trend peak price tracing [11] and the peak price tracking approach [17].

Following-the-loser strategies are opposite to Following-the-winner strategies, and they assume that the asset price will reverse to some kind of its historical average. The inverse price (IP) expects the asset price will reverse to the last day. Passive aggressive mean reversion [7] and confidence weighted mean reversion [8] adopt the IP as trend estimator but suffer from significant performance degradation if the underlying short-term reversion fails to exist. The simple moving average (SMA) and exponential moving average (EMA) [9] improve the estimation of IP. The SMA utilizes the average of the asset prices within a specified time window to estimate the future price movement. It assigns equal weight to each asset price in the specified period, effectively smoothing out any extreme prices within the time frame to derive the most recent price information from the financial market. The EMA incorporates all historical asset prices but assigns higher significance to recent prices in predicting future trends, making it generally more reflective of the actual price trends in comparison to the SMA. Recently, Gaussian weighting reversion [18] uses Gaussian weighting function to enhance the time validity. Vector autoregressive weighting reversion [19] and weighted multivariate mean reversion [20] use autoregressive moving average and multi-variate robust mean respectively to estimate the mean more precisely. The state-dependent exponential moving average [21] predicts asset returns by additionally analyzing market states.

3.2. Ensemble Learning

In recent years, the ensemble learning algorithm has been promoted more widely in improving its generalization ability and application scope. The online portfolio selection has garnered significant interest from both industry professionals and academic researchers. In this context, Guo et al. [22] introduced a novel approach termed adaptive online moving average. This technique enhances the prediction of future returns of volatile assets by integrating an adaptive decaying factor with the traditional moving average approach. Furthermore, it incorporates the concept of ensemble learning through the inclusion of a net profit maximization model. It significantly enhances the precision of return forecasts. Kumar and Segev [23] introduced a Bayesian ensembled mean-median reversion to adeptly integrate the advantages of mean and median reversion techniques. It employs Bayesian methods to amalgamate these two strategies, improving future portfolio decisions by learning from past prediction errors. Hong et al. [11] introduced a trend peak price tracing strategy in online portfolio selection to address market unpredictability by combining trend analysis and ensemble learning. It employs a dynamic price forecasting method, using linear combinations of slopes to determine stock price trends. The trend peak price tracing's three-state forecasting adapts to market conditions by incorporating peak values, exponential moving averages, or maintaining current values based on trend analysis.

4. Methodology

To take the advantages of different trends and adapt to different financial environments, we propose a passive aggressive ensemble (PAE) framework and two strategies: PAE-R and PAE-C in this section. We take four traditional trend estimators as the input of PAE:

$$\text{SMA: } \hat{\mathbf{x}}_{S,t+1}(w) = \frac{\sum_{k=0}^{w-1} \mathbf{p}_{t-k}}{w\mathbf{p}_t}, \quad (5)$$

$$\text{EMA: } \hat{\mathbf{x}}_{E,t+1}(\vartheta) = \vartheta \mathbf{1} + (1 - \vartheta) \frac{\hat{\mathbf{x}}_{E,t}}{\mathbf{x}_t}, \quad (6)$$

$$\text{IP: } \hat{\mathbf{x}}_{I,t+1} = \frac{1}{\mathbf{x}_t} = \frac{\mathbf{p}_{t-1}}{\mathbf{p}_t}, \quad (7)$$

$$\text{PP: } \hat{\mathbf{x}}_{M,t+1} = \frac{\hat{\mathbf{p}}_{t+1}}{\mathbf{p}_t}, \quad \hat{\mathbf{p}}_{t+1}^{(i)} = \max_{0 \leq k \leq w-1} \mathbf{p}_{t-k}^{(i)}, \quad (8)$$

where w is the time window size, $\vartheta \in (0,1)$ denotes the decaying factor, $\mathbf{1}$ is a m -dimensional vector whose elements are 1, and $i = 1, 2, \dots, m$ is the order of asset.

4.1. Passive Aggressive Ensemble

The construction of trend composite starts with a trend back-test step. Let $\{\hat{\mathbf{x}}_{l,t+1}\}_{l=1}^L$ denote a set of trends—e.g., $\hat{\mathbf{x}}_{1,t+1} = \hat{\mathbf{x}}_{S,t+1}$, $\hat{\mathbf{x}}_{2,t+1} = \hat{\mathbf{x}}_{E,t+1}$ and $\hat{\mathbf{x}}_{3,t+1} = \hat{\mathbf{x}}_{I,t+1}$, where L is the total number of trends. We project $\{\hat{\mathbf{x}}_{l,t+1}\}_{l=1}^L$ onto a unit simplex [24] to get feasible portfolios

$$\tilde{\mathbf{x}}_{l,t+1} = \underset{\mathbf{x} \in \Delta_m}{\operatorname{argmin}} \|\mathbf{x} - \hat{\mathbf{x}}_{l,t+1}\|^2, \quad l = 1, \dots, L. \quad (9)$$

Then we are able to measure the return-based performance [25] of each trend estimator in a certain time period

$$r_{l,t} = \tilde{\mathbf{x}}_{l,t}^\top \mathbf{x}_t, \quad \mathbf{r}_t = [r_{1,t}, r_{2,t}, \dots, r_{L,t}]^\top, \quad (10)$$

where $r_{l,t}$ is the back-tested return of l^{th} trend in t^{th} period, \mathbf{x}_t is the real relative price factor given by (2), and $\mathbf{r}_t \in \mathbb{R}^L$ is the collection of L different back-tested returns.

Drawing upon from the cross-entropy [26], we also propose an indicator to evaluate each trend estimator called cross-entropy-based performance

$$c_{l,t} = -\tilde{\mathbf{x}}_t^\top \log \tilde{\mathbf{x}}_{l,t}, \quad \mathbf{c}_t = [c_{1,t}, c_{2,t}, \dots, c_{L,t}]^\top, \quad (11)$$

where $\tilde{\mathbf{x}}_t$ is the projection of real relative price vector by (9). The cross-entropy allows us to measure the gap between the trend estimator and the true price movement.

Two ensemble targets are defined to meet the goal of our approach. We expect that our ensembled trend estimator is no worse than the average performance of any single trend estimator over the investment period, which is consistent with investing intuition. For the cross-entropy-based performance, smaller values indicate a better trend estimator, and the opposite is true for returns-based performance. Our ensemble targets are defined as follow:

$$r_{*,t} \triangleq \underset{1 \leq l \leq L}{\operatorname{argmax}} \frac{1}{w} \sum_{k=0}^{w-1} r_{l,t-k}, \quad (12)$$

$$c_{*,t} \triangleq \underset{1 \leq l \leq L}{\operatorname{argmin}} \frac{1}{w} \sum_{k=0}^{w-1} c_{l,t-k}, \quad (13)$$

where w is the window size in the most recent time. We average the back-tested performance of each trend estimator in the most recent time window and select the one with the best average performance.

Combining the passive aggressive principle [28] and online ensemble framework [13], a weighting vector $\mathbf{w}_t \in \Delta_L$ for weighted averaging is first defined. Then, we propose two new loss functions to combine all the trends and consider their performance for our PAE-R and PAE-C strategies, respectively:

$$\text{PAE-R: } \ell_{\xi_r}(\mathbf{w}; (\mathbf{r}_t, r_{*,t})) = \begin{cases} 0 & r_{*,t} - \mathbf{w}^\top \mathbf{r}_t \leq \xi_r, \\ r_{*,t} - \mathbf{w}^\top \mathbf{r}_t - \xi_r & \text{otherwise} \end{cases}, \quad (14)$$

$$\text{PAE-C: } \ell_{\xi_c}(\mathbf{w}; (\mathbf{c}_t, c_{*,t})) = \begin{cases} 0 & \mathbf{w}^\top \mathbf{c}_t - c_{*,t} \leq \xi_c, \\ \mathbf{w}^\top \mathbf{c}_t - c_{*,t} - \xi_c & \text{otherwise} \end{cases}, \quad (15)$$

where $r_{*,t}$ and $c_{*,t}$ are the performance of ensemble targets, ξ_r is the tolerance threshold between the return-based performance of ensemble target and weighted averaging trend estimators, ξ_c is the tolerance threshold between the cross-entropy-based performance of ensemble target and weighted averaging trend estimators.

We update the weighting vector at the end of each trading period through the following optimization problem

$$\mathbf{w}_{t+1} = \underset{\mathbf{w} \in \Delta_L}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{w} - \mathbf{w}_t\|^2 \quad \text{s.t.} \quad \begin{cases} \ell_{\xi_r}(\mathbf{w}; (\mathbf{r}_t, r_{*,t})) = 0 & \text{PAE-R} \\ \ell_{\xi_c}(\mathbf{w}; (\mathbf{c}_t, c_{*,t})) = 0 & \text{PAE-C} \end{cases}. \quad (16)$$

On one hand, if our weighted averaging trend estimators outperform or close to the ensemble target, our approach passively maintains the weighting vector \mathbf{w} in the next trading period. On the other hand, when the weighted averaging trend estimators perform much worse than the ensemble target, our approach aggressively adjusts \mathbf{w} to seek a better weight. The problem will be solved in Section 4.3.

4.2. Online Portfolio Selection with Multiple Trend Estimators

The next step is to decide the proportion of the wealth being allocated to each asset of the stock based on the weighting and trend estimators received from the previous step. First proposed by Li et al. [7,9,27], the portfolio selection through passive aggressive algorithms is widely used today [18,20,23], especially in single trend estimator situation. We extend the application of this algorithm to the case of multi-trend estimators. Our basic idea to form the final portfolio is to maximize the ensemble trend estimator and keep last portfolio information through a regularization term. The final portfolio optimization problem can be expressed as:

$$\mathbf{b}_{t+1} = \underset{\mathbf{b} \in \Delta_m}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{b} - \mathbf{b}_t\|^2 \quad \text{s.t.} \quad \mathbf{b}^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1} \geq \epsilon, \hat{\mathbf{X}}_{t+1} = [\hat{\mathbf{x}}_{1,t+1}, \dots, \hat{\mathbf{x}}_{L,t+1}], \quad (17)$$

where $\hat{\mathbf{X}}_{t+1} \in \mathbb{R}^{m \times L}$ is the collection of L different trends, $\mathbf{w}_{t+1} \in \Delta_L$ is the weighting vector, ϵ is the reversion threshold, and $\hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}$ is the ensemble trend estimator.

If the expected return $\mathbf{b}^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}$ is greater than the threshold, we passively maintain the original investment portfolio vector. If the expected return is less than the threshold, we aggressively adjust the investment portfolio vector to maximize our returns. $\hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}$ is capable of combining various trends via the weighting vector \mathbf{w}_{t+1} as the output of (16). We will solve this problem in subsection 4.3.

4.3. Solving Algorithm

In this subsection, we will derive the solutions for the above optimization problem and summarize the whole procedure for PAE algorithms in Algorithm 1.

Theorem 1. Without considering the non-negativity constraint, the solution of PAE-R type optimization problem (16) is

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \tau_{1,t}(\mathbf{r}_t - \bar{r}_t \mathbf{1}), \quad (18)$$

where $\bar{r}_t = \frac{\mathbf{1}^\top \mathbf{r}_t}{L}$ is the mean value of back-tested return-based performance and $\tau_{1,t}$ is computed as:

$$\tau_{1,t} = \max \left\{ 0, \frac{r_{*,t} - \mathbf{w}_t^\top \mathbf{r}_t - \xi_r}{\|\mathbf{r}_t - \bar{r}_t \mathbf{1}\|^2} \right\}. \quad (19)$$

The proof of this theorem is in Appendix A.

Theorem 2. Without considering the non-negativity constraint, the solution of PAE-C type optimization problem (16) is

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \tau_{1,t}(\bar{c}_t \mathbf{1} - \mathbf{c}_t), \quad (20)$$

where $\bar{c}_t = \frac{\mathbf{1}^\top \mathbf{c}_t}{L}$ is the mean value of back-tested cross-entropy-based performance and $\tau_{1,t}$ is computed as:

$$\tau_{1,t} = \max \left\{ 0, \frac{\mathbf{w}_t^\top \mathbf{c}_t - c_{*,t} - \xi_c}{\|\bar{c}_t \mathbf{1} - \mathbf{c}_t\|^2} \right\}. \quad (21)$$

Because the proof of theorem 2 is almost the same as theorem 1, we omit the proof.

No matter which type of loss function is considered, \mathbf{w}_{t+1} should be further projected onto the L -dimensional unit simplex [24] to satisfy the non-negativity constraint of weighted averaging

$$\mathbf{w}_{t+1} = \underset{\mathbf{w} \in \Delta_L}{\operatorname{argmin}} \|\mathbf{w} - \mathbf{w}_{t+1}\|^2. \quad (22)$$

Theorem 3. Without considering the non-negativity constraint, the solution of problem (17) is

$$\mathbf{b}_{t+1} = \mathbf{b}_t + \tau_{2,t}(\mathbf{b}_t - \mu \mathbf{1}_{(m)}), \quad (23)$$

where $\mathbf{1}_{(m)}$ denotes the m -dimensional $\mathbf{1}$ vector to distinguish with L -dimensional $\mathbf{1}$ in (19) and (21), and $\mu = \frac{1}{m} \mathbf{1}_{(m)}^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}$ is the average value of trend composite. Then $\tau_{2,t}$ is calculated as:

$$\tau_{2,t} = \max \left\{ 0, \frac{\epsilon - \mathbf{b}_t^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}}{\|\hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1} - \mu \mathbf{1}_{(m)}\|^2} \right\}. \quad (24)$$

The proof of theorem 3 is in Appendix B.

\mathbf{b}_{t+1} is further projected onto a m -dimensional unit simplex [24] to satisfy the non-negativity constraint

$$\mathbf{b}_{t+1} = \underset{\mathbf{b} \in \Delta_m}{\operatorname{argmin}} \|\mathbf{b} - \mathbf{b}_{t+1}\|^2. \quad (25)$$

Algorithm 1. The general computing process of PAE-R and PAE-C.

Algorithm 1 PAE framework

Input: Given parameters $w, \xi_r, \xi_c, \epsilon$, the trends $\{\hat{\mathbf{x}}_{l,t-k}\}_{k=-1}^{w-1}$ and the actual price relatives $\{\mathbf{x}_{t-k}\}_{k=0}^{w-1}$ in recent time window, the current weighting vector \mathbf{w}_t and the current portfolio \mathbf{b}_t .

Output: The next portfolio \mathbf{b}_{t+1} .

- 1 Calculating the feasible portfolio $\{\hat{\mathbf{x}}_{l,t-k}\}_{k=0}^{w-1}$ by (9);
- 2 Calculating the recent back-tested returns $\{r_{l,t-w}\}_{k=0}^{w-1}$ by (10) and $\{c_{l,t-w}\}_{k=0}^{w-1}$ by (11) to measure the performance of each trend;
- 3 Defining the ensemble target $r_{*,t}$ by (12) and $c_{*,t}$ by (13);

- 4 Setting parameter $\tau_{1,t}$:

$$\tau_{1,t} = \begin{cases} \max \left\{ 0, \frac{r_{s,t} - \mathbf{w}_t^T \mathbf{r}_t - \xi_r}{\|\mathbf{r}_t - \bar{r}_t \mathbf{1}\|^2} \right\} & \text{PAE-R} \\ \max \left\{ 0, \frac{\mathbf{w}_t^T \mathbf{c}_t - c_{s,t} - \xi_c}{\|\bar{c}_t \mathbf{1} - \mathbf{c}_t\|^2} \right\} & \text{PAE-C} \end{cases};$$
- 5 Updating weighting vector:

$$\mathbf{w}_{t+1} = \begin{cases} \mathbf{w}_t + \tau_{1,t}(\mathbf{r}_t - \bar{r}_t \mathbf{1}) & \text{PAE-R} \\ \mathbf{w}_t + \tau_{1,t}(\bar{c}_t \mathbf{1} - \mathbf{c}_t) & \text{PAE-C} \end{cases};$$
- 6 Normalizing the next weighting vector \mathbf{w}_{t+1} by (9);
- 7 Setting parameter $\tau_{2,t}$ by (24);
- 8 Updating $\mathbf{b}_{t+1} = \mathbf{b}_t + \tau_{2,t}(\mathbf{b}_t - \mu \mathbf{1}_{(m)})$;
- 9 Normalizing the next portfolio \mathbf{b}_{t+1} by (9).

4.4. Complexity Analysis

In the context of online portfolio selection, the running efficiency is a crucial factor to consider as the application has time constraints. The ideal candidate for high-frequency trading [29] would be a portfolio selection strategy that effectively balances wealth accumulation and computational efficiency. The main factors that affect the computational cost of PAE are the total investing periods n , the number of assets m , and the number of trends L . All the steps of PAE cost $O(m)$ except step 7, which consume $O(Lm)$. Table 1 shows the time complexity for different portfolio selection strategies. Our PAE algorithm achieves the lowest complexity when using multiple trend estimators. As the number of trends and assets increases, PAE is still efficient to compute. To be specific, PAE maintains the complexity in $O(Lmn)$, which makes it applicable to time-limited applications.

Table 1. Complexity comparison with other strategies.

Type	Strategies	Complexity
Single trend estimator	OLMAR	$O(mn)$
	PPT	$O(m^2n)$
Multiple trend estimators	AICTR	$O((m + L)mn)$
	PAE	$O(Lmn)$

5. Experiments and Results

In this section, we mainly focus on the comparison studies. First, we introduce experimental datasets and competing portfolio strategies and criteria of evaluation in turn. Then, we conduct experiments on parameter settings and ensemble effectiveness. Finally, we report and analyze the results of comparison studies, and also discuss the transaction costs and running time.

5.1. Data

To improve the reproducibility of the experiments and get close to the recent market, we conduct comparison studies on four popular traditional datasets and three new datasets. All of them are publically available. Four traditional datasets include: NYSE(O) [30], NYSE(N) [8], TSE [31], and MSCI [7]. Three new datasets include: NYSE19 [16], ZZ28 [32], and ETF23.

They contain real-world daily close price relatives from different stock and index markets, including the New York Stock Exchange (NYSE(O), NYSE(N), NYSE19), the Toronto Stock Exchange (TSE), the MSCI World Index (MSCI), and the China Securities 500 index (ZZ28). In 2023, China's ETF market achieved leapfrog development, with a total scale exceeding 1.8 trillion yuan, an increase of nearly 40% from 2022. The significant growth of this market underscores its importance, and we propose the ETF23 dataset, comprising the largest ETFs by market capitalization from 23 different industries in the Chinese ETF market. The detailed information about these datasets are listed in Table 2.

Table 2. Detailed Information of 7 datasets.

Dataset	Region	Time	Days	Assets
NYSE(O)*	US	3/7/1962 - 31/12/1984	5651	36
NYSE(N)	US	1/1/1985 - 30/6/2010	6431	23
TSE	CA	4/1/1994 - 31/12/1998	1259	88
MSCI	Global	1/4/2006 - 31/3/2010	1043	24
NYSE19	US	2/1/2015 - 4/9/2019	1167	47
ZZ28	CN	4/1/2000 - 1/4/2020	4905	28
ETF23	CN	1/2/2021-1/10/2023	647	23

* NYSE(O) was used as the parameter search.

5.2. Competing Portfolio Strategies

For comparison, we consider some benchmarks and a number of existing online portfolio management strategies (including some the state-of-the-art ones). In the following, we show these benchmarks and strategies, where the parameters of each strategy are set according to the recommendations of the corresponding studies.

- BAH: the uniform Buy-And-Hold trading strategy. The strategy invests equally in m assets at the onset and maintains this allocation throughout.
- OLMAR [9]: It takes the moving average to predict the future price. The parameters are set as: $\epsilon = 10$ and $\alpha = 0.5$.
- PPT [10]: It takes the PP in recent time window to predict the future price. The parameters are set as: $w = 5$ and $\epsilon = 10$.
- AICTR [25]: The adaptive input and composite trend representation combines three trends (SMA, EMA, PP) and market conditions through radial basis functions. The parameters are set as: $w = 5$, $\sigma_t^2 = 0.0025$, and $\epsilon = 1000$.
- SPOLC [16]: The short-term portfolio optimization with loss control strategy with the window size $w = 5$ and the mixing parameter $\gamma = 0.025$.

5.3. Evaluation Criteria

For the convenience of calculations, we assume that the initial wealth is 1 [33]. Under such a circumstance, the single-period growth rate is equal to portfolio's gross return at t^{th} period, and the cumulative growth rate is equal to cumulative wealth. Then, we evaluate the characteristics of the aforementioned portfolio strategies by two criteria and five performance metrics. The two criteria are return and risk-adjusted return. Basically, the higher the values of metrics are in return and risk-adjusted return criteria, the better the portfolio strategy performs.

- Cumulative wealth. The cumulative wealth (CW) is utilized as the key evaluation metric for the investment performance of each portfolio selection algorithm.
- Annualized Percentage Yield. The annualized percentage yield (APY) is a widely used metric for evaluating investment returns. It represents the average return of a strategy over the course of a year. APY is computed as follow:

$$APY = CW^{1/y} - 1, \quad (26)$$

where y represents the number of years according to n trading days. In this study, all datasets consist of daily prices. Therefore, y is calculated as n divided by 252, which is the average number of annual trading days.

- Sharpe ratio. In the realm of financial trading, it is often observed that higher returns are accompanied by elevated levels of risk. Thus, it is crucial for an investment algorithm to strike a balance between maximizing returns and managing risks. The Sharpe Ratio (SR) serves as a widely utilized metric for evaluating risk-adjusted returns and is defined as follows:

$$SR = \frac{\bar{r}_s - r_f}{\sigma(r_s)}, \quad (27)$$

Where \bar{r}_s is the mean of r_s , r_f is the return of a risk-free asset and is set to 0 in this paper since we do not consider a risk-free asset, $\sigma(r_s)$ is the standard deviation of return r_s estimated by the samples $r_{s,t}$ in n trading periods.

- Information ratio. The Information Ratio (IR) [34,35] is a performance evaluation metric that quantifies the excess risk-adjusted return of an investment strategy compared to a benchmark. It is defined as:

$$IR = \frac{\bar{r}_s - \bar{r}_m}{\sigma(r_s - r_m)}. \quad (28)$$

- Calmar ratio. The Calmar Ratio (CR) [36] is a comparison of the average annual compound return and the maximum drawdown (MDD) risk, which is widely adopted in fund management. The calculation formula is $CR = APY/MDD$, where $MDD = \max_{t \in [1, T]} \frac{M_t - S_t}{M_t}$, $M_t = \max_{k \in [1, t]} S_k$.

In addition, to test whether the strategy achieves its returns just by luck [34], we introduce the statistic t-test method into the measurement of the proposed strategies. Particularly, a regression model of portfolio excess returns and market excess returns is established as follows:

$$r_{s,t} = \alpha + \beta r_{m,t}. \quad (29)$$

By assuming that parameter α follows a normal distribution, we conduct a statistic t-test on α and derive the probability of achieving the excess return by luck.

5.4. Results

5.4.1. Parameter Setting

Previous studies [10,16–18] typically search for parameters across all datasets, and they carry the risks of overfitting. To finely tune hyperparameters in PAE and avoid overfitting, we tune our PAE in the oldest dataset NYSE(O) and make sure the training set ends earlier than the start of other datasets.

We take four trends ($\hat{x}_{S,t+1}$, $\hat{x}_{E,t+1}$, $\hat{x}_{I,t+1}$ and $\hat{x}_{P,t+1}$) as the input of PAE and initiate the weighting vector as $\mathbf{w} = [1/4, 1/4, 1/4, 1/4]$. The value of w is set to 5. It aligns with previous research [9–11,16,25] and is a commonly used time window size in stock and futures investment because it reflects the recent financial environment. To run trend back-test step, the PAE strategies require at least $w + 1$ days of data. For fairness, all strategies adopt their respective algorithm outputs for investment from the sixth day.

Regarding the threshold parameters ξ_r , ξ_c and ϵ , an initial approximation is made by following grid searching. As shown in Figure 1, PAE-C performs better and stable around $\epsilon > 5$ and $\xi_c > 1.4$. Thus, we recommend to choose $\epsilon = 30$ and $\xi_c = 1.5$ for PAE-C. Similarly, Figure 1 indicates that PAE-R performs better and stable around $\epsilon > 10$ and $7.5 \times 10^{-4} > \xi_r > 6.8 \times 10^{-4}$. We recommend to choose $\epsilon = 30$ and $\xi_r = 7 \times 10^{-4}$ for our PAE-R. The rest experiments adopt this parameter setting for comparison.

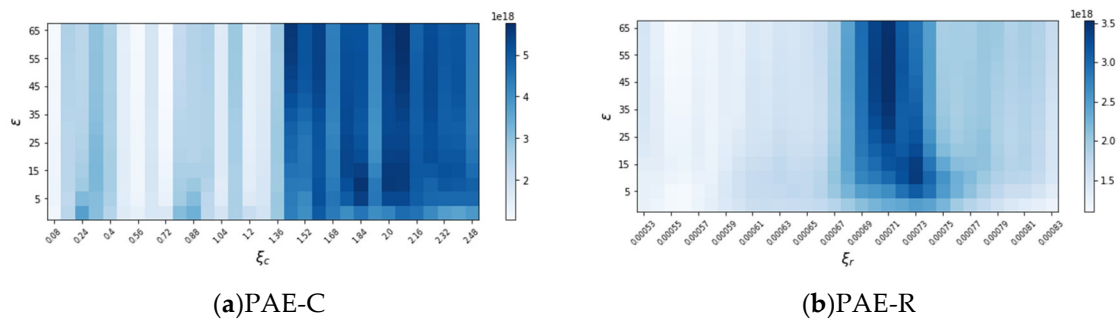


Figure 1. Parameter sensitivity of PAE strategies in terms of CW. Grid searching is used for finding the optimal threshold parameters.

5.4.2. Ensemble Effectiveness

Drawing upon from previous study [25], we compare our PAE performance against single trend situation to evaluate the effectiveness of our ensemble method in combining diverse trends for improved investment performance.

Different trend estimators exhibit significant variability in performance across various datasets. As what we initially emphasized, it is impossible to predict which trend estimator will demonstrate the best performance over the entire investment period. Therefore, we evaluate the effectiveness of our ensemble strategy from two perspectives. First, we expect the strategy to optimize the performance. Second, the baseline for evaluating this strategy is that its performance should not be inferior to that of any single trend estimator.

In Table 3, the best-performing strategy is highlighted in bold, while the two worst-performing strategies are underlined. The results from Table 3 clearly show that our PAE-C and PAE-R strategies consistently exceed the baseline method. Furthermore, they demonstrate leading performance on multiple datasets. It significantly validates the effectiveness of the PAE strategy in integrating trend estimators.

Table 3. CWs of different trend estimator. The best-performing strategy is highlighted in bold, while the two worst-performing strategies are underlined.

Trend estimator	NYSE(N)	MSCI	TSE	ZZ28	NYSE19	ETF23
PP	2.08E+09	<u>8.33</u>	<u>226.84</u>	906.7	<u>1.46</u>	1.4
EMA	4.64E+08	23.6	680.83	283.58	2.46	1.69
SMA	<u>4.26E+08</u>	14.1	<u>76.77</u>	<u>134.64</u>	<u>1.14</u>	<u>1.26</u>
IP	<u>1.16E+06</u>	<u>10.28</u>	1.39E+03	<u>185.13</u>	9.34	<u>0.84</u>
PAE-C	6.83E+08	23.63	706	348.31	2.51	1.29
PAE-R	4.15E+09	14.98	2.26E+03	867.83	2.44	1.71

5.4.3. Comparison Studies

Table 4 shows the performance of PAE and competing strategies on six real-world datasets. We evaluate the performance of portfolio strategies comprehensively with respect to return and risk-adjusted return.

Table 4. Performance of different strategies. The best score is highlighted in each row.

Dataset	Metrics	BAH	OLMAR	PPT	AICTR	SPOLC	PAE-C	PAE-R
NYSE(N))	CW	18.29	4.19E+08	2.63E+09	1.01E+09	1.99E+07	6.83E+08	4.15E+09
	APY	0.121	1.177	1.339	1.254	0.932	1.219	1.382
	SR	0.046	0.104	0.108	0.106	0.105	0.105	0.113
	IR	-0.025	0.096	0.102	0.099	0.095	0.097	0.107
	CR	0.225	1.28	1.561	1.374	1.082	1.298	1.629
MSCI	CW	0.89	14.5	7.99	12.38	7.34	23.63	14.98
	APY	-0.027	0.908	0.652	0.837	0.618	1.147	0.923
	SR	0.001	0.116	0.09	0.108	0.09	0.132	0.116
	IR	-0.036	0.169	0.136	0.158	0.132	0.193	0.17
	CR	-0.041	1.889	1.269	2.026	1.13	2.677	1.901
TSE	CW	1.56	57.79	265.05	544.47	277.1	706	2.26E+03
	APY	0.093	1.252	2.055	2.529	2.083	2.717	3.692
	SR	0.048	0.082	0.101	0.111	0.111	0.114	0.129
	IR	-0.002	0.078	0.098	0.108	0.107	0.111	0.127
	CR	0.311	1.527	2.672	3.81	4.087	4.779	5.127
ZZ28	CW	31.76	124.58	922.47	195.73	853.07	348.31	867.83
	APY	0.194	0.281	0.42	0.311	0.415	0.351	0.416
	SR	0.048	0.05	0.063	0.053	0.068	0.057	0.063
	IR	0.002	0.024	0.044	0.029	0.044	0.034	0.043
	CR	0.335	0.389	0.583	0.436	0.711	0.474	0.548
NYSE19	CW	1.37	0.98	1.5	1.79	2.45	2.51	2.44

	APY	0.07	-0.005	0.092	0.133	0.214	0.22	0.213
	SR	0.034	0.018	0.028	0.032	0.04	0.04	0.039
	IR	-0.022	0.009	0.021	0.024	0.033	0.033	0.032
	CA	0.288	-0.006	0.113	0.168	0.299	0.277	0.301
ETF23	CW	0.87	1.19	1.41	1.28	0.84	1.29	1.71
	APY	-0.053	0.07	0.143	0.102	-0.064	0.104	0.231
	SR	-0.012	0.023	0.034	0.028	-0.003	0.029	0.048
	IR	-0.005	0.036	0.051	0.043	0.005	0.043	0.067
	CR	-0.185	0.156	0.375	0.24	-0.181	0.224	0.532

In terms of return, PAE-R demonstrates the best performance for most of the datasets, especially in TSE and NYSE(N), where its cumulative wealth far exceeds other strategies. PAE-C performs well in the MSCI and ETF23 datasets but generally is inferior to PAE-R for other datasets. To provide a visual representation of the evolution of trajectory, cumulative wealth plots of various portfolio selection strategies on the benchmark datasets are presented in Figure 2. The plots reveal that the PAE algorithm consistently attains higher wealth return compared to others over a majority of time periods, demonstrating the superiority of the model in portfolio decision-making.

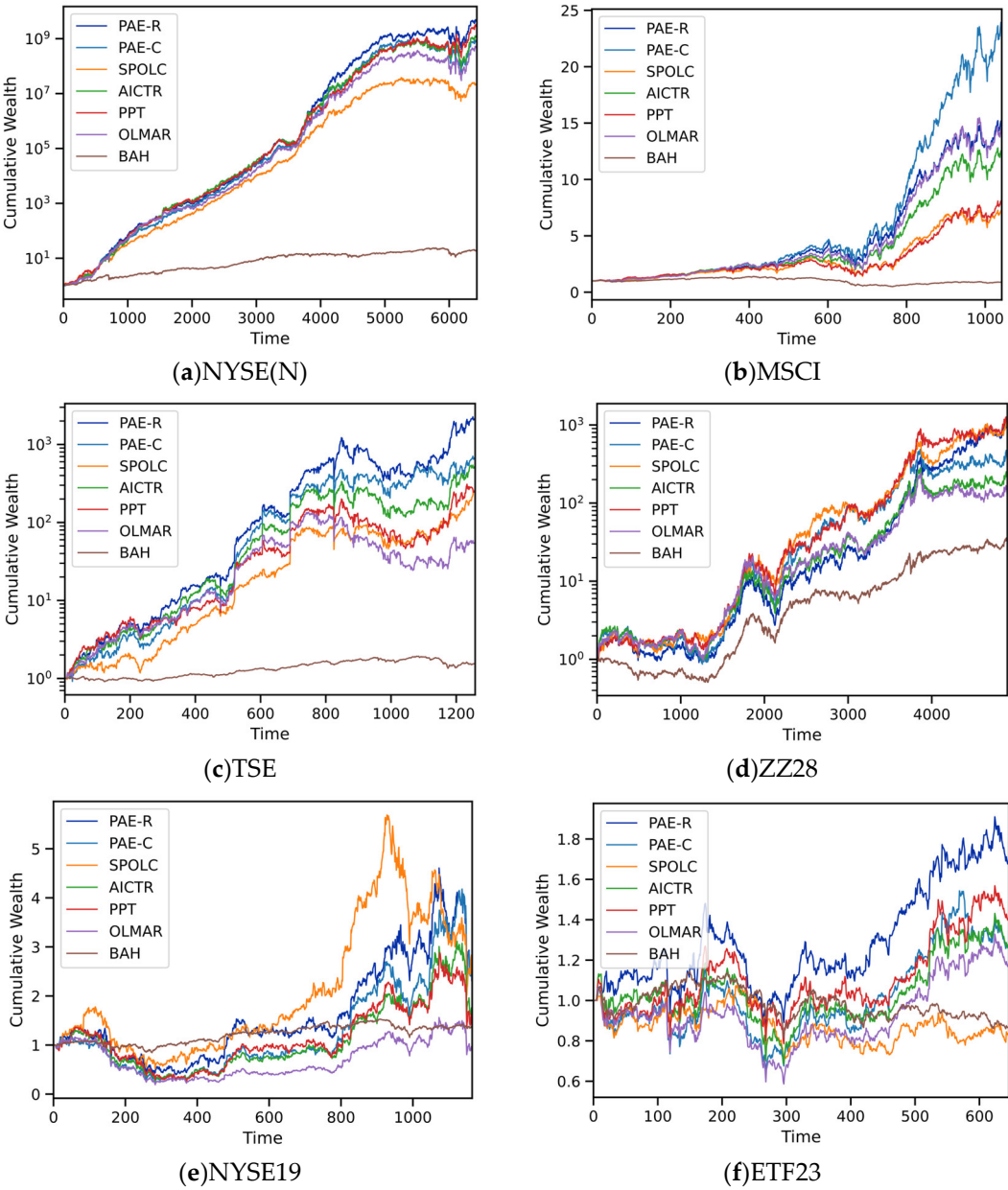


Figure 2. Cumulative wealth of different portfolio selection algorithms during the entire investments on six datasets.

Regarding the risk-adjusted returns, PAE-R outperforms other strategies for most of the datasets, especially in ETF23 and TSE. PAE-C has a higher Sharpe ratio in the MSCI dataset, indicating better performance. PAE-R exhibits higher information ratios in TSE and MSCI, indicating stronger relative outperformance against benchmarks. PAE-C, while performing well in multiple datasets, is generally not as strong as PAE-R. PAE-R almost always has the highest Calmar ratio across all datasets, particularly in ETF23 and TSE. PAE-C shows strength in the MSCI and NYSE19 datasets but is usually inferior to PAE-R for other datasets.

Besides the quantitative evaluation results, Table 5 lists the results of t -test to justify the effectiveness of the proposed PAE. Table 5 shows that except for NYSE19 and ETF23, it is almost impossible for PAE-C to produce the corresponding returns simply by luck at a high confidence level of 97%. There are also around 80% of confidence level for PAE-C not to produce their returns by luck in the rest datasets. The results also indicate that the values of α of PAE-R are significantly larger than 0 at a high confidence level of 99% on four datasets while the rest are higher than 90%.

Table 5. Statistical t -test of performance of PAE-C and PAE-R.

Statistics	NYSE(N)		MSCI		TSE		ZZ28		NYSE19		ETF23	
	PAE-C	PAE-R	PAE-C	PAE-R	PAE-C	PAE-R	PAE-C	PAE-R	PAE-C	PAE-R	PAE-C	PAE-R
α	0.0030	0.0033	0.0033	0.0029	0.0061	0.0071	0.0007	0.0009	0.0009	0.0014	0.0008	0.0012
β	1.3411	1.3561	1.2009	1.1946	2.2057	2.0434	1.0666	1.0718	1.7696	1.4239	1.1315	1.1495
t -statistics	7.3420	8.0863	6.3287	5.5531	3.6954	4.2855	2.2146	2.8055	1.3519	1.7481	1.1124	1.7443
p -value	0.0000	0.0000	0.0000	0.0000	0.0002	0.0000	0.0268	0.0050	0.1769	0.0807	0.2664	0.0816

In general, PAE show noticeable advantages over other competing strategies in terms of comprehensive performance.

5.4.4. Transaction Costs

The paramount concern in practical real-world trading is the transaction cost. When there is a transaction cost rate of r for each trade in the portfolio re-balancing process, the cumulative wealth can be determined using the proportional transaction cost model [9] as:

$$S_n^r = S_0 \prod_{t=1}^n \left[(\mathbf{b}_t^T \mathbf{x}_t) \times \left(1 - \frac{r}{2} \sum_{i=1}^m |\hat{\mathbf{b}}_t^{(i)} - \hat{\mathbf{b}}_{t-1}^{(i)}| \right) \right], \quad (30)$$

where $\hat{\mathbf{b}}_{t-1}^{(i)} = \frac{\hat{\mathbf{b}}_{t-1}^{(i)} * \mathbf{x}_{t-1}^{(i)}}{\hat{\mathbf{b}}_{t-1}^T \mathbf{x}_{t-1}}$ is the price adjusted portfolio of asset i in the $(t-1)^{th}$ period and \mathbf{b}_0 is set to $[0, \dots, 0]^T$. The term $(r/2) \sum_{i=1}^m |\hat{\mathbf{b}}_t^{(i)} - \hat{\mathbf{b}}_{t-1}^{(i)}|$ represents the transaction cost incurred from the adjustment of portfolio $\hat{\mathbf{b}}_{t-1}$ to \mathbf{b}_t through re-balancing.

To evaluate the practicality of the portfolio selection algorithms, we perform experiments on cumulative wealth while changing the transaction cost rate r between 0 and 0.5%. The findings, displayed in Figure 3, indicate that PAE-R delivers superior results on NYSE(N), MSCI, TSE and ETF23. Additionally, PAE-R is comparable to other the state-of-the-art algorithms on the remaining datasets. PAE-C shows competitive performance in MSCI and NYSE19.

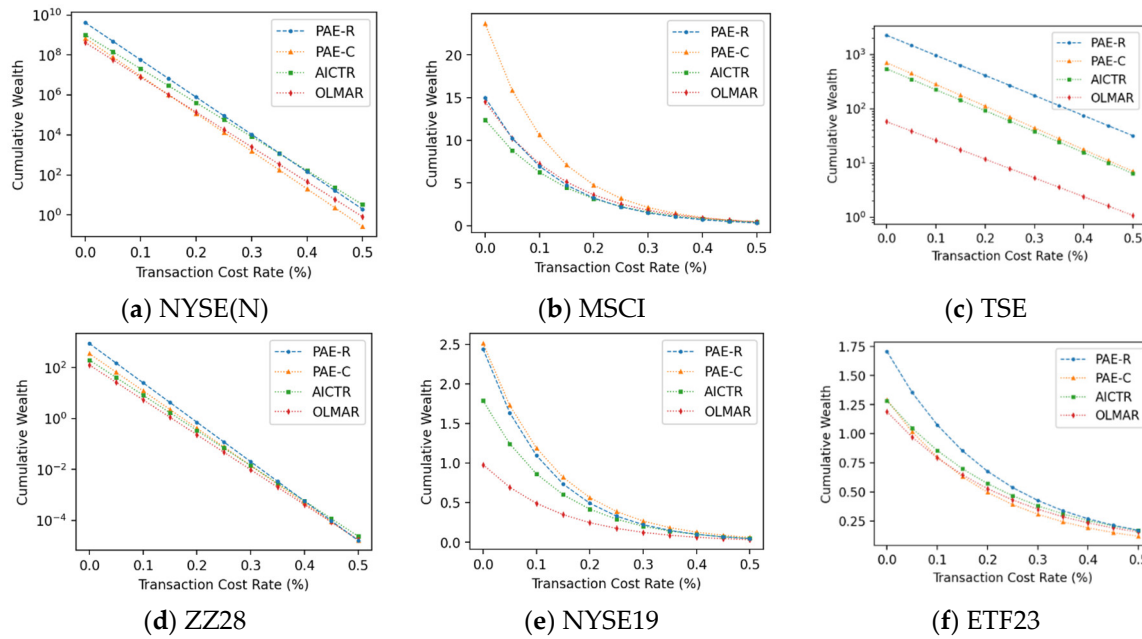


Figure 3. Cumulative wealth of different algorithms under different transaction cost rates on six benchmarks datasets.

This demonstrates that PAE is a capable framework for managing transaction costs, making it suitable for real-world financial environments.

5.4.5. Running Time

To evaluate the computational cost of the PAE, we conducted 5 iterations of the algorithms on each dataset using an AMD 3500x CPU with two 16G DDR4 3000MHz RAM. The average running times (in seconds) of PAE-C per trading period are 0.0064, 0.0064, 0.0071, 0.0064, 0.0067 and 0.0073 on NYSE(N), MSCI, TSE, ZZ28, NYSE19 and ETF23, respectively. The average running times (in seconds) of PAE-R per trading period are 0.0056, 0.0056, 0.0063, 0.0057, 0.0056 and 0.0060. It indicates that PAE has a fast running time, making it applicable to real-world financial environments that require efficient computations.

6. Conclusion and Future Work

In this study, we present a Passive Aggressive Ensemble (PAE) framework, a new approach for online portfolio selection to integrate multiple trend estimators. PAE stands out by efficiently combining different estimators and enhancing their performance through a novel weighting mechanism. Our extensive experiments across various real-world datasets demonstrate that PAE not only outperforms the competing algorithms in key evaluation metrics but also shows potential in managing transaction costs effectively. This approach is particularly notable for its adaptability to different market conditions and scalability, making it suitable for practical financial applications.

This paper envisages three primary directions for future research. First, the exploration of different loss functions: In our framework, the weighting factors of different trend estimators are determined by their relative loss to the market. Identifying more optimal loss functions to evaluate the effectiveness of trend estimators could significantly enhance the overall performance of the framework. Second, the investigation into the application of regularization techniques in ensemble methods: Regularization techniques can prevent overfitting and bolster the robustness of the ensemble approach. Finally, we acknowledge that our model has substantial room for improvement in high transaction fee environments. Therefore, refining the PAE model to improve its performance by considering transaction costs is also a pivotal aspect of our forthcoming research endeavors.

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Data Availability Statement: The dataset, *i.e.*, ETF23, is free to download from website: https://drive.google.com/file/d/1W89bkqKqQJioLC9ljf_mLFHgk9fPcSh/view?usp=sharing, or <https://pan.baidu.com/s/1j3wMPVHHWiiFKW4jJ2iS-Q?pwd=ETF0> with password ETF0.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Proof of Theorem 1. When $\ell_{\xi_r} = 0$, the constraint in (14) is satisfied by \mathbf{w}_t , and it becomes the optimal solution. For the case where $\ell_{\xi_r} \neq 0$, we can solve the optimization problem by introducing the Lagrangian of the problem in (14):

$$\mathcal{L}(\mathbf{w}, \tau_{1,t}, \lambda) = \frac{1}{2} \|\mathbf{w} - \mathbf{w}_t\|^2 + \tau_{1,t}(\mathbf{r}_{*,t} - \mathbf{w}^\top \mathbf{r}_t - \xi_r) + \lambda(\mathbf{1}^\top \mathbf{w} - 1), \quad (31)$$

where $\tau_{1,t} \geq 0$ and λ are the Lagrangian multipliers. Taking the gradient with respect to \mathbf{w} and setting it to zero, we get,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = (\mathbf{w} - \mathbf{w}_t) - \tau_{1,t} \mathbf{r}_t + \lambda \mathbf{1} = 0. \quad (32)$$

By left multiplying $\mathbf{1}^\top$ on both sides, we get

$$1 = 1 - \tau_{1,t} \mathbf{1}^\top \mathbf{r}_t + \lambda L, \quad (33)$$

$$\lambda = \tau_{1,t} \bar{r}_t, \quad (34)$$

where $\bar{r}_t = \frac{\mathbf{1}^\top \mathbf{r}_t}{L}$ is the mean value of back-tested returns. Plugging the above equation to (32), we get the update of \mathbf{w}

$$\mathbf{w} = \mathbf{w}_t + \tau_{1,t}(\mathbf{r}_t - \bar{r}_t \mathbf{1}). \quad (35)$$

Simplifying the formula after plugging (34) and (35) to Lagrangian (31), we get

$$\mathcal{L}(\tau_{1,t}) = -\frac{1}{2} \tau_{1,t}^2 \|\mathbf{r}_t - \bar{r}_t \mathbf{1}\|^2 + \tau_{1,t}(\mathbf{r}_{*,t} - \mathbf{w}_t^\top \mathbf{r}_t - \xi_r). \quad (36)$$

Taking derivative with respect to $\tau_{1,t}$ and setting it to zero, we have

$$\frac{\partial \mathcal{L}}{\partial \tau_{1,t}} = -\tau_{1,t} \|\mathbf{r}_t - \bar{r}_t \mathbf{1}\|^2 + (\mathbf{r}_{*,t} - \mathbf{w}_t^\top \mathbf{r}_t - \xi_r) = 0, \quad (37)$$

which implying,

$$\tau_{1,t} = \frac{\mathbf{r}_{*,t} - \mathbf{w}_t^\top \mathbf{r}_t - \xi_r}{\|\mathbf{r}_t - \bar{r}_t \mathbf{1}\|^2}. \quad (38)$$

Further projecting $\tau_{1,t}$ to $[0, +\infty)$, we complete the proof of (19). \square

Appendix B

Proof of Theorem 3. The Lagrangian of optimization problem (17) is

$$\mathcal{L}(\mathbf{b}, \tau_{2,t}, \eta) = \frac{1}{2} \|\mathbf{b} - \mathbf{b}_t\|^2 + \tau_{2,t}(\epsilon - \mathbf{b}^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}) + \eta(\mathbf{1}^\top \mathbf{b} - 1), \quad (39)$$

where $\tau_{2,t} \geq 0$ and η are the Lagrangian multipliers. Taking the gradient with respect to \mathbf{b} and setting it to zero, we have

$$\frac{\partial \mathcal{L}}{\partial \mathbf{b}} = (\mathbf{b} - \mathbf{b}_t) - \tau_{2,t} \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1} + \eta \mathbf{1}_{(m)} = 0, \quad (40)$$

where $\mathbf{1}_{(m)}$ denotes the m -dimensional $\mathbf{1}$ vector. By left multiplying both sides with $\mathbf{1}_{(m)}^\top$, we have

$$1 = 1 - \tau_{2,t} \mathbf{1}_{(m)}^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1} + \eta m, \quad (41)$$

$$\eta = \tau_{2,t} \mu, \quad (42)$$

where $\mu = \frac{1}{m} \mathbf{1}_{(m)}^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}$. Plugging the above equation to (40), we have the update of \mathbf{b}

$$\mathbf{b} = \mathbf{b}_t + \tau_{2,t}(\mathbf{b}_t - \mu \mathbf{1}_{(m)}). \quad (43)$$

Simplifying the formula after plugging (42) and (43) to Lagrangian (39), we have

$$\mathcal{L}(\tau_{2,t}) = -\frac{1}{2} \tau_{2,t}^2 \|\mathbf{b}_t - \mu \mathbf{1}\|^2 + \tau_{2,t}(\epsilon - \mathbf{b}_t^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}). \quad (44)$$

Taking derivative with respect to $\tau_{2,t}$ and setting it to zero, we arrive at

$$\frac{\partial \mathcal{L}}{\partial \tau_{2,t}} = -\tau_{2,t} \|\mathbf{b}_t - \mu \mathbf{1}\|^2 + (\epsilon - \mathbf{b}_t^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}) = 0, \quad (45)$$

which can be simplified to,

$$\tau_{2,t} = \frac{\epsilon - \mathbf{b}_t^\top \hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1}}{\|\hat{\mathbf{X}}_{t+1} \mathbf{w}_{t+1} - \mu \mathbf{1}_{(m)}\|^2}. \quad (46)$$

Further projecting $\tau_{2,t}$ to $[0, +\infty)$, we finally complete the proof of (24). \square

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