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Nonlinear Maccone-Pati Uncertainty Principle

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Abstract: We show that one of the two important uncertainty principles derived by Maccone and Pati *[Phys. Rev. Lett., 2014]* can be derived for arbitrary maps defined on subsets of $\mathcal{L}^p$ spaces for $1 < p < \infty$. Our main tool is the Clarkson inequalities. We also derive a nonlinear uncertainty principle for weak parallelogram spaces and Type-p Banach spaces.

Keywords: uncertainty principle; lebesgue space; clarkson inequality; parallelogram space; type of banach space

MSC: 46B20; 46E30

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint operator defined on domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the uncertainty of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h\rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h\rangle^2}.$$

In 1929, Robertson [11] derived the following mathematical form of the uncertainty principle of Heisenberg derived in 1927 [7]. Recall that for two operators $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$, we define $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$.

Theorem 1.1. *[5,7,11,13] (**Heisenberg-Robertson Uncertainty Principle**) Let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\frac{1}{2}\left(\Delta_h(A)^2 + \Delta_h(B)^2\right) \geq \frac{1}{4}\left(\Delta_h(A) + \Delta_h(B)\right)^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2}|\langle [A, B]h, h\rangle|. \tag{1}$$

In 1930, Schrodinger improved Inequality (1) [12].

Theorem 1.2. *[12] (**Heisenberg-Robertson-Schrodinger Uncertainty Principle**) Let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\Delta_h(A)\Delta_h(B) \geq |\langle Ah, Bh\rangle - \langle Ah, h\rangle\langle Bh, h\rangle| = \frac{\sqrt{|\langle [A, B]h, h\rangle|^2 + |\langle \{A, B\}h, h\rangle - 2\langle Ah, h\rangle\langle Bh, h\rangle|^2}}{2}.$$

A fundamental drawback of Inequality (1) is that if h satisfies $ABh = BAh$, then the right side is zero. In 2014, Maccone and Pati derived the following two uncertainty principles which overturned this problem [9].

Theorem 1.3. *[9] (**Maccone-Pati Uncertainty Principle**) Let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $\|h\| = 1$, we have*

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{1}{2}\left(|\langle (A + B)h, k\rangle|^2 + |\langle (A - B)h, k\rangle|^2\right), \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1, \langle h, k\rangle = 0.$$

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Theorem 1.4. *[9] (**Maccone-Pati Uncertainty Principle**) Let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $\|h\| = 1$, we have*

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq -i\langle [A, B]h, h\rangle + |\langle (A + iB)h, k\rangle|^2, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1, \langle h, k\rangle = 0.$$

In this note, we show that Theorem 1.3 can be generalized even for arbitrary maps on Lebesgue spaces using Clarkson inequalities. We also derive uncertainty principle for Banach spaces satisfying weak parallelogram law and Type-p Banach spaces.

Our main motivation comes from the sentence 'The first proof, based on the parallelogram law, was communicated to us by an anonymous referee, while the second (independent) proof was our original argument" given in [9]. Note that Clarkson inequalities are generalizations of Jordan-von Neumann parallelogram law in Hilbert space [8].

2. Nonlinear Maccone-Pati Uncertainty Principle

We first define the uncertainty for maps on Lebesgue spaces. Let $\mathcal{M} \subseteq \mathcal{L}^p(\Omega, \mu)$ be a subset and $A : \mathcal{M} \to \mathcal{L}^p(\Omega, \mu)$ be a map (need not be linear or Lipschitz). Given $f \in \mathcal{M}$ and $a \in \mathbb{C}$, we define the uncertainty at f relative to a as

$$\Delta_f(A, a) := \|Af - af\|_p.$$

To derive our first uncertainty principle we need the following breakthrough inequalities of Clarkson.

Theorem 2.1. *[4,6,10] (**Clarkson Inequalities**) Let (Ω, μ) be a measure space.*

1. *Let $2 \leq p < \infty$. Then*
$$\|f\|_p^p + \|g\|_p^p \geq \frac{1}{2^{p-1}}\left(\|f+g\|_p^p + \|f-g\|_p^p\right), \quad \forall f, g \in \mathcal{L}^p(\Omega, \mu).$$
2. *Let $1 < p \leq 2$. Then*
$$\|f\|_p^p + \|g\|_p^p \geq \frac{1}{2}\left(\|f+g\|_p^p + \|f-g\|_p^p\right), \quad \forall f, g \in \mathcal{L}^p(\Omega, \mu).$$
3. *Let $2 \leq p < \infty$ and q be the conjugate index of p. Then*
$$\|f\|_p^q + \|g\|_p^q \geq \left(\frac{1}{2}\left(\|f+g\|_p^p + \|f-g\|_p^p\right)\right)^{\frac{1}{p-1}}, \quad \forall f, g \in \mathcal{L}^p(\Omega, \mu).$$
4. *Let $1 < p \leq 2$ and q be the conjugate index of p. Then*
$$\|f\|_p^p + \|g\|_p^p \geq \left(\frac{1}{2}\left(\|f+g\|_p^q + \|f-g\|_p^q\right)\right)^{\frac{1}{q-1}}, \quad \forall f, g \in \mathcal{L}^p(\Omega, \mu).$$
5. *Let $1 < p < \infty$ and q be the conjugate index of p. Let $1 < r \leq \min\{p, q\}$ and s be the conjugate index of r. Then*
$$\|f\|_p^r + \|g\|_p^r \geq \left(\frac{1}{2}\left(\|f+g\|_p^s + \|f-g\|_p^s\right)\right)^{\frac{1}{s-1}},$$
$$\|f\|_p^s + \|g\|_p^s \geq \frac{1}{2^{s-1}}\left(\|f+g\|_p^s + \|f-g\|_p^s\right),$$
$$\|f\|_p^r + \|g\|_p^r \geq \frac{1}{2}\left(\|f+g\|_p^r + \|f-g\|_p^r\right), \quad \forall f, g \in \mathcal{L}^p(\Omega, \mu).$$

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Theorem 2.2. *(**Nonlinear Maccone-Pati Uncertainty Principle**) Let (Ω, μ) be a measure space. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{L}^p(\Omega, \mu)$ be subsets and $A : \mathcal{M} \to \mathcal{L}^p(\Omega, \mu)$, $B : \mathcal{N} \to \mathcal{L}^p(\Omega, \mu)$ be maps. Let $f \in \mathcal{M} \cap \mathcal{N}$ and $a, b \in \mathbb{C}$.*

1. *Let $2 \leq p < \infty$. Then*
$$\Delta_f(A,a)^p + \Delta_f(B,b)^p \geq \frac{1}{2^{p-1}}\left(|\phi((A+B)f))|^p + |\phi((A-B)f)|^p\right), \quad \forall \phi \in (\mathcal{L}^p(\Omega,\mu))^* \text{ satisfying } \|\phi\| \leq 1, \phi(f) = 0.$$

2. *Let $1 < p \leq 2$. Then*
$$\Delta_f(A,a)^p + \Delta_f(B,b)^p \geq \frac{1}{2}\left(|\phi((A+B)f))|^p + |\phi((A-B)f)|^p\right), \quad \forall \phi \in (\mathcal{L}^p(\Omega,\mu))^* \text{ satisfying } \|\phi\| \leq 1, \phi(f) = 0.$$

3. *$2 \leq p < \infty$ and q be the conjugate index of p. Then*
$$\Delta_f(A,a)^q + \Delta_f(B,b)^q \geq \left(\frac{1}{2}\left(|\phi((A+B)f))|^p + |\phi((A-B)f)|^p\right)\right)^{\frac{1}{p-1}}, \quad \forall \phi \in (\mathcal{L}^p(\Omega,\mu))^* \text{ satisfying } \|\phi\| \leq 1, \phi(f) = 0.$$

4. *$1 < p \leq 2$ and q be the conjugate index of p. Then*
$$\Delta_f(A,a)^p + \Delta_f(B,b)^p \geq \left(\frac{1}{2}\left(|\phi((A+B)f))|^q + |\phi((A-B)f)|^q\right)\right)^{\frac{1}{q-1}}, \quad \forall \phi \in (\mathcal{L}^p(\Omega,\mu))^* \text{ satisfying } \|\phi\| \leq 1, \phi(f) = 0.$$

5. *Let $1 < p < \infty$ and q be the conjugate index of p. Let $1 < r \leq \min\{p, q\}$ and s be the conjugate index of r. Then*
$$\begin{aligned}
\Delta_f(A,a)^r + \Delta_f(B,b)^r &\geq \left(\frac{1}{2}\left(|\phi((A+B)f))|^s + |\phi((A-B)f)|^s\right)\right)^{\frac{1}{s-1}}, \\
\Delta_f(A,a)^s + \Delta_f(B,b)^s &\geq \frac{1}{2^{s-1}}\left(|\phi((A+B)f))|^s + |\phi((A-B)f)|^s\right), \\
\Delta_f(A,a)^r + \Delta_f(B,b)^r &\geq \frac{1}{2}\left(|\phi((A+B)f))|^r + |\phi((A-B)f)|^r\right), \quad \forall \phi \in (\mathcal{L}^p(\Omega,\mu))^* \\
&\text{satisfying } \|\phi\| \leq 1, \phi(f) = 0.
\end{aligned}$$

Proof. We prove (i) and remaining are similar. Using (i) in Theorem 2.1

$$\begin{aligned}
\Delta_f(A,a)^p + \Delta_f(B,b)^p &= \|Af - af\|_p^p + \|Bf - bf\|_p^p \\
&\geq \frac{1}{2^{p-1}}\left(\|(Af - af) + (Bf - bf)\|_p^p + \|(Af - af) - (Bf - bf)\|_p^p\right) \\
&= \frac{1}{2^{p-1}}\left(\|(A+B)f - (a+b)f\|_p^p + \|(A-B)f - (a-b)f\|_p^p\right) \\
&\geq \frac{1}{2^{p-1}}\|\phi\|\left(\|(A+B)f - (a+b)f\|_p^p + \|(A-B)f - (a-b)f\|_p^p\right) \\
&\geq \frac{1}{2^{p-1}}\left(|\phi((A+B)f) - \phi((a+b)f)|^p + |\phi((A-B)f) - \phi((a-b)f)|^p\right) \\
&= \frac{1}{2^{p-1}}\left(|\phi((A+B)f))|^p + |\phi((A-B)f)|^p\right).
\end{aligned}$$

$\square$

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We next note the following extension of Theorem 2.2 (we state generalizations of only (i) and (ii) and others are similar). Recall that the collection of all Lipschitz functions $\psi : \mathcal{L}^p(\Omega, \mu) \to \mathbb{C}$ satisfying $\psi(0) = 0$, denoted by $\mathcal{L}^p(\Omega, \mu)^\#$ is a Banach space [14] w.r.t. the Lipschitz norm

$$\|\psi\|_{\mathrm{Lip}_0} := \sup_{f,g \in \mathcal{L}^p(\Omega,\mu), f \neq g} \frac{|\psi(f) - \psi(g)|}{\|f - g\|_p}.$$

Corollary 2.1. *Let (Ω, μ) be a measure space. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{L}^p(\Omega, \mu)$ be subsets and $A : \mathcal{M} \to \mathcal{L}^p(\Omega, \mu)$, $B : \mathcal{N} \to \mathcal{L}^p(\Omega, \mu)$ be maps. Let $f \in \mathcal{M} \cap \mathcal{N}$ and $a, b \in \mathbb{C}$ be such that $a + b = 1$.*

1. *Let $2 \leq p < \infty$. Then*

$$\Delta_f(A, a)^p + \Delta_f(B, b)^p \geq \frac{1}{2^{p-1}} \left(|\phi((A + B)f))|^p + |\phi((A - B)f)|^p\right), \quad \forall \phi \in (\mathcal{L}^p(\Omega, \mu))^\#$$
$$\text{satisfying } \|\phi\|_{Lip_0} \leq 1, \phi(f) = 0.$$

2. *Let $1 < p \leq 2$. Then*

$$\Delta_f(A, a)^p + \Delta_f(B, b)^p \geq \frac{1}{2} \left(|\phi((A + B)f))|^p + |\phi((A - B)f)|^p\right), \quad \forall \phi \in (\mathcal{L}^p(\Omega, \mu))^\#$$
$$\text{satisfying } \|\phi\|_{Lip_0} \leq 1, \phi(f) = 0.$$

In 1972, Bynum and Drew derived the following surprising result [2].

Theorem 2.3. *[2] For $1 < p < 2$,*

$$\|x + y\|_p^2 + (p - 1)\|x - y\|_p^2 \leq 2 \left(\|x\|_p^2 + \|y\|_p^2\right), \quad \forall x, y \in \ell^p(\mathbb{N}).$$

Theorem 2.3 promoted the notion of parallelogram law spaces by Cheng and Ross [3].

Definition 2.1. *[3] Let $C > 0$ and $1 < p < \infty$. A Banach space $\mathcal{X}$ is said to satisfy lower p-weak parallelogram law with constant C if*

$$\|x + y\|^p + C\|x - y\|^p \leq 2^{p-1} \left(\|x\|^p + \|y\|^p\right), \quad \forall x, y \in \mathcal{X}.$$

in this case, we write $\mathcal{X}$ is p-LWP(C).

For parallelogram law spaces, by following a similar computation as in the proof of Theorem 2.2 we get the following theorem.

Theorem 2.4. *Let $\mathcal{X}$ be p-LWP(C). Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets and $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be maps. Let $x \in \mathcal{M} \cap \mathcal{N}$ and $a, b \in \mathbb{C}$. Then*

$$\Delta_x(A, a)^2 + \Delta_x(B, b)^2 \geq \frac{1}{2^{p-1}} \left(|\phi((A + B)x))|^p + C|\phi((A - B)x)|^2\right), \quad \forall \phi \in \mathcal{X}^*$$
$$\text{satisfying } \|\phi\| \leq 1, \phi(x) = 0.$$

Corollary 2.2. *Let $\mathcal{X}$ be p-LWP(C). Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets and $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be maps. Let $x \in \mathcal{M} \cap \mathcal{N}$ and $a, b \in \mathbb{C}$ be such that $a + b = 1$. Then*

$$\Delta_x(A, a)^2 + \Delta_x(B, b)^2 \geq \frac{1}{2^{p-1}} \left(|\phi((A + B)x))|^2 + C|\phi((A - B)x)|^2\right), \quad \forall \phi \in \mathcal{X}^\#$$
$$\text{satisfying } \|\phi\|_{Lip_0} \leq 1, \phi(x) = 0.$$

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We now proceed to derive nonlinear uncertainty principle for a class of Banach spaces (at present, we don't know it for arbitrary Banach spaces). Recall that a Banach space $\mathcal{X}$ is said to be of Type-p, $p \in [1,2]$ [1] if there exists a constant $C > 0$ satisfying following: For every $n \in \mathbb{N}$,

$$\left(\frac{1}{2^n}\sum_{(\varepsilon_j)_{j=1}^n \in \{-1,1\}^n}\left\|\sum_{j=1}^n \varepsilon_j x_j\right\|^p\right)^{\frac{1}{p}} \leq C\left(\sum_{j=1}^n \|x_j\|^p\right)^{\frac{1}{p}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}. \tag{2}$$

In this case, we define the Type-p constant of $\mathcal{X}$ as

$$T_p(\mathcal{X}) := \inf\{C : C \text{ satisfies Inequality } (2)\}.$$

Theorem 2.5. *Let $\mathcal{X}$ be a Banach space of Type-p. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets and $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be maps. Let $x \in \mathcal{M} \cap \mathcal{N}$ and $a, b \in \mathbb{C}$. Then*

$$\Delta_x(A,a)^p + \Delta_x(B,b)^p \geq \frac{1}{2T_p(\mathcal{X})^p}\left(|\phi((A+B)x))|^p + |\phi((A-B)x)|^p\right), \quad \forall \phi \in \mathcal{X}^*$$
$$\text{satisfying } \|\phi\| \leq 1, \phi(x) = 0.$$

Proof. Using the definition of Type-p, we get

$$\begin{aligned}
\Delta_x(A,a)^p + \Delta_x(B,b)^p &= \|Ax - ax\|^p + \|Bx - bx\|^p \\
&\geq \frac{1}{2T_p(\mathcal{X})^p}\left(\|(Ax - ax) + (Bf - bf)\|^p + \|(Ax - ax) - (Bx - bx)\|^p\right) \\
&= \frac{1}{2T_p(\mathcal{X})^p}\left(\|(A+B)x - (a+b)x\|^p + \|(A-B)x - (a-b)x\|^p\right) \\
&\geq \frac{1}{2T_p(\mathcal{X})^p}\|\phi\|\left(\|(A+B)x - (a+b)x\|^p + \|(A-B)x - (a-b)x\|^p\right) \\
&\geq \frac{1}{2T_p(\mathcal{X})^p}\left(|\phi((A+B)x) - \phi((a+b)x)|^p + |\phi((A-B)x) - \phi((a-b)x)|^p\right) \\
&= \frac{1}{2T_p(\mathcal{X})^p}\left(|\phi((A+B)x))|^p + |\phi((A-B)x)|^p\right).
\end{aligned}$$

□

Corollary 2.3. *Let $\mathcal{X}$ be a Banach space of Type-p. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets and $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be maps. Let $x \in \mathcal{M} \cap \mathcal{N}$ and $a, b \in \mathbb{C}$ be such that $a + b = 1$. Then*

$$\Delta_x(A,a)^p + \Delta_x(B,b)^p \geq \frac{1}{2T_p(\mathcal{X})^p}\left(|\phi((A+B)x))|^p + |\phi((A-B)x)|^p\right), \quad \forall \phi \in \mathcal{X}^\#$$
$$\text{satisfying } \|\phi\|_{Lip_0} \leq 1, \phi(x) = 0.$$

Note that we can derive following results if we won't bother about power p.

Theorem 2.6. *Let $\mathcal{X}$ be a Banach space. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets and $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be maps. Let $x \in \mathcal{M} \cap \mathcal{N}$ and $a, b \in \mathbb{C}$. Then*

$$\Delta_x(A,a) + \Delta_x(B,b) \geq |\phi((A+B)x))|, \quad \forall \phi \in \mathcal{X}^* \text{ satisfying } \|\phi\| \leq 1, \phi(x) = 0.$$

Proof. By directly applying triangle inequality

$$\begin{aligned}\Delta_x(A,a)+\Delta_x(B,b) &= \|Ax-ax\|+\|Bx-bx\| \geq \|Ax-ax+Bx-bx\| \\ &\geq \|\phi\|\|Ax-ax+Bx-bx\| \geq |\phi((A+B)x)-\phi((a+b)x))| \\ &= |\phi((A+B)x)|.\end{aligned}$$

□

Corollary 2.4. *Let $\mathcal{X}$ be a Banach space. Let $\mathcal{M},\mathcal{N} \subseteq \mathcal{X}$ be subsets and $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be maps. Let $x \in \mathcal{M} \cap \mathcal{N}$ and $a,b \in \mathbb{C}$ be such that $a+b=1$. Then*

$$\Delta_x(A,a)+\Delta_x(B,b) \geq |\phi((A+B)x))|, \quad \forall \phi \in \mathcal{X}^\# \text{ satisfying } \|\phi\|_{Lip_0} \leq 1, \phi(x)=0.$$

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