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Towards An Info-Geometric Theory Of The Analysis Of Non-Time Dependent Queueing Systems

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Article

Towards an Info-Geometric Theory of the Analysis of Non-Time Dependent Queueing Systems

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Abstract: Information geometry (IG) seeks to characterize the structure of statistical geodesic models from a differential geometric point of view. By considering families of probability distributions as manifolds with coordinate charts determined by the parameters of each individual model, the tools of differential geometry, such as divergences and metric tensors, provide effective means of studying their characteristics. The research undertaken in this paper presents a novel approach to the modelling study of information geometrics of a queueing system. In this context, the manifold of a stable M/G/1queue is characterised from the viewpoint of IG. The Fisher Information matrix (FIM) as well as the inverse of (FIM), (IFIM) of stable M/G/1 queue manifold are devised. In addition to that, new results that uncovered the significant impact of stability of M/G/1 queue manifold on the existence of (IFIM) are obtained. The Kullback's divergence (KD), and J-divergence (JD).New result has been devised on the significant impact of both server utilization and squared coefficient of variation of the underlying M/G/1 queue manifold on both (KD) and (KD) are devised. Also, it is revealed that stable M/G/1 QM is developable (i.e., has a zero Gaussian curvature) and has a non-zero Ricci Curvature Tensor (RCT). Novel stability dynamics of M/G/1 queue manifold is revealed by discovering the mutual dual impact between the behaviour of (RCT) and the stability and the instability phases of the underlying M/G/1 queue manifold. Furthermore, a new discovery that presents the significant impact of stability of M/G/1 queue manifold and the continuity of the unique representation between M/G/1 queue manifold and Ricci Curvature Tensor (RCT). The information matrix exponential (IME) is devised. It is also shown that the obtained (IME) is unstable. Also, it is shown that stability of the devised (IME) enforces the instability of M/G/1 queue manifold. Unifying IG with Queueing Theory enables the study of dynamics of queueing system from a novel Riemannian Geometric (RG) point of view, leading to the analysis of the stable M/G/1 queue, based on the Theory of Relativity (TR).Extending the study over two new additional divergence measures, namely Rényi's and $sAB's$ together with a complete illustrative numerical results for all these measures including KD, JD. This links Queueing theory, IG with deep machine learning and metric learning. Furthermore, this reveals the revolutionary approach of queue learning. Full analytic study of Gaussian curvatures subject to both Angular and Monge techniques together with the overall stability dynamics impact on these curvatures. Full analytic study of Einestein Tensor and Stress Energy Tensor together with the overall stability dynamics impact on these curvatures. The inclusion of the definitions of Gaussian and Ricci, Ricci scalar curvatures and Einstein Tensor together with their physical interpretations; The proposed novel approach for the pioneer visualization of queueing systems via computational information geometry. The determination of new important links between classical queueing theory and other mathematical disciplines, such as IG, matrix theory Riemannian geometry and the THEORY OF RELATIVITY by providing for first time i) The full detailed derivations of the Gaussian curvature ii) The Ricci curvature tensor and iii) The full physical as well as the geometric interpretation of these new results. The provision of a novel link between Ricci Curvature (RCT) and the stability analysis of the stable M/G/1 QM. The full investigation of the newly introduced QT-IG unifiers together with the impact of stability/ instability of the underlying M/G/1 QM on them. The full investigation of the newly introduced (QIGU) unifiers together with the impact of stability/ instability of the underlying M/G/1 QM on them.

Keywords: Maximum entropy (ME); IG; SM; QM; RCT; Einstein Tensor; Stress Energy Tensor; Riemannian metric (RM),probability density function (PDF) Fisher Information matrix (FIM); Inverse Fisher Information matrix (IFIM); threshold theorem; Kullback's divergence (KD); J-divergence (JD),Rényi Divergence(RD); sAB Divergence; QT-IG unifiers; Queueing Theoretic Fisher Information Unifiers(QIGU); information matrix exponential (IME); Stability of a matrix

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1. Introduction

Information geometry (IG) is a field that applies techniques from differential geometry to statistics[1]. It aims to use geometric metrics to provide a new way to describe the probability density function, serving as a coordinate system in statistical manifolds(SMs). A manifold [2]is a mathematical concept that represents a space with certain properties. In this context, a manifold is a finite-dimensional Cartesian space, denoted as $\mathbb{R}^{\boldsymbol{n}}$, where $\mathbb{R}^{\boldsymbol{n}}$ refers to a topological space. It is important to note that although figures can be visualized, they are considered abstract geometric figures rather than concrete representations.

In the given context, IG is highlighted as being significantly important[1,3,4]. Figure 1 illustrates how parameter inference, represented by $\hat{\theta}$. Additionally, previous research has explored the geometric structures of exponential distribution families.

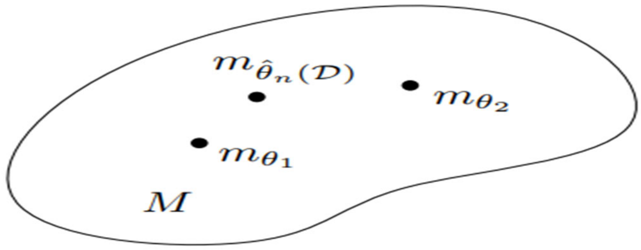

Figure 1. SM's parametrization [3].

One mathematical method for solving systems of linear differential equations is the information matrix exponential (IME). It also has significant applications in the theory of Lie groups, which are mathematical structures that have important implications in various areas of mathematics and physics [5]. Interarrival time distribution (IG) of stable M/D/1 queues was studied by using features of queue length pathways, the article introduced a geometric structure to the set of M/D/1 queues, for a more detailed survey, consult [5] . This strategy connected information matrix theories with IG, opening new insights into queueing theory. According to [3,6], Ricci curvature quantifies the distinction between the standard Euclidean metric (EM) and the Riemannian metric (RM) in the setting of the article. In contrast, the difference in volume between a geodesic ball and a Euclidean ball with the same radius is measured by scalar curvature. Figure 2 depicts the knowledge facilitation of comprehension of the geometric characteristics of spaces and how they deviate from Euclidean geometry (c.f., figure 2).

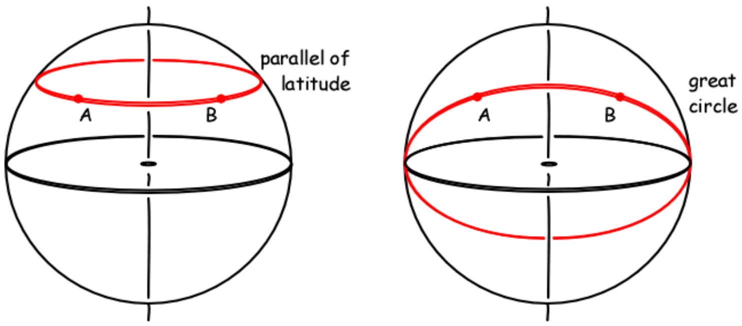

Figure 2. curved surfaces' geodesic representation [6].

- [7] states that the exponential of the Fisher Information Matrix (FIM) for the stable M/G/1 queue, a mathematical model used in queueing theory, solves $dx/dt = Ax$. Here, x represents a vector with n dimensions, and A is a nxn matrix. The second extended study by [7] builds upon their previous work [8] and introduces new contributions:

- Finding the underlying QM's KD and JD measures
- Proving that FIM of the underlying QM solves : $dx/dt = Ax$.

This current paper is an ultimate extension of both papers, with main deliverables :

- Extending the study over two new additional divergence measures, namely Rényi's and $sAB's$ together with a complete illustrative numerical result for all these measures including KD, JD. This links Queueing theory, IG with deep machine learning and metric learning. Furthermore, this reveals the revolutionary approach of queue learning.
- iv) The solenoidability(incompressibility) of the underlying queueing system is shown. This concept is analogous to the Divergence Theorem[9].
- Full analytic study of Gaussian curvatures subject to both Angular and Monge techniques together with the overall stability dynamics impact on these curvatures.
- The current paper provides a comprehensive analysis of the Einstein Tensor and Stress Energy Tensor, exploring their relationship with stability dynamics and curvatures. It also introduces the definitions and interpretations of Gaussian and Ricci curvatures, as well as the Einstein Tensor.
- Extending the study to include two new divergence measures, Rényi's and $sAB's$, along with illustrative numerical results such as KD and JD. This extension connects queueing theory, information geometry, deep machine learning, and metric learning, revealing a novel approach called queue learning.
- Additionally, the paper explores the impact of stability dynamics on Gaussian curvatures, provides a comprehensive analysis of Einsteinian and Stress Energy Tensors, and establishes a unified theorem of queueing-theoretic correlations with both special and general relativity.

The road map of this study is: The core definitions for IG are contained in Section 2. In Section 3, FIM and its inverse for the underlying QM are obtained. In Section 4, the α-connection of a stable M/G/1 queue manifold is obtained. In Section 5, the KD and JD [7], Rényi's, and sAB divergences of a stable M/G/1 QM are computed. In Section 6, methodical arguments are developed demonstrating the developability and non-zero RCT for the underlying queuing manifold system. In addition, a comprehensive examination of the recently announced QT-IG unifiers is presented in Section 6.Section 7 investigates $e^{FIM(\boldsymbol{M/G/1})}$ and how the underlying QM's stability impacts FIM's stability.

In section 8, Ricci scalar,$\mathcal{R}$, curvature of space time(einestein tensor)$\wp$, stress energy tensor, T, the corresponding threshold theorems for the underlying curvatures and the dual queueing impact on the existence of the inverse fisher information matrix(IFIM). Section 9 discusses Queueing theoretic impact on the continuity of new devised queueing-information geometric unifiers (QIGU). Section 10 is entirely devoted to closing remarks combined with next phase of research.

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2. Main Definitions

2.1. Main Definition on IG

Definition 2.1 Statistical Manifold (SM) [7]

We denote a statistical manifold, $M = \{p(x,\theta)|\theta\epsilon\Theta\}$ and $p(x,\theta)$ as a PDF. Here, $\theta = (\theta_1, \theta_2, .., \theta_n)\epsilon\Theta \subset \mathbb{R}^n$.

Definition 2.2 Potential Function [7]

The potential function$\Psi(\theta)$ denotes $(-\mathcal{L}(x;\theta) = -ln(p(x;\theta)))$ with coordinates only.

Definition 2.3 FIM, namely $[g_{ij}]$

$[g_{ij}]$(c.f., [7]) reads as

$$[g_{ij}] = \left[\frac{\partial^2}{\partial\theta^i\partial\theta^j}(\Psi(\theta))\right], i,j = 1,2,..,n \quad (2.1)$$

Definition 2.4 IFIM, namely $[g^{ij}]$ (c.f., [7])

$$[g^{ij}] = ([g_{ij}])^{-1} = \frac{adj[g_{ij}]}{\Delta}, \Delta = \det[g_{ij}] \quad (2.2)$$

The arc length is defined to be:

$$(ds)^2 = \sum_{i,j=1}^{n} g_{ij}(d\theta^i)(d\theta^j) \quad (2.3)$$

Definition 2.5 α-Connection(c.f., [7])

The α-*connection* reads as

$$\Gamma_{ij,k}^{(\alpha)} = \left(\frac{1-\alpha}{2}\right)(\partial_i\partial_j\partial_k(\Psi(\theta))), \ \partial_i = \frac{\partial}{\partial\theta_i}, \quad \alpha \text{ is real} \quad (2.4)$$

Definition 2.6 Kullback's Divergence (KD), $K(p,q)$

KD, namely $K(p,q)$[7]reads as

$$K(p,q) = E_{\theta_p}\left[ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right)\right] \quad (2.5)$$

$$= \int p(x;\theta_p) ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right) dx \quad (2.6)$$

J-divergence reads as

$$J(p,q) = ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right)^{(p(x;\theta_p)-q(x;\theta_q))} dx \quad (2.7)$$

$$= K(p,q) + K(q,p) \quad (2.8)$$

In this paper, however, we focus on the Rényi divergence [10,11],

$$D_R^{\gamma}(p||q) = \frac{1}{(\gamma-1)} ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma}(q(n))^{1-\gamma}\right) \quad (2.9)$$

used in Rényi variational inference VI [12].

$D_{s,AB}^{\gamma,\eta}(p||q)$ [13] reads as:

$$D_{s,AB}^{\gamma,\eta}(p||q) = \frac{1}{\eta(\eta+\gamma)} ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma+\eta}\right) + \frac{1}{\gamma(\eta+\gamma)} ln\left(\sum_{n=0}^{\infty}(q(n))^{\gamma+\eta}\right) - \frac{1}{\gamma\eta} ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma}(q(n))^{\eta}\right) \quad (2.10)$$

for $(\gamma,\eta)\epsilon\mathbb{R}^2$ such that $\gamma \neq 0, \eta \neq 0$ *and* $\gamma + \eta \neq 0$.

The authors [13] have presented a novel (dis)similarity measure, namely $\boldsymbol{D_{s,AB}^{\gamma,\eta}(p||q)}$ (c.f., (2.10)). Moreover, it has been illustrated [13] that $D_{s,AB}^{\gamma,\eta}(p||q)$ is potentially robust.

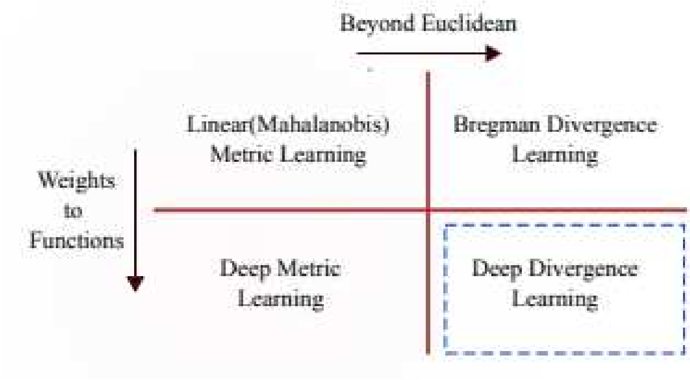

Figure 3. (c.f., [14]).

Definition 2.8

1. The $\alpha -$ curvature RiemannianTensors, $R_{ijkl}^{(\alpha)}$[7] reads

$$R_{ijkl}^{(\alpha)} = [(\partial_j \Gamma_{ik}^{s(\alpha)} - \partial_i \Gamma_{jk}^{s(\alpha)}) g_{sl} + (\Gamma_{jt,l}^{(\alpha)} \Gamma_{ik}^{t(\alpha)} - \Gamma_{it,l}^{(\alpha)} \Gamma_{jk}^{t(\alpha)})], i, j, k, l, s, t = 1,2,3, \dots, n \quad (2.11)$$

where $\Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)} g^{sk}$, i,j,k,s = 1,2,…,n

2. The $\alpha -$ Ricci curvatures (Ricci Tensors) $R_{ik}^{(\alpha)}$ reads [7]

$$R_{ik}^{(\alpha)} = R_{ijkl}^{(\alpha)} g^{jl}, i, j, k, l = 1,2,3, \dots, n \quad (2.12)$$

3.The $\alpha -$ sectional curvatures$K_{ijij}^{(\alpha)}$ reads [7]

$$K_{ijij}^{(\alpha)} = \frac{R_{ijij}^{(\alpha)}}{(g_{ii})(g_{jj}) - (g_{ij})^2}, i, j = 1,2, \dots, n \quad (2.13)$$

Potentially,

$$K^{(\alpha)} = \frac{R_{1212}^{(\alpha)}}{det(g_{ij})} \quad (2.14)$$

4. One mathematical object that can be obtained by contacting the Riemannian Tensor is the Ricci Tensor [15]. It measures the curvature of space and is employed in the study of Riemannian manifolds. To obtain the Ricci Tensor, the contraction procedure entails summing a few components of the Riemannian Tensor [7].

5. An oriented Riemannian manifold's Ricci curvature tensor (RCT) (c.f., [16]) quantifies the difference between a geodesic ball's volume on the manifold and its volume in Euclidean space. It gives details on the manifold's curvature and how it differs from flat space. knowledge the geometry and characteristics of curved spaces in connection to Euclidean geometry requires a knowledge of this topic.

6.RCT (c.f., [17]) measures how volumes change over time along geodesic paths on a Riemannian manifold. When the Ricci curvature is positive, it indicates a smaller diameter. This relationship is supported by the Bonnet-Myers theorem, which establishes a connection between the manifold's positive Ricci curvature and the curvature properties.

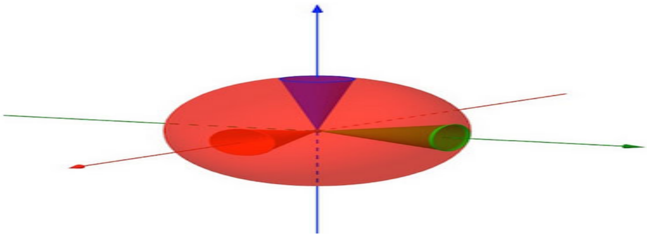

Figure 4. RCT (c.f., [18]).

Definition 2.9

1. Considering the linear system of differential equations

$$\frac{dx}{dt} = Ax \tag{2.15}$$

with x is an n-dimensional vector and A is a nxn matrix. It can be shown that (Gunawardena, 2006) the matrix exponential:

$$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \ldots \tag{2.16}$$

is the solution of (2.15).

2. If the characteristic polynomial of A is defined by

$$\Phi(\delta) = \det(A - \delta I) \tag{2.17}$$

The eigen values of A (c.f., [19]) solve:

$$\Phi(\delta) = (\delta) = 0 \tag{2.18}$$

such that:

$$Ax = \delta x \tag{2.19}$$

e^A reads as:

$$e^A = T\ e^D\ T^{-1} \tag{2.20}$$

where D is the diagonal matrix of eigen values of A, and T is matrix having of the corresponding eigen vectors of A as its columns (c.f., [19]).

Definition 2.10

Developable surfaces are a special kind of ruled surfaces, they have a Gaussian curvature equal to 0, and can be mapped onto the plane surface without distortion of curves: any curve from such a surface drawn onto the flat plane remains the same (c.f., [20]).

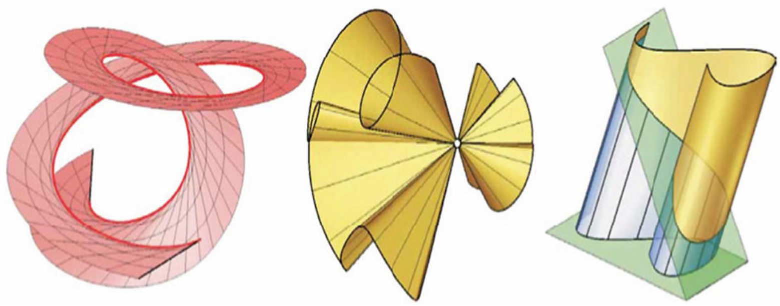

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Figure 5. Three kinds of developable surfaces: Tangential on Figure. 5a (on the left), Conical on Fig. 5b (on the centre) and Figure. 5c (on the right), Cylindrical. Note that curves in bold are directrix or base curves and straight lines in bold are directors or generating lines (curves) (c.f., [20]).

2.2. Gaussian and Mean Curvatures, $\boldsymbol{K_G}$ **and** $\boldsymbol{H}$ ***respectively*** *(c.f., [21])*

Definition 2.11(Mong Patch Technique)

1.Let K_1 and K_2 be the principal curvatures of a surface patch $\delta(u,v)$. $K_G(\delta)$ is

$$K_G = K_1K_2 \tag{2.21}$$

and its Mean Curvature is:

$$H = \frac{1}{2}(K_1 + K_2) \tag{2.22}$$

2.For a Mong patch $z = f(x,y), K_G$and H are given by

$$K_G = \frac{LN-M^2}{EG-F^2} \tag{2.23}$$

and its Mean Curvature is

$$H = \frac{1}{2}\left(\frac{LG-2MF+NE}{EG-F^2}\right) \tag{2.24}$$

with $E = \left(\frac{\partial f}{\partial x}\right)^2, F = \frac{\partial f}{\partial x \partial y}, G = \left(\frac{\partial f}{\partial y}\right)^2, L = \frac{\partial^2 f}{\partial x^2}, M = \frac{\partial f}{\partial xy}, N = \frac{\partial^2 f}{\partial y^2}$

Classification of Surface Points

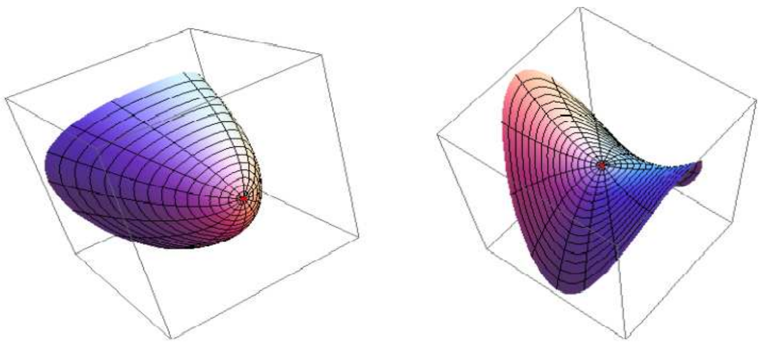

Figure 6. The elliptic paraboloids $z = x^2 + 2y^2 (to\ the\ left)\ and\ z = x^2 - 2y^2$ (to the right) (c.f., [21]).

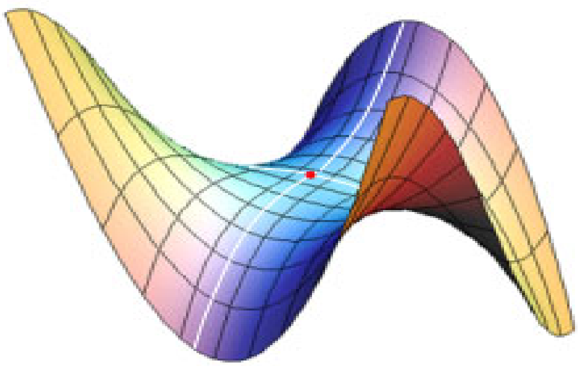

Figure 7. Planar points with quite different shapes(c.f., [21]).

A torus as shown in Figure 8

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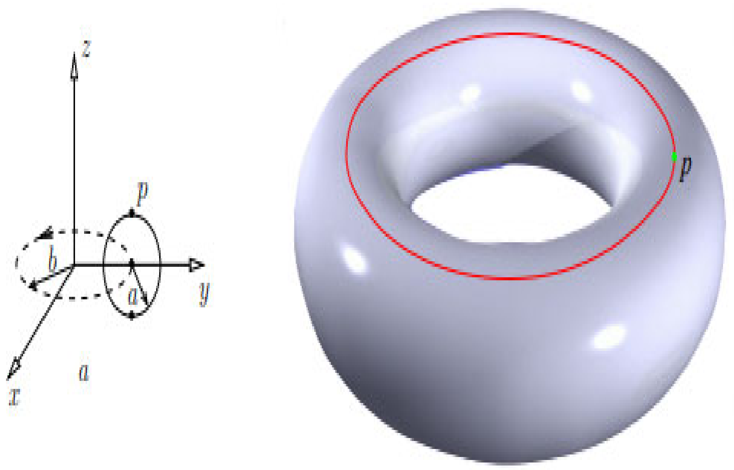

Figure 8. (c.f., [21]).

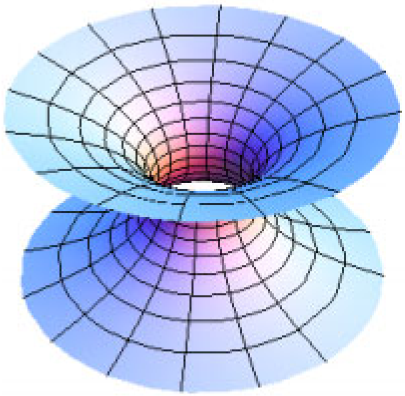

Figure 9. Catenoid (c.f., [21]).

2.3. Different Approach to Gaussian and Mean Curvatures (Angular Technique) (c.f., [22])

A new formulation (c.f., [22]) is introduced for *Gaussian Curvature* K_G and the *Mean Curvature* is H are defined as

$$K_G = K_1 K_2 \tag{2.25}$$

And

$$H = \frac{1}{2}(K_1 + K_2) \tag{2.26}$$

with K_1 and K_2 as the principal curvatures are determined by:

$$K_1 = \frac{B_{11}}{(1+(\frac{\partial f}{\partial x_1'})^2)^{\frac{3}{2}}}\ ,\ K_2 = \frac{B_{22}}{(1+(\frac{\partial f}{\partial x_2'})^2)^{\frac{3}{2}}} \tag{2.27}$$

where $x_3 = f(x_1, x_2)$ defines the shape of the surface. x_1' and x_2' are parallel to the directions of the principal curvature, which are rotated through an angle ω with respect to $x_1$andx_2, and

$$\frac{\partial f}{\partial x_1'} = cos\omega \frac{\partial f}{\partial x_1} - sin\omega \frac{\partial f}{\partial x_2} \tag{2.28}$$

$$\frac{\partial f}{\partial x_2'} = sin\omega \frac{\partial f}{\partial x_1} + cos\omega \frac{\partial f}{\partial x_2} \tag{2.29}$$

$$B_{11} = \frac{\partial^2 f}{\partial x_1^2} cos^2\omega - 2\frac{\partial^2 f}{\partial x_1 \partial x_2} sin\omega cos\omega + \frac{\partial^2 f}{\partial x_2^2} sin^2\omega = \frac{\partial^2 f}{\partial x_1'^2} \tag{2.30}$$

$$B_{22} = \frac{\partial^2 f}{\partial x_1^2} sin^2\omega + 2\frac{\partial^2 f}{\partial x_1 \partial x_2} sin\omega cos\omega + \frac{\partial^2 f}{\partial x_2^2} cos^2\omega = \frac{\partial^2 f}{\partial x_2'^2} \tag{2.31}$$

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The angle Ꞷ through which the coordinate frame is rotated to align the axes with the directions of the principal curvature at each point on the surface is given by

$$tan2Ꞷ = \frac{-2(\frac{\partial^2 f}{\partial x_1 \partial x_2})}{(\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2})} \tag{2.32}$$

A contour plot of Gaussian curvature indicates where structures occur on a surface. If both principal curvatures are non- zero, the surface is said to have double curvature [22].

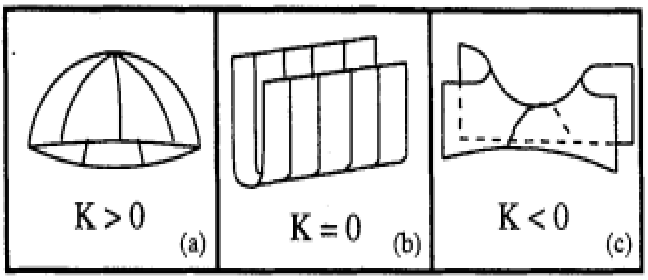

Figure 10. (c.f., [22]). K_G are slab structures- related by the three regions (a), (b), and (c).

2.4. Well Defined Functions and Bijective Functions

Definition 2.12(c.f., [23]).

1. A function is well-defined if it gives the same result when the representation of the input is changed without changing the value of the input.

Definition 2.13(c.f., [24])

1. function f is said to be one-to-one, or injective, if and only if $f(x) = f(y)$implies $x = y$ for all x, y in the domain of f. A function is said to be an injection if it is one-to-one. Alternative: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$. This is the contrapositive of the definition.

2.A function f from A to B is called onto, or surjective, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$. Alternative: all co-domain elements are covered.

3. A function f is called a bijection if it is both one-to-one (injection) and onto (surjection).

Definition 2.14(c.f., [25])

The solution of the cubic equation

$$a^*w^3 + b^*w^2 + c^*w + d^* = 0 \tag{2.33}$$

is characterized arbitrarily by

$$y = z - \frac{\varepsilon_3}{z} \tag{2.34}$$

$$w = y - \frac{b^*}{3a^*} \tag{2.35}$$

$$z = \sqrt[3]{(-\frac{\varepsilon_1}{2}) \pm \sqrt{\varepsilon_2}}, \tag{2.36}$$

$$\varepsilon_1 = \frac{2(b^*)^3}{27} + \frac{d^*}{a^*} - \frac{b^*c^*}{3(a^*)^2}, \tag{2.37}$$

$$\varepsilon_2 = \frac{(\varepsilon_1)^2}{4} + \frac{(\varepsilon_3)^3}{27}, \tag{2.38}$$

where ε_3 is given by

$$\varepsilon_3 = -\frac{2(b^*)^2}{3(a^*)^2} + \frac{c^*}{a^*} \tag{2.39}$$

ε_2 is called the *discriminant* of the cubic equation.

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Preliminary Theorem(PT) 2.15 [26].

Let *f* be a function that is defined and differentiable on an open interval (*c*,*d*).

(1) If $f'(x) > 0$ for all $x \in (c,d)$,then *f* is increasing on (c,d). (2.40)

If $f'(x) < 0$ for all $x \in (c,d)$, then *f* is decreasing on (c,d). (2.41)

2.5. Stability Analysis for Ordinary Differential Equations (ODEs) [27]

Equilibria are not always stable. Since stable and unstable equilibria play quite different roles in the dynamics of a system, it is useful to be able to classify equilibrium points based on their stability. Suppose that we have a set of autonomous ordinary differential equations, written in vector form:

$$\frac{dx}{dt} = f(x) \qquad (2.42)$$

Suppose that x^* is an equilibrium point. By definition, $f(x^*) = 0$.

Theorem 2.16 (c.f., [27]) An equilibrium point x^* of the differential equation 1 is stable if all the eigenvalues of J^*, the Jacobian evaluated at x^* , have negative real parts. The equilibrium point is unstable if at least one of the eigenvalues has a positive real part.

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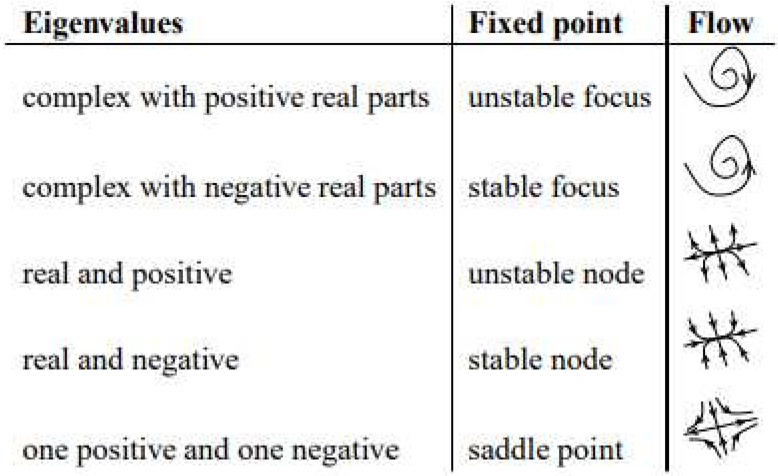

Eigenvalues	Fixed point	Flow
complex with positive real parts	unstable focus	
complex with negative real parts	stable focus	
real and positive	unstable node	
real and negative	stable node	
one positive and one negative	saddle point	

Figure 11. (c.f., [27]).

2.6. Scalar Curvature(Ricci Scalar), $\mathcal{R}$ and Einestein Tensor, ℘

The scalar curvature(Ricci Scalar), $\mathcal{R}$(c.f., [15]) measures RCT's contraction(c.f., (2.12))

$\mathcal{R} = R_{ij}^{(\alpha)} g^{ij}, i,j = 1,2,3, ...$ (2.43)

The two-dimensional Ricci Scalar, $\mathcal{R}$ [28] is twice as the Gaussian Curvature K_G (c.f., (2.25)),

$\mathcal{R} = 2K_G = 2\, K_1 K_2$ (2.44)

provided that $K_1$and K_2 are determined by (2.27).

The Ricci scalar $\mathcal{R}$ [15] has a similar meaning to K_G,

$\mathcal{R} = lim_{\epsilon \to 0} \frac{6n}{\epsilon^2}[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}]$ (2.45)

Ricci scalar completely captures the curvature of the surface.

The equations of motion of a classical theory like General Relativity can be derived directly from a suitable action by using the Euler-Lagrange equations, leading to the well-known Einstein equations [29]

$G_{ij} = R_{ij}^{(\alpha)} - \frac{\mathcal{R}}{2} g_{ij} = \frac{8\pi g \varpi_{ij}}{c^4}$ (2.46)

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where G_{ij} is the Curvature of Spacetime(Einstein tensor), $\wp$, $\mathrm{R}_{\mathrm{ij}}^{(\alpha)}$ defines spacetime − RCT, namely g_{ij}, $\mathcal{R} = R_{ij}^{(\alpha)} g^{ij}, i, j = 1,2,3, \ldots$,is the Ricci scalar or scalar curvature, g is the universal gravitational constant, c is the speed of light, and ϖ_{ij} are the components of the stress-energy tensor, ϖ ,as a descriptor of spacetime- matter-energy distributions.

2.7. Maxima and Minima of Functions of Two Variables [30]

Suppose that (a_1, b_1) is a critical point of $f(x, y)$ (i.e, $\frac{\partial f(a_1,b_1)}{\partial x} = 0 = \frac{\partial f(a_1,b_1)}{\partial y}$). Let's denote:

$$D = D(a_1, b_1) = f_{xx}(a_1, b_1) f_{yy}(a_1, b_1) - [f_{xy}(a_1, b_1)]^2 \tag{2.47}$$

This provides the critical point categories:

1. If $D > 0$ and $f_{xx}(a_1, b_1) > 0$ then there is a relative minimum at (a_1, b_1).
2. If $D > 0$ and $f_{xx}(a_1, b_1) < 0$ then there is a relative maximum at (a_1, b_1).
3. If $D < 0$ then the point (a_1, b_1) is a saddle point.
4. If $D = 0$ then the point (a_1, b_1) may be a relative minimum, relative maximum, or a saddle point. Other techniques would need to be used to classify the critical point.

2.8. Continuous Functions (c.f., [31])

Theorem 2.16

A function f is continuous at x_0 if and only if f is defined on an open interval (r, s) containing x_0 and for each $\varepsilon > 0$ there is a $\delta > 0$ such that:

$$|f(x) - f(x_0)| < \varepsilon \tag{2.48}$$

whenever $|x - x_0| < \delta$.

2.9. The Maclauren's Series of $\boldsymbol{ln(1-x)}$ *for* $\boldsymbol{x}$ ***around zero****(Shaw,2015)*

$$ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \tag{2.49}$$

3. The Fim and Its Inverse for the Stable M/G/1 QM

According to [32], the maximum entropy (ME) state probability of the generalized geometric solution of a stable M/G/1 queue (c.f., Figure. 12), subject to normalisation, mean queue length (MQL), L and server utilisation, $\rho(<1)$ is given by

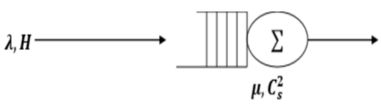

Figure 12.

$$p(n) = \begin{cases} 1 - \rho, & n = 0 \\ (1 - \rho) g x^n , & n \geq 1 \end{cases} \tag{3.1}$$

where $g = \frac{\rho^2}{(L-\rho)(1-\rho)}$, $x = \frac{L-\rho}{L}$ and $L = \frac{\rho}{2}\left(1 + \frac{1+\rho C_s^2}{1-\rho}\right)$, $\rho = 1 - p(0)$ and $\beta = C_s^2$.

$p(n)$of (3.1) reads as:

$$p(n) = \begin{cases} 1 - \rho, & n = 0 \\ \frac{2\rho\left(\frac{1+\rho\beta}{1-\rho} - 1\right)^{n-1}}{\left(\left(\frac{1+\rho\beta}{1-\rho} + 1\right)^n\right.}, & n > 0, \ \text{with} \beta = C_s^2 \end{cases} \tag{3.2}$$

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Theorem 3.1 The underlying QM satisfies:

(i) FIM reads as

$$[g_{ij}] = \begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix} \quad (3.3)$$

(ii) $(ds)^2 = (\frac{1}{(1-\rho)^2})(d\rho)^2 - \frac{1}{(\beta+1)^2}(d\beta)^2$ (3.4)

(iii) $[g^{ij}]$ reads as

$$[g^{ij}] = \frac{adj[g_{ij}]}{\Delta} = \begin{pmatrix} (1-\rho)^2 & 0 \\ 0 & -(\beta+1)^2 \end{pmatrix} \quad (3.5)$$

Proof

(i)

Case I: $p(0) = 1 - \rho$. Thus,

$\mathcal{L}(x;\theta) = ln\big(p(x;\theta)\big) = ln(1-\rho)$,

$$\theta = \theta_1 = \rho \quad (3.6)$$

$$\Psi(\theta) = -ln(1-\rho) \quad (3.7)$$

Therefore,

$\partial_1 = \frac{\partial\Psi}{\partial\rho} = \frac{1}{1-\rho}$ (3.8). $\partial_1\partial_1 = \frac{\partial^2\Psi}{\partial\rho^2} = \frac{1}{(1-\rho)^2}$ (3.9)

FIM is given by:

$$[g_{ij}] = \left[\frac{\partial^2\Psi}{\partial\rho^2}\right] = \left[\frac{1}{(1-\rho)^2}\right] \quad (3.10)$$

$[g^{ij}]$ reads as

$$[g^{ij}] = [g_{ij}]^{-1} = [(1-\rho)^2] \quad (3.11)$$

Case II: For $n > 0$, $p(n) = \frac{2\rho\left(\frac{1+\rho\beta}{1-\rho}-1\right)^{n-1}}{(\left(\frac{1+\rho\beta}{1-\rho}+1\right)^{n}}$. Therefore, the coordinate system is two-dimensional satisfying:

$$\mathcal{L}(x;\theta) = ln\big(p(x;\theta)\big) = ln(1-\rho) + ln2 - ln(\beta+1) + n\, ln(\frac{\rho(1+\beta)}{2+\rho(\beta-1)}) \quad (3.12)$$

where

$$\theta = (\theta_1, \theta_2) = (\rho, \beta), \quad with\ \beta = C_s^2 \quad (3.13)$$

We have

$$\Psi(\theta) = ln(\beta+1) - ln(1-\rho) - ln2 \quad (3.14)$$

Thus, we have

$$\partial_1 = \frac{1}{1-\rho}\partial_2 = \frac{1}{\beta+1}, \partial_{11} = \frac{1}{(1-\rho)^2}, \partial_1\partial_2 = \partial_2\partial_1 = 0, \partial_{22} = -\frac{1}{(\beta+1)^2} \quad (3.15)$$

FIM is given by $[g_{ij}] = \begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix}$ (3.16). This completes the proof of (i)

It could be verified that

$(ds)^2 =$ $(\frac{1}{(1-\rho)^2})(d\rho)^2$ - $\frac{1}{(\beta+1)^2}(d\beta)^2$ (c.f., (3.4))

Finally, after some manipulation, it could be shown that:

$$[g^{ij}] = \frac{adj[g_{ij}]}{\Delta} = \begin{pmatrix} (1-\rho)^2 & 0 \\ 0 & -(\beta+1)^2 \end{pmatrix} \quad \text{(c.f., (3.5))}$$

4. The α(. OR $\nabla^{(\alpha)}$)-Connection of the M/G/1 QM

(2.8) implies:

$$\Gamma_{11,1}^{(\alpha)} = (\frac{1-\alpha}{(1-\rho)^3}),\ \Gamma_{11,1}^{(\alpha)} = \frac{(1-\alpha)}{(1-\rho)^3} \quad (4.1)$$

$$\Gamma_{11}^{1(\alpha)} = \frac{1-\alpha}{(1-\rho)}, \Gamma_{11}^{1(0)} = \frac{1}{(1-\rho)} \quad (4.2)$$

$$\Gamma_{22}^{2(\alpha)} = -\frac{1-\alpha}{(1+\beta)},\ \Gamma_{22}^{2(0)} = -\frac{1}{(1+\beta)} \quad (4.3)$$

Engaging the same logic, RCT's remaining components can be determined.

5. Variational Inference, KD,JD,RÉNYI AND s AB DIVERGENCES OF STABLE M/G/1 QM

5.1. KD and JD Divergences of Stable M/G/1 QM

The following theorem characterizes both KD and JD of (2.5) and (2.6) respectively.

Theorem 5.1 The underlying QM satisfies:

(i) $K(p,q) = E_{\theta_p}\left[ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right)\right] = ln\,(\left(\frac{1-\rho_p}{1-\rho_q}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)[\left(\frac{\rho_q(2+\rho_q(\beta_q-1)}{\rho_p(2+\rho_p(\beta_p-1)}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)]^{L_p})$ (5.1)

L_p defines the Mean Queue Length at p.

Also,

(ii) $JD(p,q) == ln[\left(\frac{\rho_q(2+\rho_q(\beta_q-1)}{\rho_p(2+\rho_p(\beta_p-1)}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)]^{(L_p-L_q)}$ (5.2)

where L_p, L_q defines the Mean Queue Length at p and q respectively.

Proof

To show (i), the case for $n = 0$ is straightforward. For $n > 0$, It could be verified that, using (3.2), we have

After some few mathematical steps, it could be seen that:

$$ln\left(\frac{p(n)}{q(n)}\right) = ln\left(\frac{\rho_p}{\rho_q}\right) + ln\left(\frac{1-\rho_p}{1-\rho_q}\right) + (n-1)ln\left(\frac{\rho_p(1+\beta_p)}{\rho_q(1+\beta_q)}\right) + n\,ln\left(\frac{(2+\rho_q(\beta_q-1)}{(2+\rho_p(\beta_p-1)}\right) \quad (5.3)$$

By (5.4), it follows that KD will be determined by

$$K(p,q) = ln\left[\left(\frac{1-\rho_p}{1-\rho_q}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)\right] + L_p(ln(\left(\frac{\rho_q(2+\rho_q(\beta_q-1)}{\rho_p(2+\rho_p(\beta_p-1)}\right)\left(\frac{1+\beta_p}{1+\beta_q}\right))) \quad (5.4)$$

$(n = 0,1,2,\ldots,\ and\ L_p = \sum_{n=0}^{\infty} np(n))$

$$= ln\left(\frac{1-\rho_p}{1-\rho_q}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)[\left(\frac{\rho_q(2+\rho_q(\beta_q-1)}{\rho_p(2+\rho_p(\beta_p-1)}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)]^{L_p} \text{(c.f., (5.1))}$$

This completes the proof of (i).

To prove (ii), we have by (2.6) and (5.5)

Following some mathematical steps, (5.2) could be easily verified.

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Clearly it follows from (5.2), that JD is also zero if and only if $p = q$.This present a novel result which declares the compressibility of the underlying QM if and only if it is stable or when p and q are identical, (i.e., $\rho_p = \rho_q, \beta_p = \beta_q$)

It is observed by (5.1), that JD is dependent on ρ_q, β_q and the MQL of Pollaczeck-Khinchin Formula of a stable M/G/1 QM at p, L_p(which is dependent on ρ_p and β_p). To examine the impact of L_p on JD, the following experiment is introduced.

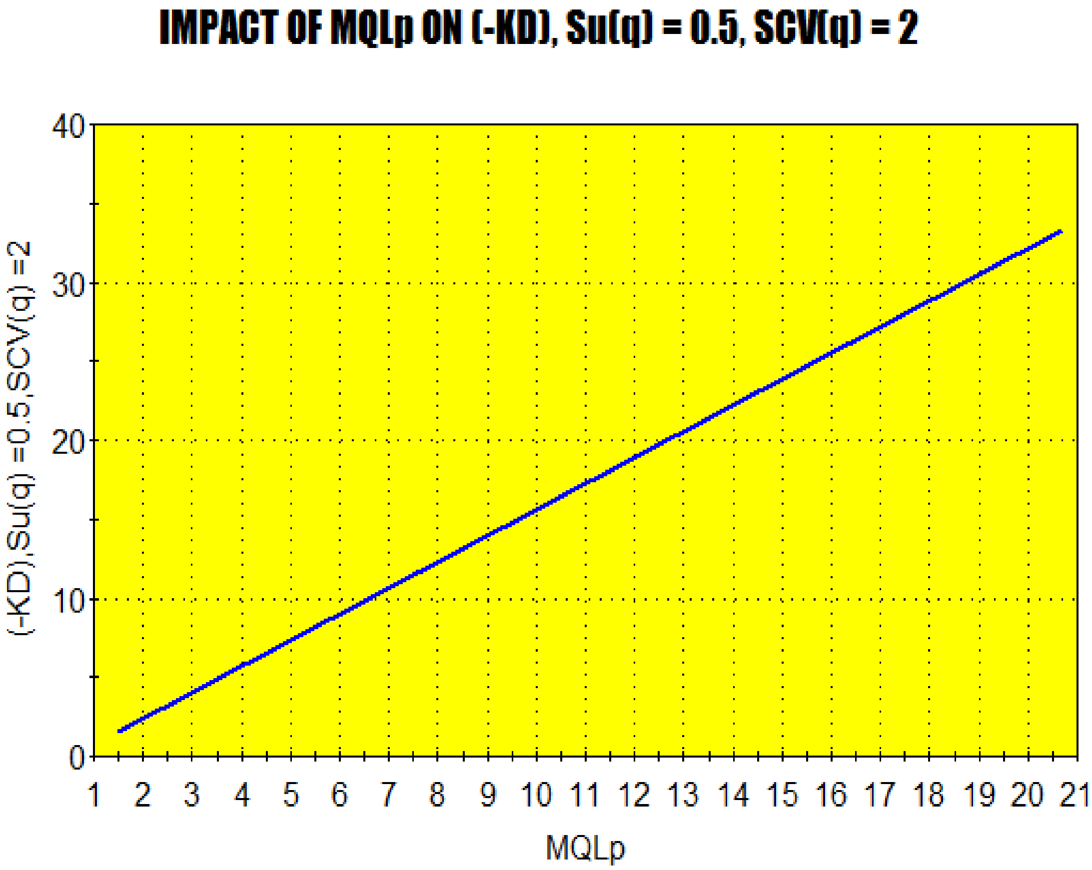

Figure 13.

Figure 13 depicts that KD is a negative decreasing function in MQL at p,L_p. This justifies that the increase of L_p will have a significant impact on the decrease of KD. In other words, the increase of MQL at p, would enforce the distance between p and q to increase in magnitude.

Compressibility of M/G/1 QM could be identified visually by presenting the following numerical experiment:

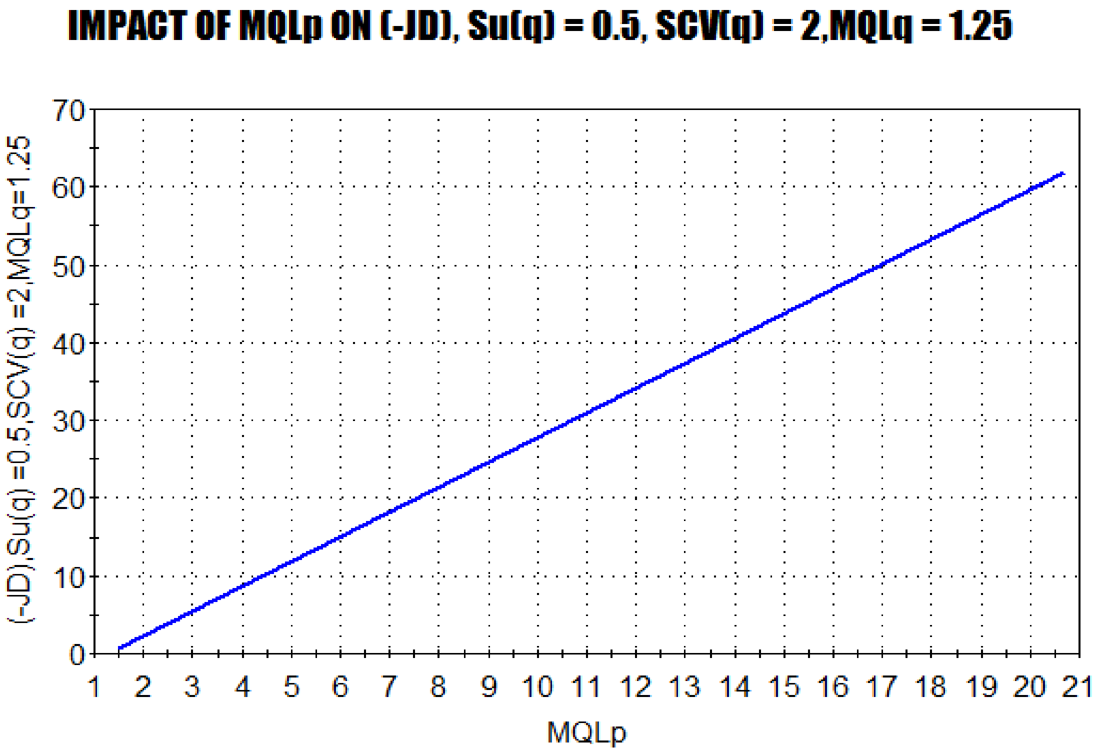

Figure 14.

The findings of figure 14, show that the increase of $MQL\ at\ p$ impacts the the stable M/G/1 QM's solenoidability of. As it observed that M/G/1 QM is solenoidal at the steady state phase of the QM. By the increase of L_p *such that* $L_p \neq L_q$, the stable M/G/1 QM is no longer solenoidal. This shows the direct impact of queueing parameters on the visualization of the regions of solenoidability of the stable M/G/1 QM.

Meanwhile, we have

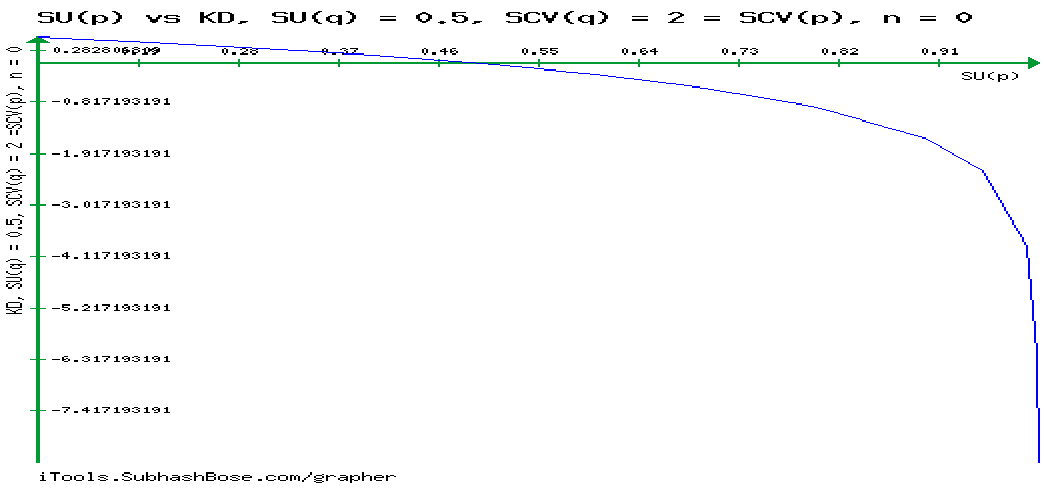

Figure 15.

As observed by Figure 15, KD decreases and vanishes at ρ_p = 0.5. by the increase of ρ_p , KD decreases and tends to $-\infty$ when the underlying M/G/1 QM approaches instability (ρ_p = 1).

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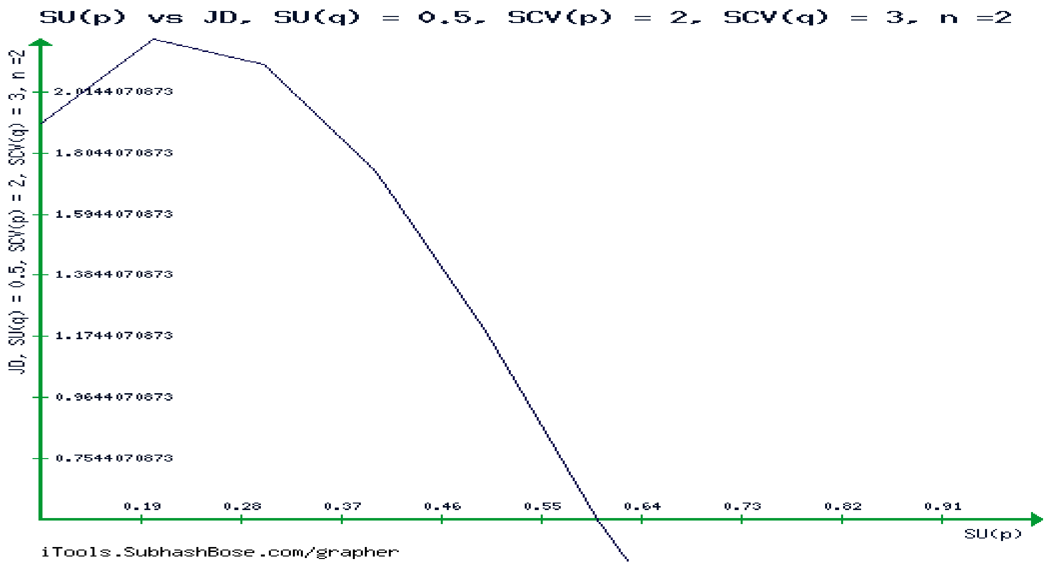

Figure 16.

As observed by Figure 16, for n =2, KD increases for permissible values of SU(p) and starts to decrease at SU(p) = as we go along. Then, it decreases when SU(p) = 0.4. Afterwards, KD decreases unsoundly , speeding rapidly to $-\infty$. In this physical interpretation, stability has a significant impact on the behaviour of KD. In principle, it is uncovered that the stability of the M/G/1 QM has a significant impact on the performance of KD.

5.2. Rényi Divergence of Stable M/G/1 QM

Theorem 5.2 The underlying queueing system satisfies:

$$RD(p,q) = D_R^{\gamma}(p||q)$$

$$= \begin{cases} \frac{1}{(\gamma-1)}(\gamma\, ln(1-\rho_p) + (1-\gamma)l\,n(1-\rho_q)), & n = 0 \\ ln\left[\left(\frac{\rho_p(1+\beta_q)(1-\rho_p)}{(1-\rho_q)(1+\beta_p)}\right)\right]^{\rho_p} + \frac{1}{(\gamma-1)} ln\left[\sum_{n=1}^{\infty} p(n)\left(\frac{\rho_p(1+\beta_p)(2+\rho_q(\beta_q-1))}{\rho_q(2+\rho_p(\beta_p-1))(1+\beta_q)}\right)\right]^{n(\gamma-1)}, & n > 0 \end{cases} \quad (5.5)$$

Proof

$$RD(p,q) = D_R^{\gamma}(p||q) = \frac{1}{(\gamma-1)} ln\,(\left((p(0))^{\gamma}(q(0)\right)^{1-\gamma}) = \frac{1}{(\gamma-1)} ln\,(((1-\rho_p)^{\gamma}(1-\rho_q)^{1-\gamma}) \quad (5.6)$$

It could be easily checked that $D_R^{\gamma}(p||q)$ of (5.6) that:

$$D_R^{\gamma}(p||q) = \frac{1}{(\gamma-1)}(\gamma\, ln(1-\rho_p) + (1-\gamma)l\,n(1-\rho_q))$$ as required.

It could be verified that for $n > 0$,

$$RD(p,q) = ln\left(\frac{(1-\rho_p)(1+\beta_q)}{(1-\rho_q)(1+\beta_p)}\right)^{\rho_p} + \frac{1}{(\gamma-1)} ln\left(\sum_{n=1}^{\infty}\left(\frac{\rho_p(1+\beta_p)(2+\rho_q(\beta_q-1))}{\rho_q(2+\rho_p(\beta_p-1)(1+\beta_q)}\right)^{n(\gamma-1)} p(n)\right) \quad \text{(c.f., (5.5))}$$

We are done.

As $\gamma \to 1$,

$$\lim_{\gamma\to1} D_R^{\gamma}(p||q) = \lim_{\gamma\to1}\frac{1}{\left(\sum_{n=0}^{\infty}(p(n))^{\gamma}(q(n))^{1-\gamma}\right)}\left(\sum_{n=0}^{\infty}[\gamma(p(n))^{\gamma-1}(q(n))^{1-\gamma} + (1-\gamma)(q(n))^{-\gamma}(p(n))^{\gamma}\right)$$

$$= \lim_{\gamma\to1}\frac{1}{\left(\sum_{n=0}^{\infty}(p(n))^{\gamma}(q(n))^{1-\gamma}\right)}\sum_{n=0}^{\infty}(1) = \infty$$

Data RÉNYI DIVERGENCE,RD

CASE ONE

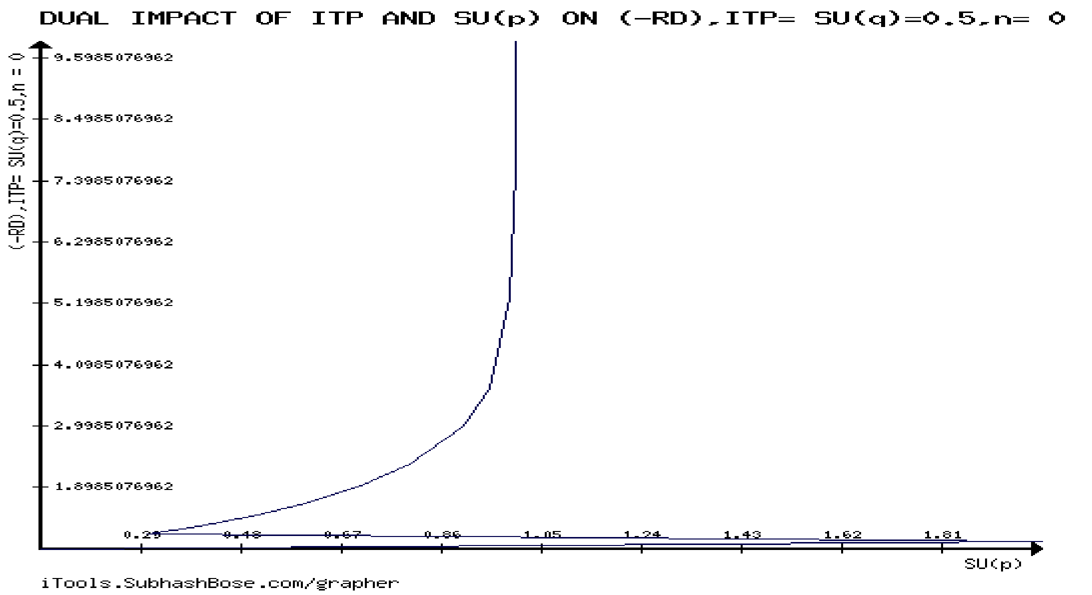

Figure 17.

It is shown by Figure 17,RD is drastically decreasing until SU(p) is greater than 1, RD becomes imaginary number, i.e., the instability of M/G/1 QM occurs , RD becomes imaginary!!!

It is observed that for $n = 0, \gamma \to 1 (shannonian\ phase), D_{R,}^{1}(p||q) = ln\left(\frac{1-\rho_p}{1-\rho_q}\right)$

CASE TWO

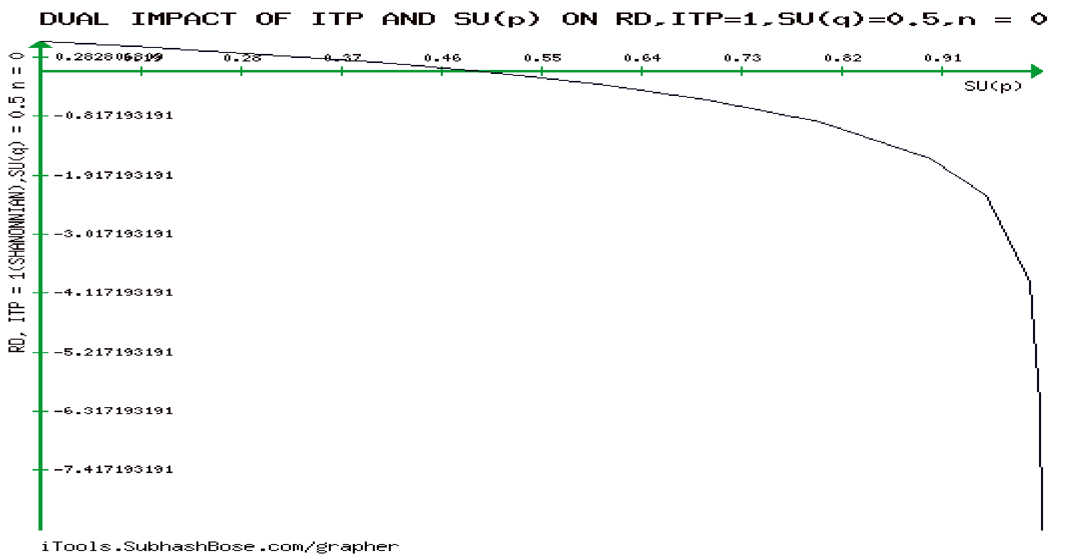

Figure 18.

Following Figure 18, The decreasability of RD is clear because of the dual impact of Su(p) and ITP on RD.

CASE TWO, RD, $n = 2$

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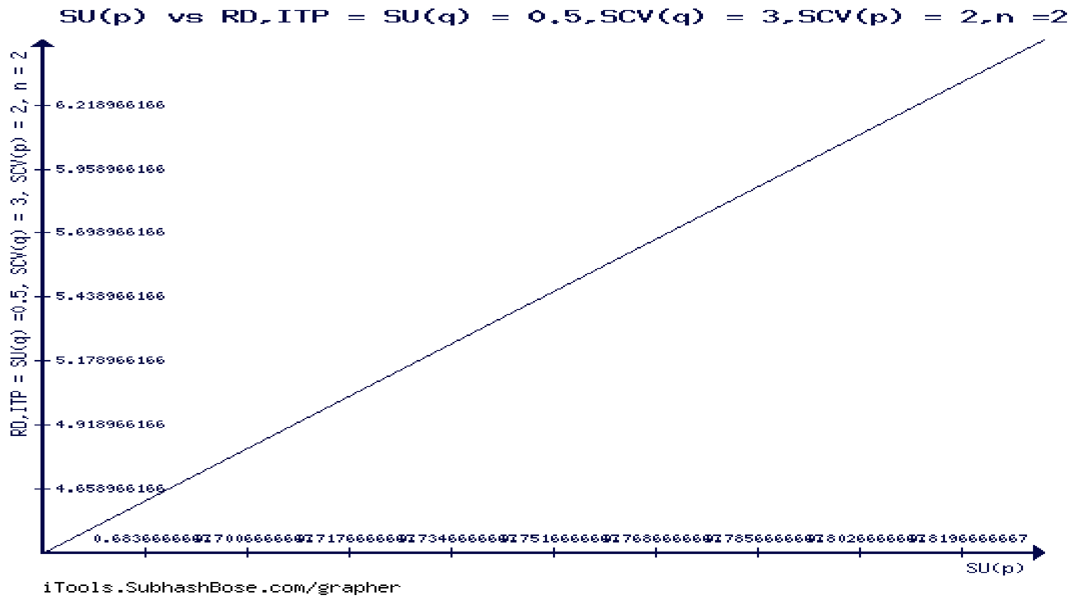

Figure 19.

As observed from figure 19, that RD increases rapidly by the increase of ρ_p. It is expected that RD approaches infinity as ρ_p approaches unity(i.e, M/G/1 QM is unstable at p). To show this, we can take the limit of (5.9) as ρ_p approaches unity. This directly implies that:

$D_R^{0.5}(p||\rho_q = 0.5,\ \beta_q = 3) \to\ ln\,0 + 3ln(3) - 2ln\,(p(1)(1-p(1))$, or

The devised corresponding absolute limiting value of RD is

$|D_R^{0.5}(p||\rho_q = 0.5,\ \beta_q = 3)| \to |\,ln\,0 + 3ln(3) - 2ln\,(p(1)(1-p(1))| = \infty$

5.3. sAB divergence, $\boldsymbol{D_{s,AB}^{\eta,\gamma}}(p||q)$ *of Stable M/G/1 QM*

Before going into details, we need to prove the following important lemma as it is needed in the proofs.

Lemma 5.3 $p(n)$ of (3.2) rewrites to

$$p(n) = \begin{cases} 1-\rho, & n = 0 \\ \frac{2\rho(1-\rho)}{(1+\beta)}\left(\frac{\rho^2(1+\beta)}{(1-\rho)L}\right)^n, & n > 0,\ \text{with } \beta = C_s^2 \end{cases} \quad (5.10)$$

where $L = \frac{\rho}{2}\left(1 + \frac{1+\rho C_s^2}{1-\rho}\right)$, $\rho = 1 - p(0)$ and $\beta = C_s^2$.

Proof For $n = 0$, it is immediate by (3.2)

As for $n > 0$ By the MQL formula, we have $L = \frac{\rho}{2}\left(1 + \frac{1+\rho\beta}{1-\rho}\right)$. Hence, $\left(2 + \rho(\beta - 1)\right) = \frac{2(1-\rho)L}{\rho}$. This implies :

$$\frac{2\rho\left(\frac{1+\rho\beta}{1-\rho} - 1\right)^{n-1}}{\left(\left(\frac{1+\rho\beta}{1-\rho} + 1\right)^n\right.} = \frac{2\rho\left(\frac{\rho(1+\beta)}{1-\rho}\right)^{n-1}}{\left(\left(\frac{(2+\rho(\beta-1))}{1-\rho}\right)^n\right.} = \frac{2\rho\left(\frac{\rho(1+\beta)}{1-\rho}\right)^{n-1}}{\left(\left(\frac{\frac{2(1-\rho)L}{\rho}}{1-\rho}\right)^n\right.} = \frac{2\rho(1-\rho)}{(1+\beta)}\left(\frac{\rho^2(1+\beta)}{(1-\rho)L}\right)^n \quad (5.11)$$

By (5.11) and (3.2), the proof follows.

Theorem 5.4 $sAB_{n=0}$ divergence vanishes,

$$D_{s,AB}^{\eta,\gamma}(p||q) = 0 \quad (5.12)$$

Proof For $n = 0, p(n) = 1 - \rho_p$, $q(n) = 1 - \rho_q$. hence, it follows by (2.10) that :

$$D_{s,AB}^{\gamma,\eta}(p||q) = 0 \quad \text{(c.f,(5.12))}$$

Theorem 5.5 For $n \neq 0$, sAB divergence is determined by

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$$D^{\eta,\gamma}_{s,AB}(p\,||\,q)= ln\left[\left(\sum_{n=1}^{\infty}\left(\frac{\rho_p^2(1+\beta_p)}{(1-\rho_p)L_p}\right)^{n(\gamma+\eta)}\right)^{\frac{1}{\eta(\eta+\gamma)}}\left(\sum_{n=1}^{\infty}\left(\frac{\rho_q^2(1+\beta_q)}{(1-\rho_q)L_q}\right)^{n(\gamma+\eta)}\right)^{\frac{1}{\gamma(\eta+\gamma)}}\left(\sum_{n=1}^{\infty}\left(\frac{\rho_p^2(1+\beta_p)}{(1-\rho_p)L_p}\right)^{n\gamma}\left(\frac{\rho_q^2(1+\beta_q)}{(1-\rho_q)L_q}\right)^{n\eta}\right)^{-\frac{1}{\gamma\eta}}\right] \quad (5.13)$$

for $(\gamma,\eta)\epsilon\mathbb{R}^2$ such that $\gamma\neq 0,\eta\neq 0\ and\ \gamma+\eta\neq 0$

Proof We have

$$p(n)=\frac{2\rho_p(1-\rho_p)}{(1+\beta_p)}\left(\frac{\rho_p^2(1+\beta_p)}{(1-\rho_p)L_p}\right)^n,\ q(n)=\frac{2\rho_q(1-\rho_q)}{(1+\beta_q)}\left(\frac{\rho_q^2(1+\beta_q)}{(1-\rho_q)L_q}\right)^n \quad \text{(c.f. (5.11) of Lemma 5.3)}$$

Following (2.10),

$$D^{\gamma,\eta}_{s,AB}(p||q)=\left[\frac{1}{\eta(\eta+\gamma)}ln\left(\sum_{n=1}^{\infty}\left(\frac{2\rho_p(1-\rho_p)}{(1+\beta_p)}\left(\frac{\rho_p^2(1+\beta_p)}{(1-\rho_p)L_p}\right)^n\right)^{\gamma+\eta}\right)+\frac{1}{\gamma(\eta+\gamma)}ln\left(\sum_{n=1}^{\infty}\left(\frac{2\rho_q(1-\rho_q)}{(1+\beta_q)}\left(\frac{\rho_q^2(1+\beta_q)}{(1-\rho_q)L_q}\right)^n\right)^{\gamma+\eta}\right)-\right.$$
$$\left.\frac{1}{\gamma\eta}ln\left(\sum_{n=1}^{\infty}\left(\frac{2\rho_p(1-\rho_p)}{(1+\beta_p)}\left(\frac{\rho_p^2(1+\beta_p)}{(1-\rho_p)L_p}\right)^n\right)^{\gamma}\left(\frac{2\rho_q(1-\rho_q)}{(1+\beta_q)}\left(\frac{\rho_q^2(1+\beta_q)}{(1-\rho_q)L_q}\right)^n\right)^{\eta}\right)\right] \quad (5.14)$$

It could be checked that RHS of (5.14) reduces after some lengthy computation to

$$D^{\gamma,\eta}_{s,AB}(p||q)= ln\left[\left(\sum_{n=1}^{\infty}\left(\frac{\rho_p^2(1+\beta_p)}{(1-\rho_p)L_p}\right)^{n(\gamma+\eta)}\right)^{\frac{1}{\eta(\eta+\gamma)}}\left(\sum_{n=1}^{\infty}\left(\frac{\rho_q^2(1+\beta_q)}{(1-\rho_q)L_q}\right)^{n(\gamma+\eta)}\right)^{\frac{1}{\gamma(\eta+\gamma)}}\left(\sum_{n=1}^{\infty}\left(\frac{\rho_p^2(1+\beta_p)}{(1-\rho_p)L_p}\right)^{n\gamma}\left(\frac{\rho_q^2(1+\beta_q)}{(1-\rho_q)L_q}\right)^{n\eta}\right)^{-\frac{1}{\gamma\eta}}\right] \quad \text{(c.f.,(5.13))}$$

Numerical experiment for $D^{\gamma,\eta}_{s,AB}(p||q)$

Data one:

Following (5.13), it can be verified after some manipulation, that for $\rho_q=0.5,\beta_p=\ \beta_q=2,n=2$ DITP = (γ,η)=(1,1),

$$D^{1,1}_{s,AB}(p||q)=\ ln\left[\frac{\sqrt{\left(\frac{(\frac{6\rho_p}{2+\rho_p})^2+(\frac{6\rho_p}{2+\rho_p})^4}{(\frac{12}{5})^2+(\frac{12}{5})^4}\right)}}{(\frac{72\rho_p}{5(2+\rho_p)})+(\frac{72\rho_p}{5(2+\rho_p)})^2}\right],$$

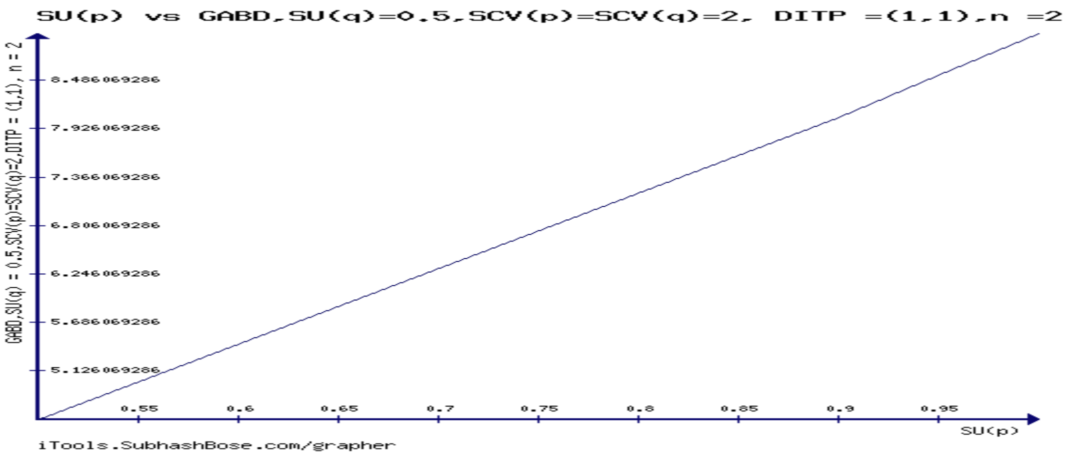

Figure 20.

The increasability of the GENERALIZED ALPHA BETA DIVERGENCE(GABD) as SU(p) increases is obvious from figure 20.

DATA TWO

After some lengthy computation, it could be verified that for the non-extensive information theoretic dual, DITP= $(\gamma,\eta)=(0.5,0.5),n=2$

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$$\left(D_{s,AB}^{(0.5,0.5)}(p||q)\right)_{(\rho_{q=0.5},\beta_{q=3},\beta_p=2)} = \mathbf{ln}\left(\frac{3\left(\sum_{n=1}^{2}\left(\frac{6\rho_p}{2+\rho_p}\right)^n\right)}{2\rho_p(1-\rho_p)\left(\sum_{n=1}^{2}\left(\frac{6\rho_p}{2+\rho_p}\right)^{\frac{n}{2}}\right)^4}\right)$$

As $\rho_p \to 1(instabilty\ phase$ of M/G/1 QM , $D_{AB}^{(0.5,0.5)}(p||q_{\rho_{q=0.5},\beta_{q=0.5}}) \to \infty$

6. Investigations of the Developability of the Stable M/G/1 QM ,RICCI CURVATURE (RCT) tENSOR and QT-IG Unifiers

6.1. Investigation of Developability of M/G/1 QM and Finding Its Ricci Curvature Tensor (RCT)

Theorem 6.1 The stable M/G/1 QM

i) Has a zero 0-Gaussian curvature, for which the stable M/G/1 QM would be developable.

ii) Has a non-zero Ricci Tensor

iii) Is non-developable minimal surface under Monge Technique, with a zero Mean Curvature

iv) Is developable under Angular Technique if and only if M/G/1 QM unstable

v) If the underlying QM is unstable, then the *Mean Curvature* is negative under Angular Technique.

The converse statement is not always true.

vi) The first principal curvature, K_1 under the Angular Technique satisfies the inequality

$$K_1 < 1 \tag{6.1}$$

vii) The second principal curvature, $K_2\ is\ \beta -$ dependent and is negative under the Angular Technique.

viiii) Under the Angular Technique, the second principal curvature, K_2 tends to zero as $\beta \to \infty$.

Proof

For i), we must prove :

$$K^{(\alpha)} = \frac{R_{1212}^{(\alpha)}}{det(g_{ij})} = 0 \tag{6.2}$$

It could be verified that, $R_{1212}^{(\alpha)} = 0$(6.3)

$det(g_{ij}) = -\frac{1}{(\beta+1)^2(1-\rho)^2} \neq 0$. Hence,

$K^{(\alpha)} = \frac{R_{1212}^{(\alpha)}}{det(g_{ij})} = 0$, which proves that the underlying QM is developable subject to $\alpha -$ Gaussian $curvature$.

ii) We must prove that:

$$R_{ik}^{(\alpha)} = R_{ijkl}^{(\alpha)} g^{jl} \text{ is non-zero.}$$

We have

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$$R^{(\alpha)}_{1212}g^{11} + R^{(\alpha)}_{1112}g^{12} + R^{(\alpha)}_{1211}g^{21} + R^{(\alpha)}_{1212}g^{22}$$

Engaging the same procedure as in (6.2), we have

$R^{(\alpha)}_{11} = R^{(\alpha)}_{12} = R^{(\alpha)}_{22} = 0$(6.3). $R^{(\alpha)}_{21} = -\frac{1-\alpha}{(1-\rho)^2}$, $R^{(0)}_{21} = -\frac{1}{(1-\rho)^2}$(6.5)

Hence, $R^{(\alpha)}_{21} \neq 0$ (6.6). The corresponding Ricci Curvature Tensor is given by

$$(\text{RCT}) = \begin{pmatrix} 0 & -\frac{1}{(1-\rho)^2} \\ 0 & 0 \end{pmatrix}$$

As $\rho \to 1$, $R^{(0)}_{21} \to -\infty$. The highlights the significant influence of instability in a specific type of the underlying QM, by providing supporting evidence on how RCT is significantly impacted by the stability analysis of the system. The also shows that ρ, represented by $R^{(0)}_{21}$,affects the behavior of RCT, and Figure 21 demonstrates that the stability phase of the M/G/1 QM causes RCT to decrease as ρ increases.

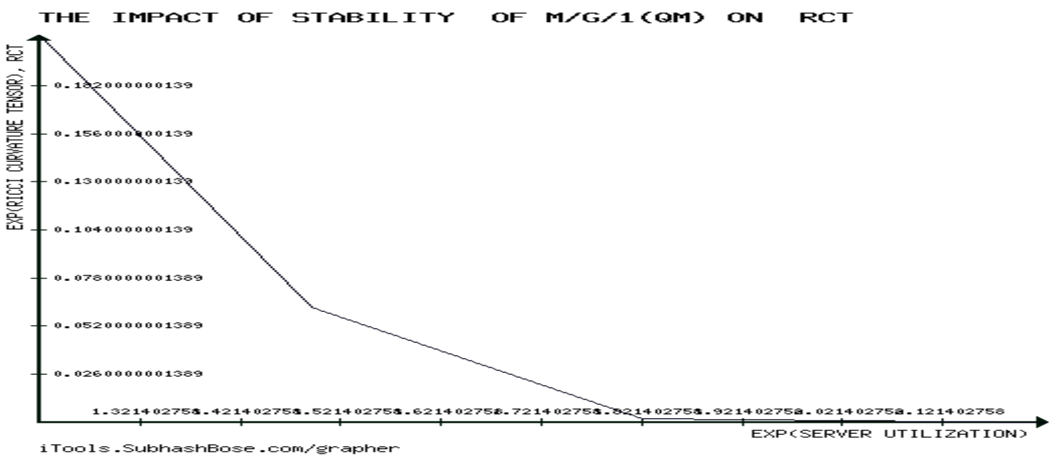

Figure 21.

whereas in Figure 22, RCT increasability in ρ is caused by the underlying QM's instability.

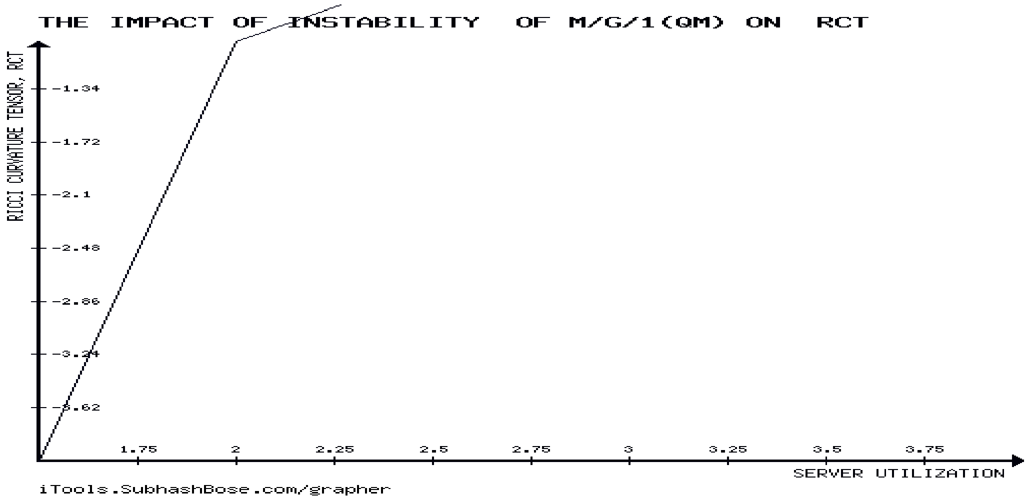

Figure 22.

iii) Following (3.14) and (3.15), it is clear that

Thus, we have $\partial_1 = \frac{1}{1-\rho}\partial_2 = \frac{1}{\beta+1}, \partial_{11} = \frac{1}{(1-\rho)^2}, \partial_1\partial_2 = \partial_2\partial_1 = 0, \partial_{22} = -\frac{1}{(\beta+1)^2}$ (3.15)

The Gaussian Curvature

$$K_G = \frac{LN-M^2}{EG-F^2} \quad \text{(c.f.,(2.23))}$$

$$E = (\partial_1)^2 = \left(\frac{1}{1-\rho}\right)^2, F = \partial_1\partial_2 = 0, G = (\partial_2)^2 = \frac{1}{(\beta+1)^2},$$

$$L = \partial_{11} = \frac{1}{(1-\rho)^2}, M = \partial_{12} = 0, N = \partial_{22} = -\frac{1}{(\beta+1)^2}$$

And the Mean Curvature is

$$H = \frac{1}{2}\left(\frac{LG-2MF+NE}{EG-F^2}\right) \quad \text{(c.f., (2.24))}$$

Therefore, it is obtained that

$$K_G = \frac{LN-M^2}{EG-F^2} = -\frac{\frac{1}{(1-\rho)^2}\frac{1}{(\beta+1)^2}}{\frac{1}{(1-\rho)^2}\frac{1}{(\beta+1)^2}} = -1, \rho \neq 1 (the\ underlying\ QM\ is\ stable), \beta \neq -1 \quad (6.7)$$

$(\beta = \ C_s^2\ is\ never\ negative\ by\ default)$

Since $K_G = -1$, it follows by the non- developability of the underlying QM and its minimal surface under Monge Technique.

The *Mean Curvature* is

$$H = \frac{1}{2}\left(\frac{LG-2MF+NE}{EG-F^2}\right) = \frac{\frac{1}{2}\left(\frac{1}{(1-\rho)^2}\frac{1}{(\beta+1)^2} - \frac{1}{(1-\rho)^2}\frac{1}{(\beta+1)^2}\right)}{\frac{1}{(1-\rho)^2}\frac{1}{(\beta+1)^2}} = 0 \quad (6.8)$$

Hence, iii) is done.

iv) Following the Angular Technique, it can be verified that the calculations of the principal curvatures K_1 and K_2 are determined by

$$K_1 = \frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}}, \ K_2 = \frac{-(1+\beta)}{(1+((1+\beta)^2)^{\frac{3}{2}}} \quad (6.9)$$

$$K_G = K_1K_2 = \frac{-(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}}\frac{(1+\beta)}{(1+((1+\beta)^2)^{\frac{3}{2}}} \quad (6.10)$$

and

$$H = \frac{1}{2}(K_1 + K_2) = \frac{1}{2}\left(\frac{(1-\rho)}{\left(1+((1-\rho))^2\right)^{\frac{3}{2}}} - \frac{(1+\beta)}{(1+((1+\beta)^2)^{\frac{3}{2}}}\right) \quad (6.11)$$

The axial rotator angle Ꞷ reads as

$$tan2Ꞷ = \frac{-2(\partial_1\partial_2)}{(\partial_{11} - \partial_{22})} = 0 \quad (6.12)$$

This implies, $Ꞷ = 0, 2\pi, 4\pi, ...$

It appears from (6.10), that $K_G = 0\ (equivalently, the\ underlying\ QM\ is\ developable)$ if and only if

$$\frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}}\frac{(1+\beta)}{(1+((1+\beta)^2)^{\frac{3}{2}}} = 0 \quad (6.13)$$

This implies that:

either $(1-\rho) = 0$ (equivalently, $\rho = 1$) or $(1+\beta) = 0$ (6.14)

The second possibility ($(1+\beta) = 0$) generates a contradiction as β is never negative.

Moreover,

$$lim_{\beta\to\infty}K_G = -\frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}} lim_{\beta\to\infty}\frac{(1+\beta)}{(1+((1+\beta)^2)^{\frac{3}{2}}} = 0 \quad (6.15)$$

Linking the findings of (6.13) and (6.15) completes the proof of iv).

As for v), it has been obtained that

$$H = \frac{1}{2}(K_1 + K_2) = \frac{1}{2}\left(\frac{(1-\rho)}{\left(1+((1-\rho))^2\right)^{\frac{3}{2}}} - \frac{(1+\beta)}{(1+((1+\beta)^2)^{\frac{3}{2}}}\right) \quad (6.11)$$

This directly implies

$$H < \frac{1}{2}\frac{(1-\rho)}{\left(1+((1-\rho))^2\right)^{\frac{3}{2}}} \quad (6.16)$$

It is clear that $1 + \left((1-\rho)\right)^2 > 0$ holds for all the possible values of ρ. Consequently, if $(1-\rho) = 0$ or $\rho = 1$ (equivalently, the underlying QM is unstable), it follows that $H < 0$

To prove the necessity condition, assume that $H < 0$. This generates two possibilities:

The first possibility, $\frac{1}{2}\frac{(1-\rho)}{\left(1+((1-\rho))^2\right)^{\frac{3}{2}}} < 0$. This implies $\rho > 1$. Hence, M/G/1 QM is unstable.

The second possibility, $\frac{1}{2}\frac{(1-\rho)}{\left(1+((1-\rho))^2\right)^{\frac{3}{2}}} > 0$. This implies $\rho < 1$. Hence, M/G/1 QM is stable.

This justifies that the converse statement is not always true. The proof of v) is complete.

To show vi), we have

$$K_1 = \frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}} \text{ (c.f., (6.9))}$$

Since the underlying M/G/1 QM is assumed to be stable. Hence, $\rho \in (0,1)$. Thus, we have

$1 > (1-\rho)^2 > 0$ or $2 > (1-\rho)^2 + 1 > 1$. Therefore, $2^{\frac{3}{2}} > (1 + ((1-\rho))^2)^{\frac{3}{2}} > 1$. Consequently,

$$\frac{1}{2^{\frac{3}{2}}} < \frac{1}{(1+((1-\rho))^2)^{\frac{3}{2}}} < 1. \text{ This implies } \frac{(1-\rho)}{2^{\frac{3}{2}}} < \frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}} < (1-\rho) < 1 \quad (6.17)$$

By (6.17), it holds that $K_1 < 1$.

vii) we have by (6.9), $K_2 = \frac{-(1+\beta)}{(1+((1+\beta)^2)^{\frac{3}{2}}}$, which is of course a $\beta -$ dependent functionThe stability of M/G/1 QMenforces the condition $\beta > 1$to hold. The negativity of $K_2$is clear.

viii) Immediate from (6.15).

6.2. Revealing Novel QT-IG Unifiers and Discovering Their Algebraic Structures

Throughout this section, the following novel unifiers between both queueueing theoretic and information geometric structures of the stable M/G/ 1 QM are established by the following two unifiers,

$$\varphi_1(\rho) = K_1 = \frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}} \quad (6.18)$$

$$\varphi_2(\beta) = K_2 = \frac{-(1+\beta)}{(1+((1+\beta))^2)^{\frac{3}{2}}} \quad (6.19)$$

Theorem 6.2. For the above devised unifiers (c.f., (6.18) and (6.19)), it holds that

i) φ_1 is a well-defined function

$$\text{ii)}\varphi_1 = \begin{cases} \textit{lies in the interval } (0,1), & \rho \in (0,1) \\ 0, & \rho = 1 \\ < 0, & \rho > 1 \end{cases} \quad (6.20)$$

iii)φ_1 is one-to –one.

iv)φ_1 is onto

v)$\varphi_1$is bijection, with animaginary inverse φ_1^{-1}determined by

$$\varphi_1^{-1}(\rho) = 1 \mp ((z-1) + \frac{(3+\frac{1}{\rho^2})}{z})^{\frac{1}{2}} \quad (6.21)$$

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where$z = \sqrt[3]{-\frac{1}{2\rho^2} \pm i\sqrt{\frac{1}{12\rho^4} + 1 + \frac{1}{2\rho^2} + \frac{1}{27\rho^6}}}, i = \sqrt{-1}$

vi) φ_2 is a well-defined function

$$\text{vii)}\varphi_2 = \begin{cases} < -\frac{2}{5^{\frac{3}{2}}}, & \beta \in (0,1) \\ -\frac{2}{5^{\frac{3}{2}}}, & \beta = 1 \\ > -\frac{2}{5^{\frac{3}{2}}}, & \beta > 1 \end{cases} \tag{6.22}$$

viii)φ_2 isone-to –one

x)φ_2 is onto

xi)$\varphi_2$is bijection, with an imaginary inverse φ_2^{-1}(i.e., a complex number)determined by

$$\varphi_2^{-1}(\beta) = -1 \mp ((z-1) + \frac{(3+\frac{1}{\beta^2})}{z})^{\frac{1}{2}} \tag{6.23}$$

where $z = \sqrt[3]{-\frac{1}{2\beta^2} \pm i\sqrt{\frac{1}{12\beta^4} + 1 + \frac{1}{2\beta^2} + \frac{1}{27\beta^6}}}, i = \sqrt{-1}$

xii) The underlying QM has inverse of ρ unifier, namely, φ_1^{-1} satisfies

$$|1 - \varphi_1^{-1}(\rho)| < 5 + \frac{(1+(\frac{101}{54})^{\frac{1}{6}}}{\rho^2} \tag{6.24}$$

xiii) The underlying of inverse of β unifier, namely, φ_2^{-1} satisfies

$$|1 - \varphi_2^{-1}(\beta)| < (1 + \left(\frac{101}{54}\right)^{\frac{1}{6}} + \frac{1}{\left(\frac{101}{54}\right)^{\frac{1}{6}}} + \frac{3\beta^2}{\left(\frac{101}{54}\right)^{\frac{1}{6}}}) \tag{6.25}$$

xiv)The increasabilty and decreasability of φ_1 in $\rho$are undecidable.

xv)$\varphi_2$is forever increasing in$\beta$and is never decreasing in β

To prove i), it is enough to show that for all $\rho_1, \rho_2 \in (0,1)$ such that $\varphi_1(\rho_1) = \varphi_1(\rho_2)$, then $\rho_1$andρ_2 should never be distinct.

Let $\varphi_1(\rho_1) = \varphi_1(\rho_2)$. After some lengthy mathematical steps, (6.26) reduces to

$$(\rho_1 - \rho_2)(\rho_1 + \rho_2 - 2)[((1-\rho_1)^2(1-\rho_2)^2((1-\rho_1)^2 + (1-\rho_2)^2 - 2))] = 0 \tag{6.28}$$

Equation (6.28) generates three possible cases:

Case 1:$\rho_1 - \rho_2 = 0.$ Hence$\rho_1 = \rho_2$ (contradiction to $\rho_1 \neq \rho_2$) (6.29)

Case 2:$\rho_1 + \rho_2 - 2 = 0.$ Hence$\rho_1 + \rho_2 = 2$ (contradiction, since M/G/1 is a stable QM,$\rho_1, \rho_2 \in (0,1)$)

(6.30)

Case 3:$[((1-\rho_1)^2(1-\rho_2)^2((1-\rho_1)^2 + (1-\rho_2)^2 - 2))] = 0$ (6.31)

By$\rho_1, \rho_2 \in (0,1)$, following mathematical analysis it holds that

Therefore, $[((1-\rho_1)^2(1-\rho_2)^2((1-\rho_1)^2 + (1-\rho_2)^2 - 2))] < 0$, which contradicts (6.29).

Based on the above analysis and by (6.29), i) follows.

ii) we have by (6.18), $\varphi_1(\rho) = K_1 = \frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}}$. Since $\rho \in (0,1)$, it could be verified that

$2 > 1 + ((1-\rho))^2 > 0$. Hence, $\frac{1}{(1+((1-\rho))^2)^{\frac{3}{2}}} \in (0,\frac{1}{2})$. Therefore, $0 < \varphi_1(\rho) = \frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}} < \frac{1}{2} < 1$ (6.33)

The case $\varphi_1(1) = 0$ is clear. Also, for $\rho > 1$, it is immediate that $\varphi_1(\rho) < 0$. This completes the proof of ii).

iii) It suffices to show that for all $\rho_1, \rho_2 \in (0,1)$ such that $\varphi_1(\rho_1) = \varphi_1(\rho_2)$, then $\rho_1 = \rho_2$ holds. The proof is clearly immediate from (6.29).

iv) From the definition, $\varphi_1(\rho) = \frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}}$. Every $\frac{(1-\rho)}{(1+((1-\rho))^2)^{\frac{3}{2}}}$ is characterized by ρ. This clearly proves the surjectivity of φ_1. Hence, iv) follows.

v) Clearly, φ_1 is a bijection. To calculate the inverse of φ_1, namelyφ_1^{-1}. Define $\varphi_1(\rho) = y$. Hence, $\frac{(1+((1-\rho))^2)^3}{y^2} = (1-\rho)^2$. Let $w = (1-\rho)^2$. Then, we have the cubic equation:

$$w^3 + 3w^2 + \left(3 - \frac{1}{y^2}\right) w + 1 = 0 \tag{6.34}$$

Following the method for solving cubic equations (c.f., definition 2.14), we have

$$a^* = 1, b^* = 3, c^* = \left(3 - \frac{1}{y^2}\right), d^* = 1 \tag{6.35}$$

The solution of (6.34) is characterized arbitrarily by

$$w = r - 1 \tag{6.36}$$

$$\gamma = z - \frac{\varepsilon_3}{z} , \tag{6.37}$$

$$z = \sqrt[3]{(-\frac{\varepsilon_1}{2}) \pm \sqrt{\varepsilon_2}}, \tag{6.38}$$

$$\varepsilon_1 = \frac{1}{y^2}, \tag{6.39}$$

The discriminant of the cubic equation

$$\varepsilon_2 = \frac{1}{4y^2} + \frac{(\varepsilon_3)^3}{27}, \tag{6.40}$$

ε_3 is given by $\varepsilon_3 = -3 - \frac{1}{y^2}$(6.41)

After some lengthy calculations, it can be verified that

$$\rho = 1 \mp ((z-1) + \frac{(3+\frac{1}{y^2})}{z})^{\frac{1}{2}} = \varphi_1^{-1}(y) \tag{6.42}$$

where $z = \sqrt[3]{-\frac{1}{2y^2} \pm i\sqrt{\frac{1}{12y^4} + 1 + \frac{1}{2y^2} + \frac{1}{27y^6}}}, i = \sqrt{-1}$

By (6.20), we have

$$1 \mp ((z-1) + \frac{(3+\frac{1}{\rho^2})}{z})^{\frac{1}{2}} = \varphi_1^{-1}(\rho) \tag{6.43}$$

where $$z = \sqrt[3]{-\frac{1}{2\rho^2} \pm i\sqrt{\frac{1}{12\rho^4} + 1 + \frac{1}{2\rho^2} + \frac{1}{27\rho^6}}}, i = \sqrt{-1}$$

This completes the proof.

To prove vi), it is enough to show that for all $\beta_1, \beta_2 \in (1, \infty)$ such that $\varphi_2(\beta_1) = \varphi_2(\beta_2)$, then β_1 and β_2 should never be distinct.

Let $\varphi_2(\beta_1) = \varphi_2(\beta_2)$. Then $\frac{(1+\beta_1)}{(1+((1+\beta_1))^2)^{\frac{3}{2}}} = \frac{(1+\beta_2)}{(1+((1+\beta_2))^2)^{\frac{3}{2}}}$, such that $\beta_1, \beta_2 \in (1, \infty), \beta_1 \neq \beta_2$. This implies

Hence,

$$(\beta_2 - \beta_1)(\beta_1 + \beta_2 + 2)[(1+\beta_1)^2(1+\beta_2)^2((1+\beta_1)^2 + (1+\beta_2)^2 + 2))] = 0 \quad (6.46)$$

Equation (6.46) generates three possible cases:

Case 1: $\beta_2 - \beta_1 = 0$. Hence $\rho_1 = \rho_2$ (contradiction to $\beta_1 \neq \beta_2$) (6.47)

Case 2: $\beta_1 + \beta_2 + 2 = 0$. Hence $\beta_1 + \beta_2 = -2$ (contradiction, since M/G/1 is a stable QM, $\beta_1, \beta_2 \in (1, \infty)$) (6.48)

Case 3: $[(1+\beta_1)^2(1+\beta_2)^2((1+\beta_1)^2 + (1+\beta_2)^2 + 2))] = 0$ (6.49)

By $\beta_1, \beta_2 \in (1, \infty)$, following mathematical analysis,

$[(1+\beta_1)^2(1+\beta_2)^2((1+\beta_1)^2 + (1+\beta_2)^2 + 2))] > 96$, which directly implies by (6.47)

$$0 > 96\ (contradiction) \quad (6.51)$$

Based on the above analysis, vi) follows.

vii) It suffices to show that for all $\beta_1, \beta_2 \in (1, \infty)$ such that $\varphi_2(\beta_1) = \varphi_2(\beta_2)$, then $\beta_1 = \beta_2$ holds. The proof is clearly immediate from (6.47).

viii) Since M/G/1 QM is stable, the condition $\beta > 1$ and a similar proof to that in ii), viii) follows.

The proof of x) is analogous to iii).

xi) We have $\varphi_1^{-1}(\rho) = 1 \mp \left((z-1) + \frac{\left(3+\frac{1}{\rho^2}\right)}{z}\right)^{\frac{1}{2}}$, for $\rho \in (0,1)$, (c.f., (6.21)).

Hence,

$$|1 - \varphi_1^{-1}(\rho)|^2 = \left|(z-1) + \frac{\left(3+\frac{1}{\rho^2}\right)}{z}\right| \leq |z| + 1 + \frac{\left(3+\frac{1}{\rho^2}\right)}{|z|} \quad (6.52)$$

By (6.43),

$$z = \sqrt[3]{-\frac{1}{2\rho^2} \pm i\sqrt{\frac{1}{12\rho^4} + 1 + \frac{1}{2\rho^2} + \frac{1}{27\rho^6}}}$$

This implies

$$|z|^6 = \left(\frac{1}{4\rho^4} + \frac{1}{12\rho^4} + 1 + \frac{1}{2\rho^2} + \frac{1}{27\rho^6}\right) < \left(\frac{1}{4\rho^6} + \frac{1}{12\rho^6} + \frac{1}{\rho^6} + \frac{1}{2\rho^6} + \frac{1}{27\rho^6}\right) = \frac{101}{54\rho^6}, or\ |z| < \frac{(\frac{101}{54})^{\frac{1}{6}}}{\rho} < \frac{(\frac{101}{54})^{\frac{1}{6}}}{\rho^2} \quad (6.53)$$

Moreover, by the above step, it is clear that

$$|z|^6 = \left(\frac{1}{4\rho^4} + \frac{1}{12\rho^4} + 1 + \frac{1}{2\rho^2} + \frac{1}{27\rho^6}\right) > 1\,, which\ directly\ implies\ |z| > 1(equivaently, \frac{1}{|z|} < 1) \quad (6.54)$$

Thus, it is obtained that

$$|1 - \varphi_1^{-1}(\rho)|^2 < \frac{(\frac{101}{54})^{\frac{1}{6}}}{\rho^2} + 2 + \left(3 + \frac{1}{\rho^2}\right) = 5 + \frac{(1 + (\frac{101}{54})^{\frac{1}{6}}}{\rho^2}$$

This completes the proof.

xii) The stability of the underlying QM implies $\beta > 1$. We have

$\varphi_2^{-1}(\beta) = -1 \mp ((z-1) + \frac{(3+\frac{1}{\beta^2})}{z})^{\frac{1}{2}}$ (c.f., (6.23))

Hence,

$$|1 + \varphi_2^{-1}(\beta)|^2 = \left|(z-1) + \frac{(3+\frac{1}{\beta^2})}{z}\right| \leq |z| + 1 + \frac{\left(3+\frac{1}{\beta^2}\right)}{|z|} \quad (6.55)$$

Hence, $z = \sqrt[3]{-\frac{1}{2\beta^2} \pm i\sqrt{\frac{1}{12\beta^4} + 1 + \frac{1}{2\beta^2} + \frac{1}{27\beta^6}}}, i = \sqrt{-1}$ (c.f., (6.23)). So, $|z| < (\frac{101}{54})^{\frac{1}{6}}$ (6.56).

Also, it can be verified that $\frac{1}{|z|} < \frac{\beta^2}{(\frac{101}{54})^{\frac{1}{6}}}$ (6.57). Consequently, xiii) will follows.

xiv) we have $\frac{\partial\varphi_1}{\partial\rho} = \frac{(2(1-\rho)^2-1)}{(1+(1-\rho)^2)^{\frac{5}{2}}}$. Hence, $\frac{\partial\varphi_1}{\partial\rho} > 0(< 0)$ if and only if $(1-\rho)^2 > \frac{1}{2}(< \frac{1}{2})$. By (PT) 2.15, φ_1 is increasing(decreasing) in ρ if and only if $(1-\rho)^2 > \frac{1}{2}(< \frac{1}{2})$. According to stability of M/G/1 QM, $\rho \in (0,1)$. Hence, it follows that $(1-\rho)^2 \in (0,1)$.Consequently, xiv) follows:

$$\varphi_2(\beta) = K_2 = \frac{-(1+\beta)}{(1+(1+\beta)^2)^{\frac{3}{2}}} \quad (6.19)$$

xv) We have $\frac{\partial\varphi_2}{\partial\beta} = \frac{2(1+\beta)^2-1}{(1+(1+\beta)^2)^{\frac{5}{2}}}$. Hence, $\frac{\partial\varphi_2}{\partial\beta} > 0(< 0)$ if and only if $(1+\beta)^2 > \frac{1}{2}(< \frac{1}{2})$. By (PT) 2.15, φ_2 is increasing(decreasing) in ρ if and only if $(1+\beta)^2 > \frac{1}{2}(< \frac{1}{2})$. According to stability of M/G/1 QM, $\beta \in (1, \infty)$. Consequently, xv) follows.

7. $e^{FIM(M/G/1)}$ &. Impact of Stability of M/G/1 QM on the Stability of Fim

7.1. Exponential Matrix of FIM

Theorem 7.1 $e^{FIM(\boldsymbol{M/G/1})}$ solves $\frac{dx}{dt}$ = Ax.

Proof

It is shown that $[g_{ij}]$ of Theorem (3.1) is:

$$[g_{ij}] = \begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix} \quad (7.1)$$

We write

$[g_{ij}] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a = \left(\frac{1}{(1-\rho)^2}\right), b = \frac{-1}{(\beta+1)^2}$(7.2)

Thus,

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$\Phi(\delta) = \det\begin{pmatrix} a-\delta & 0 \\ 0 & b-\delta \end{pmatrix} = 0$.

Therefore,

$\delta^2 - (a+b)\delta + ab$ =0, so $\delta_{1,2} = a, b.$

Hence,

$$D = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \quad (7.3)$$

For $\delta_{1,2} = a, b$

Hence, T $= T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (7.4)

Thus, $e^{FIM(\boldsymbol{M/G/1})}$ reads as:

$$e^{FIM(\boldsymbol{M/G/1})} = Te^{D}T^{-1} = \begin{pmatrix} e^{a} & 0 \\ 0 & e^{b} \end{pmatrix} \quad (7.5)$$

This proves that IME of the underlying QM solves:

$$\frac{dx}{dt} = \mathrm{A}x \quad (7.6)$$

7.2. Impact of Stability of $\boldsymbol{M/G/1}$ *QM on the stability of FIM*

Theorem 7.2 The stability of FIM of the underlying QM holds $\Leftrightarrow$ the underlying QM is unstable.

Proof Following theorem 2.16, it suffices to show that:

FIM's eigen values of FIM of the underlying QM are negative real numbers $\Leftrightarrow$ the instability of the underlying QM is satisfied.

It holds by (7.4) of theorem 7.1 that the eigen values of FIM are $\delta_{1,2} = a, b,$ $, a = \left(\frac{1}{(1-\rho)^2}\right), b = \frac{-1}{(\beta+1)^2}$. clearly, $b = \frac{-1}{(\beta+1)^2} < 0$. Therefore, the proof would be immediate if we proved that

$$a = \left(\frac{1}{(1-\rho)^2}\right) < 0 \Leftrightarrow \text{the instability of the underlying QM is satisfied} \quad (7.6)$$

We first prove the necessity condition, $a = \left(\frac{1}{(1-\rho)^2}\right) < 0 \Rightarrow$ M/G/1 QM is unstable. Assume that FIM is stable , then $a = \left(\frac{1}{(1-\rho)^2}\right) < 0$ follws. This implies $\frac{1}{(1-\rho)} = im, i = \sqrt{(-1)}, m$ is any real number. Hence, $(1-\rho) = -im$. Consequently, $\rho = 1 + im, |\rho| = \sqrt{1+m^2} > 1$. In other words, M/G/1 QM is unstable.

To prove sufficiency, let M/G/1 QM be unstable. Then, $\rho > 1$. This directly implies $, |\rho| > 1$. This rewrite ρ to be of the form $\rho = 1 + im, m$ is any real number. Clearly, this implies , $(1-\rho) = -im$, or $\frac{1}{(1-\rho)} = im$. Thus, it holds that $a = \left(\frac{1}{(1-\rho)^2}\right) = -\frac{1}{m^2} < 0$, which proves that FIM is stable.

7.3. Revealing Queue-Fisher Information Matrix Unifiers, (QFIMU)

Throughout this section, we introduce QFIMU, to be devised by the function:

$$\eta(\rho,\beta) = [g_{ij}]\begin{pmatrix} \rho \\ -\beta \end{pmatrix} = \begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix}\begin{pmatrix} \rho \\ -\beta \end{pmatrix} = \begin{pmatrix} \frac{\rho}{(1-\rho)^2} \\ \frac{-\beta}{(\beta+1)^2} \end{pmatrix} \quad (7.7)$$

where $\rho = 1 - p(0)$and $\beta = C_s^2$.

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Theorem 7.3 The functionη (c.f., (7.7)) satisfies the following:

i)η is ell defined.

ii) η is One-to-One.

iii) η is surjective.

iv)η has a unique inverse, η^{-1} determined which is characterized by

$$\left|\eta^{-1}(\rho) - \left(1 + \frac{1}{2\rho}\right)\right|^2 < \frac{2}{\rho^2} \tag{7.8}$$

and

$$\left|\eta^{-1}(\beta) + \left(1 + \frac{1}{2\beta}\right)\right|^2 < 2 \tag{7.9}$$

Proof

To prove i), it suffices to show that for all (ρ_1, β_1), (ρ_2, β_2) such that $\rho_1 \neq \rho_2$ and $\beta_1 \neq \beta_2$ and

$$\eta(\rho_1, \beta_1) = \eta(\rho_2, \beta_2) \tag{7.10}$$

By (7.10), we have

$$\frac{\rho_1}{(1-\rho_1)^2} = \frac{\rho_2}{(1-\rho_2)^2} \tag{7.11}$$

and

$$\frac{\beta_1}{(1+\beta_1)^2} = \frac{\beta_2}{(1+\beta_2)^2} \tag{7.12}$$

By (7.11), one gets:

$$(\rho_1 - \rho_2)(1 + \rho_1\rho_2) = 0 \tag{7.13}$$

(7.13) implies either $\rho_1 = \rho_2$ or $\rho_2 = \frac{-1}{\rho_1}$ (contradiction, since for example if $\rho_1 = 2$ implies that$\rho_2 = -0.5 \notin (0,1)$, i.e., enforcing instability of the underlyin stableM/G/1 QM). Therefore, the ρ branch of the QIFMU is well-defined.

Following (7.12), we have

$$(\beta_1 - \beta_2)(1 - \beta_1\beta_2) = 0 \tag{7.14}$$

(7.14) implies either $\beta_1 = \beta_2$ or $\beta_2 = \frac{1}{\beta_1}$ (contradiction, since for example if $\beta_1 = 2$ implies that$\beta_2 = 0.5 \notin (1, \infty)$, i.e., enforcing instability of the underlyin stableM/G/1 QM). Therefore, the β branch of the QIFMU is well-defined. This completes the proof of i).

As for ii), it suffices to show that all (ρ_1, β_1), (ρ_2, β_2) such that:

$$\eta(\rho_1, \beta_1) = \eta(\rho_2, \beta_2) \text{implies} (\rho_1, \beta_1) = (\rho_2, \beta_2) \tag{7.15}$$

Clearly, by (7.13) and (7.14), (7.15) is satisfied. Hence, ii) follows.

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Clearly, by (7.7), both of $\frac{\rho}{(1-\rho)^2}$ and $\frac{-\beta}{(\beta+1)^2}$ are uniquely characterized by ρ and β respectively. Thus, iii) holds.

iv) To compute η^{-1}, assume that thee exists x , y such that

$$\eta(\rho,\beta) = \begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix}\begin{pmatrix} \rho \\ -\beta \end{pmatrix} = \begin{pmatrix} \frac{\rho}{(1-\rho)^2} \\ \frac{-\beta}{(\beta+1)^2} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (7.16)$$

Therefore,

$$\frac{\rho}{(1-\rho)^2} = x \quad (7.17)$$

And

$$\frac{-\beta}{(\beta+1)^2} = y \quad (7.18)$$

Following (7.15), one gets

$$\rho = \frac{(2+\frac{1}{x})\pm\frac{1}{x}\surd(1+4x^2)}{2} \quad (7.19)$$

Using (7.16) and following a similar argument as in (7.17), we have

$$\beta = \frac{-(2+\frac{1}{y})\pm\frac{1}{y}\surd(1+4y^2)}{2} \quad (7.20)$$

Based on (7.19) and (7.20), it is determined for both ρ and branches of η^{-1} would respectively satisfy that

$$|\eta^{-1}(\rho) - (1+\frac{1}{2\rho})\,|^2 = \frac{(1+4\rho^2)}{4\rho^2} = 1+\frac{1}{4\rho^2} < \frac{1}{\rho^2}+\frac{1}{\rho^2} = \frac{2}{\rho^2} \text{ (since } \rho \in (0,1)) \quad \text{(c.f., (7.8)}$$

Following similar argument, it could be shown that

$$|\eta^{-1}(\beta) + (1+\frac{1}{2\beta})\,|^2 = \frac{(1+4\beta^2)}{4\beta^2} = 1+\frac{1}{4\beta^2} < 1+1 = 2 \text{ (since } \beta \in (1,\infty)) \quad \text{(c.f., (7.9)}$$

This completes the proof of our theorem.

8. RICCI SCALAR, $\mathcal{R}$, CURVATURE OF SPACE TIME(EINESTEIN TENSOR)$\wp$, STRESS ENERGY TENSOR, Ω, THE CORRESPONDING THRESHOLD THEOREMS FOR THE UNDERLYING CURVATURES AND THE DUAL QUEUEING IMPACT ON THE EXISTENCE OF THE INVERSE FISHER INFORMATION MATRIX (IFIM)

Theorem 8.1 The underlying QM satisfies:

i)The Ricci scalar subject to Angular Technique, $\mathcal{R}_{AT}$ is determined by

$$\mathcal{R}_{AT} = \frac{2(\rho-1)(1+\beta)}{(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}} \quad (8.1)$$

where $\rho = 1 - p(0)$and C_s^2 define server utilization and Squared coefficient of variations respectively.

ii) $\mathcal{R}_{AT} \to 0$ if and only if $\rho = 1$

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iii) $\mathcal{R}_{AT} \to 0$ if and only if $\beta \to \infty$

for all $\rho \neq 1$(equivalently, whether the undelying QM is either stable or unstable)

iv) M/G/1 QM is unstable $\Leftrightarrow$

There exists a small enough positive number ϵ, with $\epsilon \to 0$ such that $A_{curved}(\epsilon), A_{flat}(\epsilon)$ (c.f., (2.45))

must satisfy:

$$A_{curved}(\epsilon) \gtrsim A_{flat}(\epsilon) \tag{8.2}$$

v)The Spacetime curvature(Einestein Tensor) subject to Angular Technique, $\wp_{AT}$ is determined by

$$\wp_{AT} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{8.3}$$

where the components G_{11}, G_{12}, G_{21} and G_{22} are determined by

$$G_{11} = \frac{(1+\beta)}{(1-\rho)(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}} \tag{8.4}$$

$$G_{12} = 0 \tag{8.5}$$

$$G_{21} = \frac{(\alpha-1)}{(1-\rho)^2} \tag{8.6}$$

where α is the curvature parameter (c.f., definition (2.8))

$$G_{22} = \frac{(\rho-1)}{(1+\beta)(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}} \tag{8.7}$$

vi)The stress-energy tensor ϖ is devised by

$$\Omega = \begin{pmatrix} \varpi_{11} & \varpi_{12} \\ \varpi_{21} & \varpi_{22} \end{pmatrix} \tag{8.8}$$

where the components $\varpi_{11}, \varpi_{12}, \varpi_{21}$ and ϖ_{22} are determined by

$$\varpi_{11} = \frac{c^4 G_{11}}{8\pi g} \tag{8.9}$$

$$\varpi_{12} = 0 \tag{8.10}$$

$$\varpi_{21} = \frac{c^4 G_{21}}{8\pi g} \tag{8.11}$$

$$\varpi_{22} = \frac{c^4 G_{22}}{8\pi g} \tag{8.12}$$

where g is the universal gravitational constant, c is the speed of light

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vii)$\mathcal{R}_{AT}$ =

$$\begin{cases} \text{increasing in } \rho, & \rho = 1 + \frac{m}{\sqrt{2}}(instability\ phase) \\ \text{increasing in } \rho, & \rho = 1 - \frac{m}{\sqrt{2}}(stability\ phase) \end{cases} \quad (8.13)$$

provided that $m > 1$

viii)$\mathcal{R}_{AT}$ =

$$\begin{cases} \text{decreasing in } \rho, & \rho = 1 + \frac{m}{\sqrt{2}}(instability\ phase) \\ \text{decreasing in } \rho, & \rho = 1 - \frac{m}{\sqrt{2}}(instability\ phase) \end{cases} \quad (8.14)$$

provided that $1 > m > 0$

x)$\mathcal{R}_{AT}$ =

$$\begin{cases} \text{decreasing in } \beta, & \text{the underlying QM is stable} \\ \text{increasing in } \beta, & \text{the underlying QM is unstable} \end{cases} \quad (8.15)$$

xi)G_{11} =

$$\begin{cases} \text{increasing in } \rho, & \text{the underlying QM is stable or unstable, } \rho \neq 1 \\ \text{deccreasing in } \beta, & \text{the underlying QM is stable} \end{cases} \quad (8.16)$$

xii)G_{21} is forever decreasing in α (curvature parameter) whether M/G/1 QM is stable or unstable. If $\rho = 1$, the decreasability of G_{21} in α is undecidable.

xiii)G_{21} is forever increasing(decreasing) in ρ if M/G/1 QM is stable, $\alpha < 1(\alpha > 1)$.

xv)G_{21} is forever decreasing in ρ if the either one of the following branches hold:

$$\begin{cases} \rho \in (0,1), & \text{the underlying QM is stable, } \alpha > 1 \\ \rho > 1, & \text{the underlying QM is unstable, } \alpha < 1 \end{cases} \quad (8.17)$$

xvi) G_{22} is forever increasing in ρ.

xvii) G_{22} is forever increasing(decreasing) in β if and only if $\rho < 1(\rho > 1)$. If $\rho = 1$, the decision is undecidable.

Proof

i)Immediate by (6.9) and (2.44).

ii)By i), $\mathcal{R}_{AT} = \frac{2(\rho-1)(1+\beta)}{(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}}$. Hence, $\mathcal{R}_{AT} \to 0$ if and only if $(\rho-1)(1+\beta) \to 0$. Since, $\beta > 1$, the required result follows.

iii)Since M/G/1 is a stable QM, $\rho \in (0,1)$ holds. This implies for all $\rho \neq 1$ (equivalently, whether the undelying QM is either stable or unstable)

$\mathcal{R}_{AT} \to 0$ if and only if $\frac{2(\rho-1)}{(1+(1-\rho)^2)^{\frac{3}{2}}} lim_{\beta\to\infty} \frac{(1+\beta)}{(1+(1+\beta)^2)^{\frac{3}{2}}} = 0$. By $(1+(1+\beta)^2)^{\frac{3}{2}} > (1+\beta)$, the proof follows.

iv)M/G/1 QM is unstable if and only if $\rho \geq 1$, or equivalently $\mathcal{R}_{AT} \geq 0$. This holds if and only if

$$\mathcal{R}_{AT} = lim_{\epsilon\to 0} \frac{6n}{\epsilon^2}[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}] \geq 0 \quad \text{(c.f.(2.45))}$$

This is equivalent to $[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}] \gtrsim 0$. This completes the proof.

By (2.46), $G_{ij} = R_{ij}^{(\alpha)} \text{-} \frac{\mathcal{R}_{AT}}{2} g_{ij} = \frac{8\pi g \varpi_{ij}}{c^4}, i,j = 1,2$. Hence, it follows that

$$G_{11} = R_{11}^{(\alpha)} \text{-} \frac{\mathcal{R}_{AT}}{2} g_{11} = \frac{8\pi g \varpi_{11}}{c^4} \quad (8.18)$$

$$G_{12} = R_{12}^{(\alpha)} \text{-} \frac{\mathcal{R}_{AT}}{2} g_{12} = \frac{8\pi g \varpi_{12}}{c^4} \quad (8.19)$$

$$G_{21} = R_{21}^{(\alpha)} \text{-} \frac{\mathcal{R}_{AT}}{2} g_{11} = \frac{8\pi g \varpi_{21}}{c^4} \quad (8.20)$$

$$G_{22} = R_{22}^{(\alpha)} \text{-} \frac{\mathcal{R}_{AT}}{2} g_{22} = \frac{8\pi g \varpi_{22}}{c^4} \quad (8.21)$$

Using (3.3) of Theorem 3.1, (6.50 of theorem (6.1) , (8.1) together with (8.18), (8.19), (8.20) and (8.21), the proof of v) and vi) will follow.

vii)It could be verified that:

$$\frac{\partial \mathcal{R}_{AT}}{\partial \rho} = \frac{2(1-2(1-\rho)^2)(1+\beta)}{(1+(1-\rho)^2)^{\frac{5}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}} \quad (8.22)$$

Therefore,

$$\frac{\partial \mathcal{R}_{AT}}{\partial \rho} < 0 (> 0) \text{ if and only if } (1-2(1-\rho)^2) < 0 (> 0) \quad (8.23)$$

if $(1-2(1-\rho)^2) > 0$ is satisfied $\Leftrightarrow \exists\, m \in (1,\infty)$ satisfying $(1-\rho)^2 = \frac{m^2}{2}$, which $1-\rho = \pm\frac{m}{\sqrt{2}}$ or $\rho = 1 \mp \frac{m}{\sqrt{2}}$. Following the preliminary theorem (PT) 2.15, this implies $\mathcal{R}_{AT}$ is increasing in ρ if and only if $\rho = 1 \mp \frac{m}{\sqrt{2}}$. For or $\rho = 1 + \frac{m}{\sqrt{2}}$, this enforces or $\rho > 1$, which violates the underlying QM's stability, or $\rho = 1 - \frac{m}{\sqrt{2}}$, this enforces $\rho < 1$, which guarantees the stability of M/G/1 QM.

On the other hand, if $(1-2(1-\rho)^2) < 0$, it holds by (PT) 2.15, that $\mathcal{R}_{AT}$ is decreasing in ρ if and only ifthere exists $m \in (0,1)$ satisfying $(1-\rho)^2 = \frac{m^2}{2}$, which $1-\rho = \pm\frac{m}{\sqrt{2}}$. For or $\rho = 1 + \frac{m}{\sqrt{2}}$, this enforces or $\rho > 1$, which indicates the underlying QM's instability. Moreover, or $\rho = 1 - \frac{m}{\sqrt{2}}$, this enforces $\rho > 1 - \frac{1}{\sqrt{2}}$, which violates the stability of M/G/1 QM.

Following the above analytic results, vii) and viii) are immediate.

x)It could be shown that

$$\frac{\partial \mathcal{R}_{AT}}{\partial \beta} = \frac{2(2(1+\beta)^2-1)(1-\rho)}{(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{5}{2}}} \quad (8.24)$$

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Undertaking similar mathematical mechanism as in vii) and viii), the proof follows.

xi) After some mathematical manipulation, we have

$$\frac{\partial G_{11}}{\partial \rho} = \frac{(2(1-\rho)^2+1)(1-\rho)(1+\beta)}{(1-\rho)^2(1+(1-\rho)^2)^{\frac{5}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}} \tag{8.25}$$

(8.25) provides an evidence that $\frac{\partial G_{11}}{\partial \rho} > 0$ for all $\rho \neq 1$. Applying (PT) 2.15, shows that G_{11} is forever increasing in ρ. This is applicable for stable and unstable M/G/1 QM, with $\rho \neq 1$ (since $\rho = 1$, violates the continuity requirement of $\frac{\partial G_{11}}{\partial \rho}$). Furthermore, it could be proved that G_{11} is increasing in ρ when the underlying QM is in the stability phase.

Moreover,

$$\frac{\partial G_{11}}{\partial \beta} = \frac{(1-2(1+\beta)^2)(1-\rho)(1+\beta)}{(1-\rho)(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{5}{2}}} \tag{8.26}$$

Clearly from (8.26) and (PT) 2.16, , it follows that G_{11} is never increasing in β (since $\rho = 1$, violates the continuity requirement of $\frac{\partial G_{11}}{\partial \beta}$). Let us assume that G_{11} is never increasing in β. This implies:

$$(1 - 2(1+\beta)^2) > 0 \tag{8.27}$$

This is equivalent to $(1+\beta)^2) < \frac{1}{2}$ (contradiction, since stability of M/G/1 QM enforces the requirements $\rho \in (0,1)$ and $\beta \in (1,\infty)$). This suggests that the only left possibility is that G_{11} is forever increasing in β, or equivalently by (PT) 2.15, to$(1+\beta)^2) > \frac{1}{2}$, a satisfied condition by stable M/G/1 QM.

xii) $\frac{\partial G_{21}}{\partial \alpha} = \frac{-1}{(1-\rho)^2}$, implies by (PF) 2.15 that G_{21} is forever decreasing in α whether M/G/1 QM is stable or unstable. If $\rho = 1$, the decreasability of G_{21} in α is undecidable, since this means $\frac{\partial G_{21}}{\partial \alpha} \to \infty$.

Furthermore, $\frac{\partial G_{21}}{\partial \rho} = \frac{2(1-\alpha)}{(1-\rho)^3}$. Consequently, $\frac{\partial G_{21}}{\partial \rho} > 0$ if and only if M/G/1 QM is stable, $\alpha < 1$, , $\frac{\partial G_{21}}{\partial \rho} < 0$ if and only if M/G/1 QM is stable, $\alpha > 1$. By (PT) 2.15, it follows that G_{21} is forever increasing(decreasing) in ρ if M/G/1 QM is stable, $\alpha < 1 (\alpha > 1)$. This proves xiii).

Engaging the same technique proves xv).

We have

$$\frac{\partial G_{22}}{\partial \rho} = \frac{(4(1-\rho)^2+1)}{(1+\beta)(1+(1-\rho)^2)^{\frac{5}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}} \quad \text{and} \quad \frac{\partial G_{22}}{\partial \beta} = \frac{(3(1+\beta)^2+1)(1-\rho)}{(1+\beta)^2(1+(1-\rho)^2)^{\frac{5}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}} \tag{8.28}$$

Engaging our technique, the reader can easily verify that both xvi) and xvii) will hold. The completes the proof of our theorem.

Theorem 8.2 The underlying QM satisfies:

i)Ricci scalar subject to Monge Technique, $\mathcal{R}_{MT}$ is determined by

$$\mathcal{R}_{MT} = -2 \tag{8.29}$$

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ii) There exists a small enough positive number ϵ, with $\epsilon \to 0$ such that $A_{curved}(\epsilon), A_{flat}(\epsilon)$ (c.f., (2.45)) must satisfy

$$A_{curved}(\epsilon) \to (A_{flat}(\epsilon)) \quad (8.30)$$

iii)The Spacetime curvature(Einstein Tensor) subject to Angular Technique, $\wp_{AT}$ is determined by:

$$\wp_{MT} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} (8.31)$$

where the components G_{11}, G_{12}, G_{21} and G_{22} are determined by

$$G_{11} = \frac{(1-\alpha)}{(1-\rho)^2} = G_{21} (8.32)$$

where α is the curvature parameter (c.f., definition (2.8))

$$G_{12} = 0 \quad (8.33)$$

$$G_{22} = -\frac{1}{(1+\beta)^2} \quad (8.34)$$

iv) The stress-energy tensor ϖ is devised by

$$\varpi_{MT} = \begin{pmatrix} \varpi_{11} & \varpi_{12} \\ \varpi_{21} & \varpi_{22} \end{pmatrix} \quad (8.35)$$

where the components $\varpi_{11}, \varpi_{12}, \varpi_{21}$ and ϖ_{22} are determined by

$$\varpi_{11} = \frac{c^4 G_{11}}{8\pi g} (8.36)$$

$$\varpi_{12} = 0 \quad (8.37)$$

$$\varpi_{21} = \frac{c^4 G_{21}}{8\pi g} \quad (8.38)$$

$$\varpi_{22} = \frac{c^4 G_{22}}{8\pi g} \quad (8.39)$$

where g is the universal gravitational constant, c is the speed of light

$$\text{v)} \quad G_{11} = G_{21} = \begin{cases} \text{increasing in } \rho, & \text{the underlying QM is stable or unstable} \\ \text{deccreasing in } \beta, & \text{the underlying QM is stable} \end{cases} \quad (8.40)$$

vi) G_{11} is forever decreasing in α (curvature parameter) whether M/G/1 QM is stable or unstable. If $\rho = 1$, the decreasability of G_{21} in α is undecidable.

vii)G_{11} is forever increasing(decreasing) in ρ if M/G/1 QM is stable, $\alpha < 1 (\alpha > 1)$.

viii)G_{11} is forever decreasing in ρ if the either one of the following branches hold:

$$\begin{cases} \rho \in (0,1), & \text{the underlying QM is stable, } \alpha > 1 \\ \rho > 1, & \text{the underlying QM is unstable , } \alpha < 1 \end{cases} \quad (8.41)$$

x) G_{22} is forever increasing in β.

xi) G_{22} is forever increasing(decreasing) in β if and only if $\rho < 1(\rho > 1)$. If $\rho = 1$, the decision is undecidable.

Proof

i)By (6.7), we have $K_{G,MT}= -1$. Following (2.44), we have $\mathcal{R}_{MT} = 2K_{G,MT} = -2$(c.f., (8.29)).

ii)Since $\mathcal{R}_{MT} = -2$. Hence,

$$\mathcal{R}_{MT} = lim_{\epsilon\to 0}\frac{6n}{\epsilon^2}[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}] = -2 \qquad \text{(c.f.(2.45))}$$

This is equivalent to $[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}] \gtrsim -2$. This completes the proof.

By (2.46), $G_{ij} = R_{ij}^{(\alpha)} - \frac{\mathcal{R}_{MT}}{2} g_{ij} = \frac{8\pi\mathcal{g}\varpi_{ij}}{c^4}, i,j = 1,2$. Hence, it follows that:

$$G_{11} = R_{11}^{(\alpha)} - \frac{\mathcal{R}_{MT}}{2} g_{11} = \frac{8\pi\mathcal{g}\varpi_{11}}{c^4} \qquad (8.42)$$

$$G_{12} = R_{12}^{(\alpha)} - \frac{\mathcal{R}_{MT}}{2} g_{12} = \frac{8\pi\mathcal{g}\varpi_{12}}{c^4} \qquad (8.43)$$

$$G_{21} = R_{21}^{(\alpha)} - \frac{\mathcal{R}_{MT}}{2} g_{11} = \frac{8\pi\mathcal{g}\varpi_{21}}{c^4} \qquad (8.44)$$

$$G_{22} = R_{22}^{(\alpha)} - \frac{\mathcal{R}_{MT}}{2} g_{22} = \frac{8\pi\mathcal{g}\varpi_{22}}{c^4} \qquad (8.45)$$

Using (3.3) of Theorem 3.1, (6.50 of theorem (6.1) , (8.29) together with (8.42), (8.43), (8.44) and (8.45), the proof of iii) and iv) will follow.

The remaining proofs of v), vi), vii), viiii), x) and xi) are omitted since they are provable by following the same analytic mechanism undertaken in (8.13)-(8.17).

Theorem 8.3 The underlying QM satisfies:

i)$\mathcal{R}_{AT}$ has a relative minimum at $(1 \mp \frac{1}{\sqrt{2}}, -1 \pm \frac{1}{\sqrt{2}})$

ii) Both maxima and minima for all the components of the Spacetime curvature(Einestein Tensor) subject to Angular Technique, $\wp_{AT}$ is undecidable.

Proof

i)We have $\frac{\partial\mathcal{R}_{AT}}{\partial\rho} = \frac{2(1-2(1-\rho)^2)(1+\beta)}{(1+(1-\rho)^2)^{\frac{5}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}}$, $\frac{\partial\mathcal{R}_{AT}}{\partial\beta} = \frac{2(2(1+\beta)^2-1)(1-\rho)}{(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{5}{2}}}$ (c.f., (8.22) and (8.24)) respectively. Hence, $\frac{\partial\mathcal{R}_{AT}}{\partial\rho} = 0\frac{\partial\mathcal{R}_{AT}}{\partial\beta}$. The critical points are $(\rho_{critical}, \beta_{critical}) = (1 \mp \frac{1}{\sqrt{2}}, -1 \pm \frac{1}{\sqrt{2}})$.

Moreover, we have

$$\frac{\partial^2\mathcal{R}_{AT}}{\partial\rho^2} = \frac{6(3-2(1-\rho)^2)(1+\beta)}{(1+(1-\rho)^2)^{\frac{7}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}}, \frac{\partial^2\mathcal{R}_{AT}}{\partial\beta^2} = \frac{2(1-14(1+\beta)^2)(\rho-1)}{(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{7}{2}}}, \frac{\partial^2\mathcal{R}_{AT}}{\partial\rho\partial\beta} = \frac{2(1-2(1-\rho)^2)(1-2(1+\beta)^2)}{(1+(1-\rho)^2)^{\frac{5}{2}}(1+(1+\beta)^2)^{\frac{5}{2}}} \qquad (8.46)$$

Hence, by (2.47)

$D\left(1-\frac{1}{\sqrt{2}},-1+\frac{1}{\sqrt{2}}\right)$

$= \frac{\partial^2\mathcal{R}_{AT}}{\partial\rho^2}(1-\frac{1}{\sqrt{2}},-1+\frac{1}{\sqrt{2}})\frac{\partial^2\mathcal{R}_{AT}}{\partial\beta^2}(1-\frac{1}{\sqrt{2}},-1+\frac{1}{\sqrt{2}})-[\frac{\partial^2\mathcal{R}_{AT}}{\partial\rho\partial\beta}(1-\frac{1}{\sqrt{2}},-1+\frac{1}{\sqrt{2}})]^2 = \frac{(32)^3}{(81)^2} >$

0 (8.47)

Since $\frac{\partial^2\mathcal{R}_{AT}}{\partial\beta^2}\left(1-\frac{1}{\sqrt{2}},-1+\frac{1}{\sqrt{2}}\right)=-\frac{1024}{87}<0$, it holds that $\left(1-\frac{1}{\sqrt{2}},-1+\frac{1}{\sqrt{2}}\right)$is a relative minima for $\mathcal{R}_{AT}$. similarly, $\left(1+\frac{1}{\sqrt{2}},-1-\frac{1}{\sqrt{2}}\right)$is a relative minima for $\mathcal{R}_{AT}$.

ii) As for G_{11}, we have $\frac{\partial G_{11}}{\partial\rho}=\frac{(2(1-\rho)^2+1)(1-\rho)(1+\beta)}{(1-\rho)^2(1+(1-\rho)^2)^{\frac{5}{2}}(1+(1+\beta)^2)^{\frac{3}{2}}}$, $\frac{\partial G_{11}}{\partial\beta}=\frac{(1-2(1+\beta)^2)(1-\rho)(1+\beta)}{(1-\rho)(1+(1-\rho)^2)^{\frac{3}{2}}(1+(1+\beta)^2)^{\frac{5}{2}}}$ (c.f., (8.25) and (8.26)).

Hence, the critical point are $(\rho_{critical},\beta_{critical})=\left(1-\frac{1}{\sqrt{2}},0\right),(1+\frac{1}{\sqrt{2}},-2)$. Clearly, $\beta_{critical}=0,-2$ are never permissible since, M/G/1 is stable. Thus, no conclusion can be drawn for G_{11}. Engaging the same procedure, it could be verified that both maxima and minima for all the remaining components of the Spacetime curvature(Einstein Tensor) subject to Angular Technique, $\wp_{AT}$ is undecidable.

Theorem 8.4 The underlying QM satisfies:

i)Both maxima and minima of $\mathcal{R}_{AT}$ is undecidable.

ii) Both maxima and minima for all the components of the Spacetime curvature(Einstein Tensor) subject to Monge Technique, $\wp_{MT}$ is undecidable.

Proof

i)$\mathcal{R}_{MT}=-2$ (c.f., (8.29).Hence,$\frac{\partial\mathcal{R}_{AT}}{\partial\rho}=0=\frac{\partial\mathcal{R}_{AT}}{\partial\beta}$ for all ρ,β. It can be shown that D of (2.47) is zero. Hence i) follows.

ii)The proof is immediate for G_{12}, since it is zero(c.f.(8.33)). By (8.32), $G_{11}=\frac{(1-\alpha)}{(1-\rho)^2}=G_{21}$. We have $\frac{\partial G_{11}}{\partial\rho}=\frac{2(1-\alpha)}{(1-\rho)^3}$, $\frac{\partial G_{11}}{\partial\alpha}=\frac{-1}{(1-\rho)^2}$(=0 if and only if $\rho\to\infty$). Hence, both maxima and minima is undecidable for G_{11},G_{21}.Finally, $G_{22}=-\frac{1}{(1+\beta)^2}$(c.f., (8.34)). $\frac{\partial G_{22}}{\partial\beta}=\frac{2}{(1+\beta)^3}$. Therefore, $\frac{\partial G_{22}}{\partial\beta}=0$ if and only if $\beta\to\infty$. Consequently, it is not possible to decide maxima and minima for G_{22}.

The following theorem captures the impact of stability(instability) of M/G/1 QM on the increasability (decreasability) of the only non-zero component of Ricci Curvature Tensor(RCT), $R_{21}^{(0)}$(c.f.,(6.5)).

Theorem 8.5 The underlying QM satisfies:

i)$R_{21}^{(0)}$is forever increasing in ρ $\Leftrightarrow$ the underlying QM is unstable.

ii)$R_{21}^{(0)}$ forever decreases in ρ if and only if M/G/1 QM is stable.

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Proof

i)**Necessity:**

Assume that M/G/1 QM is unstable, then $\rho > 1$. We have $\frac{\partial}{\partial\rho}(R_{21}^{(0)}) = -\frac{2}{(1-\rho)^3}(8.48)$. By $\rho > 1$, $\frac{\partial}{\partial\rho}(R_{21}^{(0)}) > 0$, which directly implies by (PT) 2.15 that $R_{21}^{(0)}$is forever increasing in ρ.

Sufficiency:

Let $R_{21}^{(0)}$ be forever increasing in ρ, then by (PT) 2.15, $\frac{\partial}{\partial\rho}(R_{21}^{(0)}) = -\frac{2}{(1-\rho)^3} > 0$. This implies $(\rho-1)^3 > 0$. Hence, $\rho > 1$. This proves i).

Engaging the same procedure, ii) follows.

The following theorem captures the dual impact between (IG) and Queueing Theory. This dual impact is influencing the existence of FIM (c.f.,(3.3)) upon the behaviour of $\Delta_{[g^{ij}]}$.

Theorem 8.6 The underlying QM satisfies:

1)$\Delta_{[g^{ij}]}$is forever increasing(decreasing) in ρ $\Leftrightarrow$ the underlying QM is(stable) unstable.

2)$\Delta_{[g^{ij}]}$is forever increasing(decreasing) in β $\Leftrightarrow$ the underlying QM is(unstable) stable.

3)The inflection point of $\Delta_{[g^{ij}]}$ with respect to β is at $\rho = 1$, where $\Delta_{[g^{ij}]}$ changes its behaviour around this threshold $\rho = 1$.

Proof

1)We have $[g^{ij}] = \frac{adj[g_{ij}]}{\Delta} = \begin{pmatrix}(1-\rho)^2 & 0 \\ 0 & -(\beta+1)^2\end{pmatrix}$(c.f., (3.5) of Theorem 3.1) . Therefore,

$$\Delta_{[g^{ij}]} = -(\beta+1)^2(1-\rho)^2 \quad (8.49)$$

By (8.49), $\Delta_{[g^{ij}]} = 0$ if and only if $\rho = 0$ or $\beta = -1$(this is never permissible).

$$\frac{\partial\Delta_{[g^{ij}]}}{\partial\rho} = 2(1-\rho)(\beta+1)^2 \quad (8.50)$$

Clearly from (8.50), it follows that:

$$\frac{\partial\Delta_{[g^{ij}]}}{\partial\rho} > 0(<0) \textit{if and only if} (\rho<1)(\rho>1) (8.51)$$

By (PT) 2.15, the proof of 1) follows.

2) We have

$$\frac{\partial\Delta_{[g^{ij}]}}{\partial\beta} = -2(1-\rho)^2(\beta+1) \quad (8.52)$$

$$\frac{\partial\Delta_{[g^{ij}]}}{\partial\beta} > 0(<0) \textit{ if and only if } (\rho>1)(\rho<1) \quad (8.53)$$

By (PT) 2.15, the proof of 2) follows.

3)It is straightforward to see that

$$\frac{\partial^2(\Delta_{[g^{ij}]}(\rho,\beta))}{\partial\beta^2} = -2(1-\rho)^2 (8.54)$$

$\frac{\partial^2(\Delta_{[g^{ij}]}(\rho,\beta))}{\partial\beta^2}$ if and only if $\rho = 1$, which implies by 1) and 2) that the proof of 3) follows.

DATA $\Delta_{[g^{ij}]}$: $\Delta_{[g^{ij}]}\ for\ \beta = 2, \rho \in (0,1)$

CASE ONE:

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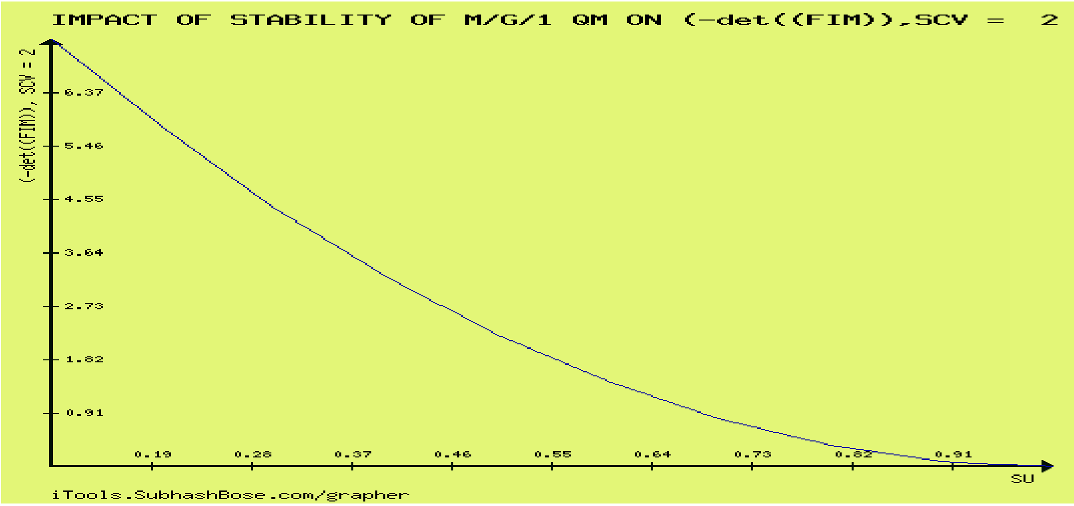

Figure 23.

As observed in Figure 23, det(IFIM) is increasing in server utilization if and only if M/G/1 QM is stable. Also, this proves how the stability of the underlying QM impacts the existence of $[g^{ij}]$.

CASE TWO: $\Delta_{[g^{ij}]} for\ \beta = 2, \rho \in [1, \infty)$

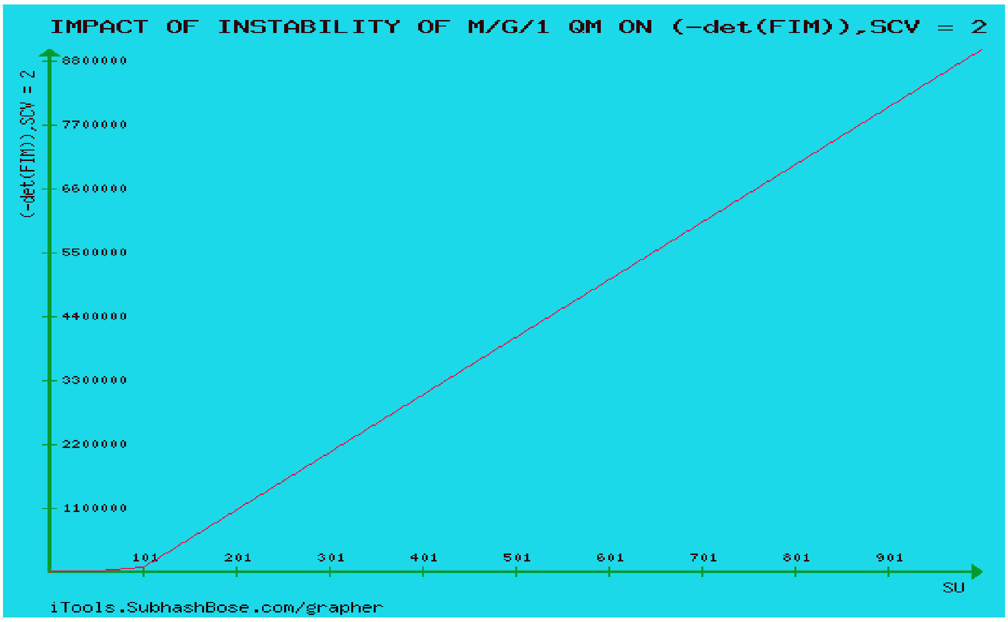

Figure 24.

As observed in Figure 24, instability of M/G/1 QM is unstable $\Leftrightarrow$ Δ decreases in ρ.

CASE THREE: $\Delta_{[g^{ij}]}$ behaviour for β in stabilitly phase of M/G/1 QM, $\rho = 0.5$

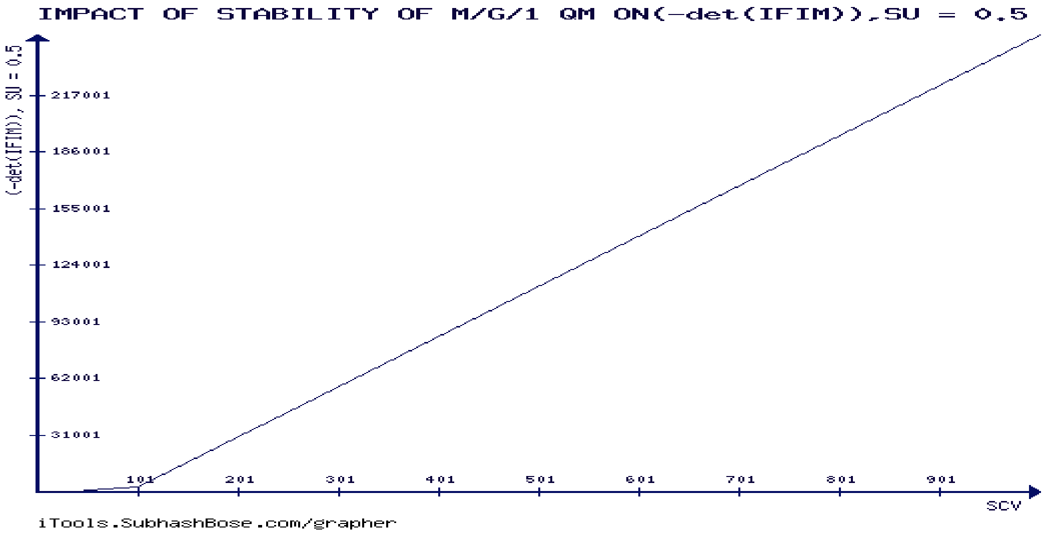

Figure 25.

As seen above in Figure 25, det(IFIM) is forever decreasing in SCV if and only if M/G/1 QM is stable.

CASE FOUR: $\Delta_{[g^{ij}]}$ *behaviour for* β *in instability phase of* MG1 QM, $\rho = 2$

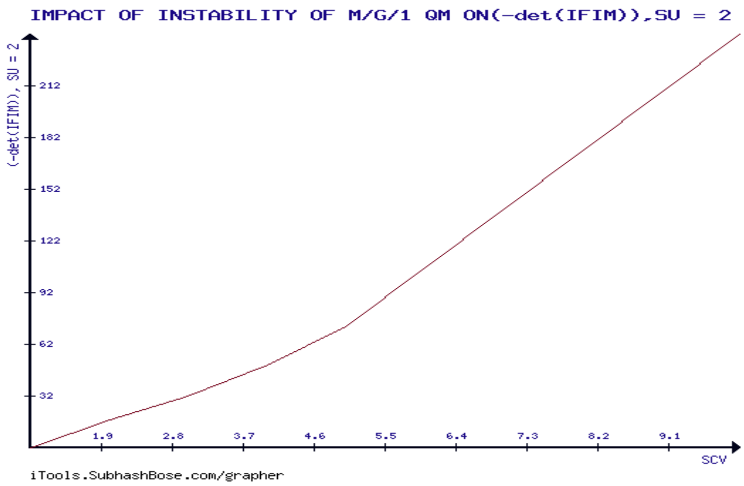

Figure 26.

As observed in Figure 26, within the instability phase of M/G/1 QM, $\beta \in (0,1)$, det(IFIM) is decreasing in β.

CASE FIVE:

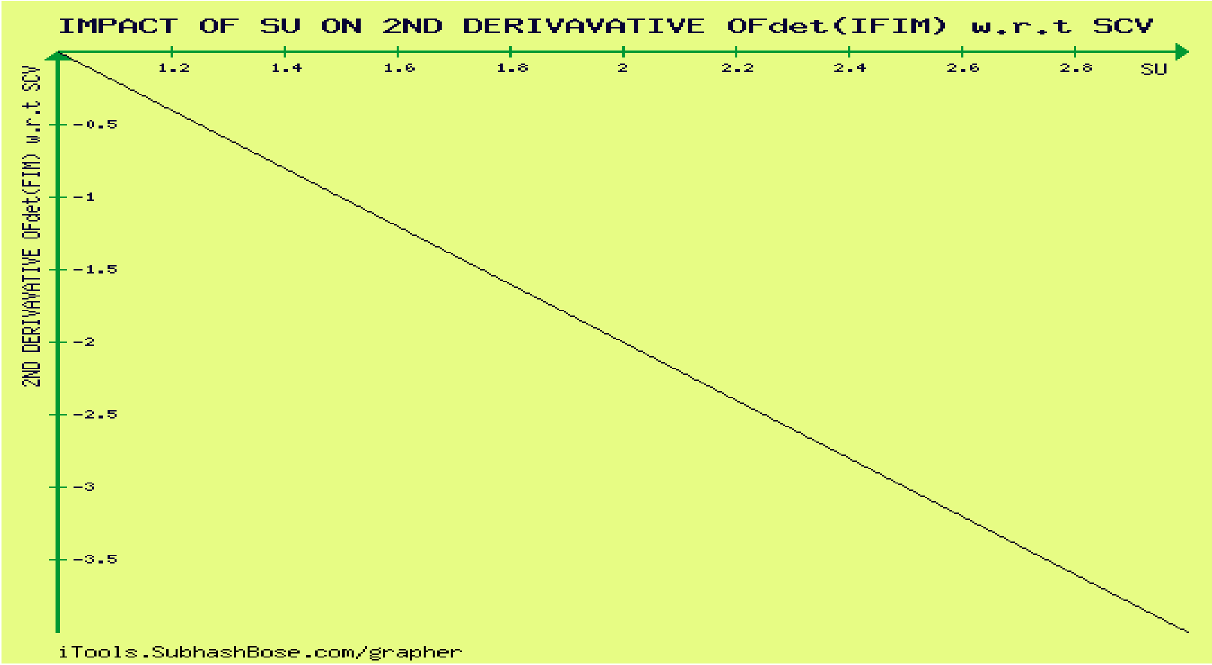

Figure 27.

Figure 27 justifies that the threshold of stability of M/G/1 QM, $\rho = 1$, would be the inflection point of det(IFIM) as well as being the decision parameter which controls the existence of inverse of (IFIM).

Define (QIGUs) by the triad functions, namely $h_1^{\text{QIGU}}, h_2^{\text{QIGU}}, h_3^{\text{QIGU}}$, with

$h_1^{\text{QIGU}}, h_2^{\text{QIGU}}$: M/G/1 QM → $[g_{ij}]$, where $[g_{ij}] = \begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix}$(c.f.,(3.3) of Theorem 3.1)

$$h_1^{\text{QIGU}}(\rho) = g_{11} = \frac{1}{(1-\rho)^2}, \rho \in (0,1) \tag{8.55}$$

$$h_2^{\text{QIGU}}(\beta) = -g_{22} = \frac{1}{(1+\beta)^2}, \beta \in (1,\infty) \tag{8.56}$$

$$h_3^{\text{QIGU}}(\rho,\beta) = \begin{cases} \Psi_{n=0}(\rho,\beta) = -l\,n(1-\rho), & n = 0 \\ \Psi_{n>0}(\rho,\beta) = l\,n(\beta+1) - l\,n(1-\rho) - ln2, & n > 0 \end{cases} \tag{8.57}$$

provided that $\Psi_{n=0}(\rho,\beta)$, $\Psi_{n>0}(\rho,\beta)$are determined by (3.7),(3.14) of Theorem 3.1 respectively.

9. Queueing Theoretic Impact on the Continuity of New Devised Queueing-Information Geometric Unifiers (QIGU)

Theorem 9.1 For the stable M/G/1 QM

1)h_1^{QIGU} is continuous, for $\rho \in (0,1)$

2)h_1^{QIGU} is well-defined.

3)h_1^{QIGU} is one-to-one.

4)h_1^{QIGU} is onto.

5) The inverse function of h_1^{QIGU} is characterized by, $(h_1^{\text{QIGU}})^{-1}(\rho) = 1 - \frac{1}{\sqrt{\rho}}, \rho \in (0,1)$

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6)h_2^{QIGU} is continuous, for $\beta \in (1, \infty)$

7)h_2^{QIGU} is well-defined.

8)h_2^{QIGU} is one-to-one.

9)h_2^{QIGU} is onto.

10) The inverse function of h_2^{QIGU} is characterized by, $(h_2^{\text{QIGU}})^{-1}(\beta)= -1+\frac{1}{\sqrt{\beta}}, \beta \in (1, \infty)$.

11)$\Psi_{n=0}(\rho, \beta)$ is well defined.

12)$\Psi_{n=0}(\rho, \beta)$is continuous if and only if M/G/1 QM is stable.

13)$\Psi_{n=0}(\rho, \beta)$ is discontinuous $\Leftrightarrow$ the instability of the underlying QM is satisfied.

14)$\Psi_{n=0}(\rho, \beta)$ is one-to-one.

15)$\Psi_{n=0}(\rho, \beta)$is onto.

16) The inverse function of $\Psi_{n=0}(\rho, \beta)$ is characterized by, $(\Psi_{n=0}(\rho, \beta))^{-1}(\rho) = 1 - e^{-\rho}, \rho \in (0,1)$.

17)$\Psi_{n>0}(\rho, \beta)$ characterizes a family of non-well defined (ρ, β) dependent functions

18)$\Psi_{n>0}(\rho, \beta)$ is continuous if and only if M/G/1 QM is stable.

19)$\Psi_{n>0}(\rho, \beta)$ is discontinuous $\Leftrightarrow$ the instability of the underlying QM is satisfied.

Proof

1)Let $\rho_1, \rho_2 \in (0,1)$such that there is a $\delta > 0$ satisfying

$$|\rho_1 - \rho_2| < \delta \tag{9.1}$$

$$|h_1^{\text{QIGU}}(\rho_1) - h_1^{\text{QIGU}}(\rho_2)| =\left|\frac{1}{(1-\rho_1)^2} - \frac{1}{(1-\rho_2)^2}\right| = \left|\frac{(\rho_1 - \rho_2)(\rho_1 + \rho_2 - 2)}{(1-\rho_1)^2(1-\rho_2)^2}\right| < \frac{\delta|(\rho_1 + \rho_2 - 2)|}{(1-\rho_1)^2(1-\rho_2)^2} = \varepsilon \tag{9.2}$$

Clearly by (2.48), the proof 1) follows.

It is also clear that if either $\rho_1 = 1$ or$\rho_2 = 1$. Then, the the underlying QM is unstable, which implies by (9.2), that $\varepsilon \to \infty$. This implies that $h_1^{\text{QIGU}}$is discontinuous at $\rho = 1$.

2) Let $\rho_1, \rho_2 \in (0,1), \rho_1 \neq \rho_2$. Assume that $h_1^{\text{QIGU}}(\rho_1) = h_1^{\text{QIGU}}(\rho_2)$. Hence, $\frac{1}{(1-\rho_1)^2} = \frac{1}{(1-\rho_2)^2}$, equivalently, $(1-\rho_1)^2 = (1-\rho_2)^2$. This means, $\rho_1 = \rho_2$ or$\rho_1 + \rho_2 = 2$. We have to reject , $\rho_1 = \rho_2$. Also, $\rho_1 + \rho_2 = 2$is impossible since $\rho_1, \rho_2 \in (0,1)$,Clearly, a contradiction follows. Therefore, 2) holds.

3)$h_1^{\text{QIGU}}(\rho_1) = h_1^{\text{QIGU}}(\rho_2)$, then following the same proof as in 2) implies that $\rho_1 = \rho_2$. This proves 3).

4) Obviously, every $\rho \in (0,1)$ uniquely characterizes $\frac{1}{(1-\rho)^2}$ such that $h_1^{\text{QIGU}}(\rho) = g_{11} = \frac{1}{(1-\rho)^2}$. Hence, h_1^{QIGU} is onto.

5) Assume $h_1^{\text{QIGU}}(\rho) = y$. Hence,

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$$(h_1^{\mathrm{QIGU}})^{-1}(\rho) = 1-\frac{1}{\sqrt{\rho}}, \rho \in (0,1) \quad (9.3)$$

which proves 5).

6)Let $\beta_1, \beta_2 \in (1,\infty)$such that there is a $\delta > 0$ satisfying

$$|\beta_1 - \beta_2| < \delta \quad (9.4)$$

$$|h_2^{\mathrm{QIGU}}(\beta_1) - h_2^{\mathrm{QIGU}}(\beta_2)| = \left|\frac{1}{(1+\beta_1)^2} - \frac{1}{(1+\beta_2)^2}\right| = \left|\frac{(\beta_1-\beta_2)(\beta_1+\beta_2-2)}{(1+\beta_1)^2(1+\beta_2)^2}\right| < \frac{\delta}{16}|(\beta_1+\beta_2-2)| = \varepsilon \quad (9.5)$$

Clearly by (2.48), the proof 6) follows.

7) Let $\beta_1, \beta_2 \in (1,\infty), \beta_1 \neq \beta_2$. Assume that $h_2^{\mathrm{QIGU}}(\beta_1) = h_2^{\mathrm{QIGU}}(\beta_2)$. Hence, $\frac{1}{(1+\beta_1)^2} = \frac{1}{(1+(\beta_2)^2}$, equivalently, $(1+\beta_1)^2 = (1+\beta_2)^2$. This means, $\beta_1 = \beta_2$ or $\beta_1 + \beta_2 = -2$. We must reject, $\beta_1 = \beta_2$. Also, $\beta_1 + \beta_2 = -2$ is impossible since $\beta_1, \beta_2 \in (1,\infty)$,, Clearly, a contradiction follows. Therefore, 7) holds.

It is also clear that even if the underlying QM is unstable(or equivalently, $\beta \in (0,1]$which implies by (9.5), that $\varepsilon \nrightarrow \infty$). This implies that h_2^{QIGU} is everywhere continuous.

Engaging the same procedure as in 3) and 4), the proofs of 8) and 9) are easily verified respectively.

10)Assume $h_2^{\mathrm{QIGU}}(\beta) = \frac{1}{(1+\beta)^2} = w, \beta \in (1,\infty)$. These leaves one choice,

$$\beta = -1 + \frac{1}{\sqrt{w}} \quad (9.6)$$

This shows that, $(h_2^{\mathrm{QIGU}})^{-1}(\beta) = 1 - \frac{1}{\sqrt{\beta}}, \beta \in (1,\infty)$which proves 10).

11)Let $\rho_1, \rho_2 \in (0,1), \rho_1 \neq \rho_2$. Assume that $\Psi_{n=0}(\rho_1) = \Psi_{n=0}(\rho_2)$. Hence, $-ln(1-\rho_1) = -ln(1-\rho_2)$, equivalently, $(1-\rho_1) = (1-\rho_2)$. This means, $\rho_1 = \rho_2$. Clearly, a contradiction follows. Therefore, 12) holds.

12) Necessity: Assume that M/G/1 QM is stable, then $\rho_1, \rho_2 \in (0,1)$

Suppose that there is a $\delta > 0$ satisfying:

$$|\rho_1 - \rho_2| < \delta \quad (9.7)$$

$$|\Psi_{n=0}(\rho_1) - \Psi_{n=0}(\rho_2)| = |ln(1-\rho_1) - ln(1-\rho_2)| \quad (9.8)$$

It is well known that Maclauren's series of $ln\,(1-\rho)$, for $\rho \in (0,1)$ is determined by

$$l\,n(1-\rho) = -\sum_{n=1}^{\infty}\frac{\rho^n}{n} \text{ (c.f., (2.49))} \quad (9.10)$$

Thus, it follows that

$$|ln(1-\rho_1) - ln(1-\rho_2)| < \delta \sum_{n=1}^{\infty}(\rho_1^{n-1} + \rho_1^{n-2} + \cdots + 1) = \frac{\delta}{(1-\rho_1)} = \varepsilon \quad (9.12)$$

This proves continuity.

Sufficiency:

By (9.12), there exists $\frac{\delta_1}{(1-\rho_1)} = \varepsilon_1 > 0$, $\frac{\delta_2}{(1-\rho_2)} = \varepsilon_2 > 0$, with $\delta = min(\delta_1, \delta_2) > 0$, $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ satisfying, (9.7) and (9.12). Consequently, $\rho_1, \rho_2 \in (0,1)$, which directly implies the underlying QM's stability.

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13) Following (9.12), discontinuity of $\Psi_{n=0}$ occurs if and if $\rho_1 \geq 1$ to enforce ε to be infinite of any negative real number. Therefore, 13) holds.

The validity of both 14) and 15) can easily be verified.

Assume $\Psi_{n=0}(\rho) = -ln\,(1-\rho) = \mathrm{m}, \rho \in (0,1)$. Thus, it holds that $1-\rho = e^{-m}$. Consequently, $\rho = 1-e^{-m}$. This shows that , $(\Psi_{n=0})^{-1}(\rho)= 1-e^{-\rho}, \rho \in (0,1)$, which proves 16).

17) Let $\rho_1, \rho_2 \in (0,1), (\beta_1, \beta_2) \in (1,\infty)$such that $\rho_1 \neq \rho_2, \beta_1 \neq \beta_2$ Assume that $\Psi_{n>0}(\rho_1, \beta_1) = \Psi_{n>0}(\rho_2, \beta_2)$. Hence,

$\frac{(\beta_1+1)}{(1-\rho_1)} = \frac{(\beta_2+1)}{(1-\rho_2)}$. This means,

$$(\beta_1 - \beta_2) + (\rho_1 - \rho_2) = (\beta_2\rho_1 - \beta_1\rho_2) \tag{9.13}$$

By (9.13), 17) holds.

It is obvious that $\Psi_{n>0}$ is continuous for all $\beta \in (1,\infty), \rho \in (0,1)$. This proves the necessity requirement. As for the sufficiency requirement, let $\Psi_{n>0}(\rho,\beta)$ be continuous. This directly implies that $l\,n(1-\rho)$ attains non- infinite real values. Consequently, $1-\rho > 0$. Hence, the underlying M/G/1 QM is stable. This proves 18).

To prove 19), it is clear that $\Psi_{n>0}(\rho,\beta)$ is discontinuous if and only if both $l\,n(\beta+1)$ and $ln(1-\rho)$ is discontinuous. This is equivalent to $\beta > -1, \rho \geq 1$. Equivalently, M/G/1 QM is unstable.

10. Closing Remarks with Next Phase Research

This paper discusses the application of information geometric concepts in queueing theory, specifically focusing on the specified QM. It introduces the Fisher information metric, the α-connection, and analyses the geometric properties of the queue manifold, such as Gaussian and Ricci curvatures. The paper also highlights the potential for further research in IG unification with existent knowledge of scientific fields.

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