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Solvability Criterion for a System Arising from Monge-Ampère Equations with Two Parameters

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Abstract: Monge-Ampère equations have important research significance in many fields such as geometry, convex geometry and mathematical physics. In this paper, under some superlinear and sublinear conditions, the existence of nontrivial solutions for a system arising from Monge-Ampère equations with two parameters is investigated based on Guo-Krasnosel'skii fixed point theorem. In the end, two examples are given to illustrate our main results.

Keywords: fixed point theorem; Monge-Ampère equations; boundary value problem

MSC: 35J60; 34A08; 47H10

1. Introduction

In this paper, we consider the existence of nontrivial solutions for the following boundary value problem:

$$\begin{cases} ((u'(s))^N)' = \lambda N r^{N-1} f(-u(s), -v(s)), \ 0 < s < 1, \\ ((v'(s))^N)' = \mu N r^{N-1} g(-u(s), -v(s)), \ 0 < s < 1, \\ u'(0) = u(1) = 0, \ v'(0) = v(1) = 0, \end{cases} \tag{1}$$

where $N \geq 1$, $f, g : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous, λ and μ are two positive parameters. The problem (1.1) arises in the study of the existence of nontrivial solutions to the following Dirichlet problem of the Monge-Ampère equations:

$$\begin{cases} \det(D^2 u) = \lambda f(-u, -v) \text{ in } B, \\ \det(D^2 v) = \mu g(-u, -v) \text{ in } B, \\ u = v = 0 \ \text{ on } \ \partial B, \end{cases}$$

where $D^2 u = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ is the Hessian matrix of u, $D^2 v = (\frac{\partial^2 v}{\partial x_i \partial x_j})$ is the Hessian matrix of v, $B = \{x \in R^N | \ |x| < 1\}$.

Monge-Ampère equations play an important role in the study of geometric problems, fluid mechanics, and various other applied fields(see[1]). Many researchers have done some investigations related to Monge-Ampère equations. Some scholars have studied the existence of nontrivial radial convex solutions for a single Monge-Ampère equation or systems of Monge-Ampère equations, utilizing the theory of topological degree, bifurcation techniques,the method of upper and lower solution, and so on. For further details, see [2-9,14-15,17-18,20-23] and the references therein.

For example, in [3], Ma and Gao investigated the following boundary value problem:

$$\begin{cases} ((u_1'(t))^n)' = \lambda n t^{n-1} f(-u(t)), \ 0 < t < 1, \\ u'(0) = u(1) = 0. \end{cases} \tag{1.2}$$

The boundary value problem (1.2) arose from the following Monge-Ampère equation:

$$\begin{cases} \det(D^2u) = \lambda f(-u) \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases} \tag{1.3}$$

where $D^2u = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ is the Hessian matrix of u, $B = \{x \in R^n | \ |x| < 1\}$. The global bifurcation technique was applied to determine the optimal intervals for parameter λ, ensuring the existence of single or multiple solutions of boundary value problem (1.2).

In [4], Wang established two solvability criteria for weakly coupled system:

$$\begin{cases} ((u_1'(t))^N)' = Nt^{N-1}f(-u_2(t)), \ 0 < t < 1, \\ ((u_2'(t))^N)' = Nt^{N-1}g(-u_1(t)), \ 0 < t < 1, \\ u_1'(0) = u_2'(0) = 0, \ u_1(1) = u_2(1) = 0, \end{cases} \tag{1.4}$$

where $N \geq 1$. The system (1.4) arose from the following Monge-Ampère equations:

$$\begin{cases} \det(D^2u_1) = f(-u_2) \text{ in } B, \\ \det(D^2u_2) = g(-u_1) \text{ in } B, \\ u_1 = u_2 = 0 \text{ on } \partial B, \end{cases}$$

where $B = \{x \in R^N | \ |x| < 1\}$, and D^2u_i is the determinant of the Hessian matrix $(\frac{\partial^2 u_i}{\partial x_m \partial x_n})$ of u_i. The existence of convex radial solutions of the weakly coupled system (1.4) in superlinear and sublinear cases was obtained based on fixed point theorems in a cone.

In [5], Wang and An discussed the following system of the Monge-Ampère equations:

$$\begin{cases} \det(D^2u_1) = f_1(-u_1, \cdots, -u_n) \text{ in } B, \\ \cdots \\ \det(D^2u_n) = f_n(-u_1, \cdots, -u_n) \text{ in } B, \\ u(x) = 0 \text{ on } \partial B, \end{cases} \tag{1.5}$$

where $D^2u_i = (\frac{\partial^2 u_i}{\partial x_i \partial x_j})$ is the Hessian matrix of u_i, $B = \{x \in R^N | \ |x| < 1\}$. The system (1.5) can be easily changed into the following boundary value problem:

$$\begin{cases} ((u_1'(r))^N)' = Nr^{N-1}f_1(-u_1, \cdots, -u_n), \ 0 < r < 1, \\ \cdots \\ ((u_n'(r))^N)' = Nr^{N-1}f_n(-u_1, \cdots, -u_n), \ 0 < r < 1, \\ u_i'(0) = u_i(1) = 0, \ i = 1, \cdots, n, \end{cases}$$

where $N \geq 1$. The existence of triple nontrivial radial convex solutions was obtained by using the Leggett-Williams fixed point theorem.

In [6], the author studied the following system:

$$\begin{cases} ((u_1'(r))^N)' = \lambda Nr^{N-1}f_1(-u_1, \cdots, -u_n), \ 0 < r < 1, \\ \cdots \\ ((u_n'(r))^N)' = \lambda Nr^{N-1}f_n(-u_1, \cdots, -u_n), \ 0 < r < 1, \\ u_i'(0) = u_i(1) = 0, \ i = 1, \cdots, n, \end{cases} \tag{1.6}$$

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where $N \geq 1$. The system (1.6) arose from the following system:

$$\begin{cases} \det(D^2u_1) = \lambda f_1(-u_1, \cdots, -u_n) \text{ in } B, \\ \cdots \\ \det(D^2u_n) = \lambda f_n(-u_1, \cdots, -u_n) \text{ in } B, \\ u_i = 0 \ \text{ on } \ \partial B, i = 1, \cdots, n, \end{cases}$$

where $D^2u_i = (\frac{\partial^2 u_i}{\partial x_i \partial x_j})$ is the Hessian matrix of u_i, $B = \{x \in R^N | \ |x| < 1\}$.

Using fixed point theorems and considering sublinear and superlinear conditions, Wang explored the existence and two nontrivial radial solutions for the system (1.6) with a carefully selected parameter.

In [7], Gao and Wang considered the following boundary value problem:

$$\begin{cases} ((u_1'(r))^N)' = \lambda_1 N r^{N-1} f_1(-u_1, -u_2, \cdots, -u_n), \\ ((u_2'(r))^N)' = \lambda_2 N r^{N-1} f_2(-u_1, -u_2, \cdots, -u_n), \\ \cdots \\ ((u_n'(r))^N)' = \lambda_n N r^{N-1} f_n(-u_1, -u_2, \cdots, -u_n), \\ u_i'(0) = u_i(1) = 0, \ \ i = 1, 2, \cdots, \ n, \ 0 < r < 1, \end{cases} \tag{1.7}$$

where $N \geq 1$. The system (1.7) arose from the following system:

$$\begin{cases} \det(D^2u_1) = \lambda_1 f_1(-u_1, \cdots, -u_n) \text{ in } B, \\ \det(D^2u_2) = \lambda_2 f_2(-u_1, \cdots, -u_n) \text{ in } B, \\ \cdots \\ \det(D^2u_n) = \lambda_n f_n(-u_1, \cdots, -u_n) \text{ in } B, \\ u_i = 0 \ \text{ on } \ \partial B, i = 1, \cdots, n, \end{cases}$$

where $D^2u_i = (\frac{\partial^2 u_i}{\partial x_i \partial x_j})$ is the Hessian matrix of u_i, and $B = \{x \in R^N | \ |x| < 1\}$ is the unit ball in R^N. The existence, multiplicity, and nonexistence of convex solutions for systems of Monge-Ampère equations with multiparameters were established via the upper and lower solutions method and the fixed point index theory.

In [18], Feng has continued to consider the existence and uniqueness of nontrivial radial convex solutions to the Monge-Ampère equation (1.3). And the author also studied the following system:

$$\begin{cases} \det(D^2u_1) = \lambda_1 f_1(-u_2) \text{ in } B, \\ \det(D^2u_2) = \lambda_2 f_2(-u_3) \text{ in } B, \\ \cdots \\ \det(D^2u_n) = \lambda_n f_n(-u_1) \text{ in } B, \\ u_1 = u_2 \cdots = u_n = 0 \ \text{ on } \ \partial B, \end{cases} \tag{1.8}$$

where $\lambda_i (i = 1, 2, \cdots, n)$ are positive parameters. By defining composite operators and using the eigenvalue theory in cones, the author obtained some new existence results of nontrivial radial convex solutions to the system (1.8), and also analyzed the asymptotic behavior of solutions to the system (1.8).

Meanwhile, in recent years, some authors have studied the existence of nontrivial solutions to other differential equations with parameters. For example, in [10], Hao et al. considered the existence of positive solutions for a system of nonlinear fractional differential equations nonlocal boundary value problems with parameters and p-Laplacian operator via the Guo-Krasnosel'skii fixed point theorem. In [11], by means of the method of upper and lower solutions and the fixed point index theory, Yang proved the existence of positive solutions for Dirichlet boundary value problem of 2m-order nonlinear differential systems with multiple different parameters. In [12], Jiang and Zhai investigated a class of

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nonlinear fourth-order systems with coupled integral boundary conditions and two parameters based on the Guo–Krasnosel'skii fixed point theorem and the Green's functions.

Inspired by literatures [4-7,10-12,18], we consider the problem (1.1). In this paper, under some different combinations of superlinearity and sublinearity of the nonlinear terms, we use the Guo-Krasnosel'skii fixed point theorem to study the existence results of the system (1.1) and establish some existence results of nontrivial solutions based on various different values values of λ and μ. Here we extend the study in literature [4], and the main results are different from literatures [4,7,18].

2. Preliminaries

In this section, we give some preliminaries that will be used to prove existence results in Section 3. For further background knowledge of cone, we refer readers to the papers [4,16] for more details.

Lemma 1. *(see [16]) Let E be a Banach space, and $P \subset E$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, the operator $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. If the following conditions are satisfied:*

$$(\mathrm{i})\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1, \|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2, or$$
$$(\mathrm{ii})\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1, \|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2,$$

In order to solve the system (1.1), we give a simple transformation $x(s) = -u(s), y(s) = -v(s)$ in the system (1.1), then the system (1.1) can be changed to the following system:

$$\begin{cases} ((-x'(s))^N)' = \lambda N s^{N-1} f(x(s), y(s)), \ \ 0 < s < 1, \\ ((-y'(s))^N)' = \mu N s^{N-1} g(x(s), y(s)), \ \ 0 < s < 1, \\ x'(0) = x(1) = 0, \ y'(0) = y(1) = 0. \end{cases} \tag{2}$$

In the following, we treat the existence of positive solutions of the system (2.1).

Let $E = C[0,1] \times C[0,1]$ with the norm $\|(x,y)\|_E = \|x\| + \|y\|$, where $\|x\| = \max\limits_{s\in[0,1]} |x(s)|$ and $\|y\| = \max\limits_{s\in[0,1]} |y(s)|$.

Define

$$P = \{(x,y) \in E : x(s) \geq 0, y(s) \geq 0, \forall s \in [0,1], \min_{s\in[\frac{1}{4},\frac{3}{4}]} (x(s) + y(s)) \geq \frac{1}{4}\|(x,y)\|_E\}.$$

Then P is a cone of E.

By literature [4], we define the operators A_1, A_2 and A as follows:

$$A_1(x,y)(s) = \int_s^1 (\int_0^u \lambda N \tau^{N-1} f(x(\tau), y(\tau)) d\tau)^{\frac{1}{N}} du, \ \ s \in [0,1],$$

$$A_2(x,y)(s) = \int_s^1 (\int_0^u \mu N \tau^{N-1} g(x(\tau), y(\tau)) d\tau)^{\frac{1}{N}} du, \ \ s \in [0,1].$$

and $A(x,y) = (A_1(x,y), A_2(x,y)), (x,y) \in E$. It is obvious that the fixed points of the operator A are solutions of the system (2.1).

Similar to the proof of Lemma 2.3 in literature [4], we have the following lemma.

Lemma 2. *$A : P \to P$ is completely continuous.*

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3. Main Results

Denote

$$f_0 = \limsup_{x+y\to 0^+} \frac{f(x,y)}{(x+y)^N}, \quad g_0 = \limsup_{x+y\to 0^+} \frac{g(x,y)}{(x+y)^N},$$

$$f_\infty = \liminf_{x+y\to \infty} \frac{f(x,y)}{(x+y)^N}, \quad g_\infty = \liminf_{x+y\to \infty} \frac{g(x,y)}{(x+y)^N},$$

$$\widehat{f}_0 = \liminf_{x+y\to 0^+} \frac{f(x,y)}{(x+y)^N}, \quad \widehat{g}_0 = \liminf_{x+y\to 0^+} \frac{g(x,y)}{(x+y)^N},$$

$$\widehat{f}_\infty = \limsup_{x+y\to \infty} \frac{f(x,y)}{(x+y)^N}, \quad \widehat{g}_\infty = \limsup_{x+y\to \infty} \frac{g(x,y)}{(x+y)^N}.$$

$$F = \int_0^1 (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} du, \quad G = \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} N\tau^{N-1} d\tau)^{\frac{1}{N}} du.$$

For $f_0, g_0, f_\infty, g_\infty \in (0,\infty)$, we define the symbols below:

$$M_1 = \frac{2^N}{G^N f_\infty}, \quad M_2 = \frac{1}{2^N F^N f_0},$$

$$M_3 = \frac{2^N}{G^N g_\infty}, \quad M_4 = \frac{1}{2^N F^N g_0}.$$

Theorem 1. *(1) Assume that $f_0, g_0, f_\infty, g_\infty \in (0,\infty), M_1 < M_2, M_3 < M_4$, then for $\lambda \in (M_1, M_2)$ and $\mu \in (M_3, M_4)$, the system (2.1) has at least one positive solution.*

(2) Assume that $f_0 = 0, g_0, f_\infty, g_\infty \in (0,\infty), M_3 < M_4$, then for $\lambda \in (M_1,\infty)$ and $\mu \in (M_3, M_4)$, the system (2.1) has at least one positive solution.

(3) Assume that $f_0, f_\infty, g_\infty \in (0,\infty), g_0 = 0, M_1 < M_2$, then for $\lambda \in (M_1, M_2)$ and $\mu \in (M_3,\infty)$, the system (2.1) has at least one positive solution.

(4) Assume that $f_0 = g_0 = 0, f_\infty, g_\infty \in (0,\infty)$, then for $\lambda \in (M_1,\infty)$ and $\mu \in (M_3,\infty)$, the system (2.1) has at least one positive solution.

(5) Assume that $f_0, g_0 \in (0,\infty), f_\infty = \infty$ or $f_0, g_0 \in (0,\infty), g_\infty = \infty$, then for $\lambda \in (0, M_2)$ and $\mu \in (0, M_4)$, the system (2.1) has at least one positive solution.

(6) Assume that $f_0 = 0, g_0 \in (0,\infty), g_\infty = \infty$ or $f_0 = 0, g_0 \in (0,\infty), f_\infty = \infty$, then for $\lambda \in (0,\infty)$ and $\mu \in (0, M_4)$, the system (2.1) has at least one positive solution.

(7) Assume that $f_0 \in (0,\infty), g_0 = 0, g_\infty = \infty$ or $f_0 \in (0,\infty), g_0 = 0, f_\infty = \infty$, then for $\lambda \in (0, M_2)$ and $\mu \in (0,\infty)$, the system (2.1) has at least one positive solution.

(8) Assume that $f_0 = g_0 = 0, g_\infty = \infty$ or $f_0 = g_0 = 0, f_\infty = \infty$, then for $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$, the system (2.1) has at least one positive solution.

Proof. Due to the similarity in the proofs of the above cases, we will demonstrate the case (1) and the case (6).

(1) For each $\lambda \in (M_1, M_2)$ and $\mu \in (M_3, M_4)$, there exists $\varepsilon > 0$ such that

$$\frac{2^N}{G^N(f_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{2^N F^N (f_0 + \varepsilon)},$$

$$\frac{2^N}{G^N(g_\infty - \varepsilon)} \leq \mu \leq \frac{1}{2^N F^N (g_0 + \varepsilon)}.$$

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By the definitions of f_0 and g_0, we know that there exists $r_1 > 0$ such that

$$f(x,y) < (f_0+\varepsilon)(x+y)^N, \ \ 0 \le x+y \le r_1,$$

$$g(x,y) < (g_0+\varepsilon)(x+y)^N, \ \ 0 \le x+y \le r_1.$$

We define the set $\Omega_1 = \{(x,y) \in E : \|(x,y)\|_E < r_1\}$, for any $(x,y) \in P \cap \partial\Omega_1$, we get

$$0 \le x(s)+y(s) \le \|x\| + \|y\| = \|(x,y)\|_E = r_1, \ \forall s \in [0,1],$$

then

$$\begin{aligned} A_1(x,y)(s) &= \int_s^1 \Big(\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau\Big)^{\frac{1}{N}} du \\ &\le \int_0^1 \Big(\int_0^1 \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau\Big)^{\frac{1}{N}} du \\ &\le \int_0^1 \Big(\int_0^1 \lambda N\tau^{N-1} (f_0+\varepsilon)(x(\tau)+y(\tau))^N d\tau\Big)^{\frac{1}{N}} du \\ &\le (f_0+\varepsilon)^{\frac{1}{N}} \int_0^1 \Big(\int_0^1 \lambda N\tau^{N-1} (\|x\|+\|y\|)^N d\tau\Big)^{\frac{1}{N}} du \\ &= (f_0+\varepsilon)^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 \Big(\int_0^1 N\tau^{N-1} d\tau\Big)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\le \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Therefore,

$$\|A_1(x,y)\| \le \frac{1}{2}\|(x,y)\|_E, \ \ \forall\ (x,y) \in P \cap \partial\Omega_1. \tag{3}$$

By applying the same method, we deduce

$$\begin{aligned} A_2(x,y)(s) &= \int_s^1 \Big(\int_0^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau\Big)^{\frac{1}{N}} du \\ &\le \int_0^1 \Big(\int_0^1 \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau\Big)^{\frac{1}{N}} du \\ &\le \int_0^1 \Big(\int_0^1 \mu N\tau^{N-1} (g_0+\varepsilon)(x(\tau)+y(\tau))^N d\tau\Big)^{\frac{1}{N}} du \\ &\le (g_0+\varepsilon)^{\frac{1}{N}} \int_0^1 \Big(\int_0^1 \mu N\tau^{N-1} (\|x\|+\|y\|)^N d\tau\Big)^{\frac{1}{N}} du \\ &= (g_0+\varepsilon)^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_0^1 \Big(\int_0^1 N\tau^{N-1} d\tau\Big)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\le \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Therefore,

$$\|A_2(x,y)\| \le \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_1. \tag{4}$$

By (3.1) and (3.2) we have

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \le \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_1. \tag{5}$$

On the other hand, considering the definitions of f_∞ and g_∞, there exists $\bar{r}_2 > 0$ such that

$$f(x,y) \ge (f_\infty-\varepsilon)(x+y)^N, \ \ x+y \ge \bar{r}_2,$$

$$g(x,y) \ge (g_\infty-\varepsilon)(x+y)^N, \ \ x+y \ge \bar{r}_2.$$

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We take $r_2 = \max\{2r_1, 4\bar{r}_2\}$ and denote $\Omega_2 = \{(x,y) \in E : \|(x,y)\|_E < r_2\}$. For any $(x,y) \in P \cap \partial\Omega_2$, we get

$$\min_{s\in[\frac{1}{4},\frac{3}{4}]} (x(s)+y(s)) \geq \frac{1}{4}\|(x,y)\|_E = \frac{1}{4}r_2 \geq \bar{r}_2,$$

then

$$\begin{aligned}
A_1(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1} (\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \lambda N\tau^{N-1} (f_\infty - \varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\
&\geq (f_\infty - \varepsilon)^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \lambda N\tau^{N-1} (\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\
&= \frac{1}{4}(f_\infty - \varepsilon)^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Therefore,

$$\|A_1(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{6}$$

In a similar manner, for any $(x,y) \in P \cap \partial\Omega_2$, we obtain

$$\begin{aligned}
A_2(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1} (\int_0^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} (g_\infty - \varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\
&\geq (g_\infty - \varepsilon)^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} (\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\
&= \frac{1}{4}(g_\infty - \varepsilon)^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Therefore,

$$\|A_2(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{7}$$

By (3.4) and (3.5) we have

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \geq \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{8}$$

From (3.3), (3.6) and Lemma 2.1, we get that A has at least one fixed point $(x,y) \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $r_1 \leq \|(x,y)\|_E \leq r_2$, so the system (2.1) has at least one positive solution. The proof of the case (1) is completed.

(6) Assume $f_0 = 0, g_0 \in (0,\infty), g_\infty = \infty$, then for each $\lambda \in (0,\infty)$ and $\mu \in (0, M_4)$, there exists $\varepsilon > 0$ such that

$$0 < \lambda < \frac{1}{2^N F^N \varepsilon}, \ \ \frac{4^N \varepsilon}{G^N} < \mu < \frac{1}{2^N F^N (g_0 + \varepsilon)}.$$

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Considering the definitions of f_0 and g_0, we know that there exists $r_3 > 0$ such that

$$f(x,y) < \varepsilon(x+y)^N, \ \ 0 \le x+y \le r_3,$$

$$g(x,y) < (g_0+\varepsilon)(x+y)^N, \ \ 0 \le x+y \le r_3.$$

We define the set $\Omega_3 = \{(x,y) \in E : \|(x,y)\|_E < r_3\}$, for any $(x,y) \in P \cap \partial\Omega_3$, we deduce

$$\begin{aligned}
A_1(x,y)(s) &= \int_s^1 (\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\le \int_0^1 (\int_0^1 \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\le \int_0^1 (\int_0^1 \lambda N\tau^{N-1} \varepsilon(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\
&\le \varepsilon^{\frac{1}{N}} \int_0^1 (\int_0^1 \lambda N\tau^{N-1} (\|x\|+\|y\|)^N d\tau)^{\frac{1}{N}} du \\
&= \varepsilon^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&< \frac{\|(x,y)\|_E}{2}.
\end{aligned} \tag{9}$$

Therefore,

$$\|A_1(x,y)\| \le \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_3.$$

Similar to the proof of (3.7), we have

$$\|A_2(x,y)\| \le \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_3,$$

then

$$\|A(x,y)\|_E \le \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_3. \tag{10}$$

On the other hand, since $g_\infty = \infty$, we know that there exists $\bar{r}_4 > 0$ such that

$$g(x,y) \ge \frac{1}{\varepsilon}(x+y)^N, \ \ x,y \ge 0, \ \ x+y \ge \bar{r}_4.$$

We take $r_4 = \max\{2r_3, 4\bar{r}_4\}$ and denote $\Omega_4 = \{(x,y) \in E : \|(x,y)\|_E < r_4\}$, for any $(x,y) \in P \cap \partial\Omega_4$, we get $\min_{s\in[\frac{1}{4},\frac{3}{4}]}(x(s)+y(s)) \ge \frac{1}{4}\|(x,y)\|_E = \frac{1}{4}r_4 \ge \bar{r}_4$, then

$$\begin{aligned}
A_2(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^1 (\int_0^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} \frac{1}{\varepsilon}(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\
&\ge (\frac{1}{\varepsilon})^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} (\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\
&= \frac{1}{4}(\frac{1}{\varepsilon})^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&> \|(x,y)\|_E.
\end{aligned}$$

Therefore,

$$\|A(x,y)\|_E \ge \|A_2(x,y)\| \ge \|(x,y)\|_E, \ \ (x,y) \in P \cap \partial\Omega_4. \tag{11}$$

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From (3.8), (3.9) and Lemma 2.1, we get that A has at least one fixed point $(x, y) \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$ such that $r_3 \le \|(x, y)\|_E \le r_4$, that is, (x, y) is a positive solution for the system (2.1), so the proof is completed. □

For $\widehat{f}_0, \widehat{g}_0, \widehat{f}_\infty, \widehat{g}_\infty \in (0, \infty)$, we define the symbols below:

$$Q_1 = \frac{2^N}{G^N \widehat{f}_0}, \quad Q_2 = \frac{1}{2^N F^N \widehat{f}_\infty},$$

$$Q_3 = \frac{2^N}{G^N \widehat{g}_0}, \quad Q_4 = \frac{1}{2^N F^N \widehat{g}_\infty}.$$

Theorem 2. *(1) Assume that $\widehat{f}_0, \widehat{g}_0, \widehat{f}_\infty, \widehat{g}_\infty \in (0, \infty), Q_1 < Q_2, Q_3 < Q_4$, then for $\lambda \in (Q_1, Q_2)$ and $\mu \in (Q_3, Q_4)$, the system (2.1) has at least one positive solution.*

(2) Assume that $\widehat{f}_0, \widehat{g}_0, \widehat{f}_\infty \in (0, \infty), \widehat{g}_\infty = 0$, and $Q_1 < Q_2$, then for each $\lambda \in (Q_1, Q_2)$ and $\mu \in (Q_3, \infty)$, the system (2.1) has at least one positive solution.

(3) Assume that $\widehat{f}_0, \widehat{g}_0, \widehat{g}_\infty \in (0, \infty), \widehat{f}_\infty = 0$, and $Q_3 < Q_4$, then for each $\lambda \in (Q_1, \infty)$ and $\mu \in (Q_3, Q_4)$, the system (2.1) has at least one positive solution.

(4) Assume that $\widehat{f}_0, \widehat{g}_0 \in (0, \infty), \widehat{f}_\infty = \widehat{g}_\infty = 0$, then for each $\lambda \in (Q_1, \infty)$ and $\mu \in (Q_3, \infty)$, the system (2.1) has at least one positive solution.

(5) Assume that $\widehat{f}_\infty, \widehat{g}_\infty \in (0, \infty), \widehat{f}_0 = \infty$ or $\widehat{f}_\infty, \widehat{g}_\infty \in (0, \infty), \widehat{g}_0 = \infty$, then for each $\lambda \in (0, Q_2)$ and $\mu \in (0, Q_4)$, the system (2.1) has at least one positive solution.

(6) Assume that $\widehat{f}_0 = \infty, \widehat{f}_\infty \in (0, \infty), \widehat{g}_\infty = 0$ or $\widehat{f}_\infty \in (0, \infty), \widehat{g}_\infty = 0, \widehat{g}_0 = \infty$, then for each $\lambda \in (0, Q_2)$ and $\mu \in (0, \infty)$, the system (2.1) has at least one positive solution.

(7) Assume that $\widehat{f}_0 = \infty, \widehat{g}_\infty \in (0, \infty), \widehat{f}_\infty = 0$ or $\widehat{g}_\infty \in (0, \infty), \widehat{g}_0 = \infty, \widehat{f}_\infty = 0$, then for each $\lambda \in (0, \infty)$ and $\mu \in (0, Q_4)$, the system (2.1) has at least one positive solution.

(8) Assume that $\widehat{f}_\infty = \widehat{g}_\infty = 0, \widehat{f}_0 = \infty$ or $\widehat{f}_\infty = \widehat{g}_\infty = 0, \widehat{g}_0 = \infty$, then for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$, the system (2.1) has at least one positive solution.

Proof. Due to the similarity in the proofs of the above cases, we will demonstrate the case (1) and the case (6).

(1) For each $\lambda \in (Q_1, Q_2)$ and $\mu \in (Q_3, Q_4)$, there exists $\varepsilon > 0$ such that

$$\frac{2^N}{G^N(\widehat{f}_0 - \varepsilon)} \le \lambda \le \frac{1}{2^N F^N(\widehat{f}_\infty + \varepsilon)},$$

$$\frac{2^N}{G^N(\widehat{g}_0 - \varepsilon)} \le \mu \le \frac{1}{2^N F^N(\widehat{g}_\infty + \varepsilon)}.$$

By the definitions of $\widehat{f}_0$ and $\widehat{g}_0$, we know that there exists $r_1 > 0$ such that

$$f(x, y) \ge (\widehat{f}_0 - \varepsilon)(x + y)^N, \quad x, y \ge 0, x + y \le r_1,$$

$$g(x, y) \ge (\widehat{g}_0 - \varepsilon)(x + y)^N, \quad x, y \ge 0, x + y \le r_1.$$

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We define the set $\Omega_1 = \{(x,y) \in E : \|(x,y)\|_E < r_1\}$, for any $(x,y) \in P \cap \partial\Omega_1$, we get

$$\begin{aligned}
A_1(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^u \lambda N\tau^{N-1}f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1}f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1}(\widehat{f}_0-\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}}du \\
&\geq (\widehat{f}_0-\varepsilon)^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1}(\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}}du \\
&= \frac{1}{4}(\widehat{f}_0-\varepsilon)^{\frac{1}{N}}\lambda^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u N\tau^{N-1}d\tau)^{\frac{1}{N}}du\cdot\|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Therefore,

$$\|A_1(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E,\ \ \forall(x,y)\in P\cap\partial\Omega_1. \tag{12}$$

In a similar manner, for any $(x,y) \in P \cap \partial\Omega_1$, we deduce

$$\begin{aligned}
A_2(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^u \mu N\tau^{N-1}g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1}g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1}(\widehat{g}_0-\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}}du \\
&\geq (\widehat{g}_0-\varepsilon)^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1}(\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}}du \\
&= \frac{1}{4}(\widehat{g}_0-\varepsilon)^{\frac{1}{N}}\mu^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u N\tau^{N-1}d\tau)^{\frac{1}{N}}du\cdot\|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Therefore,

$$\|A_2(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E,\ \ \forall(x,y)\in P\cap\partial\Omega_1. \tag{13}$$

From (3.10) and (3.11) we deduce

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \geq \|(x,y)\|_E,\ \ \forall(x,y)\in P\cap\partial\Omega_1. \tag{14}$$

Let $f^*(u) = \max_{0\leq x+y\leq u} f(x,y)$, $g^*(u) = \max_{0\leq x+y\leq u} g(x,y)$, then we have

$$f(x,y) \leq f^*(u),\ \ x,y\geq 0,\ \ x+y\leq u,$$

$$g(x,y) \leq g^*(u),\ \ x,y\geq 0,\ \ x+y\leq u.$$

Similar to the proof of [10], we have

$$\limsup_{u\to+\infty}\frac{f^*(u)}{u^N} \leq \widehat{f}_\infty,\ \ \limsup_{u\to+\infty}\frac{g^*(u)}{u^N} \leq \widehat{g}_\infty.$$

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According to the above inequality, we know that there exists $\bar{r}_2 > 0$ such that

$$\frac{f^*(u)}{u^N} \leq \limsup_{u\to+\infty} \frac{f^*(u)}{u^N} + \varepsilon \leq \widehat{f}_\infty + \varepsilon, \ \ u \geq \bar{r}_2,$$

$$\frac{g^*(u)}{u^N} \leq \limsup_{u\to+\infty} \frac{g^*(u)}{u^N} + \varepsilon \leq \widehat{g}_\infty + \varepsilon, \ \ u \geq \bar{r}_2,$$

then

$$f^*(u) \leq (\widehat{f}_\infty + \varepsilon)u^N, \ g^*(u) \leq (\widehat{g}_\infty + \varepsilon)u^N, \ u \geq \bar{r}_2.$$

We take $r_2 = \max\{2r_1, \bar{r}_2\}$ and denote $\Omega_2 = \{(x,y) \in E : \|(x,y)\|_E < r_2\}$, for any $(x,y) \in P \cap \partial\Omega_2$, we get

$$f(x(s)+y(s)) \leq f^*(\|(x,y)\|_E), \ \ g(x(s)+y(s)) \leq g^*(\|(x,y)\|_E),$$

then

$$\begin{aligned}
A_1(x,y)(s) &\leq \int_0^1 (\int_0^1 \lambda N\tau^{N-1} f^*(\|(x,y)\|_E)d\tau)^{\frac{1}{N}} du \\
&\leq \int_0^1 (\int_0^1 \lambda N\tau^{N-1} (\widehat{f}_\infty + \varepsilon)(\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\
&= (\widehat{f}_\infty + \varepsilon)^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&\leq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Therefore,

$$\|A_1(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{15}$$

In a similar manner, for any $(x,y) \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
A_2(x,y)(s) &\leq \int_0^1 (\int_0^1 \mu N\tau^{N-1} g^*(\|(x,y)\|_E)d\tau)^{\frac{1}{N}} du \\
&\leq \int_0^1 (\int_0^1 \mu N\tau^{N-1} (\widehat{g}_\infty + \varepsilon)(\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\
&= (\widehat{g}_\infty + \varepsilon)^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_0^1 (\int_0^1 N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&\leq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Therefore,

$$\|A_2(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{16}$$

From (3.13) and (3.14) we deduce

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \leq \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{17}$$

Hence, by using (3.12), (3.15) and Lemma 2.1, we conclude that A has at least one fixed point $(x,y) \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $r_1 \leq \|(x,y)\| \leq r_2$.

(6) Assume $\widehat{f}_0 = \infty, \widehat{f}_\infty \in (0,\infty), \widehat{g}_\infty = 0$, then for any $\lambda \in (0, Q_2)$ and $\mu \in (0,\infty)$, there exists $\varepsilon > 0$ such that

$$\frac{4^N\varepsilon}{G^N} < \lambda < \frac{1}{2^N F^N(\widehat{f}_\infty + \varepsilon)}, \ \ 0 < \mu < \frac{1}{2^N F^N \varepsilon}.$$

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Since $\widehat{f}_0 = \infty$, we know that there exists $r_3 > 0$ such that

$$f(x,y) \geq \frac{1}{\varepsilon}(x+y)^N, \;\; x,y \geq 0, \;\; 0 \leq x+y \leq r_3.$$

We choose the set $\Omega_3 = \{(x,y) \in E : \|(x,y)\|_E < r_3\}$, for any $(x,y) \in P \cap \partial\Omega_3$, we deduce

$$\begin{aligned} A_1(x,y)(\frac{1}{4}) &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} \lambda N \tau^{N-1} f(x(\tau), y(\tau)) d\tau)^{\frac{1}{N}} du \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} \lambda N \tau^{N-1} \frac{1}{\varepsilon}(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\ &\geq (\frac{1}{\varepsilon})^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} \lambda N \tau^{N-1} (\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\ &= \frac{1}{4}(\frac{1}{\varepsilon})^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} N \tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\geq \|(x,y)\|_E. \end{aligned}$$

Therefore,

$$\|A(x,y)\|_E \geq \|A_1(x,y)\| \geq \|(x,y)\|_E, \;\; \forall (x,y) \in P \cap \partial\Omega_3. \tag{18}$$

Let $f^*(u) = \max\limits_{0 \leq x+y \leq u} f(x,y)$, $g^*(u) = \max\limits_{0 \leq x+y \leq u} g(x,y)$. Similar to the proof of [10], we have

$$\limsup_{u \to +\infty} \frac{f^*(u)}{u^N} \leq \widehat{f}_\infty, \;\; \limsup_{u \to +\infty} \frac{g^*(u)}{u^N} = 0.$$

Hence, for above $\varepsilon > 0$, there exists $\bar{r}_4 > 0$ such that

$$\frac{f^*(u)}{u^N} \leq \limsup_{u \to +\infty} \frac{f^*(u)}{u^N} + \varepsilon \leq \widehat{f}_\infty + \varepsilon, \;\; u \geq \bar{r}_4,$$

$$\frac{g^*(u)}{u^N} \leq \limsup_{u \to +\infty} \frac{g^*(u)}{u^N} + \varepsilon = \varepsilon, \;\; u \geq \bar{r}_4,$$

then

$$f^*(u) \leq (\widehat{f}_\infty + \varepsilon)u^N, \; g^*(u) \leq \varepsilon u^N, \; u \geq \bar{r}_4.$$

We take $r_4 = \max\{2r_3, \bar{r}_4\}$ and denote $\Omega_4 = \{(x,y) \in E : \|(x,y)\|_E < r_4\}$, for any $(x,y) \in P \cap \partial\Omega_4$, we get

$$f(x(s)+y(s)) \leq f^*(\|(x,y)\|_E), \;\; g(x(s)+y(s)) \leq g^*(\|(x,y)\|_E),$$

then

$$\begin{aligned} A_1(x,y)(s) &\leq \int_0^1 (\int_0^1 \lambda N \tau^{N-1} f^*(\|(x,y)\|_E) d\tau)^{\frac{1}{N}} du \\ &\leq \int_0^1 (\int_0^1 \lambda N \tau^{N-1} (\widehat{f}_\infty + \varepsilon)(\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\ &= (\widehat{f}_\infty + \varepsilon)^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 (\int_0^1 N \tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\leq \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Therefore,

$$\|A_1(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \;\; \forall (x,y) \in P \cap \partial\Omega_4. \tag{19}$$

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In a similar manner, for any $(x,y) \in P \cap \partial\Omega_4$, we have

$$\begin{aligned} A_2(x,y)(s) &\leq \int_0^1 \Big(\int_0^1 \mu N\tau^{N-1} g^*(\|(x,y)\|_E) d\tau\Big)^{\frac{1}{N}} du \\ &\leq \int_0^1 \Big(\int_0^1 \mu N\tau^{N-1} \varepsilon(\|(x,y)\|_E)^N d\tau\Big)^{\frac{1}{N}} du \\ &= \varepsilon^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_0^1 \Big(\int_0^1 N\tau^{N-1} d\tau\Big)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\leq \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Therefore,

$$\|A_2(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_4. \tag{20}$$

From (3.17) and (3.18) we deduce

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \leq \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_4. \tag{21}$$

Hence, by using (3.16), (3.19) and Lemma 2.1, we conclude that A has at least one fixed point $(x,y) \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$ such that $r_3 \leq \|(x,y)\|_E \leq r_4$, that is, (x,y) is a positive solution for the system (2.1). □

4. Applications

Example 1. *We consider the following boundary value problem:*

$$\begin{cases} ((-x'(s))^3)' = 3\lambda s^2 f(x(s),y(s)), \ \ 0 < s < 1, \\ ((-y'(s))^3)' = 3\mu s^2 g(x(s),y(s)), \ \ 0 < s < 1, \\ x'(0) = x(1) = 0, \ y'(0) = y(1) = 0, \end{cases} \tag{22}$$

Take $f(x,y) = (x+y)^{N+2}$, $g(x,y) = (x+y)^N + (x+y)^N e^{x+y}$, where $N = 3$. By a simple calculation we get $M_4 \approx 0.0625$, and

$$f_0 = \limsup_{x+y \to 0^+} \frac{f(x,y)}{(x+y)^N} = \limsup_{x+y \to 0^+} (x+y)^2 = 0,$$

$$g_0 = \limsup_{x+y \to 0^+} \frac{g(x,y)}{(x+y)^N} = \limsup_{x+y \to 0^+} (1 + e^{x+y}) = 2,$$

$$f_\infty = \liminf_{x+y \to \infty} \frac{f(x,y)}{(x+y)^N} = \liminf_{x+y \to \infty} (x+y)^2 = \infty.$$

Then, for each $\lambda \in (0,\infty)$ and $\mu \in (0, 0.0625)$, by Theorem 3.1(6) we obtain that the system (4.1) has at least one positive solution.

Example 2. *We consider the following boundary value problem:*

$$\begin{cases} ((-x'(s))^3)' = 3\lambda s^2 f(x(s),y(s)), \ \ 0 < s < 1, \\ ((-y'(s))^3)' = 3\mu s^2 g(x(s),y(s)), \ \ 0 < s < 1, \\ x'(0) = x(1) = 0, \ y'(0) = y(1) = 0, \end{cases} \tag{23}$$

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Take $f(x,y) = \frac{(x+y)^N}{\tan(x+y)^N}$, $g(x,y) = \frac{1}{x+y}$, *where* $N = 3$. *By a simple calculation we get* $Q_2 \approx 0.1962$, *and*

$$\widehat{f}_0 = \liminf_{x+y\to 0^+} \frac{f(x,y)}{(x+y)^N} = \liminf_{x+y\to 0^+} \frac{1}{\arctan(x+y)^N} = \infty,$$

$$\widehat{g}_\infty = \limsup_{x+y\to\infty} \frac{g(x,y)}{(x+y)^N} = \limsup_{x+y\to\infty} \frac{1}{(x+y)^{N+1}} = 0,$$

$$\widehat{f}_\infty = \limsup_{x+y\to\infty} \frac{f(x,y)}{(x+y)^N} = \limsup_{x+y\to\infty} \frac{1}{\arctan(x+y)^N} = \frac{2}{\pi}.$$

Then, for each $\lambda \in (0, 0.1962)$ *and* $\mu \in (0,\infty)$, *by Theorem 3.2(6) we obtain that the system (4.2) has at least one positive solution.*

5. Conclusion

The system of Monge-Ampère equations is significant in various fields of study, including geometry, mathematical physics, materials science, and others. In this paper, by considering some combinations of superlinearity and sublinearlity of the functions f and g, we use the Guo-Krasnosel'skii fixed point theorem to study the existence of nontrivial solutions for a system of Monge-Ampère equations with two parameters and establish diverse existence outcomes for nontrivial solutions based on various values of λ and μ, which enrich the theories for the system of Monge-Ampère equations. The research in this paper is different from reference [4], and can be said to be its generalization.

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