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Abstract: In my paper, a divisibility rule named the Queen function is described, which has been generalized for every integer and encompasses all previously established divisibility rules. Subsequently, the paper discusses other areas where this function is instrumental. These include simplifying ratios, generating prime numbers, and more. Towards the end of the paper, a set of hypotheses is also presented. Among these, the most significant is the novel approach involving the application of the Queen function for prime number generation.

Keywords: arithmetic functions; discrete mathematics; primes; number theory

MSC: 11A25; 11A05; 11A41; 11N05

Introduction

I would like to start our discussion with the question, what are integers or natural numbers? Natural numbers have been a tool for measuring a reality humans have used for centuries. Integers, on the other hand, are an abstraction that includes a bit more of the abstract aspect at a point where they depart from reality. What are the characteristics of natural numbers then? Of course, one of them is whether they are divisible by each other or not. Naturally, there are other features, but the focus of this article is mostly on this aspect. When one natural number is divided by another natural number resulting in another natural number, we call this divisibility, and by using this divisibility, we generate some special numbers, such as prime numbers. I will touch on this topic shortly, but first, let's talk about the rules. What are the rules of divisibility, or in other words, the rules of being divisible? Of course, the divisibility rules, just like natural numbers, have always been intriguing throughout history. Imagine having such a power that directly gives you the divisibility of these natural numbers, which you compulsively use throughout your life, it would be nice, of course. And for centuries, many people have researched this and derived many divisibility rules. But there was a problem; none of them were generalized. Until a recent article summarized this. Of course, similar rules are also written in ancient texts, but the first known article written on this subject is "General Divisibility Criteria" by A. A. Grinberg and S. Luryi. This article generally talks about the most basic general divisibility rule. In my article, there is a group of formulas that includes all divisibility rules and also the function of these formulas in finding prime numbers. In addition, some simplification methods related to the formulas are also mentioned. These formulas are divided into two groups: "Bishop" and "Queen," and with the help of both, all divisibility rules can be used, and prime numbers can be found. Additionally, a few hypotheses provide important data.

1. Simple General Divisibility Algorithms

1.1. First General Divisibility Rule, Queen Function;

Definition 1.1. $qn : \mathbf{Z} \to \mathbf{Z}$ where $\forall x, y :\in \mathbf{Z}$, $\forall m :\in \mathbf{N}$:

$$qn_m(x, y) = S_0(x, m)S_1(y, m) - S_1(x, m)S_0(y, m)$$

1 2 3

Lemma 1.2.

$$\frac{x}{y} \in \mathbf{Z} \Rightarrow \frac{qn_m(x,y)}{y} \in \mathbf{Z}$$

.

Proof. By the definition,

$$qn_m(x,y) = S_0(x,m)S_1(y,m) - S_1(x,m)S_0(y,m)$$

$$qn_m(x,y) = (x - S_1(x,m)10^m)S_1(y,m) - S_1(x,m)(y - S_1(y,m)10^m)$$

$$qn_m(x,y) = S_1(y,m)x - S_1(x,m)S_1(y,m)10^m + S_1(x,m)S_1(y,m)10^m - S_1(x,m)y$$

$$qn_m(x,y) = S_1(y,m)x - S_1(x,m)y$$

Dividing both sides by y,

$$\frac{qn_m(x,y)}{y} = \frac{S_1(y,m)x - S_1(x,m)y}{y}$$

$$\frac{qn_m(x,y)}{y} = S_1(y,m)\frac{x}{y} - S_1(x,m)$$

then in this setup, there is only one none-integer part and it is $\frac{x}{y}$. So: $\frac{x}{y} \in \mathbf{Z} \Rightarrow \frac{qn_m(x,y)}{y} \in \mathbf{Z}$. □

Lemma 1.3.

$$\left(Gcd^4(y, S_1(y,m)) = 1 \right) \Rightarrow \left(\left(\frac{qn_m(x,y)}{y} \in \mathbf{Z} \right) \Longleftrightarrow \left(\frac{x}{y} \in \mathbf{Z} \right) \right)$$

Proof. From the first proof,

$$\frac{qn_m(x,y)}{y} = S_1(y,m)\frac{x}{y} - S_1(x,m)$$

If $Gcd(b, S_1(y,m)) = 1$, there will be 2 cases:

$\frac{x}{y} \notin \mathbf{Z}$ Then $S_1(y,m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{qn_m(x,y)}{y} \notin \mathbf{Z}$
$\frac{x}{y} \in \mathbf{Z}$ Then $S_1(y,m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{qn_m(x,y)}{y} \in \mathbf{Z}$

If we try to come from the right-hand side to the left, there will be 2 cases too:

$\frac{qn_m(x,y)}{y} \notin \mathbf{Z}$ Then only possibility is $S_1(y,m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{x}{y} \notin \mathbf{Z}$
$\frac{qn_m(x,y)}{y} \in \mathbf{Z}$ Then only possibility is $S_1(y,m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{x}{y} \in \mathbf{Z}$ because $Gcd(y, S_1(y,m)) = 1$, in equation $S_1(y,m)$ and y won't be simplified.

So all the possibilities in the if and only if equation will be accurate. □

Example 1.4. Normally in number 21's divisibility rule, we should look at if it's divisible by 3 and then 7. This is very complicated. But the reason is that its second digit has no same divisors except 1 with 21. It is very easy to use in the Queen algorithm. In Table 1 in the first column, some numbers are used with 21's algorithm. In other columns, there are results of the algorithm and in the first row there are the number of reusing of the algorithm all of the example tables have the same system:

---

1    $S_0 : \mathbf{Z} \to \mathbf{Z}$ where $S_0(x,m)$ is x's digits less than the m'th digit.
2    $S_1 : \mathbf{Z} \to \mathbf{Z}$ where $S_1(x,m)$ is x's digits greater than the m'th digit.
3    $x = S_0(x,m) + S_1(x,m)10^m$
4    $Gcd : \mathbf{Z} \to \mathbf{Z}$ where $Gcd(x,y)$ is the greatest common divisor of x and y.

Table 1

Algorithm Reapply Count ↦	1	2	3	4	5	6
126	0	0	0	0	0	0
11.214	1.113	105	0	0	0	0
93.450	9.345	9.24	84	0	0	0
398.034	39.795	3.969	378	21	0	0
194.481	19.446	1.932	189	0	0	0
693	63	0	0	0	0	0
2.310	231	21	0	0	0	0
21	0	0	0	0	0	0
441	42	0	0	0	0	0
9.261	924	84	0	0	0	0

Example 1.5. Normally 31 is a prime number so also it is so big that can't have a rule. But in the Queen algorithm it has a very easy rule that can make numbers smaller a lot.

Table 2

Algorithm Reapply Count ↦	1	2	3	4	5	6
279	0	0	0	0	0	0
31.248	3.100	310	31	0	0	0
2.687.328	268.708	26.846	2666	248	0	0
340.101	34.007	3.379	310	31	0	0
62	0	0	0	0	0	0
651	62	0	0	0	0	0
31	0	0	0	0	0	0
961	93	0	0	0	0	0

Example 1.6. 101 is a very big number for getting some divisibility rule but also it has 3 digits that can be a possibility for us in the Queen algorithm in lower variables. Our lower variable is 1 or 2. Both of them are great but if we choose to get a small result, lower variable 2 is better.

Table 3

Algorithm Reapply Count ↦	1	2	3	4	5
1.111	0	0	0	0	0
101	0	0	0	0	0
10.201	101	0	0	0	0
1.030.301	10.302	101	0	0	0
104.060.401	1.040.603	10.403	101	0	0
10.510.100.501	105.101.004	1.051.006	10.504	101	0

1.2. Queen Function For Smaller Numbers:

Lemma 1.7. *In the Queen algorithm, we should get $Gcd(y, S_1(y, m)) = 1$ for using it in any number. So if y is smaller than 10 then it won't be able to be $Gcd(y, S_1(y, m)) = 1$. Then we should use its multiples. For any natural number k, yk number also will be usable in Queen algorithm for y:*

$$\frac{x}{y} \in \mathbf{Z} \Rightarrow \frac{qn_m(x, yk)}{y} \in \mathbf{Z}$$

Proof. in **Lemma 1.2** we saw that,

$$qn_m(x, yk) = S_1(yk, m)x - S_1(x, m)yk$$

$$\frac{qn_m(x, yk)}{y} = S_1(yk, m)\frac{x}{y} - S_1(x, m)\frac{yk}{y}$$

$$\frac{qn_m(x, yk)}{y} = S_1(yk, m)\frac{x}{y} - S_1(x, m)k$$

Here we came back to our first proof. Also in here we should think like if $\frac{x}{y} \in \mathbf{Z}$, then $\frac{qn_m(x,yk)}{y} \in \mathbf{Z}$. $\square$

In this theorem, we also saw that the Queen function's result gave us the same divisors of x and y.

Lemma 1.8.

$$\left(Gcd(y, S_1(yk, m)) = 1 \right) \Rightarrow \left(\left(\frac{qn_m(x, yk)}{y} \in \mathbf{Z} \right) \iff \left(\frac{x}{y} \in \mathbf{Z} \right) \right)$$

Proof. Same as **Lemma 1.3** proof we will see 2 possibilities for each side:

$\frac{x}{y} \notin \mathbf{Z}$ Then $S_1(yk, m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{qn_m(x,yk)}{y} \notin \mathbf{Z}$ because Gcd(y, S1(y, m)) = 1, in equation, S1(y, m) and y won't be simplified.

$\frac{x}{y} \in \mathbf{Z}$ Then $S_1(yk, m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{qn_m(x,yk)}{y} \in \mathbf{Z}$

If we try to come from the right-hand side to the left, there will be 2 cases too:

$\frac{qn_m(x,yk)}{y} \notin \mathbf{Z}$ Then only possibility is $S_1(yk, m)\frac{x}{y} \notin \mathbf{Z}$, so $\frac{x}{y} \notin \mathbf{Z}$

$\frac{qn_m(x,yk)}{y} \in \mathbf{Z}$ Then only possibility is $S_1(yk, m)\frac{x}{y} \in \mathbf{Z}$, so $\frac{x}{y} \in \mathbf{Z}$ because $Gcd(y, S_1(yk, m)) = 1$, in equation, $S_1(yk, m)$ and y won't be simplified.

So all the possibilities in the if and only if equation will be accurate. $\square$

1.3. Multiple Times Applied Queen Function:

Definition 1.9. For future usage, we will use a function called multiple Queen:

$\forall k :\in \mathbf{Z}^+$

$$qn_m(x, y)^{(k)} = \underbrace{qn_m(\dots(qn_m(x, y)\dots), y)}_{\text{k pieces}}$$

$$qn_m(x, y)^{(0)} = x$$

1.4. Connection Between Queen Function and Greatest Common Divisor:

Lemma 1.10.

$$Gcd(qn_m(x, y)) = Gcd(x, y)$$

Proof. We know that for some integer of x, applying it Queen with some integer y will give us a value that multiple of common divisors x and y. By the proof in **Lemma 1.7**. This means Queen will give us all the common divisors with some different multiple that is not a divisor for y. So this will prove our claim. $\square$

1.5. More Generalized Version of Queen Algorithm:

Definition 1.11. Queen function already has 3 variables but it's nothing compared to the most generalized version of it. But first, we have the Bishop function. It has 5 variables and it can make some newborn divisibility rules.

$fl : \mathbf{Z} \to \mathbf{Z}$ where $\forall a, b, c, n :\in \mathbf{Z}, \forall m :\in \mathbf{N}$

$$bs_m^n(a, b, c) = S_0(a, m)S_1(bc, m) - S_1(a, m)(S_0(bc, m) - nb)$$

Lemma 1.12.

$$\frac{a}{b} \in \mathbf{Z} \Rightarrow \frac{bs_m^n(a,b,c)}{b} \in \mathbf{Z}$$

Proof.

$$bs_m^n(a,b,c)$$

$$S_0(a,m)S_1(bc,m) - S_1(a,m)(S_0(bc,m) - nb)$$

$$S_0(a,m)S_1(bc,m) - S_1(a,m)S_0(bc,m) + S_1(a,m)nb$$

$$qn_m(a,bc) + S_1(a,m)nb$$

If we try to divide by b:

$$\frac{qn_m(a,bc)}{b} + \frac{S_1(a,m)nb}{b}$$

$$\frac{qn_m(a,bc)}{b} + S_1(a,m)n$$

then all of the formulas turned into the same function with some additions as the Queen function. Our claim will be proved as same as the Queen's. If $\frac{qn_m(a,bc)}{b} \in \mathbf{Z}$ than $bs_m^n(a,b,c) \in \mathbf{Z}$. But we proved that the only possibility that $\frac{qn_m(a,bc)}{b} \in \mathbf{Z}$ is $\frac{a}{b} \in \mathbf{Z}$. This proves our claim. $\square$

Lemma 1.13.

$$\left(Gcd(b, S_1(b,m)) = 1 \right) \Rightarrow \left(\left(\frac{bs_m^n(a,b,c)}{b} \in \mathbf{N} \right) \iff \left(\frac{a}{b} \in \mathbf{N} \right) \right)$$

Proof. Same as other proof when it is an if and only if the situation, we will see 2 possibilities for each 2 sides. But now we don't need much work to prove it, just because it has a Queen function in it. We are looking for only the non-integer part so we can delete the integer part in the function. When deleting it we only got the $\frac{qn_m(a,bc)}{b}$ part. But we proved it before. So our claim is proved. $\square$

Example 1.14. 98 is a very big and uncomfortable number to work with. This means it is very hard for a divisibility rule. But if we use $bs_2^{-1}(a,98,2)$ then it will be much easier, $bs_2^{-1}(a,98,2) = S_0(a,2) + 2S_1(a,2)$:

Table 4

Algorithm Reapply Count $\mapsto$	1	2	3	4	5
588	98	98	98	98	98
602.112	12.054	294	98	98	98
98	98	98	98	98	98
9.604	196	98	98	98	98
941.192	18.914	392	98	98	98
92.236.816	1.844.752	36.946	784	98	98
9.039.207.968	180.784.226	3.615.710	72.324	1.470	98

Example 1.15. Normally 7 has a really simple divisibility rule. It rule can be found in Bishop as this example. $bs_1^{-1}(a,7,8) = 5S_0(a,1) + S_1(a,1)$:

Table 5

Algorithm Reapply Count $\mapsto$	1	2	3	4	5
21	7	35	28	42	14
2.310	231	28	42	14	21
45.360	4.536	483	63	21	7
7	35	28	42	14	21
49	49	49	49	49	49
343	49	49	49	49	49
2.401	245	49	49	49	49
16.807	1.715	196	49	49	49

Example 1.16. Normally 3 and 9 have very good design and simple divisibility rules and we will find them in later the article but for now, there is a simple rule too. for 3, $bs_1^{-1}(a, 3, 4) = S_0(a, 1) + S_1(a, 1)$ same as 9, $bs_1^{-1}(a, 9, 2) = S_0(a, 1) + S_1(a, 1)$. Here are some examples for 3:

Table 6

Algorithm Reapply Count $\mapsto$	1	2	3	4	5	6	7	8
21	3	3	3	3	3	3	3	3
2.310	231	24	6	6	6	6	6	6
691.200	69.120	6.912	693	72	9	9	9	9
3	3	3	3	3	3	3	3	3
9	9	9	9	9	9	9	9	9
27	9	9	9	9	9	9	9	9
81	9	9	9	9	9	9	9	9
243	27	9	9	9	9	9	9	9
729	81	9	9	9	9	9	9	9
2.178	225	27	9	9	9	9	9	9
6.534	657	72	9	9	9	9	9	9
19.602	1.962	198	27	9	9	9	9	9
58.806	5.886	594	63	9	9	9	9	9

1.6. Multiple Times Applied Bishop Function:

Definition 1.17. For future usage, we will use a function called multiple Bishop:
$$\forall k \in \mathbf{Z}^+$$

$$bs_m^n(a, b, c)^{(k)} = \underbrace{bs_m^n(\dots (bs_m^n(a, b, c) \dots), b, c)}_{\text{k pieces}}$$

$$bs_m^n(a, b, c)^{(0)} = a$$

1.7. Connection Between Bishop Function and Greatest Common Divisor:

Definition 1.18. We know that the Bishop has a Queen inside, also in the part that has no Queen there is a y factor so this equation will be accurate:

$$Gcd(bs_m^n(a, b, c), b) = Gcd(a, b)$$

2. Ratio Simplifying with General Divisibility Rules

2.1. Queen Simplifying Function:

Definition 2.1. $Sqn : \mathbf{Z} \to \mathbf{Z}$ where $\forall a, b :\in \mathbf{Z}, \forall m :\in \mathbf{N}$:

$$Sqn_m(a,b) = \frac{S_1(a,m) + \frac{qn_m(a,b)}{b}}{S_1(b,m)}$$

Lemma 2.2.

$$Sqn_m(a,b) = \frac{a}{b}$$

Proof.

$$Sqn_m(a,b)$$

$$\frac{S_1(a,m) + \frac{qn_m(a,b)}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m) + \frac{S_1(b,m)a - S_1(a,m)b}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m) + S_1(b,m)\frac{a}{b} - S_1(a,m)}{S_1(b,m)}$$

$$\frac{S_1(b,m)\frac{a}{b}}{S_1(b,m)}$$

$$\frac{a}{b}$$

$\square$

2.2. Infinite Series Version of Simplifying Function:

Lemma 2.3.

$$Sqn_m(a,b) = \sum_{n=0}^{\infty} \frac{S_1(qn_m(a,b)^{(n)}, m)}{S_1(b,m)^{n+1}}$$

Proof. In this proof, we will open out simplifying function's ratio part:

$$Sqn_m(a,b)$$

$$\frac{S_1(a,m) + \frac{qn_m(a,b)}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{\frac{qn_m(a,b)}{b}}{S_1(b,m)}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{\frac{S_1(qn_m(a,b),m) + \frac{qn_m(a,b)^{(2)}}{b}}{S_1(b,m)}}{S_1(b,m)}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{S_1(qn_m(a,b),m)}{S_1(b,m)^2} + \frac{\frac{qn_m(a,b)^{(2)}}{b}}{S_1(b,m)^2}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{S_1(qn_m(a,b),m)}{S_1(b,m)^2} + \frac{\frac{S_1(qn_m(a,b)^{(2)},m) + \frac{qn_m(a,b)^{(3)}}{b}}{S_1(b,m)}}{S_1(b,m)^2}$$

$$\frac{S_1(a,m)}{S_1(b,m)} + \frac{S_1(qn_m(a,b),m)}{S_1(b,m)^2} + \frac{S_1(qn_m(a,b)^{(2)},m)}{S_1(b,m)^3} \cdots$$

$$\sum_{n=0}^{\infty} \frac{S_1(qn_m(a,b)^{(n)},m)}{S_1(b,m)^{n+1}}$$

□

2.3. Inverse Function of Queen Function:

Definition 2.4. $qn^{(-1)} : \mathbf{Z} \to \mathbf{Z}$ where $\forall a,b,n :\in \mathbf{Z}, \forall m :\in \mathbf{N}$: $qn_m(a,b)_n^{(-1)}$ is the n'th value of inverse function of Queen. This is because there is no one answer to the inverse function of Queen.

Lemma 2.5.

$$qn_m(a,b)_n^{(-1)} = \frac{nb+a}{S_1(b,m)}$$

Proof. We will start by putting the value of $qn_m(a,b)_n^{(-1)}$ to simplifying function:

$$\frac{qn_m(a,b)_n^{(-1)}}{b} = \frac{S_1(qn_m(a,b)_n^{(-1)},m) + \frac{qn_m(qn_m(a,b)_n^{(-1)},b)}{b}}{S_1(b,m)}$$

$$\frac{qn_m(a,b)_n^{(-1)}}{b} = \frac{S_1(qn_m(a,b)_n^{(-1)},m) + \frac{a}{b}}{S_1(b,m)}$$

$$qn_m(a,b)_n^{(-1)} = \frac{S_1(qn_m(a,b)_n^{(-1)},m)b + a}{S_1(b,m)}$$

Here we can see the only unknown part on the left-hand side is $S_1(qn_m(a,b)_n^{(-1)},m)$. This is because we can choose whatever we want so with a little bit change of variable. We can say that,

$$qn_m(a,b)_n^{(-1)} = \frac{nb+a}{S_1(b,m)}$$

□

2.4. Multiple Times Applied Inverse Queen Function:

Definition 2.6. For future usage, we will use a function called multiple inverse Queen:
$\forall k :\in \mathbf{Z}^+, \forall R \subset \mathbf{Z}$

$$qn_m(x,y)_R^{(-k)} = \underbrace{qn_m(\ldots(qn_m(x,y)_{R_{\lfloor\frac{k}{2}\rfloor}}^{(-1)}\ldots)_{R_{k-1}}^{(-1)},y)_{R_k}^{(-1)}}_{k \text{ pieces}}$$

Here we can also find the Bishop function version of all of them but they will be the same proofs as these proofs so there is no need to prove non-necessary functions.

2.5. Infinite Repeated Inverse Function of Queen:

Lemma 2.7.

$$\lim_{h\to\infty} qn_m(a,b)_R^{-h} = \sum_{i=0}^{\infty} \frac{R_i b}{S_1(b,m)^{i+1}}$$

Proof.

$$\lim_{h\to\infty} qn_m(a,b)_R^{(-h)}$$

$$\lim_{h\to\infty} \frac{bR_1 + \frac{bR_2 + \frac{bR_3 + \dots}{S_1(b,m)}}{S_1(b,m)}}{S_1(b,m)}$$

$$\lim_{h\to\infty} \frac{bR_1}{S_1(b,m)} + \frac{bR_2}{S_1(b,m)^2} + \frac{bR_3}{S_1(b,m)^3} + \dots$$

$$\lim_{h\to\infty} \sum_{i=0}^{h} \frac{bR_i}{S_1(b,m)^{i+1}}$$

5

$$\sum_{i=0}^{\infty} \frac{bR_i}{S_1(b,m)^{i+1}}$$

□

Conjecture: The pattern of R set is the pattern of multiple times applied Queen function of infinitely applied reverse Queen function.

3. Full Versions of General Divisibility Rules

3.1. Full Version of Queen Algorithm:

Definition 3.1. normally we use $S()$ function in Queen algorithm. But we can also use the $D()$ and $Decimal()$ functions. And the formula will be like this:$\forall a, b \in \mathbf{Z}, \forall m \in \mathbf{N}$,

$$Mqn_m(a,b) = \sum_{i=0}^{Decimal(a)-1} (-1)^{i6\ 7} D_i(a,m) S_1(b,m)^{Decimal(a)-i} S_0(b,m)^i$$

Lemma 3.2.

$$\frac{a}{b} \in \mathbf{N} \Rightarrow \frac{Mqn_m(a,b)}{b} \in \mathbf{N}$$

Proof.

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) S_1(b,m)^{Decimal(a)-i} S_0(b,m)^i$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m)(b - 10^m S_1(b,m))^i S_1(b,m)^{Decimal(a)-i}$$

8 9 10

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right) S_1(b,m)^{Decimal(a)-i} \right) \right) \quad (3.1)$$

---

5    $\forall x : |x| < 1 | \sum_{i=0}^{\infty} x^{-i} = \frac{1}{1-\frac{1}{x}}$[4]

8    $\forall x, y \in \mathbf{R}, \forall n \in \mathbf{N}^+ | (x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$[5]

9    $\forall x, y \in \mathbf{N} | \binom{x}{y} = \frac{x!}{y!(x-y)!}$

10   $\forall x \in \mathbf{N} | x! = x(x-1)(x-2)\cdots 3 \cdot 2 \cdot 1$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(b,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.2}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. S_1(b,m)^{Decimal(a)-i} \right) \right.$$
$$\left. + S_1(b,m)^{Decimal(a)-i}(-10^m S_1(b,m))^i \right) \tag{3.3}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. S_1(b,m)^{Decimal(a)-i} \right) \right.$$
$$\left. + S_1(b,m)^{Decimal(a)}(-10^m)^i \right) \tag{3.4}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(b,m)^{Decimal(a)-i} \right) \right) \right)$$
$$+ \sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) S_1(b,m)^{Decimal(a)}(-10^m)^i \tag{3.5}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(b,m)^{Decimal(a)-i} \right) \right) \right)$$
$$+ \sum_{i=0}^{Decimal(a)-1} D_i(a,m) S_1(b,m)^{Decimal(a)} 10^{mi} \tag{3.6}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(b,m)^{Decimal(a)-i} \right) \right) \right)$$
$$+ S_1(b,m)^{Decimal(a)} \sum_{i=0}^{Decimal(a)-1} D_i(a,m) 10^{mi} \tag{3.7}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j}(-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(b,m)^{Decimal(a)-i} \right) \right) \right)$$
$$+ S_1(b,m)^{Decimal(a)} a \tag{3.8}$$

$$b \sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j-1}(-10^m S_1(b,m))^j \right. \right.\right.$$
$$\left.\left.\left. S_1(b,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.9}$$
$$+ a S_1(b,m)^{Decimal(a)}$$

Here we have two sides of the equation one of them is the right side of the plus sign which has a factor called a and other is the left side of the plus sign which has a factor called y. Also, other factors is integers. So if we try to divide it by y:

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left(b^{i-j-1}(-10^m S_1(b,m))^j \right. \right.\right.$$
$$\left.\left.\left. S_1(b,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.10}$$
$$+ \frac{a S_1(b,m)^{Decimal(a)}}{b}$$

In here if $\frac{a}{b} \in \mathbf{Z}$ then the whole equation will be an integer. This proves our claim. □

Lemma 3.3.

$$\left(Gcd(b, S_1(b,m)) = 1 \right) \Rightarrow \left(\left(\frac{Mqn_m(a,b)}{b} \in \mathbf{N} \right) \iff \left(\frac{a}{b} \in \mathbf{N} \right) \right)$$

Proof. Same as all of the if and only if proofs we see a $Gcd(b, S_1(b,m)) = 1$ situation. In here only thing we should think is that if $Gcd(b, S_1(b,m)) = 1$ then for every natural number x, $Gcd(b, S_1(b,m)^x) = 1$. So all the proof is coming from **Lemma 1.3**. □

Example 3.4. In 101 we see the same result as the normal 101 divisibility rule:

Table 7

Algorithm Reapply Count $\mapsto$	1	2	3	4	5	6	7	8
101	0	0	0	0	0	0	0	0
10.201	0	0	0	0	0	0	0	0
1.030.301	11	0	0	0	0	0	0	0
104.060.401	0	0	0	0	0	0	0	0

Example 3.5. In 11 we found the same result as the normal 11 divisibility rule:

Table 8

Algorithm Reapply Count $\mapsto$	1	2	3	4	5	6	7	8
564.537.600	0	0	0	0	0	0	0	0
209	11	0	0	0	0	0	0	0
1.331	0	0	0	0	0	0	0	0
14.641	0	0	0	0	0	0	0	0
161.051	0	0	0	0	0	0	0	0
1.771.561	0	0	0	0	0	0	0	0
19.487.171	0	0	0	0	0	0	0	0
40909	22	0	0	0	0	0	0	0

3.2. Full Version of Bishop Algorithm:

Definition 3.6. normally we use $S()$ function in Bishop algorithm. But we can also use the $D()$ and *Decimal* functions. And the formula will be like this:$\forall a, b, c, n \in \mathbf{Z}, \forall m \in \mathbf{N}$,

$$Mbs_m^n(a, b, c)$$

$$= \sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a, m) S_1(bc, m)^{Decimal(a)-i} (S_0(bc, m) - bn)^i$$

Lemma 3.7.

$$\frac{a}{b} \in \mathbf{N} \Rightarrow \frac{Mbs_m^n(a, b, c)}{b} \in \mathbf{N}$$

Proof.

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a, m) S_1(bc, m)^{Decimal(a)-i} (S_0(bc, m) - bn)^i$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a, m) (bc - bn - 10^m S_1(bc, m))^i S_1(bc, m)^{Decimal(a)-i}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a, m) \left(\sum_{j=0}^{i} \binom{i}{j} \left((bc - bn)^{i-j} (-10^m S_1(bc, m))^j \right) \right. \right.$$
$$\left. \left. S_1(bc, m)^{Decimal(a)-i} \right) \right) \tag{3.11}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a, m) \left(\sum_{j=0}^{i} \binom{i}{j} \left((bc - bn)^{i-j} (-10^m S_1(bc, m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(bc, m)^{Decimal(a)-i} \right) \right) \right) \tag{3.12}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a, m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left((bc - bn)^{i-j} (-10^m S_1(bc, m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(bc, m)^{Decimal(a)-i} \right) \right. \right.$$
$$\left. \left. + S_1(bc, m)^{Decimal(a)-i} (-10^m S_1(bc, m))^i \right) \right) \tag{3.13}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a, m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left((bc - bn)^{i-j} (-10^m S_1(bc, m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(bc, m)^{Decimal(a)-i} \right) \right. \right.$$
$$\left. \left. + S_1(bc, m)^{Decimal(a)} (-10^m)^i \right) \right) \tag{3.14}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left((bc-bn)^{i-j} (-10^m S_1(bc,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(bc,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.15}$$

$$+ \sum_{i=0}^{Decimal(a)-1} (-1)^i D_i(a,m) S_1(bc,m)^{Decimal(a)} (-10^m)^i$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left((bc-bn)^{i-j} (-10^m S_1(bc,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(bc,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.16}$$

$$+ \sum_{i=0}^{Decimal(a)-1} D_i(a,m) S_1(bc,m)^{Decimal(a)} 10^{mi}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left((bc-bn)^{i-j} (-10^m S_1(bc,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(bc,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.17}$$

$$+ S_1(bc,m)^{Decimal(a)} \sum_{i=0}^{Decimal(a)-1} D_i(a,m) 10^{mi}$$

$$\sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \left((bc-bn)^{i-j} (-10^m S_1(b,m))^j \right. \right. \right.$$
$$\left. \left. \left. S_1(bc,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.18}$$

$$+ S_1(bc,m)^{Decimal(a)} a$$

$$(bc-bn) \sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \right. \right.$$
$$\left((bc-bn)^{i-j-1} (-10^m S_1(bc,m))^j \right.$$
$$\left. \left. \left. S_1(bc,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.19}$$

$$+ a S_1(bc,m)^{Decimal(a)}$$

Here we have two sides of the equation one of them is the right side of the plus sign which has a factor called *a* and the other is the left side of the plus sign which has a factor called *b*. Also, other factors is integers. So if we try to divide it by *b*:

$$\frac{(bc-bn)}{b} \sum_{i=0}^{Decimal(a)-1} (-1)^i \left(D_i(a,m) \left(\sum_{j=0}^{i-1} \binom{i}{j} \right. \right.$$
$$\left. \left. \left((bc-bn)^{i-j-1} (-10^m S_1(bc,m))^j S_1(bc,m)^{Decimal(a)-i} \right) \right) \right) \tag{3.20}$$

$$+ \frac{a S_1(bc,m)^{Decimal(a)}}{b}$$

$$(c-n)\sum_{i=0}^{Decimal(a)-1}(-1)^i\left(D_i(a,m)\left(\sum_{j=0}^{i-1}\binom{i}{j}\right.\right.$$

$$\left.\left.\left((bc-bn)^{i-j-1}(-10^m S_1(b,m))^j S_1(bc,m)^{Decimal(a)-i}\right)\right)\right)$$

$$+\frac{aS_1(bc,m)^{Decimal(a)}}{b}$$

(3.21)

In here if $\frac{a}{b}\in\mathbf{Z}$ then the whole equation will be an integer. This proves our claim. □

Lemma 3.8.

$$\left(Gcd(b,S_1(bc,m))=1\right)\Rightarrow\left(\left(\frac{Mbs_m^n(a,b,c)}{b}\in\mathbf{N}\right)\iff\left(\frac{a}{b}\in\mathbf{N}\right)\right)$$

Proof. Same as all of the if and only if proofs we see a $Gcd(b,S_1(bc,m))=1$ situation. In here only thing we should think is that if $Gcd(b,S_1(b,m))=1$ then for every natural number x, $Gcd(b,S_1(bc,m)^x)=1$. So all the proof is coming from **Lemma 1.3**. □

Example 3.9. In 3 and 9 we get the same result as the normal divisibility rule. In 3:

Table 9

Algorithm Reapply Count $\mapsto$	1	2	3	4	5	6	7	8
9	9	9	9	9	9	9	9	9
42	6	6	6	6	6	6	6	6
84.240	18	9	9	9	9	9	9	9
81	9	9	9	9	9	9	9	9
729	18	9	9	9	9	9	9	9
6.561	18	9	9	9	9	9	9	9
59.049	18	9	9	9	9	9	9	9
531.441	18	9	9	9	9	9	9	9
4.782.969	45	9	9	9	9	9	9	9
43.046.721	27	9	9	9	9	9	9	9
39.999.999	57	12	3	3	3	3	3	3

4. Finding Prime Numbers With General Divisibility Rules

4.1. Main Theorems for Proofs:

Lemma 4.1. $\forall x,n:\in\mathbf{N}$ *if for* $\forall m\in\mathbf{Z}^+$ *that* $m\leq n$, [11]$Gcd(x,P_m)=1$ *and* $x<P_{n+1}^2$, *then a is a prime number.*

Proof. In this proof, we should know that if the natural number x has no other same divisors other than 1 with all the equal or smaller primes than P_n. Then this means the minimum prime divisor that can be in x's divisors is the first prime that is bigger than P_n. And it is P_{n+1}. But the minimum integer that is a multiple of P_{n+1} and not a prime and usable in this situation is P_{n+1}^2. Then this means if $x<P_{n+1}^2$, x must be a prime. □

Lemma 4.2. $\forall x,n\in\mathbf{N}$ *if* [12]$Gcd(x,P_n\#)=1$ *and* $x<P_{n+1}^2$, *then x is a prime number.*

---

[11]　P : set of elements that has no other same divisors that are smaller than it other than 1.
[12]　$P\#:\mathbf{N}\to\mathbf{N}$ where $P_n\#$ multiple of all the primes that equal or smaller than P_n

Proof. If a $Gcd(x, P_n\#) = 1$ then x has no other same divisors than 1 with all the equal or smaller primes than P_n. From the **Lemma 4.1**, also if $x < P_{n+1}^2$, x must be a prime. $\square$

4.2. Finding Prime Numbers With Queen Function:

Lemma 4.3. *Queen is a divisibility rule for every number. Also, it is a $Gcd()$ simpler. With these truths, we can use to find prime numbers like this by using **Lemma 4.2**:*
$\forall a, n \in \mathbf{N}, \forall m \in \mathbf{N}$ *that* $Gcd(P_n\#, S_1(P_n\#, m)) = 1$. *If* $Gcd(a, P_n\#) = 1, |qn_m(a, P_n\#)|^{13} < P_{n+1}^2$ *then,*

$$|qn_m(a, P_n\#)| \in \mathbf{P}$$

Proof. We know that $Gcd(a, P_n\#) = Gcd(qn_m(a, P_n\#))$ from **Lemma 4.1**. And also using the knowledge of **Lemma 4.2** we can say that If $Gcd(a, P_n\#) = 1, |qn_m(a, P_n\#)| < P_{n+1}^2$ then, $|qn_m(a, P_n\#)| \in \mathbf{P}$. So proof finishes by itself. $\square$

Example 4.4. We can use the $P_n\#$ type of numbers. For example 2310:

Table 10

Algorithm Reapply Count $\mapsto$	$\|qn_2(a, 2310)\| \downarrow$
703	1
101	13
401	17
503	19
1	23
403	29
1307	31
601	37
1103	41
701	47
2011	53
103	59
1007	61
1409	67
907	71
2213	79
1711	83
2113	89
$\vdots$	$\vdots$

Example 4.5. We can use not only the $P_n\#$ type of numbers. Also, we can use their multiples. For example 120:

---

[13] $|| : \mathbf{R} \to \mathbf{R}$ where $|x|$ is the positive version of x.

Table 11

Algorithm Reapply Count $\mapsto$	$\lvert qn_2(a, 120)\rvert \downarrow$
101	19
103	17
107	13
113	7
127	7
131	11
137	17
133	13
139	19
143	23
149	29
151	31
157	37
161	41
163	43
167	47

4.3. Finding Prime Numbers With Bishop Function:

Lemma 4.6. *Queen is a divisibility rule for every number. Also, it is a $Gcd()$ simpler. With these truths, we can use to find prime numbers like this by using **Lemma 4.2**:*
$\forall a, c, n, k \in \mathbf{N}, \forall m \in \mathbf{N}$ *that* $Gcd(P_n\#, S_1(P_n\#, m)) = 1$. *If* $Gcd(a, P_n\#) = 1, \lvert bs_m^n(a, P_n\#, c)\rvert < P_{n+1}^2$ *then,*

$$\lvert bs_m^n(a, P_n\#, c)\rvert \in \mathbf{P}$$

Proof. We know that $Gcd(a, P_n\#) = Gcd(bs_m^n(a, P_n\#, c))$ from **Lemma 4.1**. And also using the knowledge of **Lemma 4.2** we can say that If $Gcd(a, P_n\#) = 1, \lvert bs_m^n(a, P_n\#, c)\rvert < P_{n+1}^2$ then, $\lvert bs_m^n(a, P_n\#, c)\rvert \in \mathbf{P}$. So proof finishes by itself. □

4.4. Checkmate Theorem:

Lemma 4.7.
$$bs_m^1(a, b, 1) - qn_m(a, b) = S_1(a, m)b$$

Proof.

$$bs_m^1(a, b, 1) - qn_m(a, b)$$

$$qn_m(a, b) + S_1(a, m)b - qn_m(a, b)$$

$$S_1(a, m)b$$

  □

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