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Posted Date: 2 February 2024

doi: 10.20944/preprints202402.0038.v1

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Article

# On the Trio Unification of Information Geometry, Einsteinian Relativity, and Transient Queues the Parthasarathian Case

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**Abstract:** The current study characterizes the transient M/M/1 queue manifold info-geometrically, through devising Fisher Information matrix(FIM) and its inverse(IFIM). Additionally, the impact of stability on the existence of IFIM and explore the geodesic equations of motion has been revealed. More potentially, the paper discusses the relationship between stability and the Gaussian curvature, as well as the connections between queueing theory, information geometry, , Riemannian geometry, and the Theory of Relativity.

**Keywords:** Transient M/M/1 queueing system IG; statistical manifold (SM); QM; Geodesic equations of motion; Riemannian metric (RM); Fisher Information matrix (FIM); Inverse Fisher Information matrix (IFIM); threshold theorem

## 1. Introduction

Numerous study areas, including statistical sciences, have extensively used IG [1]. In other words, the goal of IG is to use statistics to apply the methods of differential geometry (DG), which indicates the major goal of IG is to use stochastic processes and probability theoretic in the applications of methodologies of non-Euclidean geometry.

A manifold is a topological finite-dimensional Cartesian space,  $\mathbb{R}^n$ , in which an infinite-dimensional manifold exists[2]. Additionally, IG supports SMs' descriptions that are based on intuitive reasoning. One might have a greater understanding of the crucial significance of IG[3,4]. Parametrization of a Statistical Manifold (SM) is visualized by figure 1[4].

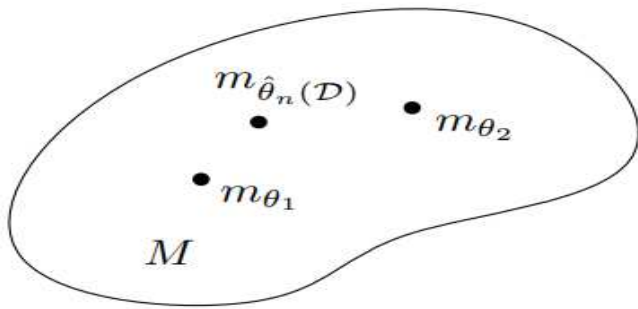


Figure 1. SM's parametrization(Nielsen 2020).

According to the literature, a paper by[5] examined info-geometrically the stable M/D/1 queue on the basis of queue length routes 'characteristics, was the real motivator for the current study.

The main deliverables of this paper are described below.

- Revealing the new discovery of the geodesic equations of motion of the coordinates of the transient M/M/  $\infty$ .
- In the context of the research paper, a novel  $\alpha$ -connection [6] is introduced, which maps each coordinate to a value. The paper also discusses the concept of incompressibility or solenoidality in the transient M/M/1 queue, where closed surfaces have no net flux [7], and includes definitions and physical interpretations of Gaussian and Ricci curvatures.

➤ Revolutionary relativistic and transient queueing theoretic connections are established.

The remainder of the paper is divided into the following sections: Preliminary definitions of Information Geometry (IG) are given in Section 2. The transient M/M/1 QM's FIM and its IFIM are introduced in Section 3. While Section 5 gives the Information Geometric Equations of Motion (IMEs) for the coordinates of the transient M/M/1 QM, Section 4 addresses the  $\alpha$ -connection of the queue manifold. The potential function, divergence measurements, stability, and curvature of the M/M/1 QM are all discussed in detail in Sections 6 to 13. Section 14 wraps up the report and provides a plan for further research.

## 2. Main Definitions

### 2.1. Main Definition on IG

**Definition 1. Statistical Manifold (SM)**  $M = \{p(x, \theta) | \theta \in \Theta\}$  is called an SM [8] if  $x$  is a random variable in sample space  $X$  and  $p(x, \theta)$  is the probability density function, which satisfies certain regular conditions. Here,  $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \Theta$  is an  $n$ -dimensional vector in some open subset  $\Theta \subset \mathbb{R}^n$ , and  $\theta$  can be viewed as the coordinates on manifold  $M$ .

**Definition 2. Potential Function** The potential function  $\Psi(\theta)$  (c.f., (2.1)) [8] is the distinguished negative function of the coordinates alone of  $(\mathcal{L}(x; \theta) = \ln(p(x; \theta)))$ .

**Definition 3. Fisher's Information Matrix (FIM)** The FIM (or, Fisher's metric)  $[g_{ij}]$  [6] is given by the Hessian (the  $n \times n$  matrix of the partial derivatives of the potential function  $\Psi(\theta)$  with respect to the coordinates) i.e.,

$$[g_{ij}] = \left[ \frac{\partial^2}{\partial \theta^i \partial \theta^j} (\Psi(\theta)) \right], i, j = 1, 2, \dots, n \quad (1)$$

with respect to natural coordinates.

**Definition 4. IFIM** Given the FIM, the inverse matrix of  $[g_{ij}]$  is defined by [8]

$$[g^{ij}] = ([g_{ij}])^{-1} = \frac{adj[g_{ij}]}{\Delta} \quad (2)$$

$$\Delta = \det[g_{ij}] \quad (3)$$

**Definition 5.  $\alpha$ -Connection** For each  $\alpha \in \mathbb{R}$ , the  $\alpha$  (or  $\nabla^{(\alpha)}$ )-connection [6] is the torsion-free affine connection with components:

$$\Gamma_{ij,k}^{(\alpha)} = \left( \frac{1-\alpha}{2} \right) (\partial_i \partial_j \partial_k (\Psi(\theta))) \quad (4)$$

where  $\Psi(\theta)$  is the potential function and  $\partial_i = \frac{\partial}{\partial \theta^i}$ .

**Definition 6. (1)** The geodesic equations of manifold  $M$  with coordinate system  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$  read as [8]

$$\frac{d^2 \theta^k}{dt^2} + \Gamma_{ij}^{k(0)} \left( \frac{d\theta^i}{dt} \right) \left( \frac{d\theta^j}{dt} \right) = 0, i, j = 1, 2, \dots, n, \Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)} g^{sk} \quad (5)$$

By the above definition, it is clear that the geodesic equations are interpreted physically as the information geometric equations of motion, shortly (IGEMs), or the relativistic equations of motion (REMs), or the Riemannian equations of motion.

At this stage, the current study provides a ground-breaking discovery of the IG analysis of transient queueing systems in comparison to that of non-time dependent queueing systems, namely IG analysis of stable queues [9,10].

**Definition 7. 1.** The  $\alpha$ -curvature Riemannian Tensors,  $R_{ijkl}^{(\alpha)}$  [6] read as:

$$R_{ijkl}^{(\alpha)} = \left[ (\partial_j \Gamma_{ik}^{s(\alpha)} - \partial_i \Gamma_{jk}^{s(\alpha)}) g_{sl} + (\Gamma_{j\beta,l}^{(\alpha)} \Gamma_{ik}^{\beta(\alpha)} - \Gamma_{i\beta,l}^{(\alpha)} \Gamma_{jk}^{\beta(\alpha)}) \right], i, j, k, l, s, \beta$$

$$= 1, 2, 3, \dots, n$$

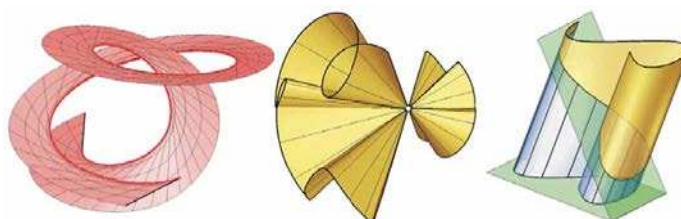
Where

$$\Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)} g^{sk} \quad (7)$$

2.  $K_{1212}^{(\alpha)} = K^{(\alpha)}$  is  $\alpha$  – Gaussian curvature and reads as [6]

$$K^{(\alpha)} = \frac{R_{1212}^{(\alpha)}}{\det(g_{ij})} \quad (8)$$

**Definition 8.** (i) Developable surfaces are a specific class of ruled surfaces that may be mapped onto a flat surface without the curves being altered in any way; each curve is generated from such a surface and keeps its original shape [11].



**Figure 2.** Three kinds of developable surfaces: Tangential on Figure. 5a (on the left), Conical on Fig. 5b (on the centre) and Figure. 5c (on the right), Cylindrical. Note that curves in bold are directrix or base curves and straight lines in bold are directors or generating lines (curves) [11].

**Definition 9.** In mathematics, a function is considered well-defined if it consistently produces the same output regardless of how the input is represented. This means that even if the input is expressed differently, if the value remains the same, the function will yield the same result. This concept ensures the reliability and consistency of mathematical operations and calculations.

**Definition 10 [13].** 1. function  $f$  is said to be one-to-one, or injective, if and only if  $f(x) = f(y)$  implies  $x = y$  for all  $x, y$  in the domain of  $f$ . A function is said to be an injection if it is one-to-one. Alternative: A function is one-to-one if and only if  $f(x) \neq f(y)$ , whenever  $x \neq y$ . This is the contrapositive of the definition.

2. A function  $f$  from  $A$  to  $B$  is called onto, or surjective, if and only if for every  $b \in B$  there is an element  $a \in A$  such that  $f(a) = b$ . Alternative: all co-domain elements are covered.

3. A function  $f$  is called a bijection if it is both one-to-one (injection) and onto (surjection).

**Theorem 1 [14].** Let  $f$  be a function that is defined and differentiable on an open interval  $(c, d)$ .

If  $f'(x) > 0 (< 0)$  for all  $x \in (c, d)$ , then  $f$  is increasing (decreasing) on  $(c, d)$ . (9)

Taylor expansions [15]) are widely used to approximate functions by expansions. We have for all  $x$  around zero,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad (10)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad (11)$$

## 2.2. Important Inequalities [16]

### 1. Chebyshev inequality

$$\frac{x}{1+x} \leq \ln(1+x) \leq x \text{ for } x \geq -1 \quad (12)$$

(12) can be rewritten as

$$1 - \frac{1}{x} \leq \ln(x) \leq x - 1 \text{ for } x \geq -1 \quad (13)$$

2. Lehmer inequality

$$\frac{x^n}{n!} < \frac{x^n}{n!} + 1 \leq e^x \text{ for all } x, n > 0 \quad (14)$$

### 3. Fim and Ifim of the Transient M/M/1QM

A single-channel model with exponential inter-arrival times, service times, and FIFO queue discipline is the simplest probabilistic queueing model that can be addressed analytically. In queueing theory, this is referred to as the M/M/1 queue [17].

Based on the introduction of the function, Parthasarathy[18] suggested a straightforward method for the transient solution of the M/M/1 system to read as:

$$p_n(t) = \rho p_0(t) e^{-(\lambda+\mu)t} \left[ \sum_{m=1}^n r_m(t) + e^{(\lambda+\mu)t} \right], \quad n = 0, 1, 2, \dots \quad (15)$$

where  $\lambda$  stands for the mean rate per unit time at which arrival instants occur,  $\mu$  is the mean rate of service time,  $\rho = \frac{\lambda(t)}{\mu(t)}$  defines the traffic intensity or utilisation factor,

$$r_n(t) = \mu \beta_2^{n-a_1} (1 - \delta_{0a_1}) [I_{n-a_1}(\beta_1 t) - I_{n+a_1}(\beta_1 t)] + \lambda \beta_2^{n-1-a_1} (I_{n+1-a_1}(\beta_1 t) - I_{n-1-a_1}(\beta_1 t)) \quad (16)$$

with  $\beta_1 = 2\sqrt{\lambda\mu}$ ,  $\beta_2 = 2\sqrt{\frac{\lambda}{\mu}}$ ,  $I_n(x)$  [19] is the modified Bessel function.

And

$$p_0(t) = \int_0^t r_1 e^{-(\lambda+\mu)y} dy + \delta_{0a_1}$$

**Theorem 2.** The underlying queue of (15) has:

(i) FIM reads as

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & 0 & h \\ c & h & l \end{bmatrix} \quad (17)$$

where

$$a = \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \quad (18)$$

$$b = t \quad (19)$$

$$c = \frac{\rho^*}{\rho^2} + \frac{\rho^*}{(1-\rho)^2} + t\mu^* + \mu\rho^* \quad (20)$$

$$h = (1 + \rho) + t\rho^* \quad (21)$$

$$l = \frac{\rho^{*2} - \rho\rho^*}{\rho^2} + \frac{\rho^{*2} + (1-\rho)\rho^*}{(1-\rho)^2} + ((1+\rho)t\mu^* + \mu + 2\rho\mu^* + 2\mu\rho^* + 2\rho\mu + \mu t\rho^*) \quad (22)$$

$$\Delta = \det([g_{ij}]) \quad (23)$$

$$= (a(-h^2) - b(bl - ch) + c(bh)) \quad (24)$$

Provided that, . refers to the temporal derivative  $\frac{d}{dt}$ .

(iii)  $[g^{ij}]$  reads as

$$[g^{ij}] = \frac{adj[g_{ij}]}{\Delta} = \begin{bmatrix} A & B & L \\ B & E & F \\ L & F & H \end{bmatrix} \quad (25)$$

where

$$A = \frac{(-h^2)}{\Delta} \quad (26)$$

$$B = \frac{(ch - bl)}{\Delta} \quad (27)$$

$$L = \frac{(bh)}{\Delta} \quad (28)$$

$$E = \frac{(al - c^2)}{\Delta} \quad (29)$$

$$F = \frac{(bc - ah)}{\Delta} \quad (30)$$

$$H = \frac{(-b^2)}{\Delta} \quad (31)$$

**Proof.** (i) Following (15), we have:

$$\begin{aligned} \mathcal{L}(x; \theta) &= \ln(p_n(x; \theta)) \\ &= \ln(\rho p_0(t)e^{-(\lambda+\mu)t}) + \ln(\rho p_0(t)e^{-(\lambda+\mu)t} [\sum_{m=1}^n r_m(t) + e^{(\lambda+\mu)t}]) \end{aligned} \quad (32)$$

$$\theta = (\theta_1, \theta_2, \theta_3) = (\rho, \mu, t) \quad (33)$$

We have

$$\Psi(\theta) = -l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t, \quad p_0(t) = 1 - \rho \quad (34)$$

Thus,

$$\partial_1 = \frac{\partial \Psi}{\partial \rho} = -\frac{1}{\rho} + \frac{1}{(1-\rho)} + \mu t, \quad \partial_2 = \frac{\partial \Psi}{\partial \mu} = (1 + \rho)t, \quad \partial_3 = \frac{\partial \Psi}{\partial t} = -\frac{\rho}{\rho} + \frac{\rho}{(1-\rho)} + (1 + \rho)t\mu + \mu(1 + \rho) + \mu t\rho \quad (35)$$

$$\begin{aligned} \partial_1 \partial_1 &= \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \quad \partial_1 \partial_2 = \partial_2 \partial_1 = t, \quad \partial_2 \partial_2 = 0, \quad \partial_1 \partial_3 = \frac{\rho}{\rho^2} + \frac{\rho}{(1-\rho)^2} + t\mu + \mu\rho = \partial_3 \partial_1 \\ , \quad \partial_2 \partial_3 &= (1 + \rho) + t\rho = \partial_3 \partial_2, \quad \partial_3 \partial_3 = \frac{\rho^2 - \rho\rho}{\rho^2} + \frac{\rho^2 + (1-\rho)\rho}{(1-\rho)^2} + (1 + \rho)t\mu + \mu + 2\rho\mu + 2\mu\rho + \mu t\rho \end{aligned} \quad (36)$$

Therefore, the Fisher Information Matrix, FIM, is obtained (c.f., (17)).

Thus (ii) follows.

$[g^{ij}]$  reads as:

$$[g^{ij}] = [g_{ij}]^{-1} = \frac{adj[g_{ij}]}{\Delta} = \frac{Transpose(Cov[g_{ij}])}{\Delta} = \frac{1}{\Delta} transpose(Cov \begin{bmatrix} a & b & c \\ b & 0 & h \\ c & h & l \end{bmatrix}) \quad (37)$$

$$= \begin{bmatrix} (-h^2) & (ch - bl) & (bh) \\ (ch - bl) & (al - c^2) & (bc - ah) \\ (bh) & (bc - ah) & (-b^2) \end{bmatrix} \quad (38)$$

Thus,

$$[g^{ij}] = \frac{1}{\Delta} transpose(Cov \begin{bmatrix} (-h^2) & (ch - bl) & (bh) \\ (ch - bl) & (al - c^2) & (bc - ah) \\ (bh) & (bc - ah) & (-b^2) \end{bmatrix}) = \begin{bmatrix} A & B & L \\ B & E & F \\ L & F & H \end{bmatrix} \quad (39)$$

It is notable that the Fisher Information Matrix, FIM should satisfy the symmetry requirement.

In the following section, the components of  $\alpha$  (or  $\nabla^{(\alpha)}$ )- connection are obtained. These calculated expressions are needed to obtain the corresponding Geodesic Equations (GEs) of the parametric coordinates of M/M/1 QM.

#### 4. Connection of the Transient M/M/1 QM

##### 4.1. The Obtained $\Gamma_{ij,k}^{(\alpha)}$ Expressions (c.f., Equation (4)) of the Transient M/M/1 QM

From (4), the reader can check that:

$$\Gamma_{11,1}^{(\alpha)} = -(1 - \alpha) \left( \frac{1}{(1-\rho)^3} + \frac{1}{\rho^3} \right) \quad (40)$$

$$0 = \Gamma_{22,2}^{(\alpha)} = \Gamma_{12,2}^{(\alpha)} = \Gamma_{22,1}^{(\alpha)} = \Gamma_{21,2}^{(\alpha)} = \Gamma_{11,2}^{(\alpha)} = \Gamma_{12,1}^{(\alpha)} = \Gamma_{21,1}^{(\alpha)} = \Gamma_{22,3}^{(\alpha)} = \Gamma_{23,2}^{(\alpha)} = \Gamma_{32,2}^{(\alpha)} \quad (41)$$

$$\Gamma_{31,1}^{(\alpha)} = (1 - \alpha)\rho \cdot \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) \quad (42)$$

$$\Gamma_{13,1}^{(\alpha)} = \Gamma_{11,3}^{(\alpha)} = (1 - \alpha)\rho \cdot \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) \quad (43)$$

$$\Gamma_{13,2}^{(\alpha)} = \Gamma_{12,3}^{(\alpha)} = \frac{(1 - \alpha)}{2} \quad (44)$$

$$\Gamma_{13,3}^{(\alpha)} = \frac{(1 - \alpha)}{2} \left[ \frac{\rho^*}{\rho^2} + \frac{\rho^*}{(1 - \rho)^2} + \frac{2\rho^2}{(1 - \rho)^3} + t\mu^* + 2\mu^* \right] \quad (45)$$

$$\Gamma_{23,1}^{(\alpha)} = \Gamma_{21,3}^{(\alpha)} = \frac{(1 - \alpha)\rho^*}{2} \quad (46)$$

$$\Gamma_{23,3}^{(\alpha)} = \frac{(1 - \alpha)(1 + 2\rho^* + t\rho^*)}{2} \quad (47)$$

$$\Gamma_{31,2}^{(\alpha)} = \Gamma_{32,1}^{(\alpha)} = \frac{(1 - \alpha)}{2} \quad (48)$$

Engaging the same procedure, the remaining components can be determined.

## 5 THE IMEs OF THE COORDINATES THE TRANSIENT M/M/1 QM

5.1. The **IMEs** of the **server utilization** coordinate,  $\rho$  of the transient M/M/1 QM

The IMEs (c.f., Equation (5)) corresponding to the **arrival rate** coordinate,  $\rho$  of the transient M/M/1 QM are:

$$\frac{d^2\theta^1}{dt^2} + \Gamma_{ij}^{1(0)} \left( \frac{d\theta^i}{dt} \right) \left( \frac{d\theta^j}{dt} \right) = 0, i, j = 1, 2, 3 \quad (49)$$

Now, we are in a situation of trying to find the path of motion of family of families of IMEs corresponding to the server utilization coordinate,  $\rho$ .



$$\begin{aligned}
& \left[ \frac{d^2 \rho}{dt^2} + \left[ \frac{L}{2} + \frac{L\rho}{2} \right] \left( \frac{d\rho}{dt} \right) \left( \frac{d\mu}{dt} \right) + \left[ \frac{1}{2} \left( 2A\rho \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + B + L \left[ \frac{\rho''}{\rho^2} + \frac{\rho''}{(1-\rho)^2} + \frac{2\rho'^2}{(1-\rho)^3} + \right. \right. \right. \\
& \left. \left. \left. t\mu' + 2\mu' \right] \right) + \frac{1}{2} \left( 2A\rho \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + B + \left( \frac{\rho\rho'' - 2\rho'^2}{\rho^3} + \frac{2\rho'^2 + (1-\rho)\rho''}{(1-\rho)^3} + t\mu' + \rho'\mu' + \right. \right. \right. \\
& \left. \left. \left. \mu' + \mu\rho'' \right) L \right] \right] \left( \frac{d\rho}{dt} \right) \left( \frac{d\mu}{dt} \right) + \left[ \rho' L \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) \right] \left( \frac{d\rho}{dt} \right)^2 + \left[ \frac{1}{2} \left( A\rho' + B(1 + 2\rho' + t\rho'') \right) + \right. \\
& \left. \frac{1}{2} \left( A + (\rho' + t\rho'') L \right) \right] \left( \frac{d\mu}{dt} \right) \left( \frac{d\mu}{dt} \right) + \left[ \frac{1}{2} \left( \left( \frac{\rho\rho'' - 2\rho'^2}{\rho^3} + \frac{2\rho'^2 + (1-\rho)\rho''}{(1-\rho)^3} + t\mu' + \rho'\mu' + \mu' + \right. \right. \right. \\
& \left. \left. \left. \mu\rho'' \right) A + (\rho' + t\rho'') B + \left[ \frac{(\rho'\rho'' - \rho\rho''')\rho - 2\rho'(\rho'^2 - \rho\rho'')}{\rho^3} + \right. \right. \right. \\
& \left. \left. \left. \frac{(\rho'\rho'' + (1-\rho)\rho''')(1-\rho) + 2\rho'(\rho'^2 + (1-\rho)\rho'')}{(1-\rho)^3} + (1+\rho)t\mu'' + (1+\rho)\mu'' + 3\rho'\mu'' + \mu'' + 2\rho''\mu'' + \right. \right. \right. \\
& \left. \left. \left. 2\mu\rho'' + 4\mu'\rho' + 2\rho\mu'' + \mu t\rho''' + \mu\rho'' + \mu' t\rho'' \right] L \right] \left( \frac{dt}{dt} \right)^2 \right] = 0
\end{aligned} \tag{50}$$

It can be verified that one of the closed form solutions of (5.1) is determined by the paths of motion:

$$\rho(t) = \text{constant}, \mu(t) = \text{constant} \text{ such that } \rho \in (0,1), \mu > 1 \tag{51}$$

## 5.2. The **IMEs** of the **Mean service Rate** coordinate, $\mu$ of the transient M/M/1 QM

The IMEs (c.f., Equation (5)) corresponding to the **mean service rate** coordinate,  $\mu$  of the transient M/M/1 QM are

$$\frac{d^2 \theta^2}{dt^2} + \Gamma_{ij}^{1(0)} \left( \frac{d\theta^i}{dt} \right) \left( \frac{d\theta^j}{dt} \right) = 0, i, j = 1, 2, 3$$

Now, we are in a situation of trying to find the path of motion of family of families of IMEs corresponding to the mean service rate,  $\mu$ .

$$\begin{aligned}
& \left[ \frac{d^2 \mu}{dt^2} + \left[ \frac{F}{2} + \frac{F\rho}{2} \right] \left( \frac{d\rho}{dt} \right) \left( \frac{d\mu}{dt} \right) + \left[ \frac{1}{2} \left( 2B\rho \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + E + F \left[ \frac{\rho''}{\rho^2} + \frac{\rho''}{(1-\rho)^2} + \frac{2\rho'^2}{(1-\rho)^3} + \right. \right. \right. \\
& \left. \left. \left. t\mu' + 2\mu' \right] \right) + \frac{1}{2} \left( 2B\rho \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + E + \left( \frac{\rho\rho'' - 2\rho'^2}{\rho^3} + \frac{2\rho'^2 + (1-\rho)\rho''}{(1-\rho)^3} + t\mu' + \rho'\mu' + \right. \right. \right. \\
& \left. \left. \left. \mu' + \mu\rho'' \right) F \right] \right] \left( \frac{d\mu}{dt} \right) \left( \frac{d\mu}{dt} \right) + \left[ \rho' \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) F \right] \left( \frac{d\rho}{dt} \right)^2 + \left[ \frac{1}{2} \left( B\rho' + E(1 + 2\rho' + t\rho'') \right) + \right. \\
& \left. \frac{1}{2} \left( B + (\rho' + t\rho'') F \right) \right] \left( \frac{d\rho}{dt} \right) \left( \frac{d\mu}{dt} \right) + \left[ \frac{1}{2} \left( \left( \frac{\rho\rho'' - 2\rho'^2}{\rho^3} + \frac{2\rho'^2 + (1-\rho)\rho''}{(1-\rho)^3} + t\mu' + \rho'\mu' + \mu' + \right. \right. \right. \\
& \left. \left. \left. \mu\rho'' \right) B + (\rho' + t\rho'') E + \left[ \frac{(\rho'\rho'' - \rho\rho''')\rho - 2\rho'(\rho'^2 - \rho\rho'')}{\rho^3} + \right. \right. \right. \\
& \left. \left. \left. \frac{(\rho'\rho'' + (1-\rho)\rho''')(1-\rho) + 2\rho'(\rho'^2 + (1-\rho)\rho'')}{(1-\rho)^3} + (1+\rho)t\mu'' + (1+\rho)\mu'' + 3\rho'\mu'' + \mu'' + 2\rho''\mu'' + \right. \right. \right. \\
& \left. \left. \left. 2\mu\rho'' + 4\mu'\rho' + 2\rho\mu'' + \mu t\rho''' + \mu\rho'' + \mu' t\rho'' \right] F \right] \left( \frac{dt}{dt} \right)^2 \right] = 0
\end{aligned} \tag{52}$$

Let  $\rho = \text{constant}$ , then (52) reduces to

$$\left[ \frac{d^2\mu}{dt^2} + \left[ \frac{1}{2} (E + F[t\mu^- + 2\mu^-]) + E + (t\mu^- + \mu^-) F \right] \left( \frac{d\mu}{dt} \right) + \left[ \frac{1}{2} ((t\mu^- + \mu^-)B + \right. \right. \quad (53)$$

$$\left. \left. [(1+\rho)t\mu^- + (1+\rho)\mu^- + \mu^- + 2\rho\mu^-]F \right) \right] = 0$$

$$\Delta = \det([g_{ij}]) = \left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right) \quad (54)$$

$$B = \frac{(t\mu^-(1-\rho) - t[(1+\rho)t\mu^- + \mu^-])}{\Delta} \quad (55)$$

$$= \frac{(t\mu^-(1-\rho) - t[(1+\rho)t\mu^- + \mu^-])}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)}$$

$$E = \frac{(al - c^2)}{\Delta} = \frac{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) [(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-] - (t\mu^-)^2 \right)}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)} \quad (56)$$

$$F = \frac{(bc - ah)}{\Delta} = \frac{(t(t\mu^-) - \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (1+\rho))}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)} \quad (57)$$

Following the above done calculations, we have

$$\left[ \frac{d^2\mu}{dt^2} + \left[ \frac{1}{2} \left( \frac{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) [(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-] - (t\mu^-)^2 \right)}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)} + \right. \right. \quad (58)$$

$$\left. \left. \frac{(t(t\mu^-) - \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (1+\rho))}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)} [t\mu^- + 2\mu^-] \right) + \right.$$

$$\left. \frac{(t(t\mu^-) - \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (1+\rho))}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)} + (t\mu^- + \mu^-) \right] \left( \frac{d\mu}{dt} \right) + \left[ \frac{1}{2} ((t\mu^- + \right.$$

$$\mu^-) \frac{(t\mu^-(1-\rho) - t[(1+\rho)t\mu^- + \mu^-])}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)} + [(1+\rho)t\mu^- + (1+\rho)\mu^- + \mu^- +$$

$$2\rho\mu^-] \frac{(t(t\mu^-) - \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (1+\rho))}{\left( \left( \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right) (-(1+\rho))^2 - (t^2[(1+\rho)t\mu^- + \mu^- + 2\rho\mu^-]) \right)} \left. \right] = 0$$

It is obvious that for arbitrary constant values of  $\rho, \mu$ , we have a closed form solution for (58). As time becomes sufficiently large, i.e.,  $t \rightarrow \infty$ , (58) reduces to

$$\left[ \frac{3\mu}{2} - \frac{3\mu^2}{2(1+\rho)} - \frac{\mu\mu''}{\mu'} \right] = 0 \quad (59)$$

We propose the closed form solution,  $\mu = \frac{2\gamma t}{(1+\gamma t)}$ , for some arbitrary non-zero constant  $\gamma$ . substituting in (59) implies:

$$18\gamma^3 + \frac{6\gamma^2}{(1+\gamma t)(1+\rho)} + 6\gamma^4 = 0 \quad (60)$$

As  $t \rightarrow \infty$ ,  $\gamma = 0, 0, -3$ . The only accepted value is  $\gamma = -3$ ,  $\mu = \frac{6t}{(3t-1)}$ , which tends to the value  $\mu = 2$  as  $t \rightarrow \infty$ .

### 5.3. The **IMEs** of the **temporal** coordinate, **t** of the transient M/M/1 QM

The IMEs (c.f., Equation (5)) corresponding to the **time** coordinate, **t** of the transient M/M/1 QM are

$$\begin{aligned} & + \Gamma_{ij}^{1(0)} \left( \frac{d\theta^i}{dt} \right) \left( \frac{d\theta^j}{dt} \right) = 0, i, j = 1, 2, 3 \\ & \left[ \frac{d^2 t}{dt^2} + \left[ \frac{1}{2} \left( \left( \frac{\rho\rho' - 2\rho^2}{\rho^3} + \frac{2\rho^2 + (1-\rho)\rho''}{(1-\rho)^3} + t\mu'' + \rho'\mu' + \mu' + \mu\rho'' \right) L \right. \right. \right. \\ & \quad + (\rho' + t\rho'')F \\ & \quad + \left[ \frac{(\rho\rho'' - \rho\rho''')\rho - 2\rho(\rho^2 - \rho\rho')}{\rho^3} \right. \\ & \quad + \frac{(\rho\rho'' + (1-\rho)\rho''')(1-\rho) + 2\rho(\rho^2 + (1-\rho)\rho')}{(1-\rho)^3} \\ & \quad + (1+\rho)t\mu'' + (1+\rho)\mu' + 3\rho\mu'' + \mu' + 2\rho'\mu' + 2\mu\rho'' + 4\mu\rho' \\ & \quad \left. \left. \left. + 2\rho\mu'' + t\rho\rho'' + \mu\rho'' + \mu't\rho'' \right] H \right] \right] \left( \frac{dt}{dt} \right)^2 \\ & \quad + \left[ \rho H \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) \right] \left( \frac{d\rho}{dt} \right)^2 + \left[ \frac{H}{2} + \frac{H\rho}{2} \right] \left( \frac{d\rho}{dt} \right) \left( \frac{d\mu}{dt} \right) \\ & \quad + \left[ \frac{1}{2} (L\rho' + F(1 + 2\rho' + t\rho'')) \right. \\ & \quad \left. + \frac{1}{2} (L + (\rho' + t\rho'')H) \right] \left( \frac{d\mu}{dt} \right) \left( \frac{dt}{dt} \right) \\ & \quad + \left[ \frac{1}{2} \left( 2L\rho' \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + F \right. \right. \\ & \quad \left. \left. + H \left[ \frac{\rho''}{\rho^2} + \frac{\rho''}{(1-\rho)^2} + \frac{2\rho^2}{(1-\rho)^3} + t\mu'' + 2\mu' \right] \right) \right. \\ & \quad \left. + \frac{1}{2} \left[ (2L\rho' \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + F + \left( \frac{\rho\rho'' - 2\rho^2}{\rho^3} \right. \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{2\rho^2 + (1-\rho)\rho''}{(1-\rho)^3} + t\mu'' + \rho'\mu' + \mu' + \mu\rho'' \right) H \right] \right] \left( \frac{d\rho}{dt} \right) \left( \frac{dt}{dt} \right) \right] = 0 \end{aligned} \quad (61)$$

For constant server utilization,  $\rho$ , (61) reduces to

$$\begin{aligned}
& \left[ \left( (t\mu' + \mu') \frac{(t(1+\rho))}{\left(\left[\frac{1}{\rho^2} - \frac{1}{(1-\rho)^2}\right](- (1+\rho)^2) - t((1+\rho)t\mu' + \mu + 2\rho\mu') - t\mu'(1+\rho)) + t\mu'(t(1+\rho))\right)} + \right. \right. \\
& \quad \left. \left. [(1+\rho)t\mu'' + (1+\rho)\mu'' + \mu' + 2\rho\mu'] \frac{(-t^2)}{\left(\left[\frac{1}{\rho^2} - \frac{1}{(1-\rho)^2}\right](- (1+\rho)^2) - t((1+\rho)t\mu' + \mu + 2\rho\mu') - t\mu'(1+\rho)) + t\mu'(t(1+\rho))\right)} \right] + \right. \\
& \quad \left[ \frac{(t(t\mu') - [\frac{1}{\rho^2} - \frac{1}{(1-\rho)^2}](1+\rho))}{\left(\left[\frac{1}{\rho^2} - \frac{1}{(1-\rho)^2}\right](- (1+\rho)^2) - t((1+\rho)t\mu' + \mu + 2\rho\mu') - t\mu'(1+\rho)) + t\mu'(t(1+\rho))\right)} + \right. \\
& \quad \left. \left. \frac{(t(1+\rho))}{\left(\left[\frac{1}{\rho^2} - \frac{1}{(1-\rho)^2}\right](- (1+\rho)^2) - t((1+\rho)t\mu' + \mu + 2\rho\mu') - t\mu'(1+\rho)) + t\mu'(t(1+\rho))\right)} \right] \left( \frac{d\mu}{dt} \right) = 0
\end{aligned} \tag{62}$$

As time reaches infinity, (62) re-writes to

$$\frac{\mu'''}{\mu''} = 0 \tag{63}$$

By (63), we have

$$\mu''' = 0 \tag{64}$$

The closed form solution of (64) is characterized by the family of families of temporal curves :

$$\mu(t) = \zeta_1 + \zeta_2 t + \zeta_3 t^2 \quad \text{for some arbitrary non-zero constants } \zeta_1, \zeta_2, \zeta_3, \quad \zeta_1 > 1 \tag{65}$$

## 7. THE THRESHOLD THEOREMS FOR THE POTENTIAL FUNCTION OF EQUATION (34).

### 7.1. The Threshold Theorem for the Potential Function, TTPF (c.f., (34) of Theorem 3.1)

Communicating the threshold theorem for the potential function of Equation (34) corresponding to each coordinate is devised.

**Theorem 2.** For the obtained potential function  $\Psi(\theta)$ , the following are satisfied:

i)  $\Psi(\theta)$  is forever increasing in  $\rho \Leftrightarrow$

$$t > \frac{1 - 2\rho}{\mu\rho(1 - \rho)} \tag{66}$$

ii)  $\Psi(\theta)$  is forever decreasing in  $\rho \Leftrightarrow$

$$t < \frac{1 - 2\rho}{\mu\rho(1 - \rho)} \tag{67}$$

iii)  $\Psi(\theta)$  is forever increasing in  $\mu$

iv)  $\Psi(\theta)$  is never increasing in  $\mu$

v)  $\Psi(\theta)$  is forever increasing in  $t$  if and only if

$$t > \frac{\rho \left( \frac{1 - 2\rho}{\rho(1 - \rho)} \right) - \mu(1 + \rho)}{((1 + \rho)\mu' + \mu\rho')} \tag{68}$$

vi)  $\Psi(\theta)$  is forever decreasing in  $t$  if and only if

$$t < \frac{\rho \left( \frac{1-2\rho}{\rho(1-\rho)} \right) - \mu(1+\rho)}{((1+\rho)\mu + \mu\rho)} \quad (69)$$

### Proof

i) We have

$$\partial_1 = \frac{\partial \Psi}{\partial \rho} = -\frac{1}{\rho} + \frac{1}{(1-\rho)} + \mu t, \partial_2 = \frac{\partial \Psi}{\partial \mu} = (1+\rho)t, \partial_3 = \frac{\partial \Psi}{\partial t} = -\frac{\rho}{\rho} + \frac{\rho}{(1-\rho)} + (1+\rho)t\mu + \mu(1+\rho) + \mu t\rho$$

It holds that  $\partial_1 > 0$  if and only if

$$-\frac{1}{\rho} + \frac{1}{(1-\rho)} + \mu t > 0 \quad (70)$$

Hence, (i) follows.

Similarly, (ii) holds.

(iii)  $\partial_2 = \frac{\partial \Psi}{\partial \mu} = (1+\rho)t$ . Therefore,  $\partial_2 = \frac{\partial \Psi}{\partial \mu} = (1+\rho)t > 0$  if and only if  $(1+\rho) > 0$  or  $t > 0$ ,

which holds forever. Hence, (iii) holds.

(iv)  $\partial_2 = \frac{\partial \Psi}{\partial \mu} = (1+\rho)t < 0$  if and only if one of the following statements hold:

1.  $(1+\rho) > 0$  and  $t < 0$

2.  $(1+\rho) < 0$  and  $t > 0$

It is a fact that both the above statements are impossible. Hence, (iv) follows.

(v)  $\partial_3 = \frac{\partial \Psi}{\partial t} = -\frac{\rho}{\rho} + \frac{\rho}{(1-\rho)} + (1+\rho)t\mu + \mu(1+\rho) + \mu t\rho > 0$  if and only if

$[(1+\rho)\mu + \mu\rho]t > -\mu(1+\rho) + \frac{\rho}{\rho} - \frac{\rho}{(1-\rho)} \Leftrightarrow t > \frac{(\frac{\rho}{\rho} - \frac{\rho}{(1-\rho)}) - \mu(1+\rho)}{[(1+\rho)\mu + \mu\rho]}$ , which proves (v) by the preliminary theorem.

A similar argument to (v) proves (vi).

## 7.2. Numerical Experiments on the potential function, PF

In what follows,  $SU = \text{Server Utilization} = \rho$ ,  $MSR = \mu = \text{Mean Service Rate}$ ,  $t = \text{time}$ ,  $PF = \text{Potential Function} = \Psi(\theta)$

## 7.3. Numerical Experiments on the potential function, PF

In what follows,  $SU = \text{Server Utilization} = \rho$ ,  $MSR = \mu = \text{Mean Service Rate}$ ,  $t = \text{time}$ ,  $PF = \text{Potential Function} = \Psi(\theta)$

### 7.3.1. Numerical Experiment One

$$\Psi(\theta) = -l n(\rho) - l n(1-\rho) + \mu(1+\rho)t, \mu = 2, t = 1$$

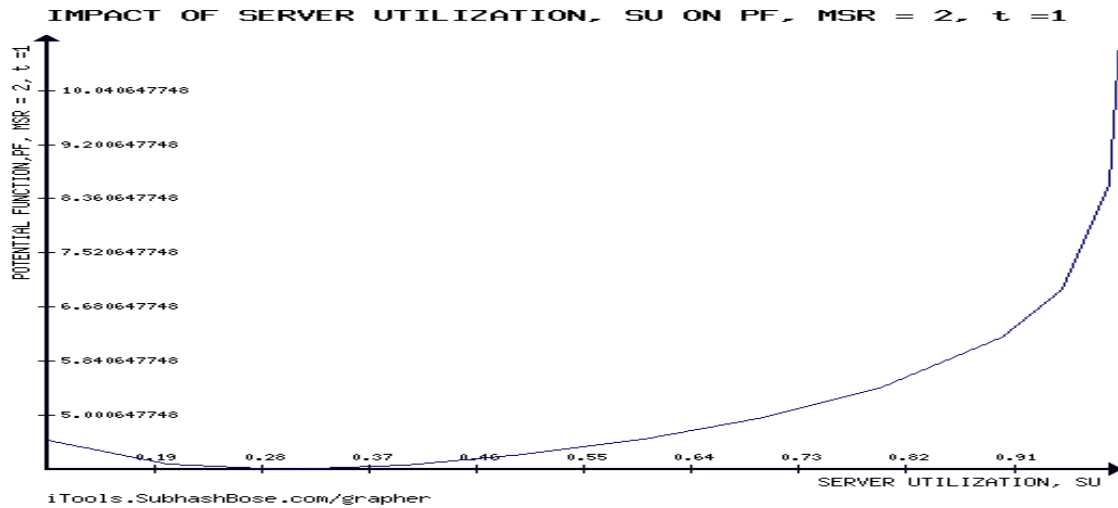


Figure 3.

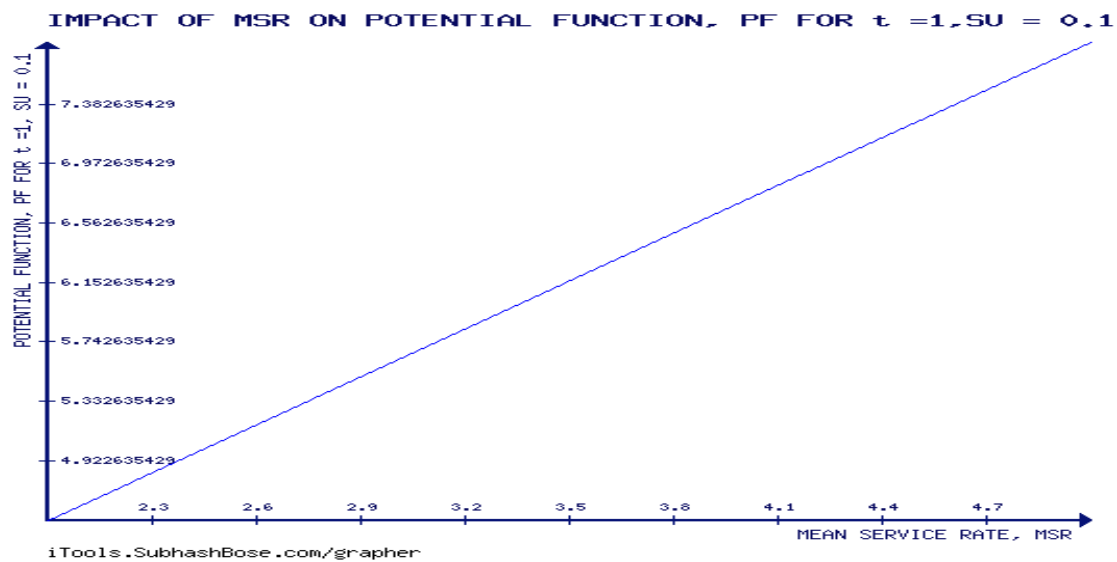


Figure 4.

Communicating figure 3, the agreement of the experimental findings and the analytic results (c.f., (i) and (ii) of Theorem 7.1). The  $\rho_{Threshold} = \frac{1-2\rho}{2\rho(1-\rho)}$  starts to be higher than the proposed temporal value,  $t = 1$  which enforces the decreasability of the potential function in server utilization. As the server utilization increases, the  $\rho_{Threshold} = \frac{1-2\rho}{2\rho(1-\rho)}$  starts to be lower than the proposed temporal value,  $t = 1$  which enforces the increasability of the potential function in server utilization. This perfectly matches with the analytic results. Figure 4 shows that the potential function is forever increasing in mean service rate,  $\mu$ . This matches with the analytic results.

More potentially,

$$\Psi(\theta) = -\ln(\rho) - \ln(1 - \rho) + \mu(1 + \rho)t, \mu = 2 + t, \rho = 0.1$$

$$= -\ln(0.099) + (1.1)(2 + t)t, \quad t_{Threshold} = \frac{\rho \left( \frac{1-2\rho}{\rho(1-\rho)} \right) - \mu(1+\rho)}{((1+\rho)\mu + \mu\rho)} = \frac{-(1.1)(t+2)}{1.1} = -(t + 2)$$

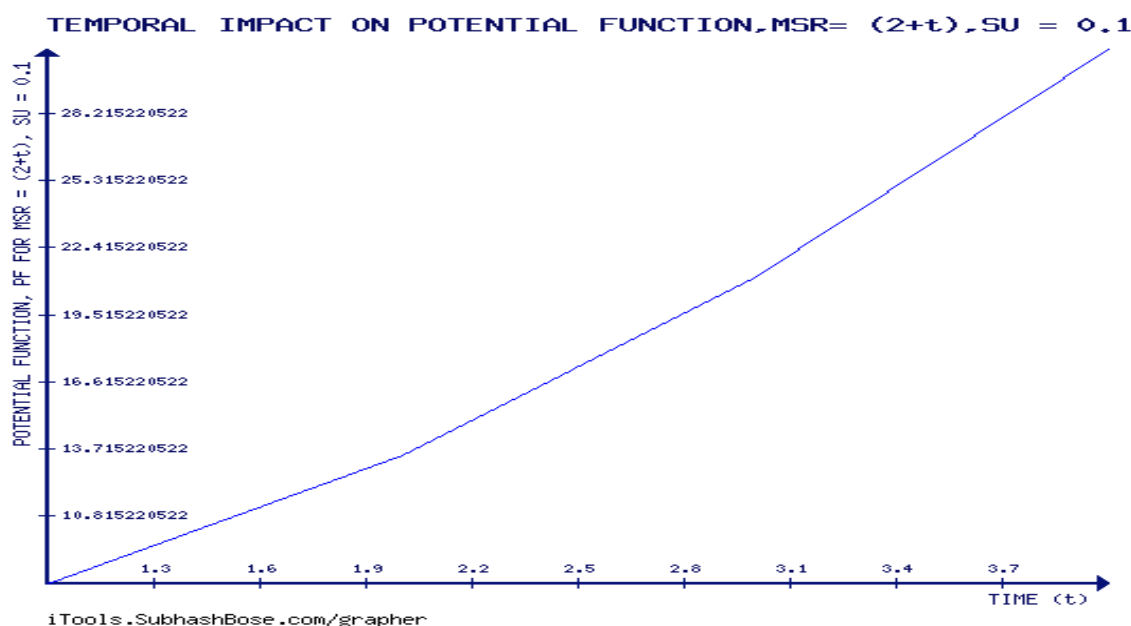


Figure 5.

As observed from figure 5 that the potential function is forever increasing in time, since time is greater than the calculated value of the  $t_{\text{Threshold}}$ . This shows that the experimental values agree with the obtained analytic results from Theorem 7.1.

### 9. SOME ALGEBRAIC PROPERTIES OF THE POTENTIAL FUNCTION, $\Psi(\theta) = -\ln(\rho) - \ln(1 - \rho) + \mu(1 + \rho)t$ (c.f., (3.20))

**Theorem 4.** The three-dimensional potential function  $\Psi(\theta) = -\ln(\rho) - \ln(1 - \rho) + \mu(1 + \rho)t$  (c.f., (34)) is generally not well-defined.

**Proof.** Let  $\rho_i, \mu_i, t_i$ ,  $i = 1, 2$  be such that  $\rho_1 \neq \rho_2$ ,  $\mu_1 \neq \mu_2$ ,  $t_1 \neq t_2$ . Let  $\Psi(\rho_1, \mu_1, t_1) = \Psi(\rho_2, \mu_2, t_2)$ . This implies

$$-\ln(\rho_1) - \ln(1 - \rho_1) + \mu_1(1 + \rho_1)t_1 = -\ln(\rho_2) - \ln(1 - \rho_2) + \mu_2(1 + \rho_2)t_2 \quad (71)$$

(71) directly implies:

$$\ln\left(\frac{\rho_2(1 - \rho_2)}{\rho_1(1 - \rho_1)}\right) = \mu_2(1 + \rho_2)t_2 - \mu_1(1 + \rho_1)t_1 \quad (72)$$

Assuming that  $\rho_1 = \rho_2$ . This transforms (72) to

$$0 = (1 + \rho_1)[\mu_2 t_2 - \mu_1 t_1] \quad (73)$$

Clearly (73) is satisfied if and only if

$$\mu_2 t_2 = \mu_1 t_1 \quad (\text{since } (1 + \rho_1) \neq 0) \quad (74)$$

It can be verified that (74) is generated for values  $\mu_1 \neq \mu_2, t_1 \neq t_2$ , which proves that  $\Psi(\theta)$  is not well-defined in general.

Several emerging important special cases of Theorem 9.1 are obtained in the following theorems.

**Theorem 2.** For constant values of  $\mu, t$ , the three-dimensional potential function  $\Psi_\rho(\theta) = -\ln(\rho) - \ln(1 - \rho) + \mu(1 + \rho)t$  (c.f., (34)) is not well-defined in general.

**Proof.** For constant values of  $\mu, t$ , define  $\rho_i, i = 1, 2$  be such that  $\rho_1 \neq \rho_2$ . Thus,  $\Psi(\rho_1, \mu, t) = \Psi(\rho_2, \mu, t)$  implies:

$$-l n(\rho_1) - l n(1 - \rho_1) + \mu(1 + \rho_1)t = -l n(\rho_2) - l n(1 - \rho_2) + \mu(1 + \rho_2)t \quad (75)$$

Hence, it follows that

$$l n\left(\frac{\rho_2(1 - \rho_2)}{\rho_1(1 - \rho_1)}\right) = \mu t(\rho_2 - \rho_1) \quad (76)$$

Consequently, it follows from (76) that

$$\left(\frac{\rho_2(1 - \rho_2)}{\rho_1(1 - \rho_1)}\right) = e^{\mu t(\rho_2 - \rho_1)} \quad (77)$$

One of the infinitely many solutions of (77) is  $\rho_1 = \rho_2$ . This means that  $\Psi_\rho$  is not generally well-defined.

**Theorem 3.** For constant  $\rho, \mu$ , the potential function  $\Psi_t(\theta) = -l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t$  (c.f., (34)) satisfies the following:

- (1)  $\Psi_t(\theta)$  is well-defined.
- (2)  $\Psi_t(\theta)$  is onto.
- (3)  $\Psi_t(\theta)$  is one-to-one.
- (4)  $\Psi_t(\theta)$  has a unique inverse,  $\Psi_t^{-1}$  given by

$$\Psi_t^{-1}(\rho, \mu, t) = \frac{t + l n(\rho(1 - \rho))}{\mu(1 + \rho)} \quad (78)$$

**Proof.**(1) For constant  $\rho, \mu$  and  $t_1 \neq t_2$ , let

$$-l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t_1 = -l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t_2 \quad (79)$$

This implies

$$\mu(1 + \rho)t_1 = \mu(1 + \rho)t_2 \quad (80)$$

Clearly, (80) implies:

$$\mu(1 + \rho)(t_1 - t_2) = 0 \quad (81)$$

(82) holds if and only if  $\mu = 0$  (impossible, since  $\mu > 1$ ) or  $(1 + \rho) = 0$  (impossible since  $\rho \in (0, 1)$ ) or

$t_1 - t_2 = 0$  which implies that  $t_1 = t_2$ . Hence, the proof of (1) follows.

(2) It is clear that for every  $-l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t$ , There is a unique triad  $(\rho, \mu, t)$  such that

$$\psi(\rho, \mu, t) = -l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t \quad (82)$$

This proves (2).

To prove (3), we need to show that



$$(\rho, \mu, t_1) = \psi(\rho, \mu, t_2) \text{ if and only if } t_1 = t_2 \quad (83)$$

The proof of (9.12) is straightforward.

To prove (4), let  $\psi(\rho, \mu, t) = -l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t = y$ . This implies that  $t = \frac{y + l n(\rho(1 - \rho))}{\mu(1 + \rho)}$ , which directly implies

$$\Psi_t^{-1}(\rho, \mu, t) = \frac{t + l n(\rho(1 - \rho))}{\mu(1 + \rho)}$$

**Theorem 4.** For constant  $\rho, t$ , the potential function  $\Psi_\mu(\theta) = -l n(\rho) - l n(1 - \rho) + \mu(1 + \rho)t$  (c.f., (34)) satisfies the following:

- (1)  $\Psi_\mu(\theta)$  is well-defined.
- (2)  $\Psi_\mu(\theta)$  is onto.
- (3)  $\Psi_\mu(\theta)$  is one-to-one.
- (4)  $\Psi_\mu(\theta)$  has a unique inverse,  $\Psi_\mu^{-1}$  given by

$$\Psi_\mu^{-1}(\rho, \mu, t) = \frac{\mu + l n(\rho(1 - \rho))}{t(1 + \rho)} \quad (84)$$

**Proof.** The proofs are similar to Theorem 9.4.

## 10. The Threshold Theorems Of The Derived Inverses Of Potential Function, Ipfs

**Theorem 10.1** For the inverse potential function,

$$\Psi_t^{-1}(\rho, \mu, t) = \frac{t + l n(\rho(1 - \rho))}{\mu(1 + \rho)} \quad (85)$$

The following hold

- i)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever increasing in  $t$
- ii)  $\Psi_t^{-1}(\rho, \mu, t)$  is never decreasing in  $t$
- iii)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever increasing in  $\mu$  if and only if

$$t < l n\left(\frac{1}{\rho(1 - \rho)}\right) \quad (86)$$

- iv)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever decreasing in  $\mu$  if and only if

$$t > l n\left(\frac{1}{\rho(1 - \rho)}\right) \quad (87)$$

- v)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever increasing in  $\rho$  if and only if

$$t < \left[ \frac{1 - \rho - 2\rho^2}{\rho(1 - \rho)} + l n\left(\frac{1}{\rho(1 - \rho)}\right) \right] \quad (88)$$

- vi)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever decreasing in  $\rho$  if and only if

$$t > \left[ \frac{1 - \rho - 2\rho^2}{\rho(1 - \rho)} + l n\left(\frac{1}{\rho(1 - \rho)}\right) \right] \quad (89)$$

**Proof.**

- i) We have

$$\frac{\partial \Psi_t^{-1}}{\partial t} = \frac{1}{\mu(1 + \rho)} \quad (90)$$

By the preliminary theorem,  $\Psi_t^{-1}$  is forever increasing since  $\mu > 1, \rho \in (0,1)$ . Following (10.5), it is implied that i) holds.

The proof of ii) is immediate by (10.5).

iii) It can be shown that

$$\frac{\partial \Psi_t^{-1}}{\partial \mu} = - \frac{t + \ln(\rho(1-\rho))}{\mu^2(1+\rho)} \quad (91)$$

Engaging the same approach,  $\frac{\partial \Psi_t^{-1}}{\partial \mu} > 0$  if and only if

$$t + \ln(\rho(1-\rho)) < 0 \quad (92)$$

(92) holds if and only if

$$< -\ln(\rho(1-\rho)) \quad (93)$$

Hence, iii) follows.

Following the same approach, iv) holds.

v) We have

$$\frac{\partial \Psi_t^{-1}}{\partial \rho} = \frac{1}{\mu} \left[ \frac{(1-\rho-2\rho^2) - \rho(1-\rho)(t + \ln(\rho(1-\rho)))}{\rho(1-\rho)(1+\rho)^2} \right] \quad (94)$$

Engaging the same approach,  $\frac{\partial \Psi_t^{-1}}{\partial \rho} > 0$  if and only if

$$(1-\rho-2\rho^2) - \rho(1-\rho)(t + \ln(\rho(1-\rho))) > 0 \quad (95)$$

Clearly, by (95) and the Preliminary Theorem (PT), v) holds.

A similar approach proves vi).

**Theorem 5.** For the inverse potential function,

$$\Psi_\mu^{-1}(\rho, \mu, t) = \frac{\mu + \ln(\rho(1-\rho))}{t(1+\rho)}$$

The following hold

i)  $\Psi_\mu^{-1}(\rho, \mu, t)$  is forever decreasing in  $t$

ii)  $\Psi_\mu^{-1}(\rho, \mu, t)$  is never increasing in  $t$

iii)  $\Psi_\mu^{-1}(\rho, \mu, t)$  is forever increasing in  $\mu$

iv)  $\Psi_\mu^{-1}(\rho, \mu, t)$  is never decreasing in  $\mu$

v)  $\Psi_\mu^{-1}(\rho, \mu, t)$  is forever increasing in  $\rho$  if and only if

$$\mu < \left[ \frac{1-\rho-2\rho^2}{\rho(1-\rho)} + \ln\left(\frac{1}{\rho(1-\rho)}\right) \right] \quad (96)$$

vi)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever decreasing in  $\rho$  if and only if

$$\mu > \left[ \frac{1-\rho-2\rho^2}{\rho(1-\rho)} + \ln\left(\frac{1}{\rho(1-\rho)}\right) \right] \quad (97)$$

**Proof.**

The proofs are analogous to Theorem 10.1.

## 11. NUMERICAL EXPERIMENTS ON THE THRESHOLD THEOREMS OF THE DERIVED INVERSES OF POTENTIAL FUNCTION

### 11.1. Numerical Experiment on Theorem 11.1

We have, the inverse of the potential function, IPF=  $\Psi_t^{-1}(\rho, \mu, t) = \frac{t + \ln(\rho(1-\rho))}{\mu(1+\rho)}$  (c.f., (85)).

**Theorem 6.** For the inverse potential function,

$$\Psi_t^{-1}(\rho, \mu, t) = \frac{t + \ln(\rho(1-\rho))}{\mu(1+\rho)}$$

The following hold

- i)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever increasing in  $t$
- ii)  $\Psi_t^{-1}(\rho, \mu, t)$  is never decreasing in  $t$
- iii)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever increasing in  $\mu$  if and only if

$$t < \ln\left(\frac{1}{\rho(1-\rho)}\right) \quad (98)$$

- iv)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever decreasing in  $\mu$  if and only if

$$t > \ln\left(\frac{1}{\rho(1-\rho)}\right) \quad (99)$$

- v)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever increasing in  $\rho$  if and only if

$$t < \left[ \frac{1-\rho-2\rho^2}{\rho(1-\rho)} + \ln\left(\frac{1}{\rho(1-\rho)}\right) \right] \quad (100)$$

- vi)  $\Psi_t^{-1}(\rho, \mu, t)$  is forever decreasing in  $\rho$  if and only if

$$t > \left[ \frac{1-\rho-2\rho^2}{\rho(1-\rho)} + \ln\left(\frac{1}{\rho(1-\rho)}\right) \right] \quad (101)$$

It can be seen that the temporal performance behaviour,

$$\Psi_t^{-1}(\rho, \mu, t) = \frac{t + \ln(\rho(1-\rho))}{\mu(1+\rho)}$$

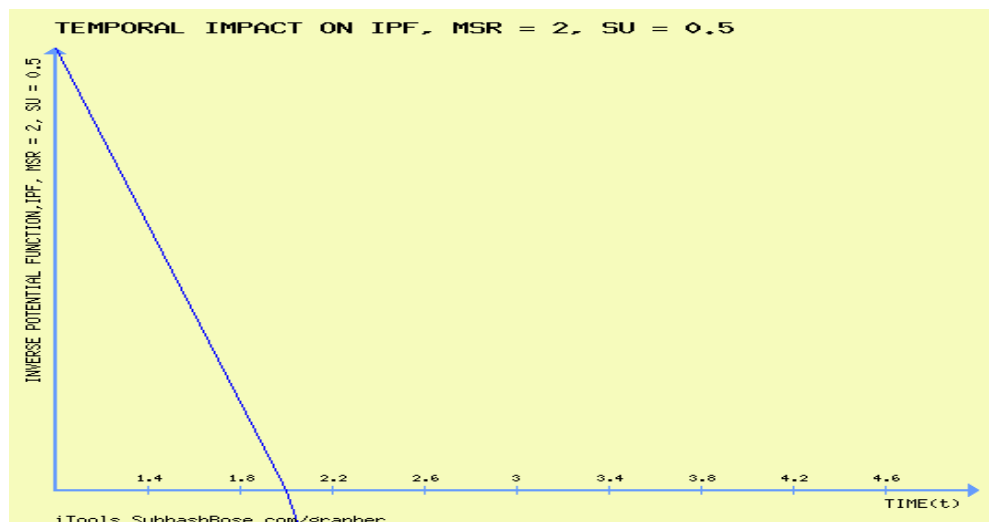


Figure 6.

It is observed from figure 6 that  $\Psi_t^{-1}(\rho, \mu, t)$  is forever decreasing in time and never increase in time.

As for the performance behaviour in Mean service Time,

Let  $t = 1, \rho = 0.5$ ,  $\ln\left(\frac{1}{\rho(1-\rho)}\right) = 1.386294361 > 1 = t$

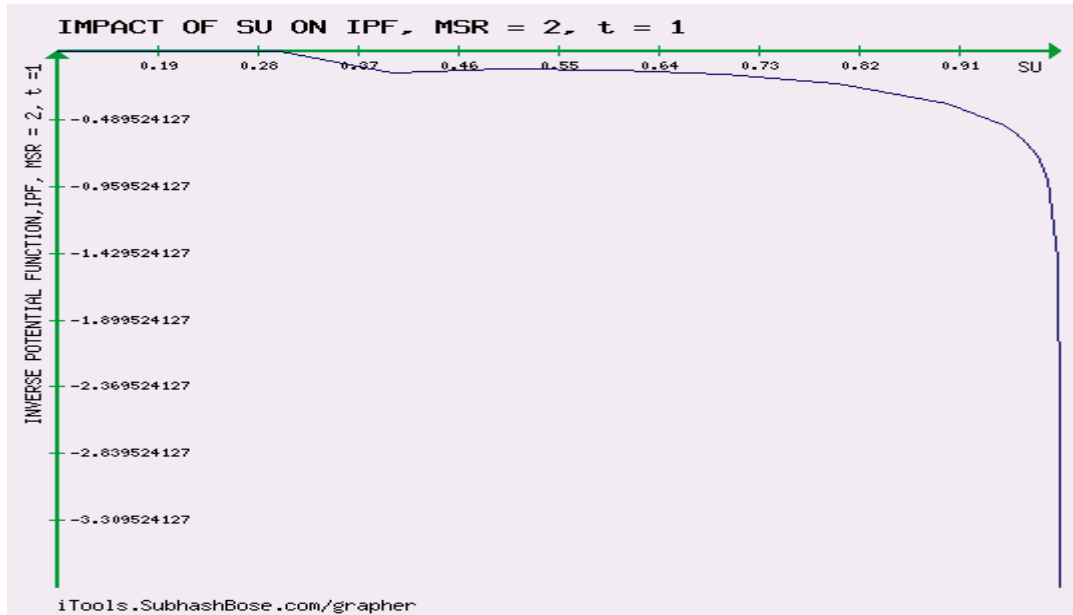


Figure 7.

It is observed from figure 10 that as time is lower than  $\rho_{Threshold} = \left[\frac{1-\rho-2\rho^2}{\rho(1-\rho)} + \ln\left(\frac{1}{\rho(1-\rho)}\right)\right]$ ,  $\Psi_t^{-1}(\rho, \mu, t) = \frac{t + \ln(\rho(1-\rho))}{\mu(1+\rho)}$  increases with respect to server utilization until times becomes greater than  $\rho_{Threshold}$ , the inverse potential function starts to decrease against the increase of server utilization. This shows that the numerical results numerically match the analytical results of v) and vi) Theorem 10.1. To make this clearer, the reader is advised to look at figures 7-10.

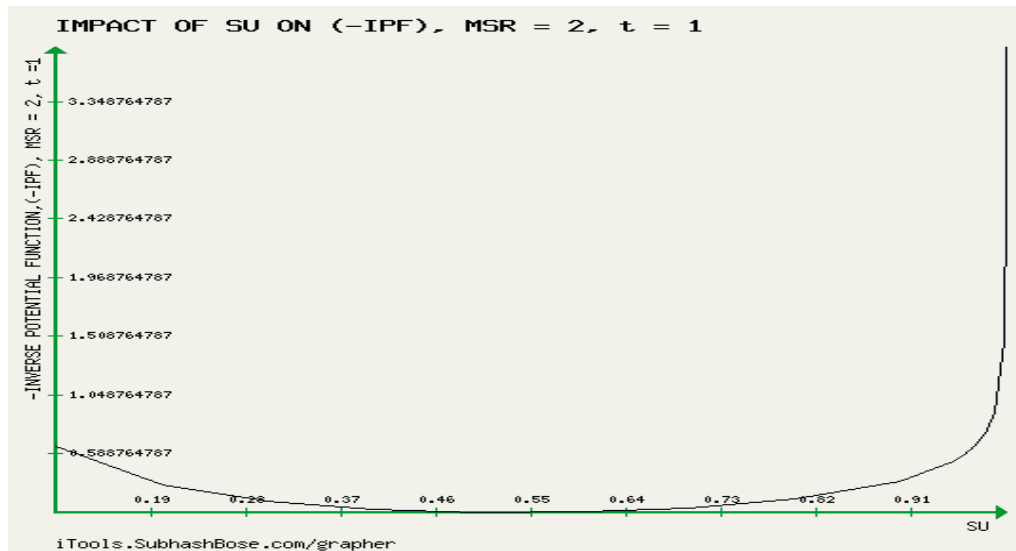


Figure 8.

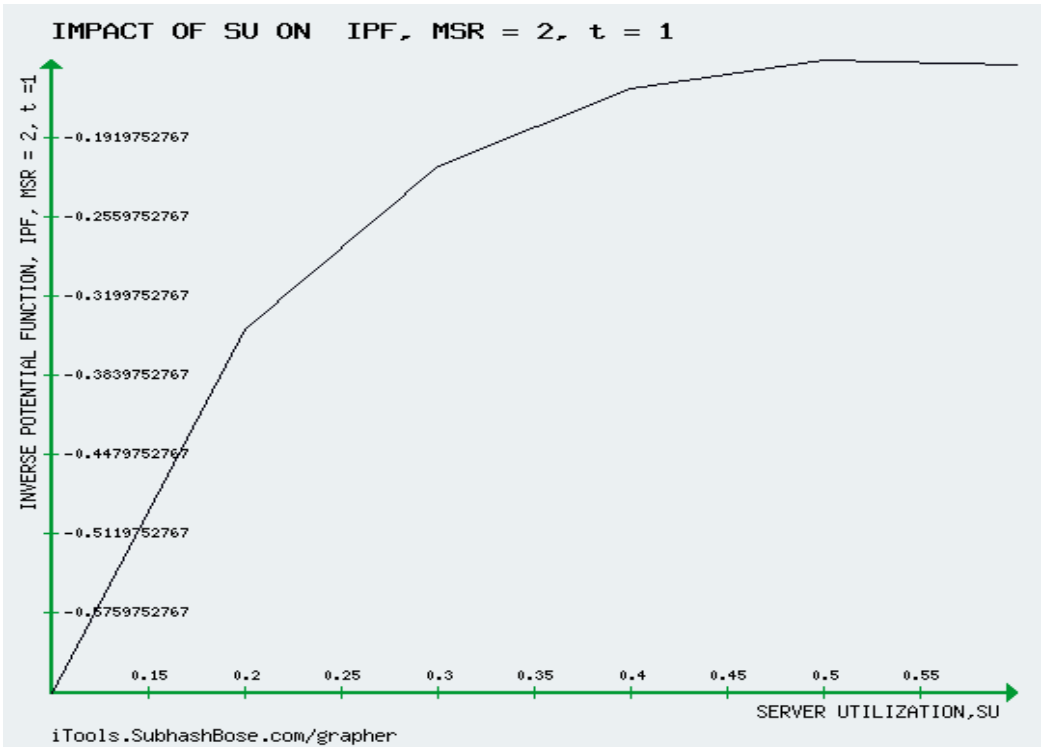


Figure 9.

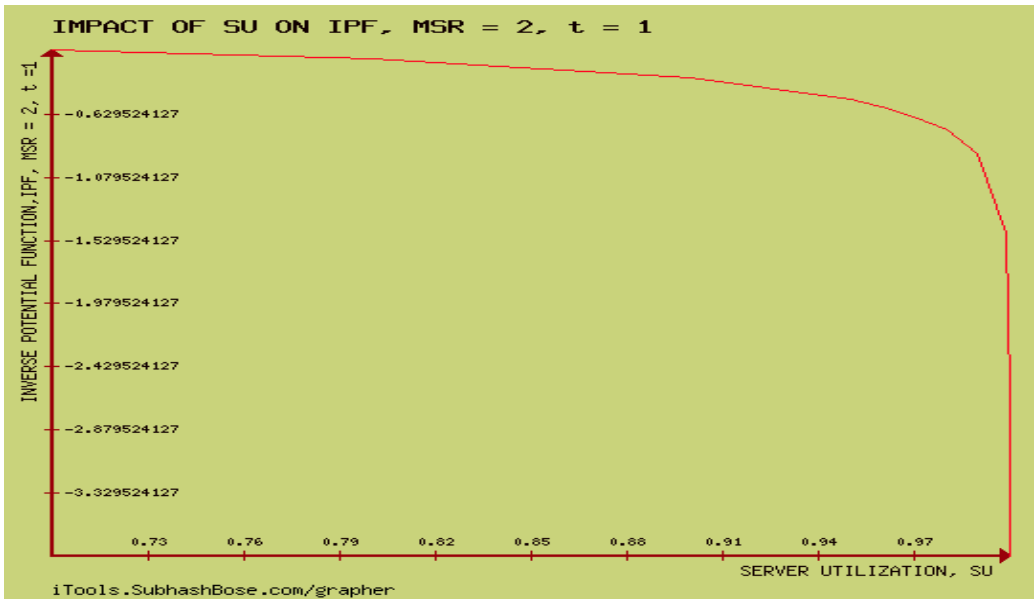


Figure 10.

12. THE  $\alpha$  – GAUSSIAN CURVATURE,  $K_0^{(\alpha)}$  OF (2.14) AS TIME APPROACHES INFINITY

**Theorem 12.1** The  $\alpha$  – Gaussian Curvature,  $K_0^{(\alpha)}$  as time approaches zero is devised by

$$K_0^{(\alpha)} = \left[ \frac{(1-\alpha) \left[ \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right] \rho}{2(1+\rho)^3} \right] \tag{10}$$

**Proof.**

We have

$$R_{ijkl}^{(\alpha)} = \left[ (\partial_j \Gamma_{ik}^{s(\alpha)} - \partial_i \Gamma_{jk}^{s(\alpha)}) g_{sl} + (\Gamma_{j\beta,l}^{(\alpha)} \Gamma_{ik}^{\beta(\alpha)} - \Gamma_{i\beta,l}^{(\alpha)} \Gamma_{jk}^{\beta(\alpha)}) \right]$$

where  $\Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)} g^{sk}$ ,  $i, j, k, s = 1, 2, \dots, n$

Therefore,

$$\begin{aligned} &= \left[ (\partial_2 \Gamma_{11}^{s(\alpha)} - \partial_1 \Gamma_{21}^{s(\alpha)}) g_{sl} + (\Gamma_{j\beta,l}^{(\alpha)} \Gamma_{11}^{\beta(\alpha)} - \Gamma_{i\beta,l}^{(\alpha)} \Gamma_{21}^{\beta(\alpha)}) \right] \\ &= \left[ \left( \frac{\partial}{\partial \mu} [\Gamma_{11}^{1(\alpha)} + \Gamma_{11}^{2(\alpha)} + \Gamma_{11}^{3(\alpha)}] - \frac{\partial}{\partial \rho} [\Gamma_{21}^{1(\alpha)} + \Gamma_{21}^{2(\alpha)} + \Gamma_{21}^{3(\alpha)}] \right) (g_{12} + g_{22} + g_{32}) \right. \\ &\quad \left. + ([\Gamma_{21,2}^{(\alpha)} \Gamma_{11}^{1(\alpha)} + \Gamma_{22,2}^{(\alpha)} \Gamma_{11}^{2(\alpha)} + \Gamma_{23,2}^{(\alpha)} \Gamma_{11}^{3(\alpha)}] - [\Gamma_{11,2}^{(\alpha)} \Gamma_{21}^{1(\alpha)} + \Gamma_{12,2}^{(\alpha)} \Gamma_{21}^{2(\alpha)} + \Gamma_{13,2}^{(\alpha)} \Gamma_{21}^{3(\alpha)}]) \right] \end{aligned} \quad (10)$$

$$= \quad (3)$$

$$\left[ \left( \frac{\partial}{\partial \mu} \left[ (1-\alpha)\rho \cdot L \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + (1-\alpha)\rho \cdot \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) F + (1-\alpha)\rho \cdot \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) \right. \right. \right. \\ \left. \left. \left. - \frac{\partial}{\partial \rho} \left[ \frac{(1-\alpha)L\rho}{2} + \frac{(1-\alpha)F\rho}{2} + \frac{(1-\alpha)H\rho}{2} \right] \right) + \left( -\left[ \frac{(1-\alpha)}{2} \frac{(1-\alpha)H\rho}{2} \right] \right) \right]$$

Recall that

$$L = \frac{(bh)}{\Delta} = \frac{t[(1+\rho)+t\rho]}{\Delta}$$

$$F = \frac{(bc-ah)}{\Delta} = \frac{(t[\frac{\rho}{\rho^2} + \frac{\rho}{(1-\rho)^2} + t\mu + \mu\rho] - [\frac{1}{\rho^2} - \frac{1}{(1-\rho)^2}][ (1+\rho)+t\rho ]}{\Delta}$$

$$H = \frac{(-b^2)}{\Delta} = \frac{-t^2}{\Delta}$$

Where

$$\begin{aligned} &= \left( \left[ \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right] (-[(1+\rho) + t\rho]^2) - t \left( t \left( \frac{\rho^2 - \rho\rho}{\rho^2} + \frac{\rho^2 + (1-\rho)\rho}{(1-\rho)^2} + ((1+\rho)t\mu + \mu + 2\rho\mu + \right. \right. \right. \\ &\quad \left. \left. \left. 2\rho\mu + \mu t\rho \right) - \left[ \frac{\rho}{\rho^2} + \frac{\rho}{(1-\rho)^2} + t\mu + \mu\rho \right] [(1+\rho) + t\rho] \right) + \left[ \frac{\rho}{\rho^2} + \frac{\rho}{(1-\rho)^2} + t\mu + \mu\rho \right] (t[(1+\rho) + t\rho]) \right) \end{aligned} \quad (10)$$

$$\rho) + t\rho])$$

It could be verified that:

$$R_{1212}^{(\alpha)} L_{t \rightarrow 0} = 0 = H_{t \rightarrow 0}, F_{t \rightarrow 0} = \frac{1}{(1+\rho)} \quad (10)$$

$$5)$$

Hence, it follows that:

$$R_{1212,t \rightarrow 0}^{(\alpha)}$$

$$\begin{aligned} & \left( (1-\alpha)\rho L \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) + (1-\alpha)\rho \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) F + (1-\alpha)\rho \left( \frac{1}{(1-\rho)^3} - \frac{1}{\rho^3} \right) H \right) \left( (1+\rho) \right. \\ & \quad \left. - \frac{\partial}{\partial \rho} \left[ \frac{(1-\alpha)L\rho}{2} + \frac{(1-\alpha)F\rho}{2} + \frac{(1-\alpha)H\rho}{2} \right] \right) \\ & \quad + \left( - \left[ \frac{(1-\alpha)}{2} \frac{(1-\alpha)H\rho}{2} \right] \right) \\ & \quad \frac{(1-\alpha)\rho}{2(1+\rho)} \end{aligned} \quad (10)$$

Clearly, it follows from the definition that

$$K_{t \rightarrow 0}^{(\alpha)} = \frac{R_{1212,t \rightarrow 0}^{(\alpha)}}{\Delta_{t \rightarrow 0}} = \frac{(1-\alpha) \left[ \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right] \rho}{2(1+\rho)^3} \quad (10)$$

In the following theorem, the zeros of the  $\alpha$  – Gaussian Curvature  $K_0^{(\alpha)}$  (c.f., (107)) are determined. Based on this, the paths of motion of the coordinates at which the underlying QM is looked at as a developable surface are obtained. The following theorem presents a novel approach which unifies Information Geometry with Riemannian Geometry, the theory of developable surfaces and the theory of time -dependent queueing systems.

**Theorem 10.** The underlying M/M/1 QM is developable on the following trajectories:

$$, \quad \text{o r} \quad \rho = \text{constant} \quad \text{o r} \quad \rho = \frac{1}{2} \quad \text{o r} \quad \rho \rightarrow \infty \text{ (instability phase)} \quad (10)$$

**Proof.**

$K_0^{(\alpha)} = 0$  if and only if

$$\frac{(1-\alpha) \left[ \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \right] \rho}{2(1+\rho)^3} = 0 \quad (10)$$

$\Leftrightarrow$  one of what follows is satisfied:

$$((1-\alpha) = 0 \Rightarrow \alpha = 1) \quad (11)$$

$$(1-\alpha) = 0 \Rightarrow \alpha = 1 \quad (11)$$

$$\frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} = 0 \Rightarrow (1-\rho)^2 = \rho^2$$

$$\Rightarrow (1-\rho) = \pm \rho \quad (11)$$

$$\Rightarrow \rho = \frac{1}{2}, \text{ where } 1 + \rho \neq 0 \quad (2)$$

or  $1 = 0$  (contradiction)

$$(iv) \frac{[1-2\rho]}{(1-\rho)^2 \rho^2 (1+\rho)^3} = 0. \text{ Let } \rho = \frac{1}{x}, \text{ then } \frac{[x^7-2x^6]}{(x-1)^2 (x+1)^3} = 0 \text{ which is equivalently, } x \rightarrow 0, \text{ or } \rho \rightarrow \infty.$$

### 13. the $\alpha$ – Sectional Gaussian Curvatures of M/M/1 QM

Recall that

$$a = \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} \quad (11)$$

3)

$$b = t \quad (11)$$

4)

$$c = \frac{\rho}{\rho^2} + \frac{\rho}{(1-\rho)^2} + t\mu + \mu\rho \quad (11)$$

5)

$$h = (1 + \rho) + t\rho \quad (11)$$

6)

$$l = \frac{\rho^2 - \rho\rho}{\rho^2} + \frac{\rho^2 + (1-\rho)\rho}{(1-\rho)^2} + ((1+\rho)t\mu + \mu + 2\rho\mu + 2\mu\rho + 2\rho\mu + \mu t\rho) \quad (11)$$

7)

$$\Delta = \det([g_{ij}]) = (a(-h^2) - b(bl - ch) + c(bh))$$

We have

$$R_{1111,t \rightarrow 0}^{(\alpha)} = \left[ (\partial_1 \Gamma_{11}^{s(\alpha)} - \partial_1 \Gamma_{11}^{s(\alpha)}) g_{s1} + \left( \Gamma_{1\beta,1}^{(\alpha)} \Gamma_{11}^{\beta(\alpha)} - \Gamma_{1\beta,1}^{(\alpha)} \Gamma_{11}^{\beta(\alpha)} \right) \right] = 0 \quad (11)$$

8)

$$R_{1211}^{(\alpha)} = \left[ (\partial_2 \Gamma_{11}^{s(\alpha)} - \partial_1 \Gamma_{21}^{s(\alpha)}) g_{s1} + \left( \Gamma_{2\beta,l}^{(\alpha)} \Gamma_{11}^{\beta(\alpha)} - \Gamma_{1\beta,l}^{(\alpha)} \Gamma_{21}^{\beta(\alpha)} \right) \right] \quad (11)$$

9)

Hence,

$$R_{1211,t \rightarrow 0}^{(\alpha)} = \left[ (\partial_2 \Gamma_{11}^{s(\alpha)} - \partial_1 \Gamma_{21}^{s(\alpha)}) g_{s1} + \left( \Gamma_{2\beta,l}^{(\alpha)} \Gamma_{11}^{\beta(\alpha)} - \Gamma_{1\beta,l}^{(\alpha)} \Gamma_{21}^{\beta(\alpha)} \right) \right] \quad (12)$$

12)

$$\left[ \partial_2 (\Gamma_{11}^{1(\alpha)} + \Gamma_{11}^{2(\alpha)} + \Gamma_{11}^{3(\alpha)}) - \partial_1 (\Gamma_{21}^{1(\alpha)} + \Gamma_{21}^{2(\alpha)} + \Gamma_{21}^{3(\alpha)}) \right] g_{s1} + \left[ (\Gamma_{21,1}^{(\alpha)} \Gamma_{11}^{1(\alpha)} + \Gamma_{22,1}^{(\alpha)} \Gamma_{11}^{2(\alpha)} + \Gamma_{23,1}^{(\alpha)} \Gamma_{11}^{3(\alpha)}) - (\Gamma_{11,1}^{(\alpha)} \Gamma_{21}^{1(\alpha)} + \Gamma_{12,1}^{(\alpha)} \Gamma_{21}^{2(\alpha)} + \Gamma_{13,1}^{(\alpha)} \Gamma_{21}^{3(\alpha)}) \right] \quad (0)$$

0)



$$K_{1111,t \rightarrow 0}^{(\alpha)} = 0 = K_{1121,t \rightarrow 0}^{(\alpha)} = K_{1113,t \rightarrow 0}^{(\alpha)} \quad (12)$$

3)

$$K_{1211,t \rightarrow 0}^{(\alpha)} = \frac{R_{1211,t \rightarrow 0}^{(\alpha)}}{\Delta_{t \rightarrow 0}} = \left[ \frac{(1-\alpha)}{2(1+\rho)^2} \left[ \frac{1}{\rho^2} - \frac{1}{(1-\rho)^2} + \frac{\rho}{\rho^2} + \frac{\rho}{(1-\rho)^2} + \mu\rho \right] \right] \quad (12)$$

4)

The remaining  $\alpha$  – Sectional Curvatures can be obtained by following the same procedure.

#### 14. Closing Remarks With Next Phase Research

This study offers a revolutionary info-geometrics of transient  $M/M/1$  QM. For this queue, FIM and IFIM are established. The geodesic equations of motion for the queue's coordinates are established. More potentially, it is shown that the underlying queue has  $RCT \neq 0$  and a zero Gaussian curvature. The paper shows how Riemannian Geometric (RG) analysis, and the Theory of Relativity (TR) may be utilised to transient queues, providing a full analytical investigation and the potential for future applications to various queueing systems.

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