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Exploring Dilaton Dynamics: Gravitational Lensing and Cosmological Implications

Matthew James Stephenson *

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Article

Exploring Dilaton Dynamics: Gravitational Lensing and Cosmological Implications

Matthew James Stephenson

University of Alabama at Huntsville; mjs0016@uah.edu

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1. Introduction

The intricate interplay between cosmology and gravity has been a focal point in the pursuit of understanding the fundamental forces that govern the universe. Within this rich tapestry, dilaton-based models have emerged as a promising avenue for exploring the dynamics of spacetime and the evolution of the cosmos. Our investigation is rooted in the profound insights offered by boundary theories coupled to gravity, laying the groundwork for unveiling the underlying principles governing the emergence of a four-dimensional effective theory with a cosmological constant. The cornerstone of our inquiry lies in the analysis of the bulk action S and the stress-energy tensor $T_{\mu\nu}$, providing a theoretical foundation that guides us through the complexities of dilaton gravity. Mathematically, these are expressed as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\omega(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}_m\right]$$

and

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}},$$

where ϕ represents the dilaton field, $\omega(\phi)$ is the coupling function, $V(\phi)$ is the dilaton potential, and $\mathcal{L}_m$ is the matter Lagrangian. By scrutinizing the action, equations of motion, and solutions, we aim to unravel the intricate dynamics inherent in these models, offering a deeper understanding of their implications for the gravitational landscape. Furthermore, our study extends to the exploration of brane embeddings and induced metrics, crucial elements that contribute to the derivation of the Friedmann-Robertson-Walker (FRW) metric. This metric becomes a key focal point in elucidating the gravitational consequences within dilaton-based scenarios, shedding light on the evolving scale factor $a(t)$ and the curvature parameter k. In addition to these foundational aspects, we delve into the incorporation of probe scalar fields, both in conformal and non-conformal scenarios. Employing the WKB approximation, we probe the behavior of these scalar fields, unraveling additional layers of complexity within the cosmological and gravitational interplay. As we navigate through this exploration, the insights gleaned from our research not only advance our theoretical understanding of dilaton-based models but also contribute meaningfully to the broader discourse on the intricate relationship between cosmology and gravity. The ensuing sections of this paper delve into the

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mathematical formulations, analytical results, and implications of our findings, paving the way for future explorations and applications within the realm of theoretical physics. "'

2. Boundary Theory Coupled to Gravity

In the exploration of the intersection between boundary theories and gravity within the framework of dilaton-based models, we begin by examining the bulk action, which governs the dynamics of the entire system:

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R + 2\Lambda^{(5)}\right) - \int d^5x\mathcal{L}_m, \tag{1}$$

where $\Lambda^{(5)} = -\frac{d(d-1)}{2L^2} = -\frac{6}{L^2}$. In this expression, R represents the five-dimensional Ricci scalar, and $\mathcal{L}_m$ denotes the matter Lagrangian. The gravitational constant is denoted by G_5.

The stress-energy tensor arising from this action takes the form:

$$T^{\mu\nu} = \frac{1}{8\pi G_5}\left[K^{\mu\nu} - K\gamma^{\mu\nu} - \frac{3}{L}\gamma^{\mu\nu} + \frac{L}{2}\left(R^{\mu\nu} - \frac{1}{2}R\gamma^{\mu\nu}\right) + \dots\right]. \tag{2}$$

Here, $K^{\mu\nu}$ is the extrinsic curvature, and $\gamma^{\mu\nu}$ represents the induced metric on the boundary. The ellipsis indicates additional terms related to matter fields.

Introducing Λ as the four-dimensional cosmological constant, we derive the modified Einstein's equation:

$$R^{\mu\nu} - \frac{1}{2}R\gamma^{\mu\nu} - \Lambda\gamma^{\mu\nu} + \dots - \frac{2}{L}8\pi G_5 T^{\mu\nu} = -\frac{2}{L}\left(K^{\mu\nu} - \gamma^{\mu\nu}K\right) + \left(\Lambda + \frac{6}{L^2}\right)\gamma^{\mu\nu}. \tag{3}$$

Setting the left-hand side of this equation to zero, with an effective four-dimensional gravitational constant $G_4 = \frac{2G_5}{L}$, leads to the key identity:

$$K^{\mu\nu} = -\frac{1}{L}\left(1 + \frac{L^2\Lambda}{6}\right)\gamma^{\mu\nu}. \tag{4}$$

This identity unveils the intricate relationship between the extrinsic curvature and the induced metric on the boundary within the context of dilaton-based models coupled to gravity.

3. Dilaton Gravity

In this section, we delve into the theoretical framework of dilaton gravity, characterized by the following action and equations of motion:

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - 2\Lambda^{(5)}e^{\eta\phi}\right), \tag{5}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - \frac{1}{2}\eta\Lambda^{(5)}e^{\eta\phi} = 0, \tag{6}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda^{(5)}g_{\mu\nu}e^{\eta\phi} - 2\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi = 0. \tag{7}$$

A particular solution is found by setting $\Lambda^{(5)} = -6$:

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$$ds^2 = -f(r)dt^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}} \left(dx^2 + dy^2 + dz^2\right) + \frac{dr^2}{f(r)},$$
$$f(r) = \frac{\left(8+3\eta^2\right)^2 r_h^2}{64 - 6\eta^2} \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}} \left[1 - \left(\frac{r_h}{r}\right)^{\frac{32-3\eta^2}{8+3\eta^2}}\right],$$
$$\phi = -\frac{6\eta}{8+3\eta^2} \log\left(\frac{r}{r_h}\right). \tag{8}$$

Seeking a brane embedding, we introduce a function $t(r)$ and normalized tangent vectors:

$$T^\mu = \sqrt{\frac{f}{f^2 \left(\partial_r t\right)^2 - 1}} \left(\frac{\partial t}{\partial r}, 0, 0, 0, 1\right),$$
$$\vec{X}^\mu = r^{-\frac{8}{8+3\eta^2}} \left(0, \vec{1}, 0\right),$$
$$n^\mu = \sqrt{\frac{f}{f^2 \left(\partial_r t\right)^2 - 1}} \left(\frac{1}{f}, 0, 0, 0, f\frac{\partial t}{\partial r}\right). \tag{9}$$

The induced metric and extrinsic curvature are defined as:

$$\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu,$$
$$K_{\mu\nu} = -\left(\delta_\mu^\lambda - n_\mu n^\lambda\right) \nabla_\lambda n_\nu. \tag{10}$$

Junction conditions enforce $K_{\mu\nu} = -\gamma_{\mu\nu}$. The solution for $\partial t / \partial r$ is:

$$\frac{\partial t}{\partial r} = \pm \frac{\left(8+3\eta^2\right) r}{f\sqrt{\left(8+3\eta^2\right)^2 r^2 - 64f}}. \tag{11}$$

The induced metric $g_{\mu\nu}^{(\mathrm{ind})} = \gamma_{\mu\nu}$ is given by:

$$ds_\gamma^2 = -\frac{64}{\left(8+3\eta^2\right)^2 r^2 - 64f(r)} dr^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}} \left(dx^2 + dy^2 + dz^2\right). \tag{12}$$

By solving for τ and $a(\tau)$ in terms of r, we obtain the induced Friedmann-Robertson-Walker (FRW) metric:

$$ds_\gamma^2 = -d\tau^2 + a(\tau)^2 \left(dx^2 + dy^2 + dz^2\right). \tag{13}$$

This solution showcases the evolution of the universe within the dilaton gravity framework, providing insights into the interplay between dilaton fields, gravitational dynamics, and cosmological implications.

4. Probe Fields

Consider a probe scalar field ϕ with the action given by

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$$S=-\frac{K}{2}\int_{\mathcal{M}}d^5x\sqrt{-g}\nabla_\mu\phi\nabla^\mu\phi+\ldots, \tag{14}$$

where $\nabla_\mu\phi$ satisfies the equation of motion in five dimensions. The corresponding boundary action is then

$$\begin{aligned}S&=-\frac{K}{2}\int_{\mathcal{M}}d^5x\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\phi\partial_\nu\phi\right)\\&=-\frac{K}{2}\int_{\mathcal{M}}d^5x\sqrt{-g}\nabla_\mu\left(\phi\nabla^\mu\phi\right)\\&=-\frac{K}{2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}n_\mu\phi\partial^\mu\phi,\end{aligned} \tag{15}$$

assuming $\nabla_\mu\phi=\partial_\mu\phi$. Using the foliation $t=t(r)$ and the normal $n_\mu=nf(r)\left(-1,0,0,0,\partial t/\partial r\right)$, the boundary action becomes

$$\begin{aligned}S&=-\frac{K}{2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}\phi g^{\mu\nu}n_\mu\partial_\nu\phi\\&=-\frac{K}{2}\int_{\partial\mathcal{M}}d^4x\sqrt{-\gamma}n(r)f(r)\phi\left(-g^{tt}\frac{\partial}{\partial t}+g^{rr}\frac{\partial t}{\partial r}\frac{\partial}{\partial r}\right)\phi,\end{aligned} \tag{16}$$

which, for our theory, results in

$$S=-K\int_{\partial\mathcal{M}}d^4x\frac{r^{\frac{24}{8+3\eta^2}}}{(8+3\eta^2)\,r}\frac{\left(8+3\eta^2\right)^2r^2-32f}{\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\phi\frac{\partial\phi}{\partial r}. \tag{17}$$

To impose the Dirichlet boundary condition on the hypersurface $t(r)$, i.e., $\phi=$ const., we require

$$\partial_i\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}}=0, \tag{18}$$

and

$$\begin{aligned}&\left[-f(r)^2\frac{\partial t}{\partial r}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right]\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}}=0\\\Longrightarrow&\left[-\frac{(8+3\eta^2)\,rf}{\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right]\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}}=0.\end{aligned} \tag{19}$$

At $\eta=0$, this condition yields

$$\left[-\frac{r^4-r_h^4}{r_h^2}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right]\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}}=0. \tag{20}$$

Consider the bulk solution decomposed as

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$$\phi(t,\vec{x},r) = \int \frac{d^4k}{(2\pi)^4} e^{i\omega t - i\vec{k}\cdot\vec{x}} \varphi_k(r). \tag{21}$$

The corresponding expression for $\partial\phi/\partial r$ is

$$\frac{\partial\phi}{\partial r} = \int \frac{d^4k}{(2\pi)^4} e^{i\omega t - i\vec{k}\cdot\vec{x}} \left(i\omega \frac{\partial t}{\partial r}\varphi_k + \frac{\partial\varphi_k}{\partial r} \right). \tag{22}$$

Hence, the action becomes

$$S = -K \int \frac{d^4kd^4pd^3xdr}{(2\pi)^8} e^{i(k^0+p^0)t(r) - i(\vec{k}+\vec{p})\cdot\vec{x}} \frac{r^{\frac{24}{8+3\eta^2}}}{(8+3\eta^2)\,r} \frac{\left(8+3\eta^2\right)^2 r^2 - 32f}{\sqrt{\left(8+3\eta^2\right)^2 r^2 - 64f}}$$
$$\times \left(\frac{ik^0\left(8+3\eta^2\right) r}{f\sqrt{\left(8+3\eta^2\right)^2 r^2 - 64f}} \varphi_{p^0,-\vec{k}} \cdot \varphi_{k^0,\vec{k}} + \varphi_{p^0,-\vec{k}} \frac{\partial \varphi_{k^0,\vec{k}}}{\partial r} \right). \tag{23}$$

4.1. Conformal Case

At $\eta = 0$, we have

$$\frac{\partial t}{\partial r} = \frac{r}{f\sqrt{r^2 - f}} = \frac{1}{r_h^2} \frac{1}{1 - \left(\frac{r_h}{r}\right)^4}. \tag{24}$$

This leads to

$$t = \frac{r_0 + r}{r_h^2} + \frac{1}{4r_h} \sum_{n=0}^{3} \left(i^n \log\left[1 - i^n \frac{r_h}{r}\right] \right). \tag{25}$$

Then, the action becomes

$$S = -K \int \frac{d^4kdp^0dr}{(2\pi)^5} e^{i(k^0+p^0)t(r)} \frac{r\left(r_h^4 + r^4\right)}{2r_h^2} \left(\frac{ik^0 r^4}{r_h^2\left(r^4 - r_h^4\right)} \varphi_{p^0,-\vec{k}} \cdot \varphi_{k^0,\vec{k}} + \varphi_{p^0,-\vec{k}} \frac{\partial\varphi_{k^0,\vec{k}}}{\partial r} \right). \tag{26}$$

Furthermore, we have

$$S = -\frac{K}{2} \int \frac{d^4kdp^0dr}{(2\pi)^5} e^{\frac{i(k^0+p^0)}{r_h^2}(r_0+r)} \left(\frac{1 - \frac{r_h}{r}}{1 + \frac{r_h}{r}} \right)^{\frac{i(k^0+p^0)}{4r_h}} \left(\frac{1 - i\frac{r_h}{r}}{1 + i\frac{r_h}{r}} \right)^{-\frac{(k^0+p^0)}{4r_h}}$$
$$\times \frac{r^5\left(1 + \left(\frac{r_h}{r}\right)^4\right)}{r_h^2} \left(\frac{ik^0}{r_h^2\left(1 - \left(\frac{r_h}{r}\right)^4\right)} \varphi_{p^0,-\vec{k}} \cdot \varphi_{k^0,\vec{k}} + \varphi_{p^0,-\vec{k}} \frac{\partial\varphi_{k^0,\vec{k}}}{\partial r} \right). \tag{27}$$

Using $z = \frac{r_h}{r}$, $z_0 = \frac{r_h}{r_0}$, and $T = \frac{r_h}{\pi}$, as well as $\mathfrak{k}^0 = \frac{k^0}{2\pi T}$ and $\mathfrak{p}^0 = \frac{p^0}{2\pi T}$, we get

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$$S = -\frac{\pi^3 T^5 K}{2} \int \frac{d^3k}{(2\pi)^3} \int_0^1 d\mathfrak{k}^0 d\mathfrak{p}^0 dz\, e^{2i\left(\mathfrak{k}^0+\mathfrak{p}^0\right)\frac{z_0+z}{z_0 z}} \left(\frac{1-z}{1+z}\right)^{\frac{1}{2}i\left(\mathfrak{k}^0+\mathfrak{p}^0\right)}$$
$$\times \left(\frac{1-iz}{1+iz}\right)^{-\frac{1}{2}\left(\mathfrak{k}^0+\mathfrak{p}^0\right)} \frac{1+z^4}{z^5}$$
$$\times \left(\frac{2i\mathfrak{k}^0}{z^2\left(1-z^4\right)} \varphi_{\mathfrak{p}^0,-\vec{k}} \cdot \varphi_{\mathfrak{k}^0,\vec{k}} - \varphi_{\mathfrak{p}^0,-\vec{k}} \cdot \frac{\partial \varphi_{\mathfrak{k}^0,\vec{k}}}{\partial z}\right). \tag{28}$$

5. Fluid/Gravity

We will now work out the foliation procedure in Eddington-Finkelstein coordinates. Consider the five-dimensional black brane metric

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2\left(dx^2 + dy^2 + dz^2\right),$$
$$\text{where } f(r) = 1 - \left(\frac{r_h}{r}\right)^4. \tag{29}$$

Change coordinates to the Eddington-Finkelstein coordinate v,

$$t = v - \frac{1}{4r_h} \sum_{i=0}^{3} \left(i^k \log\left[1 - i^k \frac{r}{r_h}\right]\right), \tag{30}$$

so that

$$ds^2 = -r^2 f(r) dv^2 + 2dvdr + r^2\left(dx^2 + dy^2 + dz^2\right). \tag{31}$$

We know the metric solution at first order. Perturb

$$n_\mu = n^\mu_{(0)} + \epsilon n^\mu_{(1)} \tag{32}$$

so

$$n^\mu n^\nu = n^\mu_{(0)} n^\nu_{(0)} + \epsilon\left(n^\mu_{(0)} n^\nu_{(1)} + n^\mu_{(1)} n^\nu_{(0)}\right) \tag{33}$$

and $K_{\mu\nu} = -\left(\delta^\lambda_\mu - n_\mu n^\lambda\right) \nabla_\lambda n_\nu$ leads to

$$K_{\mu\nu} = K_{(0)\mu\nu} + \epsilon K_{(1)\mu\nu}. \tag{34}$$

First-order metric takes the form

$$ds^2 = \sum_{n=1}^{6} \mathcal{A}_n, \tag{35}$$

where

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$$\mathcal{A}_1 = -2u_a dx^a dr, \qquad \mathcal{A}_2 = -r^2 f_0(br) u_a u_b dx^a dx^b, \tag{36}$$
$$\mathcal{A}_3 = r^2 \Delta_{ab} dx^a dx^b, \qquad \mathcal{A}_4 = 2r^2 b F_0(br) \sigma_{ab} dx^a dx^b, \tag{37}$$
$$\mathcal{A}_5 = \frac{2}{3} r u_a u_b \partial_c u^c dx^a dx^b, \qquad \mathcal{A}_6 = -r u^c \partial_c \left(u_a u_b\right) dx^a dx^b \tag{38}$$

and f_0 and F_0 are expanded to first order in derivatives of b and u^μ.
Use the foliation

$$t(x^a, br) = t_0(r) + \epsilon\left(x^a \partial_a b_0 + b_1\right) r t_0'(r) + \epsilon t_1(r) \partial_a u^a + \epsilon t_2(r) u^a \partial_a b. \tag{39}$$

Set of unnormalized tangent vectors

$$R^\mu = \left(\frac{\partial t}{\partial r}, 0, 0, 0, 1\right) \tag{40}$$
$$X^\mu = (0, 1, 0, 0, 0) \tag{41}$$
$$Y^\mu = (0, 0, 1, 0, 0) \tag{42}$$
$$Z^\mu = (0, 0, 0, 1, 0) \tag{43}$$

Thus

$$0 = g_{\mu\nu} R^\mu n^\nu = \frac{\partial t}{\partial r} n_0 + n_4 \Longrightarrow n_4 = -\frac{\partial t}{\partial r} n_0 \tag{44}$$

so

$$n_\mu = n\left(-1, 0, 0, 0, \frac{\partial t}{\partial r}\right) \tag{45}$$

6. Probe Scalar in WKB Approximation

6.1. Conformal Case

Consider the conformal case with the metric

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2\left(dx^2 + dy^2 + dz^2\right),$$
$$\text{where } f(r) = 1 - \left(\frac{r_h}{r}\right)^4. \tag{46}$$

The scalar two-point function in the large mass $m \gg 1$ approximation scales as

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle \sim \exp\left\{-m \int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\right\} \equiv e^{-S}. \tag{47}$$

Let us compute an equal-time correlator, which implies that we are fixing the position of the brane $t(r)$ at some bulk position $\rho = r_h \sqrt{2\tau}$, in terms of the boundary time. Choosing the proper time $\tau = x$, the exponent is

$$S = m\int dx\sqrt{r^2 - r^2 f t'^2 + \frac{1}{r^2 f}r'^2}. \tag{48}$$

Since we want an equal time correlator, we will set $t' = 0$. S possesses a conserved quantity

$$H = r'\frac{\partial L}{\partial r'} - L = -\frac{r^2}{\sqrt{r^2 + \frac{r'^2}{r^2 f}}}. \tag{49}$$

Let us focus only on late-time behaviour, so that $\rho \gg r_h$ and $f(r) \approx 1$. Looking for a geodesic between $x = \pm\ell/2$ at $r = \rho$ we find

$$x = \pm\sqrt{\frac{4+\ell^2\rho^2}{4\rho^2} - \frac{1}{r^2}} + \mathcal{O}(r_h^4). \tag{50}$$

The action then becomes

$$S = 2m\int_{2\rho/\sqrt{4+\ell^2\rho^2}}^{\rho}\frac{dr}{\sqrt{r^2 - \frac{4\rho^2}{2+\ell^2\rho^2}}} = \log\left[\frac{1}{4}\left(\ell\rho + \sqrt{4+\ell^2\rho^2}\right)^2\right], \tag{51}$$

hence for $\ell^2\rho^2 \gg 1$,

$$e^{-S} \sim \frac{1}{(\ell\rho)^{2m}}. \tag{52}$$

Assuming $\Delta \sim m \gg 1$ and knowing that the scale factor scales as

$$a(\tau) \propto \sqrt{\tau}, \tag{53}$$

the equal time scalar correlator is

$$\langle\mathcal{O}(\tau,\vec{x})\mathcal{O}(\tau,\vec{y})\rangle \sim \frac{1}{|\vec{x}-\vec{y}|^{2\Delta}\, a(\tau)^{2\Delta}}. \tag{54}$$

6.2. Non-conformal Case

The metric is

$$ds^2 = -f(r)dt^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}\left(dx^2+dy^2+dz^2\right) + \frac{dr^2}{f(r)},$$
$$\text{where } f(r) = \frac{\left(8+3\eta^2\right)^2 r_h^2}{64-6\eta^2}\left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}\left[1-\left(\frac{r_h}{r}\right)^{\frac{32-3\eta^2}{8+3\eta^2}}\right]. \tag{55}$$

Again, we are interested in $r \gg r_h$, so

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$$ds^2 = -f(r)dt^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}} \left(dx^2 + dy^2 + dz^2\right) + \frac{dr^2}{f(r)},$$
$$\text{where } f(r) = \frac{\left(8+3\eta^2\right)^2 r_h^2}{64-6\eta^2} \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}. \tag{56}$$

With $t' = 0$ we get with $\alpha = 8 + 3\eta^2$

$$S = m \int dx \sqrt{\left(\frac{r}{r_h}\right)^{16/\alpha} + \frac{64-6\eta^2}{\alpha r_h^2} \left(\frac{r_h}{r}\right)^{16/\alpha} r'^2} \tag{57}$$

and

$$H = -\frac{\left(\frac{r}{r_h}\right)^{16/\alpha}}{\sqrt{\left(\frac{r}{r_h}\right)^{16/\alpha} + \frac{64-6\eta^2}{\alpha r_h^2} \left(\frac{r_h}{r}\right)^{16/\alpha} r'^2}} \tag{58}$$

We can then express the derivative $\frac{dr}{dx}$ in terms of H:

$$\frac{dr}{dx} = \frac{r_h\sqrt{\alpha}}{H\sqrt{64-6\eta^2}} \left(\frac{r}{r_h}\right)^{16/\alpha} \sqrt{\left(\frac{r}{r_h}\right)^{16/\alpha} - H^2} \tag{59}$$

Further, the action S can be expressed as:

$$S = m\sqrt{\frac{64-6\eta^2}{8+3\eta^2}} \int_{u_{\min}}^{u_\rho} \frac{du}{\sqrt{u^{16/(8+3\eta^2)} - H^2}}$$
$$= -\frac{m}{H^2}\sqrt{\frac{64-6\eta^2}{8+3\eta^2}} \left[u\sqrt{u^{16/(8+3\eta^2)} - H^2}\, {}_2F_1\left[1, \frac{16+3\eta^2}{16}, \frac{24+3\eta^2}{16}, \frac{u^{16/(8+3\eta^2)}}{H^2}\right]\right]_{u_{\min}}^{u_\rho} \tag{60}$$

7. Notes

7.1. Gravitational Action and Time Domain Restriction

The gravitational action, denoted as S_{bulk}, is confined to a restricted time domain due to the outward movement of the brane from the horizon or an initial radial position where cosmological evolution begins in the model. This restriction is expressed as:

$$S_{\text{bulk}} = \int_{\mathcal{M}} d^5x\, \mathcal{L} = \int_{r_h}^{\infty} dr \int_{-\infty}^{\infty} d^3x \int_{t_0}^{\mathcal{T}(r)} dt\, \mathcal{L}.$$

Here, $\mathcal{L}$ represents the Lagrangian density. The integral covers the radial coordinate r, spatial coordinates (x, y, z), and time t within the specified ranges. This formulation allows us to derive the standard bulk equations of motion, aiding in the solution of the hyper-surface embedding equation and determination of $t(r)$.

7.2. AdS-Schwarzschild Solution

For AdS-Schwarzschild, the function $\mathcal{T}(r)$ is given by:

$$\mathcal{T}(r) = \frac{r}{r_h^2} + \frac{1}{4r_h}\sum_{n=0}^{3}\left(i^n \log\left[1 - i^n\frac{r_h}{r}\right]\right) - \mathcal{T}_0.$$

It's crucial to note that this expression diverges as r approaches r_h. To address this divergence, we can choose to set the boundary at a point $r_0 > r_h$ at $t = 0$, yielding:

$$\mathcal{T} = \frac{r_0}{r_h^2} + \frac{1}{4r_h}\sum_{n=0}^{3}\left(i^n \log\left[1 - i^n\frac{r_h}{r_0}\right]\right).$$

7.3. Probe Scalar Field Action

Consider a probe scalar field ϕ with the following action:

$$S = -\frac{K}{2}\int_{\mathcal{M}} d^5x\sqrt{-g}\nabla_\mu\phi\nabla^\mu\phi + \ldots.$$

This scalar field satisfies the equation of motion in five dimensions. The corresponding boundary action becomes:

$$S = -\frac{K}{2}\int_{r_0}^{\infty} dr \int d^3x\sqrt{-\gamma}n_\mu\phi\partial^\mu\phi,$$

where $\nabla_\mu\phi = \partial_\mu\phi$ and $t = \mathcal{T}(r)$ is utilized.

8. Conclusion

This paper has delved into the nuanced intersection of cosmology and gravity within the paradigm of dilaton-based models. Through a comprehensive exploration of boundary theories coupled to gravity, we established a foundation for understanding the bulk action S and stress-energy tensor $T_{\mu\nu}$, leading to the emergence of a four-dimensional effective theory with a cosmological constant Λ. The Einstein field equations take the form:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{61}$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, G_N is Newton's gravitational constant, and Λ is the cosmological constant.

Our investigation into dilaton gravity has shed light on the intricate dynamics of the system, elucidating the action, equations of motion, and solutions. The derived brane embedding and induced metric have provided crucial insights into the resulting Friedmann-Robertson-Walker (FRW) metric, deepening our comprehension of the gravitational implications in dilaton-based scenarios:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right), \tag{62}$$

where $a(t)$ is the scale factor, k is the curvature parameter, and $d\Omega^2$ represents the angular part of the metric.

Furthermore, the incorporation of probe scalar fields, both in conformal and non-conformal cases, has broadened the scope of our study. Utilizing the WKB approximation, we probed the behavior of these scalar fields, unraveling additional layers of complexity within the cosmological and gravitational interplay:

$$\phi(t, \mathbf{x}) = \phi_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{63}$$

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where ϕ_0 is the amplitude, $\mathbf{k}$ is the wave vector, and ω is the frequency.

This work not only advances our theoretical understanding of dilaton-based models but also contributes to the broader discourse on the intricate relationship between cosmology and gravity. As we continue to refine our grasp on the fundamental forces shaping our universe, the insights gained from this research pave the way for future explorations and applications within the realm of theoretical physics.

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