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Article

Restricted Singular Value Decomposition for a Tensor Triplet Under t-Product and Its Applications

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Abstract: We investigate and discuss in detail the structure of the restricted singular value decomposition for a tensor triplet under t-product (T-RSVD). The algorithm is provided with a numerical example illustrating the main result. For applications, we consider color image watermarking processing with T-RSVD.

Keywords: multilinear algebra; restricted singular value decomposition; tensor SVD

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1. Introduction

A great number of problems lead to decompositions of higher-order tensors; examples include color image processing, genomic signals, higher-order statistics, pattern recognition, chemometrics, aerospace engineering, etc . (see e.g., [2,5,8,9,17–20] and references therein). There have been many papers discussing tensor decompositions under various tensor products ([3,8,12,15,21,22]).

Some well-known multiplications of tensors are the n-mode, Kronecker, Khatri–Rao and Einstein products. The t-product is a new type of tensor multiplication that can be used in imaging (see e.g., [6,8,10,13]). Kilmer and Martin [11] investigated the tensor singular value decomposition under t-product (T-SVD) for third-order tensors. Martin *et al.* [13] extended the T-SVD to order-p tensors. Very recently, He *et al.* [8] considered the generalized singular value decompositions for two tensors via the t-product.

To our knowledge, there is little information on the decompositions for three tensors under t-product. Motivated by the wide applications of tensor decompositions in order to improve the theoretical development of tensor decompositions, we consider the restricted singular value decomposition (T-RSVD) for three tensors under t-product. One goal of this article is to investigate and discuss the structure and algorithm of T-RSVD. Another goal is to give an application in color image watermarking processing.

The remainder of the paper is organized as follows. In Section 2, we review some definitions, notations and background used throughout the paper. In Section 3, we derive the structure and properties of T-RSVD. We present an algorithm and a numerical example to illustrate the main result. We give an application on color image watermarking processing in Section 4.

2. Preliminaries

An order three dimension $n_1 \times n_2 \times n_3$ tensor $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ is a multidimensional array with $n_1 n_2 n_3$ entries. Let $\mathbb{C}^{n_1 \times n_2 \times n_3}$ be the set of all order three dimension $n_1 \times n_2 \times n_3$ tensors over the complex number field $\mathbb{C}$.

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If $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ with $n_1 \times n_2$ frontal slices $A_1, \dots, A_{n_3}$, then

$$\text{unfold}(\mathcal{A}) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n_3} \end{bmatrix}, \tag{1}$$

$$\text{bcirc}(\mathcal{A}) = \begin{bmatrix} A_1 & A_{n_3} & A_{n_3-1} & \cdots & A_2 \\ A_2 & A_1 & A_{n_3} & \cdots & A_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n_3} & A_{n_3-1} & A_{n_3-2} & \cdots & A_1 \end{bmatrix}, \tag{2}$$

where $A_i = \mathcal{A}(:,:,i)$ for $i = 1, \dots, n_3$. The operations $\text{fold}(\cdot)$ and $\text{bcirc}^{-1}(\cdot)$ take (1) and (2) back to an $n_1 \times n_2 \times n_3$ tensor. That means

$$\text{fold}(\text{unfold}(\mathcal{A})) = \mathcal{A}, \quad \text{bcirc}^{-1}(\text{bcirc}(\mathcal{A})) = \mathcal{A}.$$

Note that

$$(F_{n_3} \otimes I_{n_1})\,\text{bcirc}(\mathcal{A})(F_{n_3}^* \otimes I_{n_2}) = \text{diag}(D_1, \dots, D_{n_3}), \tag{3}$$

where $F_{n_3}^*$ is the conjugate transpose of $n_3 \times n_3$ normalized discrete Fourier transformation (DFT) matrix F_{n_3}, and $\otimes$ denotes the Kronecker product of matrices. The matrices $D_1, \dots, D_{n_3}$ are diagonal (sub-diagonal, upper-triangular, lower-triangular) if and only if the matrices $A_1, \dots, A_{n_3}$ are diagonal (sub-diagonal, upper-triangular, lower-triangular) [14], and

$$A_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} D_j, \quad i = 1, \dots, n_3, \tag{4}$$

where $\omega = e^{-2\pi \mathrm{i}/n_3}$ is a primitive n_3th root of unity.

The definitions of t-product of two tensors, identity tensor, conjugate transpose, inverse tensor and unitary tensor are given as follows.

Definition 1 (t-product of third-order tensors [11]). *Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times l \times n_3}$ be given. Then the t-product $\mathcal{A} * \mathcal{B} \in \mathbb{C}^{n_1 \times l \times n_3}$ is defined as*

$$\mathcal{A} * \mathcal{B} = \text{fold}(\text{bcirc}(\mathcal{A})\,\text{unfold}(\mathcal{B})). \tag{5}$$

Definition 2 (Identity tensor [11]). *The identity tensor $\mathcal{I} \in \mathbb{C}^{m \times m \times n}$is defined to be a tensor whose first frontal slice is the $m \times m$ identity matrix and whose other frontal sliced are zero matrices, i.e.,* $\text{bcirc}(\mathcal{I}) = I_{mn}$.

Definition 3 (Conjugate transpose [11]). *For a given tensor $\mathcal{A} \in \mathbb{C}^{l \times m \times n}$, $\mathcal{A}^*$ is the $m \times l \times n$ tensor obtained by conjugate transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through n, i.e.,*

$$\mathcal{A}^*(:,:,1) = A_1^*, \quad \mathcal{A}^*(:,:,i) = A_{n+2-i}^*, \quad i = 2, \dots, n,$$

where $A_k = \mathcal{A}(:,:,k)$, $k = 1, \dots, n_3$.

Definition 4 (Tensor inverse [11]). *An $m \times m \times n$ tensor $\mathcal{A}$ has an inverse $\mathcal{B} \in \mathbb{C}^{m \times m \times n}$, provided that $\mathcal{A} * \mathcal{B} = \mathcal{I}$ and $\mathcal{B} * \mathcal{A} = \mathcal{I}$, i.e.,* $\text{bcirc}(\mathcal{B}) = \text{bcirc}(\mathcal{A})^{-1}$.

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Definition 5 (Unitary tensor [11]). *An $m \times m \times n$ tensor $\mathcal{A}$ is unitary, provided that $\mathcal{A} * \mathcal{A}^* = \mathcal{I}$ and $\mathcal{A}^* * \mathcal{A} = \mathcal{I}$, i.e.,* $\mathrm{bcirc}(\mathcal{A})$ *is a unitary matrix.*

3. Restricted singular value decomposition for three tensors

In this section, we consider the restricted singular value decomposition for three tensors under t-product (T-RSVD). The following lemma gives the restricted singular value decomposition (RSVD) for matrices.

Lemma 1 (RSVD for matrices [4]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$ be given. There exist unitary matrices $U \in \mathbb{C}^{q\times q}$ and $V \in \mathbb{C}^{p\times p}$, and nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$A = PS_aQ, \quad B = PS_bV, \quad C = US_cQ, \tag{6}$$

where

$$\begin{pmatrix} S_a & S_b \\ S_c & \end{pmatrix} = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ \\ \\ r_2 \\ r_4 \\ r_6 \end{array} \begin{array}{c} \begin{array}{cccccccccc} r_1 & r_2 & r_3 & r_4 & r_6 & & & r_3 & r_4 & r_5 \end{array} \\ \left(\begin{array}{cccccc|cccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & S_{abc} & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & I & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & I & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & I & 0 & & & & \end{array}\right) \end{array}, \tag{7}$$

and $S_{abc} = \mathrm{diag}\{\sigma_1, \ldots, \sigma_{r_4}\}$ is square nonsingular diagonal with positive diagonal elements, $\sigma_1, \ldots, \sigma_{r_4}$ are defined to be the non-trivial restricted singular values of the matrix triplet A, B, C. Expressions for the integers $r_1, \ldots, r_6$ are the following:

$$r_1 = r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} - r(B) - r(C), \tag{8}$$

$$r_2 = r(A,\ B) + r(C) - r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \tag{9}$$

$$r_3 = r(B) + r\begin{pmatrix} A \\ C \end{pmatrix} - r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \tag{10}$$

$$r_4 = r(A) + r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} - r\begin{pmatrix} A \\ C \end{pmatrix} - r(A,\ B), \tag{11}$$

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$$r_5 = r(A,\ B) - r(A), \quad r_6 = r\begin{pmatrix} A \\ C \end{pmatrix} - r(A), \tag{12}$$

where the symbol $r(A)$ stands for the rank of the matrix A.

The following theorem presents the RSVD for three tensors under t-product.

Theorem 1 (T-RSVD)**.** *Let $\mathcal{A} \in \mathbb{C}^{m\times p\times n_3}$, $\mathcal{B} \in \mathbb{C}^{m\times q\times n_3}$, and $\mathcal{C} \in \mathbb{C}^{t\times p\times n_3}$. Then $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ can be factored as*

$$\mathcal{A} = \mathcal{P} * \mathcal{S}_a * \mathcal{Q}, \quad \mathcal{B} = \mathcal{P} * \mathcal{S}_b * \mathcal{V}, \quad \mathcal{C} = \mathcal{U} * \mathcal{S}_c * \mathcal{Q}, \tag{13}$$

where $\mathcal{U} \in \mathbb{C}^{t\times t\times n_3}$ and $\mathcal{V} \in \mathbb{C}^{q\times q\times n_3}$ are unitary, $\mathcal{P} \in \mathbb{C}^{m\times m\times n_3}$, $\mathcal{Q} \in \mathbb{C}^{p\times p\times n_3}$ are invertible, $\mathcal{S}_a$ is a $m \times p \times n_3$ f-diagonal tensor, $\mathcal{S}_b$ and $\mathcal{S}_c$ are $m \times q \times n_3$ and $t \times p \times n_3$ tensors whose frontal slices are, respectively, block lower-triangular and block upper-triangular matrices with the following forms

$$(S_b)_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} \begin{pmatrix} 0 & 0 \\ 0 & I_{s_j} \\ 0 & 0 \end{pmatrix},\ (S_c)_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s'_j} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{s''_j} & 0 \end{pmatrix},\ i = 1\ldots,n_3.$$

Proof. Let $\widetilde{A} = \mathrm{bcirc}(\mathcal{A}) \in \mathbb{C}^{(mn_3)\times(pn_3)}$, $\widetilde{B} = \mathrm{bcirc}(\mathcal{B}) \in \mathbb{C}^{(mn_3)\times(qn_3)}$, and $\widetilde{C} = \mathrm{bcirc}(\mathcal{C}) \in \mathbb{C}^{(tn_3)\times(pn_3)}$. It follows from (3) that

$$(F_{n_3} \otimes I_m)\widetilde{A}(F^*_{n_3} \otimes I_p) = \mathrm{diag}(\widetilde{A_1}, \ldots, \widetilde{A_{n_3}}),$$

$$(F_{n_3} \otimes I_m)\widetilde{B}(F^*_{n_3} \otimes I_q) = \mathrm{diag}(\widetilde{B_1}, \ldots, \widetilde{B_{n_3}}),$$

$$(F_{n_3} \otimes I_t)\widetilde{C}(F^*_{n_3} \otimes I_p) = \mathrm{diag}(\widetilde{C_1}, \ldots, \widetilde{C_{n_3}}),$$

where F_{n_3} is $n_3 \times n_3$ normalized DFT matrix, $\widetilde{A_i}$, $\widetilde{B_i}$, and $\widetilde{C_i}$ are $m \times p$, $m \times q$, and $t \times p$ matrices $i = 1,\ \ldots,\ n_3$. Observe that $\widetilde{A_i}$ and $\widetilde{B_i}$ have the same number of columns, meanwhile $\widetilde{A_i}$ and $\widetilde{C_i}$ have the same number of rows. Applying RSVD (Lemma 1) to each matrix triplet $\widetilde{A_i}$, $\widetilde{B_i}$, and $\widetilde{C_i}$ gives

$$\begin{pmatrix} \widetilde{A_1} & & \\ & \ddots & \\ & & \widetilde{A_{n_3}} \end{pmatrix} = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_{n_3} \end{pmatrix} \begin{pmatrix} \Omega_{A_1} & & \\ & \ddots & \\ & & \Omega_{A_{n_3}} \end{pmatrix} \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_{n_3} \end{pmatrix}, \tag{14}$$

$$\begin{pmatrix} \widetilde{B_1} & & \\ & \ddots & \\ & & \widetilde{B_{n_3}} \end{pmatrix} = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_{n_3} \end{pmatrix} \begin{pmatrix} \Omega_{B_1} & & \\ & \ddots & \\ & & \Omega_{B_{n_3}} \end{pmatrix} \begin{pmatrix} V_1 & & \\ & \ddots & \\ & & V_{n_3} \end{pmatrix}, \tag{15}$$

and

$$\begin{pmatrix} \widetilde{C_1} & & \\ & \ddots & \\ & & \widetilde{C_{n_3}} \end{pmatrix} = \begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_{n_3} \end{pmatrix} \begin{pmatrix} \Omega_{C_1} & & \\ & \ddots & \\ & & \Omega_{C_{n_3}} \end{pmatrix} \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_{n_3} \end{pmatrix}, \tag{16}$$

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where the forms of Ω_{A_i}, Ω_{B_i}, and Ω_{C_i} are given in (7), U_i and V_i are unitary, and P_i and Q_i are nonsingular. We multiply $(F_{n_3}^* \otimes I_m)$, $(F_{n_3}^* \otimes I_m)$, and $(F_{n_3}^* \otimes I_t)$ to the left of each of the block diagonal matrices in (14)–(16), respectively, and $(F_{n_3} \otimes I_p)$, $(F_{n_3} \otimes I_q)$, and $(F_{n_3} \otimes I_p)$ to the right of each of the block diagonal matrices in (14)–(16). Then, we obtain

$$\begin{aligned}\text{bcirc}(\mathcal{A}) &= \left[(F_{n_3}^* \otimes I_m)\begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_{n_3}\end{pmatrix}(F_{n_3} \otimes I_m)\right] \\ &\quad \left[(F_{n_3}^* \otimes I_m)\begin{pmatrix} \Omega_{A_1} & & \\ & \ddots & \\ & & \Omega_{A_{n_3}}\end{pmatrix}(F_{n_3} \otimes I_p)\right] \\ &\quad \left[(F_{n_3}^* \otimes I_p)\begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_{n_3}\end{pmatrix}(F_{n_3} \otimes I_p)\right] \\ &= \text{bcirc}(\mathcal{P})\,\text{bcirc}(\mathcal{S}_a)\,\text{bcirc}(\mathcal{Q}), \end{aligned} \tag{17}$$

$$\begin{aligned}\text{bcirc}(\mathcal{B}) &= \left[(F_{n_3}^* \otimes I_m)\begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_{n_3}\end{pmatrix}(F_{n_3} \otimes I_m)\right] \\ &\quad \left[(F_{n_3}^* \otimes I_m)\begin{pmatrix} \Omega_{B_1} & & \\ & \ddots & \\ & & \Omega_{B_{n_3}}\end{pmatrix}(F_{n_3} \otimes I_q)\right] \\ &\quad \left[(F_{n_3}^* \otimes I_q)\begin{pmatrix} V_1 & & \\ & \ddots & \\ & & V_{n_3}\end{pmatrix}(F_{n_3} \otimes I_q)\right] \\ &= \text{bcirc}(\mathcal{P})\,\text{bcirc}(\mathcal{S}_b)\,\text{bcirc}(\mathcal{V}), \end{aligned} \tag{18}$$

and

$$\begin{aligned}\text{bcirc}(\mathcal{C}) &= \left[(F_{n_3}^* \otimes I_t)\begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_{n_3}\end{pmatrix}(F_{n_3} \otimes I_t)\right] \\ &\quad \left[(F_{n_3}^* \otimes I_t)\begin{pmatrix} \Omega_{C_1} & & \\ & \ddots & \\ & & \Omega_{C_{n_3}}\end{pmatrix}(F_{n_3} \otimes I_p)\right] \\ &\quad \left[(F_{n_3}^* \otimes I_p)\begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_{n_3}\end{pmatrix}(F_{n_3} \otimes I_p)\right] \\ &= \text{bcirc}(\mathcal{U})\,\text{bcirc}(\mathcal{S}_c)\,\text{bcirc}(\mathcal{Q}), \end{aligned} \tag{19}$$

which yields

$$\mathcal{A} = \mathcal{P} * \mathcal{S}_a * \mathcal{Q}, \tag{20}$$

$$\mathcal{B} = \mathcal{P} * \mathcal{S}_b * \mathcal{V}, \tag{21}$$

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and

$$\mathcal{C} = \mathcal{U} * \mathcal{S}_c * \mathcal{Q}. \tag{22}$$

Note that $\mathcal{U}$ and $\mathcal{V}$ are unitary, $\mathcal{P}$ and $\mathcal{Q}$ are invertible. It follows from (4) that the frontal slices of $\mathcal{S}_a$, $\mathcal{S}_b$, and $\mathcal{S}_c$ have the following forms

$$(S_a)_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} \Omega_{A_j}, \quad (S_b)_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} \Omega_{B_j},$$

$$(S_c)_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} \Omega_{C_j}, \quad i = 1 \ldots, n_3. \quad \square$$

We provide the algorithm for T-RSVD.

Algorithm 1: Compute the T-RSVD of tensors $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$

Input: $\mathcal{A} \in \mathbb{C}^{m \times p \times n_3}$, $\mathcal{B} \in \mathbb{C}^{m \times q \times n_3}$, $\mathcal{C} \in \mathbb{C}^{t \times p \times n_3}$.

Output: $\mathcal{S}_a \in \mathbb{C}^{m \times p \times n_3}$, $\mathcal{S}_b \in \mathbb{C}^{m \times q \times n_3}$, $\mathcal{S}_c \in \mathbb{C}^{t \times p \times n_3}$, $\mathcal{U} \in \mathbb{C}^{t \times t \times n_3}$, $\mathcal{V} \in \mathbb{C}^{q \times q \times n_3}$, $\mathcal{P} \in \mathbb{C}^{m \times m \times n_3}$, $\mathcal{Q} \in \mathbb{C}^{p \times p \times n_3}$.

1. $\widetilde{A} = \text{bcirc}(\mathcal{A})$, $\widetilde{B} = \text{bcirc}(\mathcal{B})$, $\widetilde{C} = \text{bcirc}(\mathcal{C})$.
2. $(F_{n_3} \otimes I_m)\widetilde{A}(F_{n_3}^* \otimes I_p) = \text{diag}(\widetilde{A_j})$, $(F_{n_3} \otimes I_m)\widetilde{B}(F_{n_3}^* \otimes I_q) = \text{diag}(\widetilde{B}_j)$, $(F_{n_3} \otimes I_t)\widetilde{C}(F_{n_3}^* \otimes I_p) = \text{diag}(\widetilde{C}_j)$, $j = 1, 2, \ldots, n_3$.
3. **for** $j = 1, \ldots, n_3$ **do**
   % Give the RSVD of $\widetilde{A_j}$, $\widetilde{B}_j$, and $\widetilde{C}_j$ by the method in [1],
   $\widetilde{A_j} = P_j \Omega_{A_j} Q_j$,
   $\widetilde{B_j} = P_j \Omega_{B_j} V_j$,
   $\widetilde{C_j} = U_j \Omega_{C_j} Q_j$,
   end for
4. $\Omega_A = \text{diag}(\Omega_{A_j})$, $\Omega_B = \text{diag}(\Omega_{B_j})$, $\Omega_C = \text{diag}(\Omega_{C_j})$, $U = \text{diag}(U_j)$, $V = \text{diag}(V_j)$, $P = \text{diag}(P_j)$, $Q = \text{diag}(Q_j)$.
5. $\widetilde{S}_a = (F_{n_3}^* \otimes I_m)\Omega_A(F_{n_3} \otimes I_p)$, $\widetilde{S}_b = (F_{n_3}^* \otimes I_m)\Omega_B(F_{n_3} \otimes I_q)$, $\widetilde{S}_c = (F_{n_3}^* \otimes I_t)\Omega_C(F_{n_3} \otimes I_p)$, $\widetilde{U} = (F_{n_3}^* \otimes I_t)U(F_{n_3} \otimes I_t)$, $\widetilde{V} = (F_{n_3}^* \otimes I_q)V(F_{n_3} \otimes I_q)$, $\widetilde{P} = (F_{n_3}^* \otimes I_m)P(F_{n_3} \otimes I_m)$, $\widetilde{Q} = (F_{n_3}^* \otimes I_p)Q(F_{n_3} \otimes I_p)$.
6. $\mathcal{S}_a = \text{bcirc}^{-1}(\widetilde{S}_a)$, $\mathcal{S}_b = \text{bcirc}^{-1}(\widetilde{S}_b)$, $\mathcal{S}_c = \text{bcirc}^{-1}(\widetilde{S}_c)$, $\mathcal{U} = \text{bcirc}^{-1}(\widetilde{U})$, $\mathcal{V} = \text{bcirc}^{-1}(\widetilde{V})$, $\mathcal{P} = \text{bcirc}^{-1}(\widetilde{P})$, $\mathcal{Q} = \text{bcirc}^{-1}(\widetilde{Q})$.

The cost of Algorithm 1 is not less than $\mathcal{O}\left((n_1 \times n_2 \times n_3)^3\right)$.

Example 1. *Let $\mathcal{A} \in \mathbb{C}^{5 \times 4 \times 3}$, $\mathcal{B} \in \mathbb{C}^{5 \times 6 \times 3}$ and $\mathcal{C} \in \mathbb{C}^{7 \times 4 \times 3}$ be tensors with the following forms:*

$$A_1 = \begin{pmatrix} 0.5303+0.5248\mathbf{i} & 0.4221+0.7424\mathbf{i} & 0.5356+0.7391\mathbf{i} & 0.6216+0.6644\mathbf{i} \\ 0.7312+1.0116\mathbf{i} & 0.1918+0.7801\mathbf{i} & 1.0096+0.6131\mathbf{i} & 0.6995+1.0930\mathbf{i} \\ 0.7828+0.9208\mathbf{i} & 0.5561+0.4061\mathbf{i} & 0.8372+0.1792\mathbf{i} & 0.8905+0.9073\mathbf{i} \\ 0.7793+0.9972\mathbf{i} & 1.2191+0.6961\mathbf{i} & 0.3713+0.5136\mathbf{i} & 1.1552+1.0565\mathbf{i} \\ 0.5657+0.2247\mathbf{i} & 0.4331+0.7622\mathbf{i} & 0.5833+0.8704\mathbf{i} & 0.6561+0.4125\mathbf{i} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.7495+1.2373\mathbf{i} & 0.1073+0.3358\mathbf{i} & 1.0968-0.0209\mathbf{i} & 0.6810+1.1587\mathbf{i} \\ 0.4769-0.2104\mathbf{i} & 1.1669+0.9282\mathbf{i} & -0.0650+1.2320\mathbf{i} & 0.8769+0.0870\mathbf{i} \\ 0.5336+0.7049\mathbf{i} & 0.0139+0.7114\mathbf{i} & 0.8243+0.6365\mathbf{i} & 0.4596+0.8100\mathbf{i} \\ 0.5427+1.2123\mathbf{i} & -0.2579+0.5388\mathbf{i} & 1.0273+0.2411\mathbf{i} & 0.3576+1.1957\mathbf{i} \\ 0.5218+0.8433\mathbf{i} & 1.1630+0.5181\mathbf{i} & 0.0079+0.3463\mathbf{i} & 0.9135+0.8730\mathbf{i} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.8390+0.3249\mathbf{i} & -0.0225+0.4260\mathbf{i} & 1.3270+0.4157\mathbf{i} & 0.7047+0.4016\mathbf{i} \\ 0.6379+0.3782\mathbf{i} & 1.4253+0.5669\mathbf{i} & 0.0071+0.5724\mathbf{i} & 1.1182+0.4880\mathbf{i} \\ 0.8143+0.7469\mathbf{i} & 0.1163+0.2199\mathbf{i} & 1.1919+0.0088\mathbf{i} & 0.7398+0.7044\mathbf{i} \\ 0.7343+0.6969\mathbf{i} & 0.2639+0.6815\mathbf{i} & 0.9643+0.6020\mathbf{i} & 0.7312+0.7945\mathbf{i} \\ 0.2706-0.1724\mathbf{i} & 0.7472+0.8606\mathbf{i} & -0.0959+1.1341\mathbf{i} & 0.5319+0.1001\mathbf{i} \end{pmatrix},$$

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$$B_1 = \begin{pmatrix} 0.9678+1.0857i & 1.9658+2.4999i & 1.8365+2.4539i & 2.3199+2.7399i & 1.8102+2.4752i & 0.8597+1.3350i & 1.2342+1.4315i \\ 1.5437+1.4667i & 2.7275+3.0925i & 2.5159+2.8494i & 2.7909+4.0341i & 2.1110+3.0183i & 0.4634+1.7009i & 1.1811+2.3560i \\ 1.4646+1.0428i & 2.9047+2.0090i & 2.7081+1.7155i & 3.3544+3.0897i & 2.6059+1.9291i & 1.1449+1.1409i & 1.7323+1.9562i \\ 1.1014+1.3758i & 2.8630+2.8554i & 2.7241+2.5984i & 4.0353+3.8373i & 3.2509+2.7793i & 2.3711+1.5791i & 2.6126+2.2774i \\ 1.0417+0.8922i & 2.0977+2.2487i & 1.9584+2.3345i & 2.4568+2.0245i & 1.9140+2.2563i & 0.8852+1.1671i & 1.2936+0.8882i \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1.6302+1.1994i & 2.7995+2.1430i & 2.5751+1.6989i & 2.7673+3.7497i & 2.0757+2.0271i & 0.3098+1.2518i & 1.0915+2.4986i \\ 0.4485+0.7437i & 1.7340+2.2485i & 1.6849+2.5580i & 2.9096+1.2503i & 2.4047+2.3085i & 2.2296+1.1078i & 2.1604+0.1857i \\ 1.1941+1.1835i & 1.9959+2.6001i & 1.8307+2.4705i & 1.9049+3.1328i & 1.4161+2.5553i & 0.1049+1.4102i & 0.6937+1.7458i \\ 1.3604+1.3769i & 2.0418+2.6560i & 1.8506+2.2706i & 1.6530+4.0758i & 1.1719+2.5509i & -0.3966+1.5076i & 0.3426+2.5781i \\ 0.5517+1.0956i & 1.9021+2.2274i & 1.8386+1.9933i & 3.0645+3.1096i & 2.5188+2.1601i & 2.2291+1.2407i & 2.2119+1.8821i \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1.9015+0.6399i & 3.1403+1.4586i & 2.8768+1.4222i & 2.9489+1.6319i & 2.1830+1.4419i & 0.0829+0.7815i & 1.0316+0.8654i \\ 0.6725+0.8131i & 2.3250+1.8862i & 2.2478+1.8607i & 3.7500+2.0357i & 3.0827+1.8697i & 2.7315+1.0049i & 2.7087+1.0514i \\ 1.7713+0.7406i & 3.0417+1.3392i & 2.7979+1.0752i & 3.0065+2.2965i & 2.2550+1.2699i & 0.3361+0.7787i & 1.1856+1.5190i \\ 1.5120+1.1491i & 2.7359+2.5126i & 2.5295+2.3793i & 2.8771+3.0553i & 2.1902+2.4674i & 0.5972+1.3649i & 1.2812+1.7124i \\ 0.2089+0.7056i & 0.9802+2.1076i & 0.9596+2.3849i & 1.7398+1.2161i & 1.4483+2.1608i & 1.4224+1.0417i & 1.3392+0.2141i \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 7.1881+6.9753i & 5.1342+6.9579i & 7.6688+6.1966i & 8.1884+7.9916i \\ 4.8046+4.6507i & 2.4018+4.0024i & 5.8412+3.3377i & 5.0574+5.1448i \\ 6.5231+6.3144i & 4.6489+5.9511i & 6.9664+5.1761i & 7.4267+7.1342i \\ 5.0711+4.5700i & 5.6075+5.0339i & 4.0314+4.6524i & 6.5785+5.3729i \\ 5.6683+6.0800i & 3.7989+5.4458i & 6.2208+4.6295i & 6.3563+6.7875i \\ 3.9585+3.9067i & 3.1749+4.4012i & 3.9818+4.0993i & 4.6496+4.6212i \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 4.8070+4.3309i & 4.0376+4.7113i & 4.7088+4.3351i & 5.7199+5.0747i \\ 4.7909+3.8380i & 3.6841+4.9274i & 4.9292+4.7796i & 5.5635+4.7139i \\ 5.0686+4.9549i & 2.5796+4.3957i & 6.1302+3.7199i & 5.3538+5.5192i \\ 6.0918+6.4247i & 4.4663+5.7944i & 6.4191+4.9416i & 6.9861+7.1838i \\ 4.6481+4.4668i & 2.9568+3.9805i & 5.2112+3.3757i & 5.1483+4.9807i \\ 5.6523+5.9783i & 3.4304+4.8733i & 6.4517+3.9518i & 6.1939+6.5352i \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 5.8659+5.3457i & 4.4210+6.0571i & 6.0977+5.6524i & 6.7756+6.3334i \\ 5.4631+5.1432i & 2.8010+5.0694i & 6.5931+4.4929i & 5.7788+5.8750i \\ 7.2234+7.1743i & 6.2645+7.0710i & 6.9389+6.2668i & 8.6748+8.1950i \\ 5.5242+5.4475i & 6.0286+5.3725i & 4.4471+4.7627i & 7.1340+6.2235i \\ 4.8612+5.0317i & 2.0329+3.6799i & 6.1858+2.8003i & 4.9566+5.3788i \\ 5.3460+5.1352i & 4.7360+5.8883i & 5.0663+5.5168i & 6.4605+6.1042i \end{pmatrix}.$$

It follows from Theorem 1 and Algorithm 1 that

$$\mathcal{A} = \mathcal{P} * \mathcal{S}_a * \mathcal{Q}, \quad \mathcal{B} = \mathcal{P} * \mathcal{S}_b * \mathcal{V}, \quad \mathcal{C} = \mathcal{U} * \mathcal{S}_c * \mathcal{Q}, \tag{23}$$

where $\mathcal{U} \in \mathbb{C}^{t\times t\times n_3}$ *and* $\mathcal{V} \in \mathbb{C}^{q\times q\times n_3}$ *are unitary,* $\mathcal{P} \in \mathbb{C}^{m\times m\times n_3}$, $\mathcal{Q} \in \mathbb{C}^{p\times p\times n_3}$ *are invertible, and*

$$P_1 = \begin{pmatrix} 0.6408-0.5877i & -0.0936-0.0107i & -0.4060+0.5214i & 2.2421+5.0044i & 0.2949+0.0390i \\ -0.1613+0.2793i & 0.4269-0.0810i & 0.4228-1.2971i & 2.6692+5.3346i & 0.3346+0.0419i \\ 0.4351-0.4090i & -0.6897+0.1941i & -0.7972+0.1545i & 2.3592+5.8345i & -0.6301-0.0906i \\ 0.4874-0.7717i & -0.9595+0.3726i & -0.2353+1.3777i & 2.0832+8.1176i & 0.1698+0.0205i \\ 0.1609-0.1738i & 0.8428-0.1838i & 0.9707-0.3759i & 1.4278+6.5459i & -0.4527-0.0595i \end{pmatrix},$$

$$P_2 = \begin{pmatrix} -0.2512+0.3184i & -0.8487+0.5414i & -1.1722-1.0526i & 3.0836+7.1109i & -0.1229-0.0429i \\ -0.3248+0.4861i & 1.0281-0.4627i & 1.7064+1.6331i & 5.3474+7.9779i & -0.0898-0.0413i \\ -0.2539+0.3055i & -0.0275-0.3210i & -0.4593-0.2713i & 3.7814+5.1409i & -0.0982+0.0136i \\ -0.3560+0.0728i & -0.4230-0.1031i & -0.7313-1.1984i & 4.1175+6.0276i & -0.0816-0.0287i \\ 0.3311-0.3674i & -0.4491+0.6748i & 0.5431+0.8102i & 2.2330+5.9796i & -0.0161+0.0261i \end{pmatrix},$$

$$P_3 = \begin{pmatrix} -0.3059+0.3610i & 0.5091-0.4685i & -0.5184-0.9038i & 4.2031+6.9941i & 0.1701+0.0145i \\ 0.6811-0.5568i & -0.3334+0.5769i & 0.9122+1.3152i & 2.8736+7.2549i & 0.1871+0.0129i \\ -0.1097+0.1755i & -0.3939+0.0771i & -0.8263-0.9132i & 3.7283+8.3036i & 0.1302+0.0584i \\ -0.5188+0.2754i & 0.1895-0.4533i & -0.0753-0.1377i & 2.3868+9.6501i & 0.1436+0.0154i \\ -0.5290+0.5041i & 0.9653-0.4482i & 1.4892+1.1565i & 3.9351+5.1095i & -0.0641+0.0167i \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} -0.0152+0.0008i & 0.0052-0.0003i & 0.0042-0.0002i & 0.0085-0.0004i \\ 0.3822-0.8686i & -0.5113+0.7359i & 0.0025+0.6981i & 0.1330-0.5357i \\ -1.3744+0.1132i & 2.2744-2.3223i & -0.5762+3.5012i & -0.3227-0.8922i \\ 16.3877-9.0211i & 15.0754-5.6503i & 14.7485-9.8723i & 18.7357-10.3581i \end{pmatrix},$$

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$$Q_2 = \begin{pmatrix} -0.0159+0.0167i & 0.0055-0.0058i & 0.0044-0.0047i & 0.0089-0.0093i \\ -0.1900-0.4386i & 0.2039+0.4691i & 0.0773+0.1913i & -0.0979-0.2206i \\ -0.0433-0.1733i & 0.7916+0.1304i & -1.0174+0.1131i & 0.2912-0.1051i \\ 17.1512-7.2539i & 15.0898-4.4501i & 16.2790-8.6406i & 19.4136-8.4856i \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} -0.0155-0.0167i & 0.0054+0.0058i & 0.0043+0.0047i & 0.0087+0.0093i \\ -0.2820-0.3994i & 0.2413+0.4149i & 0.2093+0.1722i & -0.1650-0.1771i \\ 0.4003+0.0073i & 0.4382+0.0189i & -1.5852+0.0118i & 0.5705+0.0154i \\ 19.0878-3.7427i & 17.2905-0.7168i & 17.6647-5.5544i & 22.0257-4.3068i \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0.1054+0.2152i & 0.0949-0.0731i & 0.0209-0.0258i & 0.2220+0.0214i & 0.0051-0.1728i & -0.0446+0.1700i \\ -0.1830-0.0106i & 0.3933-0.2579i & 0.0145+0.1753i & -0.0275+0.1724i & -0.1577-0.2441i & 0.1679+0.0948i \\ -0.1066-0.1453i & -0.0150+0.0924i & -0.0448+0.0555i & -0.0960+0.0102i & -0.0641-0.1135i & 0.2924+0.0744i \\ 0.1422+0.0776i & 0.1950-0.1345i & 0.2179-0.0819i & -0.1872+0.0065i & 0.2229+0.3532i & 0.1096+0.1109i \\ 0.3893+0.1686i & 0.0423+0.0307i & -0.0311-0.0038i & -0.1386-0.0652i & -0.1125+0.0171i & -0.1152+0.1255i \\ -0.1001+0.0303i & 0.3373-0.1433i & -0.1972-0.1863i & 0.0041+0.0017i & 0.0424+0.0948i & 0.2399+0.0643i \end{pmatrix},$$

$$U_2 = \begin{pmatrix} -0.0271+0.0574i & -0.4738+0.1835i & 0.1470-0.1615i & -0.0016+0.0967i & 0.0458+0.0943i & 0.4059+0.0489i \\ -0.1138+0.0479i & 0.2263-0.0749i & 0.0146+0.1956i & 0.3062-0.1033i & 0.0124+0.1959i & -0.0104+0.1229i \\ 0.0058-0.1376i & -0.1306+0.0788i & 0.0147+0.0614i & 0.1488-0.1841i & -0.0683-0.1907i & 0.2169+0.0885i \\ -0.3325-0.0111i & -0.0530-0.0575i & 0.0370+0.1065i & 0.1376-0.1857i & 0.1034-0.1806i & -0.0858+0.1292i \\ -0.0897+0.0322i & -0.2217-0.0040i & -0.2887+0.2487i & -0.2355+0.0075i & -0.1306-0.0206i & 0.2770+0.0877i \\ 0.0872-0.0300i & -0.0489+0.0535i & -0.3116-0.0649i & -0.1771+0.2345i & -0.0770-0.2311i & -0.1628+0.2071i \end{pmatrix},$$

$$U_3 = \begin{pmatrix} 0.2309+0.0505i & -0.1479+0.1096i & -0.1576+0.2048i & 0.2842-0.0579i & 0.0791+0.0871i & -0.1743+0.1854i \\ 0.2495-0.1127i & -0.1168+0.1075i & 0.2459-0.0555i & 0.1508+0.0639i & -0.1420-0.2668i & -0.0037+0.1078i \\ -0.4449-0.2074i & -0.2231+0.0297i & 0.1255-0.0258i & -0.2640+0.1617i & 0.0398+0.2650i & -0.3123+0.2548i \\ 0.2953+0.0149i & 0.1418+0.0568i & -0.0295+0.1790i & -0.2309+0.0448i & 0.2169+0.2944i & 0.1615+0.1433i \\ 0.0385+0.1177i & 0.1669-0.0418i & 0.2708-0.3069i & -0.2373+0.0416i & -0.2154-0.2440i & -0.0079+0.1162i \\ -0.0826-0.1074i & -0.0056-0.0327i & -0.0647-0.3652i & 0.3285-0.2423i & 0.1011+0.1815i & 0.0755+0.0775i \end{pmatrix},$$

$$V_1 = \begin{pmatrix} -0.3415+0.1010i & 0.0300-0.2416i & 0.1807+0.1893i & 0.0558-0.1649i & -0.0886+0.1240i & -0.1395-0.1271i & 0.0734+0.2405i \\ 0.2890+0.0317i & -0.3653-0.0220i & -0.0635-0.1062i & 0.1117+0.1213i & 0.2928+0.0604i & -0.0117+0.0348i & -0.1816-0.1726i \\ 0.0874+0.2393i & 0.2122-0.0524i & 0.0238-0.0688i & 0.0848-0.0480i & -0.4611+0.0309i & 0.3490+0.0988i & -0.1682-0.0256i \\ 0.3112-0.2922i & -0.0508-0.0055i & 0.2356+0.2536i & -0.3086+0.0646i & -0.2476-0.1863i & 0.1613-0.0736i & 0.3172+0.0636i \\ 0.1966-0.1029i & 0.0407+0.1393i & -0.0874+0.3287i & 0.2014-0.4036i & -0.1963+0.2617i & -0.3328+0.1715i & 0.0904-0.3965i \\ -0.1763+0.3055i & -0.0640+0.3116i & 0.0264+0.2972i & -0.1110-0.1227i & 0.1530-0.0536i & 0.3092-0.5074i & -0.0026-0.3738i \\ 0.0089-0.1151i & 0.0295-0.2376i & 0.0286-0.2216i & 0.0459-0.2883i & 0.0381-0.2235i & 0.0328-0.1129i & 0.0325-0.1580i \end{pmatrix},$$

$$V_2 = \begin{pmatrix} -0.1297-0.1675i & 0.1617-0.0854i & -0.0165+0.0411i & -0.1585+0.1909i & 0.0221+0.0319i & -0.1052-0.0653i & 0.2073-0.1539i \\ -0.1460-0.0398i & 0.3177+0.0404i & -0.2142-0.0586i & -0.1842-0.1175i & 0.1187+0.1507i & -0.1224-0.1249i & 0.1848+0.1410i \\ 0.2523+0.0383i & 0.1733-0.0432i & -0.1905-0.0448i & -0.0875+0.1668i & -0.0881-0.0385i & 0.2080+0.0938i & -0.0417-0.2169i \\ -0.0506+0.1462i & 0.0356-0.1132i & -0.1192-0.0120i & 0.0142-0.0373i & 0.1457+0.0954i & -0.0786+0.0005i & -0.0253+0.0135i \\ -0.0858-0.0843i & -0.0128-0.0194i & 0.0478+0.0345i & -0.1027-0.0854i & 0.0893+0.0816i & 0.1421+0.1412i & -0.0558-0.0372i \\ -0.0696-0.0422i & -0.0683-0.0288i & -0.0686-0.0144i & 0.0534-0.0184i & 0.0216+0.0195i & 0.1322+0.0649i & 0.1099+0.0117i \\ 0.0581-0.0545i & 0.1136-0.1148i & 0.1071-0.1088i & 0.1218-0.1328i & 0.0985-0.1081i & 0.0376-0.0536i & 0.0579-0.0704i \end{pmatrix},$$

$$V_3 = \begin{pmatrix} -0.1006+0.1861i & 0.4325+0.0891i & -0.2664-0.1355i & -0.0683-0.0691i & -0.0932-0.0146i & 0.0766+0.1185i & -0.0017-0.0174i \\ -0.0680-0.1072i & 0.1545+0.0189i & -0.3050+0.0072i & -0.0057+0.0969i & 0.2728-0.0343i & -0.1181-0.0068i & -0.0462-0.0838i \\ 0.0460-0.1893i & 0.0507-0.0294i & -0.2515+0.1265i & 0.0121-0.0096i & 0.2423+0.0099i & -0.0580-0.1324i & -0.0801+0.1028i \\ 0.1875-0.0357i & 0.0615+0.1229i & -0.0588-0.0883i & -0.3116+0.0288i & 0.1080-0.0421i & -0.0381+0.0434i & 0.2962-0.0591i \\ -0.0765-0.0494i & 0.0254-0.0175i & 0.1073+0.0083i & -0.1587-0.0325i & 0.1164+0.0431i & 0.1268+0.0862i & -0.1286-0.0023i \\ -0.0533-0.0247i & -0.1041-0.0263i & -0.1371-0.0283i & 0.1006+0.0248i & -0.0428+0.0062i & 0.0822+0.0482i & 0.1601+0.0455i \\ 0.1279+0.0821i & 0.2705+0.1885i & 0.2540+0.1794i & 0.3278+0.2372i & 0.2590+0.1932i & 0.1360+0.1169i & 0.1811+0.1386i \end{pmatrix},$$

$$(S_a)_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4583 & 0 & 0 \\ 0 & 0 & 0.1069 & 0 \\ 0 & 0 & 0 & 0.0161 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(S_a)_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.0940-0.0121i & 0 & 0 \\ 0 & 0 & -0.0106+0.0028i & 0 \\ 0 & 0 & 0 & -0.0070+0.0002i \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(S_a)_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.0940+0.0121i & 0 & 0 \\ 0 & 0 & -0.0106-0.0028i & 0 \\ 0 & 0 & 0 & -0.0070-0.0002i \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(S_b)_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (S_b)_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (S_b)_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

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$$(S_c)_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \ (S_c)_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \ (S_c)_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}.$$

As special cases of T-RSVD, we obtain T-PSVD and T-QSVD for two tensors.

Corollary 1 (T-PSVD [8]). *Let $\mathcal{A} \in \mathbb{C}^{m\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{h\times n_2\times n_3}$. Then $\mathcal{A}$ and $\mathcal{B}$ can be factored as*

$$\mathcal{A} = \mathcal{U} * \mathcal{S}_a * \mathcal{P}, \quad \mathcal{B} = \mathcal{V} * \mathcal{S}_b * \mathcal{P}^{-*},$$

where $\mathcal{U} \in \mathbb{C}^{m\times m\times n_3}$ and $\mathcal{V} \in \mathbb{C}^{h\times h\times n_3}$ are unitary, and $\mathcal{P} \in \mathbb{C}^{n_2\times n_2\times n_3}$ is invertible, $\mathcal{S}_a$ is a $m \times n_2 \times n_3$ f-diagonal tensor, $\mathcal{S}_b$ is a $h \times n_2 \times n_3$ tensor whose frontal slices have the following form

$$(\mathcal{S}_b)_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} \begin{pmatrix} S_{r_{1j}} & 0 & 0 & 0 \\ 0 & 0 & I_{r_{3j}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, n_3.$$

Corollary 2 (T-QSVD [8]). *Let $\mathcal{A} \in \mathbb{C}^{m\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{h\times n_2\times n_3}$. Then $\mathcal{A}$ and $\mathcal{B}$ can be factored as*

$$\mathcal{A} = \mathcal{U} * \mathcal{S}_a * \mathcal{P}, \quad \mathcal{B} = \mathcal{V} * \mathcal{S}_b * \mathcal{P},$$

where $\mathcal{U} \in \mathbb{C}^{m\times m\times n_3}$ and $\mathcal{V} \in \mathbb{C}^{h\times h\times n_3}$ are unitary, $\mathcal{P} \in \mathbb{C}^{n_2\times n_2\times n_3}$ is invertible, $\mathcal{S}_a$ is a $m \times n_2 \times n_3$ f-diagonal tensor, $\mathcal{S}_b$ is a $h \times n_2 \times n_3$ tensor whose frontal slices have the following forms

$$(\mathcal{S}_b)_i = \sum_{j=1}^{n_3} \omega^{(j-1)(i-1)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{r_j} & 0 \end{pmatrix}, \quad i = 1 \ldots, n_3.$$

4. An application from color image watermarking processing

In this section, we use a third-order tensor $\mathcal{A} \in \mathbb{C}^{m\times n\times 3}$ stand for color image, where the number "3" represents the three channels of RGB, and m and n denote hight and width of the image. Then we present the T-RSVD-based color image watermarking schemes. At present, the research on the algorithm of adding a gray or color watermark to a color image is very rich. Moreover, B. Harjito *et al.* proposed the method to add two gray watermarks to a gray image [7]. However, to our knowledge, there has been little work on adding three color watermarks to one color image. Based on T-RSVD, we implant three color watermarks into a color image at the same time, and only save two keys to ensure extraction process.

Let $\mathcal{F}$ be a color host image of size $M \times M \times 3$. Three color watermark images of the same size $N \times N \times 3$, i.e., $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, are simultaneously inserted into the color host image. The formal procedures of the T-RSVD-based color watermark embedding are given as follows.

A1. T-RSVD-based decomposes three color watermark images $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$,

$$\mathcal{A} = \mathcal{P} * \mathcal{S}_a * \mathcal{Q},$$

$$\mathcal{B} = \mathcal{P} * \mathcal{S}_b * \mathcal{V},$$

$$\mathcal{C} = \mathcal{U} * \mathcal{S}_c * \mathcal{Q},$$

where $\mathcal{P}$ and $\mathcal{U}$ are the secret keys and saved to extract the implanted watermarks.

A2. Calculate the main components of each color watermark image,

$$\mathcal{A}_w \Leftarrow \mathcal{S}_a * \mathcal{Q}, \quad \mathcal{B}_w \Leftarrow \mathcal{S}_b * \mathcal{V}, \quad \mathcal{C}_w \Leftarrow \mathcal{S}_c * \mathcal{Q}.$$

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A3. Orthogonal transformation divides the color host image $\mathcal{F}$ into several non-overlapping color image blocks of size $N \times N \times 3$, and $\mathcal{T}_{ij} \in \mathbb{C}^{N \times N \times 3}$ is the orthogonal transformed image on block position (i, j), $i, j = 1, \ldots, m$, where $m = \frac{M}{N}$.

A4. Implant the main components $\mathcal{A}_w$, $\mathcal{B}_w$ and $\mathcal{C}_w$ into the transformed color host image blocks $\mathcal{T}_{11}$, $\mathcal{T}_{21}$ and $\mathcal{T}_{12}$:

$$\mathcal{T}_{11} \Leftarrow \mathcal{T}_{11} + \alpha \mathcal{A}_w,$$

$$\mathcal{T}_{21} \Leftarrow \mathcal{T}_{21} + \alpha \mathcal{B}_w.$$

$$\mathcal{T}_{12} \Leftarrow \mathcal{T}_{12} + \alpha \mathcal{C}_w,$$

where α is a scale factor used to control watermarking strength [7]. When the value of α is actually confirmed, the larger α is, the stronger the robustness of watermark is, and the weaker the invisibility of the watermark is. The balance between robustness and invisibility needs to be considered. Save $\mathcal{T}_{11}$, $\mathcal{T}_{21}$ and $\mathcal{T}_{12}$ to guarantee the extraction.

A5. Implement inverse orthogonal transformation for all image blocks $\mathcal{T}_{ij}$, and gain the watermarked color image $\mathcal{F}_w$.

The formal procedure of the T-RSVD-based color watermark extraction process is given as follows:

B1. Split the watermarked color image $\mathcal{F}_w$ into several non-overlapping color image blocks $\hat{\mathcal{T}}_{ij}$ with size $N \times N \times 3$, $i, j = 1, \ldots, m$.

B2. Extract the main components of each color watermark image as follows:

$$\hat{\mathcal{A}}_w \Leftarrow \frac{1}{\alpha}(\hat{\mathcal{T}}_{11} - \mathcal{T}_{11}),$$

$$\hat{\mathcal{B}}_w \Leftarrow \frac{1}{\alpha}(\hat{\mathcal{T}}_{21} - \mathcal{T}_{21}),$$

$$\hat{\mathcal{C}}_w \Leftarrow \frac{1}{\alpha}(\hat{\mathcal{T}}_{12} - \mathcal{T}_{12}).$$

B3. Calculate the extracted watermarks $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, and $\hat{\mathcal{C}}$:

$$\hat{\mathcal{A}} \Leftarrow \mathcal{P} * \hat{\mathcal{A}}_w,$$

$$\hat{\mathcal{B}} \Leftarrow \mathcal{P} * \hat{\mathcal{B}}_w,$$

$$\hat{\mathcal{C}} \Leftarrow \mathcal{U} * \hat{\mathcal{C}}_w.$$

The imperceptibility of the proposed schemes can be measured by Peak-Signalto-Noise-Ratio (PSNR) [7], when the value of PSNR is greater than 30, the signal distortion is less. Now, we present the basic framework for implanting three color watermarks concurrently into a color image, as well as a frame diagram for the extraction of the color watermarked image in Figures 1 and 2.

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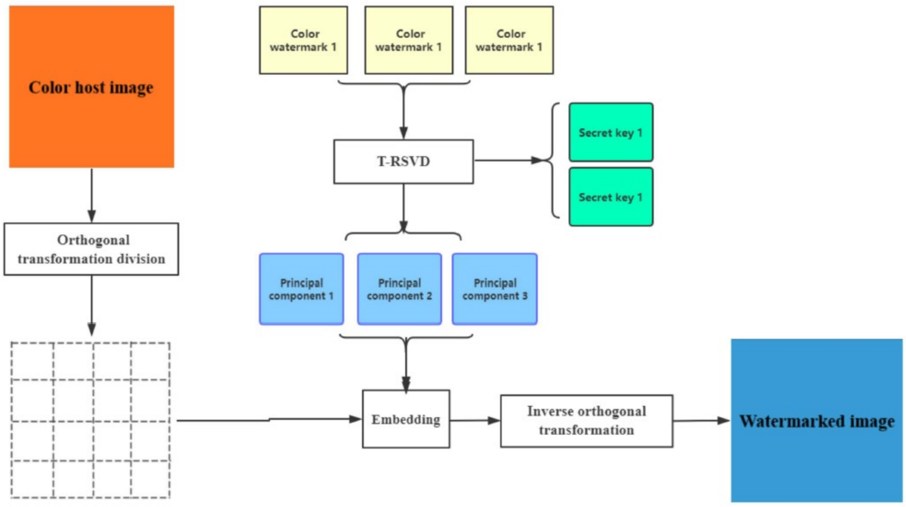

Figure 1. Schematic diagram for watermark embedding.

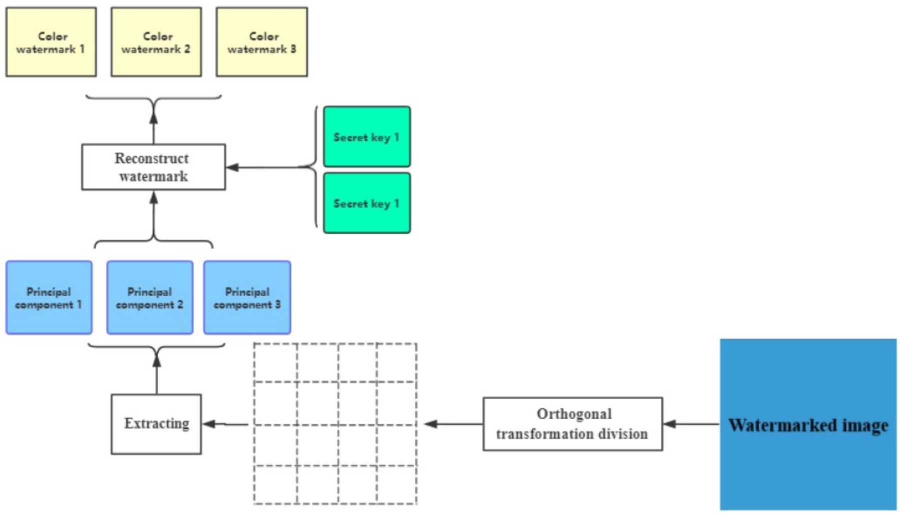

Figure 2. Schematic diagram for watermark extraction.

In this experimentation, we use the color images *trees* of 512×512 pixels as the host images, and three color images as the watermarks to be implanted, which have the same size 128×128. Choosing scaling factor $\alpha = 0.05$ for T-RSVD-based method, we use the above frameworks to deal with the actual color image watermarking problem by the discrete wavelet transform (DWT) [16]. The experimental result of the T-RSVD-based color image watermarking shows in Figure 3.

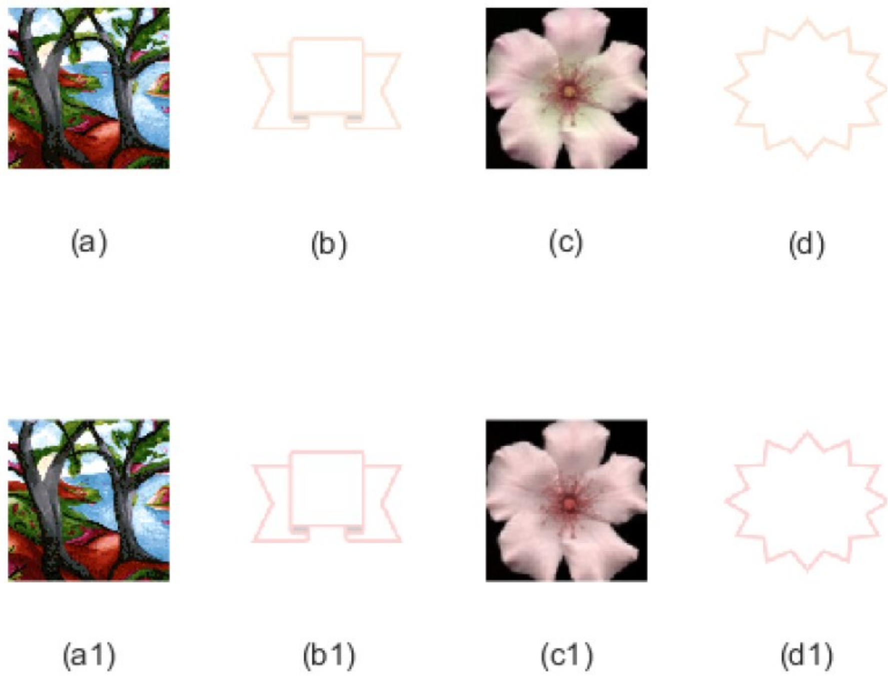

Figure 3. Result of the T-RSVD-based color image watermarking with $\alpha = 5$. (a) Host color image, (b) color watermark image 1, (c) color watermark image 2, (d) color watermark image 3, (a1) watermarked color image (PSNR=38.8733), (b1) extracted watermark 1 (PSNR=39.8408), (c1) extracted watermark image 2 (PSNR=32.7493), (c1) extracted watermark image 3 (PSNR=40.5018).

PSNR values indicate that the T-RSVD-based color image watermarking have the dependable imperceptibility and security. Furthermore, as can be seen from Figure 3, the information before and after processing is very consistent. These results clearly show that the proposed methods fully satisfy the requirements for imperceptibility and security.

The color watermark processing framework based on T-RSVD has the following advantages. First, three color watermarks can be handled concomitantly. Secondly, only two keys need to be stored, and three color watermarks can be extracted. Furthermore, through the numerical experiment, it can be seen that the proposed methods have the responsible imperceptibility and security.

5. Conclusions

We have derived the restricted singular value decomposition (T-RSVD) for three tensors under t-product. We have demonstrated the algorithms, as well as an application from color watermark processing.

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