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On the Kullback-Leibler Divergence Formalism (Kldf) of the Stable Mg1 Queue Manifold, Its Information Geometric Structure And Its Matrix Exponential

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Abstract: The paper explores the Kullback-Leibler divergence formalism (KLDF) applied to the stable MG1 queue manifold. It explores the analytic forms of state probabilities and their maximization based on entropy functionals, subject to normalization and mean value constraints. The credibility of KLDF is justified through consistency axioms, and the application of Rényi's and Tsallis's formalisms on a stable M/G/1 queue is examined, resulting in novel state probabilities and insights into information theory of Queue Learning.

Keywords: queue; server utilisation (SU); short-range interactions; Ricci Curvature Tensor (RCT)

1. Introduction

1.1. Early Dawn of Minimum Relative Entropy(MRE)

In the context of probabilistic inverse approaches[1,] it has become conventional to treat both measurable data and unknown parameters for the model being uncertain. This method provides deeper understanding of the uncertainty associated with the measured data and model parameters. [2-10]. KLD [11-16] is a method used to compare two probability distributions. In Probability and Statistics, when we need to simplify complex distributions or approximate observed data, KL Divergence helps us quantify the amount of information lost in the process of choosing an approximation. KLD measures the difference between the two distributions and assists us with comprehending the trade-off between accuracy and simplicity in statistical modelling.

Shannonian entropic measure[9,17] ,namely $H(p)$ reads as

$$H(p) = -\sum_{n=0}^{\infty} p(n) ln(p(n)) \tag{1}$$

to define "The minimum number of bits it would take us to encode our information".

KLD is commonly written as:

$$D_{KL}(p||r) = \sum_{n=0}^{\infty} p(n) ln \frac{p(n)}{r(n)} \tag{2}$$

With KL divergence we can calculate exactly how much information is lost when we approximate one distribution with another.

1.2. Information geometry(IG)

Many domains, including statistical inference, system control, and neural networks, have made extensive use of information geometry. In other words, IG seeks to apply differential geometry techniques to statistics.

A manifold [18-21] is a topological finite-dimensional Cartesian space, $\mathbb{R}^n$, where an infinite-dimensional manifold exists. In Fig. 1, model parameter inference from data is depicted as a decision-making problem. Information geometry offers a differential-geometric manifold structure M that may be used to create decision rules.

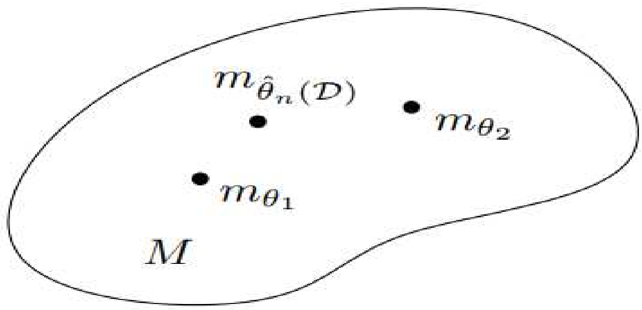

Figure 1. Parametrization of a statistical manifold (c.f., [19]).

The matrix exponential is a concept that holds significance in the study of Lie groups[22], which are mathematical structures used to analyze continuous symmetries. In the context of the given text, there is a research paperthat explores the geometry of M/D/1 queues, a type of queuing system, by introducing a geometric structure based on the characteristics of queue length routes. This innovative approach aims to provide a new perspective on analysing and understanding these queues. In the context of the study, a geometric approach is used in analogy to Information Theory because it allows for the examination of figure invariance and equivariancewithout relying on specific coordinates. This means that geometric methods provide a coordinate-free manner to analyze and understand the properties of figures, which is beneficial in certain applications. Ricci curvature measures how the Riemannian metric deviates from the standard Euclidean metric, while scalar curvature quantifies the difference in volume between a geometric ball and a Euclidean ball of the same radius, as illustrated by figure 2(c,f., [23]).

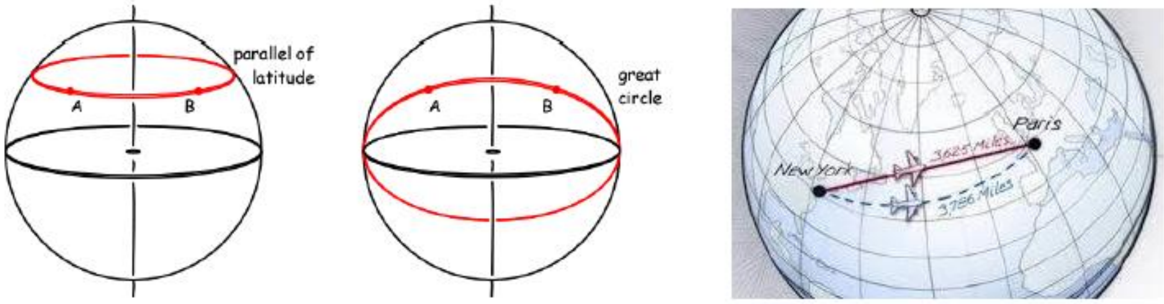

Figure 2. Geometric illustration of curved surfaces' geodesics.

This revolutionary paper contributes to:

i)The provision of both FIM and IFIM for KLDF manifold.

ii)First-time calculation of α -connection[26]. iii) KLD and JD are calculated for the underlying queue iv) The compressibility (non-solenoidality) of KLDF manifold [27]. v)First time unification between queueing and matrix theories.

vi) The Rényi divergence (RD) , $D_R^{\gamma}(p||q)$ and the S, AB-divergence, $D_{s,AB}^{\gamma,\eta}(p||q)$ for KLDF are devised. Notably, computed divergence measure in this paper set the foundation towards the information theory of Queue Learning(QL)

vi)Numerical Experiments on The Rényi divergence (RD) , $D_R^{\gamma}(p||q)$ and the S, AB-divergence, $D_{s,AB}^{\gamma,\eta}(p||q)$ of the KLDF manifold to illustrate how both analytic and numerical experiments agree.

vii) A giant step towards the unification of KLDF of stable M/G/ 1 QM and its Information geometric structure is devised. viii) Introducing novel well-defined statistical queueing functionals, SQFs and investigate their algebraic structures.

This paper provides a roadmap of its contents, starting with early definitions in section 2 and an overview of consistency axioms in section 3. In the remaining sections, it introduces the KL formalism of the stable M/G/1 queue and devises service time distribution and cumulative functions. The paper also obtains the threshold theorem of KLDF, introduces the Fisher's information matrix and metric, explores the α-connection of the queue manifold, and reveals the compressibility and developability of the manifold. Additionally, it discusses the Scalar Curvature, Einstein Tensor, and stress-energy tensor in relation to information theory, queuing theory, and General Relativity. The paper concludes with the introduction of Rényi divergence and S, AB-divergence, numerical experiments, unification of KLDF and its information geometric structure, and the introduction of statistical queueing functionals.

2. Main Definitions in Information Geometry

Definition 2.1 [28]

1We call $M = \{p(x,\theta)|\theta\epsilon\Theta\}$ a statistical manifold if x is a random variable in sample space X and $p(x,\theta)$ is the probability density function, which satisfies certain regular conditions. Here, $\theta = (\theta_1, \theta_2, .., \theta_n) \in \Theta \subset \mathbb{R}^n$.

Definition 2.2 [28]. Having defined the function

$$\mathcal{L}(x;\theta) = lnp(x,\theta) \tag{3}$$

Following (3), the potential function $\Psi(\theta)$ to be the distinguished negative function of the coordinates alone.

Definition 2.3 [26]. FIM, namely, $[g_{ij}]$) reads as

$$[g_{ij}] = \left[\frac{\partial^2}{\partial\theta^i\partial\theta^j}(\Psi(\theta))\right], i,j = 1,2,\dots,n \tag{4}$$

Definition 2.4[29]. $[g_{ij}]$ reads as

$$[g^{ij}] = ([g_{ij}]))^{-1} = \frac{adj[g_{ij}]}{\Delta},\ \Delta = \det[g_{ij}] \tag{5}$$

Definition 2.5 [26]. α-Connection

For each $\alpha\epsilon\mathbb{R}$, the α(or $\nabla^{(\alpha)}$)-*connection* is the torsion-free affine connection with components:

$$\Gamma^{(\alpha)}_{ij,k} = (\frac{1-\alpha}{2})(\partial_i\partial_j\partial_k(\Psi(\theta))),\ \partial_i = \frac{\partial}{\partial\theta_i} \tag{6}$$

Definition 2.6[29].

1. Assume $p(x;\theta_p)$ and $q(x;\theta_q)$ are two points on the manifold M, the Kullback's divergence $K(p,q)$is defined by

$$K(p,q) = \int p(x;\theta_p) l\,n\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right) dx \tag{7}$$

and the J-divergence reads as

$$J(p,q) = \int (p(x;\theta_p) - q(x;\theta_q)\ )\ l\,n\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right) dx \tag{8}$$

KLD (cf., [30-34]), may provide a reasonably straightforward to optimise target. The Rényi divergence [34,35] reads as

$$D_R^\gamma(p||q) = \frac{1}{(\gamma-1)} ln\left(\sum_{n=0}^{\infty}(p(n))^\gamma (q(n))^{1-\gamma}\right) \tag{9}$$

used in Rényi variational inference VI [31-39].

More potentially, we would lose the promises about divergence minimization in that case. Consider f the AB-divergence's scale-invariant as our primary interest. introduced this topic briefly:

$$D_{s,AB}^{\gamma,\eta}(p||q) = \frac{1}{\eta(\eta+\gamma)} ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma+\eta}\right) + \frac{1}{\gamma(\eta+\gamma)} ln\left(\sum_{n=0}^{\infty}(q(n))^{\gamma+\eta}\right) - \frac{1}{\gamma\eta} ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma} (q(n))^{\eta}\right) \tag{10}$$

for $(\gamma,\eta)\epsilon\mathbb{R}^2$ such that $\gamma \neq 0, \eta \neq 0$ *and* $\gamma + \eta \neq 0$.

The authors have presented a novel (dis)similarity measure, namely $\boldsymbol{D}_{s,AB}^{\gamma,\eta}(\boldsymbol{p}||\boldsymbol{q})$ (c.f., (2.10). Moreover, it has been illustrated that $D_{s,AB}^{\gamma,\eta}(p||q)$ is potentially robust.

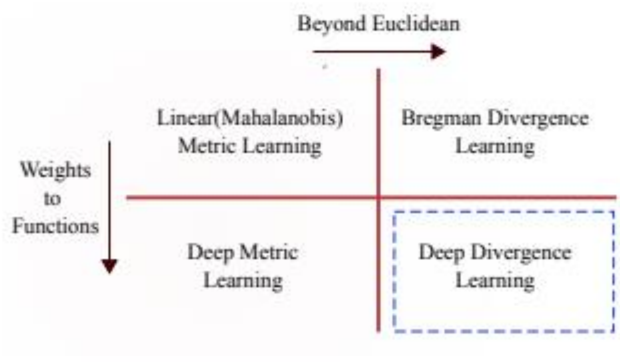

Figure 3. (c.f., [41]).

2.7. Well Defined Functions and Bijective Functions

Definition 2.12(c.f., [42])

1. When the representation of the input is modified without impacting the value of the input, the function is well-defined.

2. function f is said to be one-to-one, or injective (c.f. [43]), ⟺

$f(x) = f(y)$ implies $x = y$ for all x, y in the domain of (11)

A function is said to be an injection if it is one-to-one. Alternative: A function is one-to-one ⟺ $f(x) \neq f(y)$, whenever $x \neq$ (12)

This is the contrapositive of the definition.

3.A function f from A to B is called onto **(c.f.** [43]), or surjective, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$. Alternative: all co-domain elements are covered.

4. A function f is called a bijection (c.f. [43]) if it is both one-to-one (injection) and onto (surjection).

Definition 2.8

1. $R_{ijkl}^{(\alpha)}$ (Riemannian Tensors) are defined by

$$R_{ijkl}^{(\alpha)} = [(\partial_j \Gamma_{ik}^{s(\alpha)} - \partial_i \Gamma_{jk}^{s(\alpha)})g_{sl} + (\Gamma_{jt,l}^{(\alpha)} \Gamma_{ik}^{t(\alpha)} - \Gamma_{it,l}^{(\alpha)} \Gamma_{jk}^{t(\alpha)})] \quad (13)$$

where $\Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)} g^{sk}$

2. $\alpha -$ RCTs, $R_{ik}^{(\alpha)}$ are given by

$$R_{ik}^{(\alpha)} = R_{ijkl}^{(\alpha)} g^{jl} \quad (14)$$

3.[29]The $\alpha -$ sectional curvatures $K_{ijij}^{(\alpha)}$ are defined by

$$K_{ijij}^{(\alpha)} = \frac{R_{ijij}^{(\alpha)}}{(g_{ii})(g_{jj}) - (g_{ij})^2} \quad (15)$$

The $\alpha -$ sectional curvatures $K_{1212}^{(\alpha)} = K^{(\alpha)}$ is called the $\alpha -$ Gaussian curvature and

$$K^{(\alpha)} = \frac{R_{1212}^{(\alpha)}}{det(g_{ij})} \quad (16)$$

4.RCT is basically a smaller version of the Riemann's Tensor (see, for example, [29], [44]).

5. Oriented Riemannian manifold RCT M signifies the difference between a geodesic ball on the surface and Euclidean geodesic ball. Additionally, it reduces volume increase in a geodesic flow. According to the Bonnet Myers theorem, when RCT is positive, the diameter of the Riemannian manifold will be smaller[46].

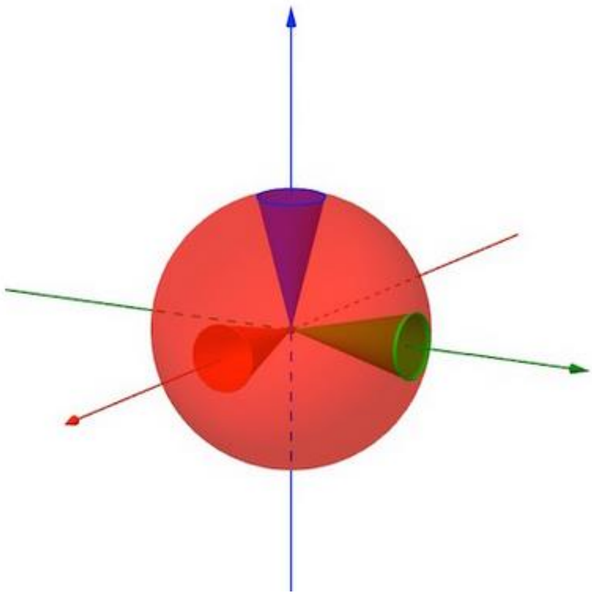

Figure 4. Depiction of how conical sections in the manifold differ in volume from corresponding conical locations in Euclidean space (c.f., [47]).

Definition 2.9[48]

(1) The matrix exponential

$e^{A} = \sum_{i=0}^{\infty} \frac{A^{i}}{i!} = I + A + \frac{A^{2}}{2!} + \cdots + \frac{A^{k}}{k!} + \ldots.$ (17)

solves

$\frac{dx}{dt} = Ax$ (18)

(2) If the characteristic polynomial of A is defined by

$\Phi(\delta) = \det(A - \delta I)$(2.18)

Then, the set of eigen values of A will is defined to be the set of all the roots of the equation:

$\Phi(\delta) = (\delta) = \det(A - \delta I) = 0,$ (19)

and corresponding eigen vectors x assigned to each eigen value δ are defined to satisfy the equation:

$Ax = \delta x$ (20)

e^{A} rewrites to

$e^{A} = T\ e^{D}\ T^{-1}$ (21)

where D is the diagonal matrix of eigen values of A, and T is matrix having of the corresponding eigen vectors of A as its columns.

Definition 2.10 [49]

(i)Developable surfaces(DSs) are a special kind of ruled surfaces: they have a Gaussian curvature equal to 0 and can be mapped onto the plane surface without distortion of curves: any curve from such a surface drawn onto the flat plane remains the same. Figure 5 as in below demonstrates DSs.

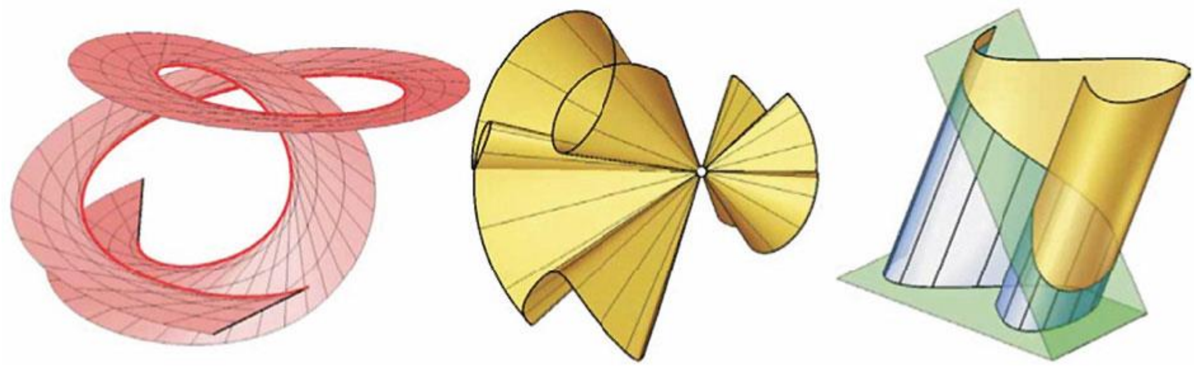

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Figure 5. The three kinds of developable surfaces: a, left) tangential; b, centre) conical; c, right) cylindrical. Curves in bold are directrix or base curves; straight lines in bold are directors or generating lines (curves) (c.f., [49]).

According to Kouvatsos [50], the maximum entropy state probability of the generalized geometric solution of a stable M/G/1 queue, subject to normalisation, mean queue length (MQL), L and server utilisation, $\rho(<1)$ is given by

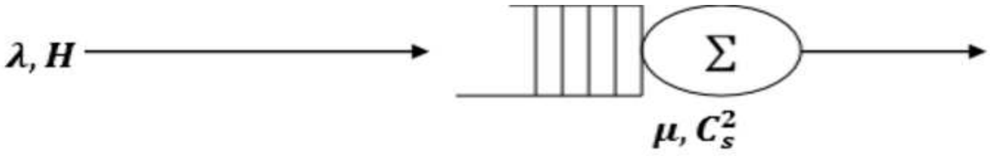

Figure 6. A Stable M/G/1 queue.

$$p(n) = \begin{cases} 1-\rho, & n=0 \\ (1-\rho)gx^n, & n \geq 1 \end{cases}, \qquad (22)$$

where $g = \frac{\rho^2}{(L-\rho)(1-\rho)}$, $x = \frac{L-\rho}{L}$ and $L = \frac{\rho}{2}\left(1+\frac{1+\rho C_s^2}{1-\rho}\right)$ (MQL for the underlying queue), $\rho(= 1 - p(0))$ and C_s^2 is the squared coefficient of variations). Obviously, $p(n)$ (c.f., (22)) reads as:

$$p(n) = \begin{cases} 1-\rho, & n=0 \\ \frac{2\rho(\frac{1+\rho\beta}{1-\rho}-1)^{n-1}}{((\frac{1+\rho\beta}{1-\rho}+1)^n}, & n \geq 1 \end{cases} \qquad (23)$$

where $\beta = C_s^2$. The reader can observe the difference between our novel approach and that in (22) and (23). Moreover, it is notable that the newly obtained KL formalism in the current paper is more general which reduces to (22) as a special case.

2.11. Scalar Curvature(Ricci Scalar), $\mathcal{R}$ and Einestein Tensor, $\wp$

The scalar curvature(Ricci Scalar), $\mathcal{R}$ (c.f., [44]) is the contraction of Ricci Tensor(c.f., (14))

$$\mathcal{R} = R_{ij}^{(\alpha)} g^{ij} \qquad (24)$$

The Ricci scalar $\mathcal{R}$ (c.f., [44]) reads as:

$$\mathcal{R} = lim_{\epsilon \to 0} \frac{6n}{\epsilon^2}[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}] \qquad (25)$$

Ricci scalar completely captures the curvature of the surface.

The equations for the motion of a classical theory, such as General Relativity, can be instantly derived from a suitable action using the Euler-Lagrange equations, which leads to the well-known Einstein equations [51].

$$G_{ij} = R_{ij}^{(\alpha)} - \frac{\mathcal{R}}{2} g_{ij} = \frac{8\pi g \varpi_{ij}}{c^4} \qquad (26)$$

where G_{ij} is the Curvature of Spacetime(Einstein tensor), $\wp$, $R_{ij}^{(\alpha)}$ is the Ricci tensor of the spacetime represented by the metric g_{ij}, $\mathcal{R} = R_{ij}^{(\alpha)} g^{ij}$, $i, j = 1,2,3, \ldots$, is the Ricci scalar or scalar curvature, g is the universal gravitational constant, c is the speed of light, and ϖ_{ij} A are the components of the stress-energy tensor, ϖ ,describing generically the matter-energy distributions in the spacetime.

3.1. NME Formalisms and EME Consistency Axioms

evaluated the credibility of Tsallis' NME formalism in terms of the four consistency axioms for large systems. Tsallis' formalism, although satisfying the consistency axioms of uniqueness, invariance, and subset independence, defied the axiom of system independence, as it should, due to the existence of 'long-range' interactions.

The credibility of Rényi's NME formalism as a method of inductive reasoning is also investigated in this context in terms of the four EME consistency axioms in Appendix A. Because of the presence

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of long-range interactions, the joint NME state probability distribution of two independent non-extensive systems Q and V challenges the assumption of system independence (cf., [6]). As a result, these NME formalisms are obviously adequate for quantitative analyses of non-extensive dynamic systems with long queue tails and asymptotic power law behaviour.

The devised analytic proof on the credibility of Rényi's NME formalism follows a similar methodology to the one employed for Tsallis's NME formalism by Kouvatsos and Assi and it can be seen in Appendix A.

3.2. A Stable M/G/1 Queue with Long-Range Interactions

Throughout this paper we shall use QM as an acronym for queueing manifold, IG for Information Geometry and QT for queueing theory.

3.3. Background:Shannon's EME State Probability of a Stable M/G/1 Queue

It has been shown (c.f., [53]) that the EME state steady probability of a stable M/G/1 queue that maximises Shannon's entropy function (c.f.,[17]),

$$H(p_{1,s}) = -\sum_{n=0}^{\infty} p(n)\ln(p(n)) \qquad (27)$$

with requirements,

- Normalization, $\sum_{n=0}^{\infty} p(n) = 1$ (28)
- SU, $p_{1,s}(0) = \sum_{n=0}^{\infty} h(n)p(n) = 1 - \rho = \frac{\lambda}{\mu}$ (29)

  where $h(n) = 1$ for $n = 0$ and $h(n) = 0$ for $n = 1, 2, \ldots$ (30)
- P-K MQL, $< n > = \sum_{n=0}^{\infty} np(n) = \frac{\rho}{2}(1 + \frac{1+\rho C_s^2}{1-\rho})$ (31)

  reads

$$p(n) = \begin{cases} p(0), & n = 0 \\ p(0)\tau_s x^n & n > 0 \end{cases} \qquad (32)$$

where $p(0) = 1 - \rho$, $\tau_s = 2/(1+ C_{s,}^2)$ and $x = \frac{<n>- \rho}{<n>}$.

3.4. KLDF

Theorem 1

KLDF, namely, $p_{KL}(n)$,under (28)-(31) reads as

$$p_{KL}(n) = \begin{cases} p_{KL}(0) & n = 0 \\ \frac{p_{KL}(0)\tau_s\, y^n}{q(0)} & n > 0 \end{cases} \qquad (33, KL)$$

where the initial state probabilities $p_{KL}(0)$ satisfy the SU constraint (3.3), namely:

$$p_{KL}(0) = 1 - \rho \qquad (34)$$

Such that

$$\tau_s = \frac{2}{1+ C_s^2} \qquad (35)$$

$$y = \frac{<n>-\rho}{<n>} = \frac{\rho}{\rho+(\tau_{KL})p_{KL}(0)},\ y^n = q(n)x^n, MQL = < n > = \frac{\rho}{2}\left(1 + \frac{\rho C_{s,KL}^2}{1-\rho}\right) \qquad (36)$$

with

$$\frac{\rho(1-y)}{(1-\rho)y} = \frac{\tau_s}{q(0)} = \tau_{KL} \qquad (37)$$

Proof

The Lagrangian, follows by maximising KLDF under (28)-(31) to satisfy:

$$[\sum_{n=0}^{\infty} p_{KL}(n)\, ln\left(\frac{p_{KL}(n)}{q(n)}\right) - \alpha(\sum_{n=0}^{\infty} h(n)) + (\beta - 1)(\sum_{n=0}^{\infty} 1) - \gamma(\sum_{n=0}^{\infty} n)] = 0 \qquad (38)$$

Hence,

$$ln\left(\frac{p_{KL}(n)}{q(n)}\right) + 1 + (\beta - 1) - \alpha h(n) + \gamma n = 0$$

Therefore, it is implied that $p_{Kl}(n)$ will be:

$$p_{KL}(n) = q(n)(e^{-\beta})\,(e^{\alpha h(n)})\,(e^{-\gamma n}) \quad (39)$$

$n = 0$, (39)translates to

$$p_{KL}(0) = q(0)(e^{-\beta})\,(e^{\alpha}) \quad (40)$$

Linking (39) with (40) implies

$$p_{KL}(n) = \frac{p_{KL}(0)q(n)\tau_s x^n}{q(0)}\ , x = e^{-\gamma} \in (0,1), \tau_s = e^{-\alpha} \in (0,1) \quad (41)$$

Hence, we have

Define , $y^n = q(n)x^n, y = \frac{<n>-\rho}{<n>}$, implies that $p_{KL}(n)$ will get the form, namely:

$$p_{Kl}(n) = \begin{cases} p_{KL}(0) & n = 0 \\ \frac{p_{KL}(0)\tau_s y^n}{q(0)} & n > 0 \end{cases} \quad (33, KL)$$

Clearly holds that:

$1 = p_{KL}(0) + p_{KL}(0)\tau_s \frac{y}{(1-y)q(0)}$, implying $\frac{\rho(1-y)}{(1-\rho)y} = \tau_{KL}$, which clearly implies $y = \frac{\rho}{\rho+(\tau_{Kl})p_{KL}(0)}$. This proves the required results.

Clearly, from the above devised result

$$p_{KL}(n) = \begin{cases} p_{KL}(0) & n = 0 \\ \frac{p_{KL}(0)\tau_s y^n}{q(0)} & n > 0 \end{cases} \quad (33, KL)$$

It is known that $q(0) \in (0,1)$. it is evident that as $q(0) \to 1$, the KL formalism in (3.7,KL) will get the form, namely:

$$p_{Sh}(n) = \begin{cases} p_{KL}(0) & n = 0 \\ p_{KL}(0)\tau_s y^n & n > 0 \end{cases}$$

which is the Shannonian formalism of the stable MG1 queueing system obtained as in [53].This shows the strength of the newly devised KL formalism.

3.5. Exact KL NME State Probabilities with Distinct GE_{KL}-type Service Time Distributions

Theorem 2

The KL NME formalism, $p_{KL}(n)$, are exact when PDFs of service time read as

$$f_{s,KL}(t) = (1-\tau_{KL})u_0(t) + \mu(\tau_{KL})^2 e^{-\mu\tau_{KL}} \quad (42)$$

where $u_0(t)$ reads as

$$u_0(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad (43)$$

such that $\int_{-\infty}^{\infty} u_0(t) = 1$ and $\tau_s = 2/(1+C_s^2)$, $\tau_{KL} = \frac{\tau_s}{q(0)}$

Proof

$Q_{KL}(z)$(c.f., [54])for $p_{KL}(n)$ (c.f., 43, KL) reads as

$$Q_{KL}(z) = \sum_{n=0}^{\infty} p_{KL}(n)z^n, \ |z| < 1 \quad (44)$$

Hence,by replacing $p_{KL}(n)$ of (33,KL) into (44), it follows that:

$$Q_{KL}(z) = \sum_{n=0}^{\infty} p_{KL}(n)z^n = p_{KL}(0) + \frac{p_{KL}(0)\tau_{KL}yz}{(1-yz)} = \frac{p_{Kl}(0)(1-yz(1-\tau_{KL}))}{1-yz} (3.21)$$

Following [41],

$$Q_{Kl}(z) = \frac{p_{KL}(0)(1-z)(F^*_{s,KL}(\lambda-\lambda z))}{F^*_{s,KL}(\lambda-\lambda z)-z} \quad (45)$$

where

$$F^*_{s,Kl}(\theta) = E[e^{-\theta s}] = \int_0^{\infty} e^{-\theta t} f_{s,Kl}(t)dt$$

(46)

Hence,

$$\frac{(1-z)(F^*_{s,KL}(\lambda-\lambda z))}{F^*_{s,KL}(\lambda-\lambda z)-z} = \frac{(1-yz(1-\tau_{KL}))}{1-yz} \quad (47)$$

Following (47), yields

$$F^*_{s,KL}(\lambda-\lambda z) = \frac{z((1-yz(1-\tau_{KL}))}{((1-yz(1-\tau_{KL}))-(1-z)(1-yz))} = \frac{(1-yz(1-\tau_{KL}))}{y\tau_{KL}-yz+1} \quad (48)$$

Define, $\lambda - \lambda z = \theta$. Therefore, it holds that $z = 1 - \frac{\theta}{\lambda}$ (49). Combining (48) with (49), the reader can check that after few algebraic steps that

$$F^*_{s,KL}(\theta) = \frac{\mu\tau_{KL}+\theta(1-\tau_{KL})}{\theta+\mu\tau_{KL}} = (1-\tau_{KL}) + \frac{\mu(\tau_{KL})^2}{\theta+\mu\tau_{KL}} \quad (50)$$

By inverting Laplace-StieltjesTransform, $F^*_{s,KL}(\theta)$the GEKI-type *pdf* $f_{s,KL}(t)$(c.f., (34)) follows.

It is observed that as $q(0) \to 1$, $\tau_{KL} \to \tau_s$, which reduces to the Shannonian limiting case obtained (c,f., [53]).

Corollary 2.1

The CDF $F_{s,KL}(t)$o f the GEKL type of service time with the PDFs $f_{s,KL}(t)$ of Theorem 2 is captured fully by $F_{s,Kl}(t)\}$, which reads as

$$F_{s,KL}(t) = 1 - \tau_{KL} e^{-\mu\tau_{KL}t} \quad (51)$$

where $\tau_s = 2/(1+ C^2_{s,})$, $\tau_{KL} = \frac{\tau_s}{q(0)}$.

Proof

We have

$$F_{s,KL}(t) = \int_0^t f_{s,KL}(x)\,dx = \int_0^t (1-\tau_{KL})u_0(x)\,dx + \mu\tau_{KL}\int_0^t e^{-\mu\tau_{KL}x}\,dx$$

$$= (1-\tau_{KL}) + \frac{\mu\tau_{KL}}{\mu\tau_{KL}}(1 - e^{-\mu\tau_{KL}t})$$

$$= 1 - \tau_{KL}e^{-\mu\tau_{KL}t}$$ QED (c.f., (51))

For $q(0) \to 1$, the novel derivation (51) reduces to the formula in [53], $F_s(t) = 1 - \tau_s e^{-\tau_s\mu t}$ with $\tau_s = \frac{2}{C_s^2+1}$.

Corollary 2.2

Following (42), we have

$$E(s_{Kl}) = \frac{1}{\mu} \quad (52)$$

$$E(s^2_{Kl}) = \frac{2}{\mu^2\tau_{KL}} \quad (53)$$

$$C^2_{s,KL} = \frac{E(s^2_{KL})}{(E(S_{KL}))^2} - 1 = \frac{(2-\tau_{KL})}{\tau_{KL}} \quad (54)$$

where $\tau_{KL} = \frac{\tau_s}{q(0)}$, $\tau_s = 2/(1+ C_s^2)$.

Proof

The mean of $S_{KL}$is given by

$$E(S_{Kl}) = \int_0^\infty t\,f_{s.KL}(t)dt = \int_0^\infty t\mu\tau_{KL}\,e^{-\mu\tau_{KL}t}dt = \mu\tau_{KL}\int_0^\infty t\,e^{-\mu\tau_{KL}t}dt \quad (55)$$

Introducing

$$\Gamma(m) = \int_0^\infty w^{m-1}\,e^{-w}dw \quad (56)$$

and substituting $w = \mu\tau_{KL}t$ on (56) and since $\Gamma(2) = 1$, it is implied that $E(s_{Kl}) = \frac{\mu\tau_{KL}}{\mu^2(\tau_{KL})^2}\Gamma(2) = \frac{1}{\mu}$.

Moreover,

$$E(s^2_{Kl}) = \int_0^\infty t^2\,f_{s,KL}(t)dt = \int_0^\infty t^2\mu\tau_{KL}\,e^{-\mu\tau_{KL}t}dt = \mu\tau_{KL}\int_0^\infty t^2\,e^{-\mu\tau_{KL}t}dt \quad (57)$$

and setting $w = \mu\tau_{KL}t$, $E(s^2_{Kl})$ is given by (57) and subsequently, $C^2_{s,KL}$by (54).

As $q(0) \to 1$, the new derivations (52)-(54) reduce to that in [53], namely:

$$E(S) = \frac{1}{\mu} \quad (58)$$

$$E(S^2) = \frac{2}{\mu^2\tau_{KL}} \quad (59)$$

$$C^2_{s,KL} = \frac{E(S^2)}{(E(S))^2} - 1 = \frac{(2-\tau_{KL})}{\tau_{KL}} \quad (60)$$

with $\tau_{KL} = \frac{\tau_s}{q(0)}$, $\tau_s = 2/(1+C_s^2)$.

4. THE THRESHOLD THEOREMS OF KL FORMALISM, $F_{s,Kl}(t)$and $C^2_{s,KL}$ FOR THE UNDERLYING MANIFOLD

Preliminary Theorem 4.1[55].

For a well-defined and differentiable function *f* on an open interval (*c*,*d*).

If $f'(x) > 0$ (< 0) for all $x \in (c,d)$,then f is increasing(decreasing) on (c,d). (61)

4.1. The Threshold Theorem of KL Formalism of The Stable MG1 Queueing System

Theorem 4.2 $\mathrm{p_{KL}(n)}$of (33,KL) is
(i)Forever increasing in τ_s ($\tau_s \in (0,1)$).
(ii)Forever decreasing in n.
(iii)Forever increasing in $p_{Kl}(0)$ ($p_{Kl}(0) \in (0,1)$).
(v)Forever decreasing in $q(0)$ ($q(0) \in (0,1)$).
(iv)Forever increasing in $q(\mathrm{n})$ ($q(\mathrm{n}) \in (0,1)$).
Proof
(i)By the preliminary theorem, it suffices to show that $\frac{\partial \mathrm{p_{KL}(n)}}{\partial \tau_s} > 0 \; for\; all\; \tau_s$ ($\tau_s \in (0,1)$). We have

$$\frac{\partial \mathrm{p_{KL}(n)}}{\partial \tau_s} = \frac{p_{Kl}(0)\mathrm{y^n}}{\mathbf{q(0)}} = \frac{\mathrm{p_{KL}(n)}}{\tau_s} > 0 \qquad (62)$$

and (i) follows.
(ii)By the preliminary theorem, it suffices to show that $\frac{\partial \mathrm{p_{Kl}(n)}}{\partial n} < 0 \; for\; all\; n = 1,2,3,..$ (63)
The reader can see that $\frac{\partial \mathrm{p_{Kl}(n)}}{\partial n} = \mathrm{p_{Kl}(n)}\, lny$ (4.5). Since $lny < 0$ $(because\; 0 < y < 1)$, it is implied by (63) that :

$$\frac{\partial \mathrm{p_{Kl}(n)}}{\partial n} < 0 \qquad (64)$$

and(ii)follows.
Engaging the same procedure proves (iii) and (iv).
As for (v), we can see that:
$\frac{\partial \mathrm{p_{KL}(n)}}{\partial q(0)} = -\frac{p_{Kl}(0)\tau_s \mathrm{y^n}}{(q(0))^2} = -\frac{\mathrm{p_{KL}(n)}}{q(0)} < 0$ (4.7), which directly by Theorem (4.1) proves (v).

4.2 The Threshold Theorem of $\boldsymbol{F_{s,Kl}(t)}$of The Stable MG1 Queueing System

Theorem 4.3 $F_{s,Kl}(t)$of (51) is
(i)Forever increasing in t ($t \geq 0$).
(ii)Forever increasing in τ_{KL} if and only if:

$$t > \frac{1}{\mu \tau_{KL}} \qquad (65)$$

And is forever decreasing in τ_{KL} if and only if:

$$t < \frac{1}{\mu \tau_{KL}} \qquad (66)$$

(iii)Forever increasing in μ ($\mu > 1$).
Proof
(i)By the preliminary theorem (4.1), we must prove:

$$\frac{\partial \mathrm{F_{s,KL}(t)}}{\partial t} > 0 \; for\; all\; \infty > t \geq 0 \qquad (67)$$

We have

$$\frac{\partial \mathrm{F_{s,KL}(t)}}{\partial t} = \mu(\tau_{KL})^2 e^{-\mu\tau_{KL}} > 0 \qquad (68)$$

and (i) follows.
(ii)By the preliminary theorem (4.1), it suffices to show that:

$$\frac{\partial \mathrm{F_{s,KL}(t)}}{\partial \tau_{KL}} > 0 \;\; \text{if and only if } t > \frac{1}{\mu \tau_{KL}}$$

The reader can check that:

$$\frac{\partial \mathrm{F_{s,KL}(t)}}{\partial \tau_{KL}} = e^{-\mu\tau_{KL}t}(\mu t \tau_{KL} - 1) \qquad (69)$$

Since $e^{-\mu\tau_{KL}t} > 0$, then it holds that by (69) that:

$$\frac{\partial \mathrm{F_{s,KL}(t)}}{\partial \tau_{KL}} > 0 \; if\; and\; only\; if\; (\mu t \tau_{KL} - 1) > 0 \qquad (70)$$

which by the preliminary theorem (4.1), the proof follows.

Engaging the same procedure, the remaining proof is immediate.

It is observed that Theorem (4.3), part (ii) presents a novel temporal threshold for $F_{s,KL}(t)$. This temporal threshold is significantly dependent on μ and τ_{KL}, which is , $\tau_s = \frac{2}{1+C_s^2}$ impcated , the initial boundary steady state probaility of the comparable distribution q(0).

Also, it is clearly obvious that this novel templural thrshold is influenced by the newly devised $C^2_{s,KL}$, $\tau_{KL} = \frac{2}{1+C^2_{s,KL}}$

In the following section, it is obtained that $C^2_{s,KL}$ of (60) is forever decreasing in τ_{KL}.

4.2. The Threshold Theorem of $C^2_{s,KL}$ of The Stable MG1 Queueing System

Theorem 4.4 $C^2_{s,KL}$ is forever decreasing in τ_{KL} for all $\tau_{KL}\epsilon(0,1)$

Proof

By the preliminary theorem (4.1), it is enough to show that:

$$\frac{\partial C^2_{s,KL}}{\partial \tau_{KL}} < 0 \text{ for all } \tau_{KL}\epsilon(0,1) \quad (71)$$

By Corollary (2.2), it is obtained that:

$$C^2_{s,KL} = \frac{E(s^2_{KL})}{(E(S_{KL}))^2} - 1 = \frac{(2-\tau_{KL})}{\tau_{KL}} = \frac{2}{\tau_{KL}} - 1 \quad (72)$$

Hence, it holds that:

$$\frac{\partial C^2_{s,KL}}{\partial \tau_{KL}} = -\left(\frac{2}{(\tau_{KL})^2}\right) < 0 \quad (73)$$

By (73), the required result follows.

5. FIM and IFIM for KLDF manifold

According to Kouvatsos [53], the maximum entropy state probability of the generalized geometric solution of a stable M/G/1 queue, subject to normalisation, mean queue length (MQL), L and server utilisation, ρ(<1) is given by:

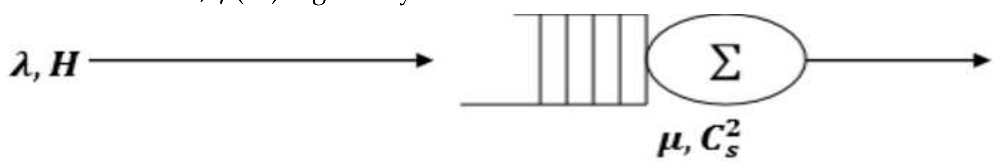

Figure 6. A Stable M/G/1 queue.

Recall that $p_{KL}(n)$ (c.f., (33, KL) is:

$$p_{KL}(n) = \begin{cases} p_{KL}(0) & n = 0 \\ \frac{p_{KL}(0)\tau_s\, y^n}{q(0)} & n > 0 \end{cases} \quad (33, KL)$$

where the initial state probabilities $p_{KL}(0)$ satisfy the SU constraint (3.3), namely:

$$p_{KL}(0) = 1 - \rho \quad (74)$$

Such that

$$\tau_s = \frac{2}{1+C_s^2} \quad (75)$$

$$y = \frac{<n>-\rho}{<n>} = \frac{\rho}{\rho+(\tau_{KL})p_{KL}(0)}, \quad y^n = q(n)x^n \quad (76)$$

with

$$\frac{\rho(1-y)}{(1-\rho)y} = \frac{\tau_s}{q(0)} = \tau_{KL} \quad (77)$$

Theorem 5.1. The underlying manifold satisfies:

(i) $[g_{ij}] = \begin{pmatrix} \frac{-1}{(1-\rho_p)^2} & 0 & 0 \\ 0 & \frac{1}{(1-\rho_q)^2} & 0 \\ 0 & 0 & \frac{-1}{(\tau_s)^2} \end{pmatrix}$ (78)

(ii) IFIM reads as

$$[g^{ij}] = ([g_{ij}])^{-1} = \frac{adj[g_{ij}]}{\Delta} = \begin{pmatrix} -(1-\rho_p)^2 & 0 & 0 \\ 0 & (1-\rho_q)^2 & 0 \\ 0 & 0 & -(\tau_s)^2 \end{pmatrix} \quad (79)$$

Proof

(i) We have

Case I: $p_{KL}(0) = 1 - \rho$

$$\mathcal{L}(x;\theta) = ln(p(x;\theta)) = ln(1-\rho), \theta = \theta_1 = \rho \quad (80)$$

$\Psi(\theta) = -ln(1-\rho)$ (81)

Thus,

$\partial_1 = \frac{\partial\Psi}{\partial\rho} = \frac{1}{1-\rho}$,

(83)

$\partial_1\partial_1 = \frac{\partial^2\Psi}{\partial\rho^2} = \frac{1}{(1-\rho)^2}$ (84)

Therefore, $[g_{ij}]$ is:

$[g_{ij}] = \left[\frac{\partial^2\Psi}{\partial\rho^2}\right] = \left[\frac{1}{(1-\rho)^2}\right]$ (85)

IFIM is:

$[g^{ij}] = [g_{ij}]^{-1} = [(1-\rho)^2]$

(86)

Moreover, it follows that:

$\Gamma^{(\alpha)}_{11,1} = (\frac{1-\alpha}{2})(\partial_1\partial_1\partial_1(\Psi(\theta))) = (\frac{1-\alpha}{(1-\rho)^3})$

(87)

$\Gamma^{1(\alpha)}_{11} = \Gamma^{(\alpha)}_{11,1}(g^{11}) = (\frac{1-\alpha}{(1-\rho)^3})((1-\rho)^2) = \frac{1-\alpha}{(1-\rho)}$ (88)

$\Gamma^{1(0)}_{11} = \frac{1}{(1-\rho)}$ (89)

<u>Case II:</u> Following similar steps to <u>Case I</u>, when $n \geq 1$, $\frac{p_{KL}(0)\tau_s y^n}{q(0)}$. We have

$$\mathcal{L}(y;\theta) = ln(p(y;\theta)) = ln(1-\rho_p) - ln(1-\rho_q) + ln(\tau_s) + (n)ln(y), \quad (5.13)$$

where $\theta = (\theta_1, \theta_2.\theta_3) = (\rho_p, \rho_q, \tau_s)$, $1-\rho_p = p_{KL}(0), 1-\rho_q = q(0)$.

To this end, we have

$\Psi(\theta) = ln\left(\frac{(1-\rho_q)}{\tau_s(1-\rho_p)}\right)$ (90)

In analogy to the above proof, after some algebraic manipulation, it clearly follows that the Fisher Information Matrix is given by:

$$[g_{ij}] = \begin{pmatrix} \frac{-1}{(1-\rho_p)^2} & 0 & 0 \\ 0 & \frac{1}{(1-\rho_q)^2} & 0 \\ 0 & 0 & \frac{-1}{(\tau_s)^2} \end{pmatrix} \quad (91)$$

Hence, expression (ii) follows.

(iii) Similarly, after some lengthy analytic derivations, we have $\Delta = \det [g_{ij}] = \frac{1}{(1-\rho_q)^2(1-\rho_p)^2(\tau_s)^2} \neq 0$. Hence, IFIM. To this end, after some algebraic steps, it follows that IFIM reads:

$$[g^{ij}] = \begin{pmatrix} -(1-\rho_p)^2 & 0 & 0 \\ 0 & (1-\rho_q)^2 & 0 \\ 0 & 0 & -(\tau_s)^2 \end{pmatrix} \quad (92)$$

This proves (iii).

6 The α-connection of the KLDF manifold

We have

$\Gamma_{11,1}^{(\alpha)} = \frac{(\alpha-1)}{(1-\rho_p)^3}$ (93)

Engaging the same procedure, the reader can check that:

$\Gamma_{22,2}^{(\alpha)} = \frac{(1-\alpha)}{(1-\rho_q)^3}$ (94)

$\Gamma_{33,3}^{(\alpha)} = \frac{(1-\alpha)}{(\tau_s)^3}$ (95)

In a similar fashion, the remaining components are computed, and they are equal to zero.

By using (13), the following expressions follow:

$\Gamma_{11}^{1(\alpha)} = \Gamma_{11,1}^{(\alpha)} g^{11} + \Gamma_{11,2}^{(\alpha)} g^{21}$, which is after some lengthy calculations:

$\Gamma_{11}^{1(\alpha)} = \frac{1-\alpha}{(1-\rho_p)}, \Gamma_{11}^{1(0)} = \frac{1}{(1-\rho_p)}$ (96)

$\Gamma_{22}^{2(\alpha)} = \Gamma_{22,1}^{(\alpha)} g^{21} + \Gamma_{22,2}^{(\alpha)} g^{22}$ which is

$\Gamma_{22}^{2(\alpha)} = \frac{1-\alpha}{(1-\rho_q)}, \Gamma_{22}^{2(0)} = \frac{1}{(1-\rho_q)}$ (97)

$\Gamma_{33}^{3(\alpha)} = \frac{\alpha-1}{(\tau_s)}, \Gamma_{22}^{2(0)} = -\frac{1}{(\tau_s)}$ (98)

Engaging the same approach, the remaining components are computed, and they are equal to zero.

7. The Compressibility (Non-Solonoidability) of KL Formalism of the Stable M/G/1 QM

Theorem (7.1) Subject to the KL Formalism, the stable M/G/1 QM could be compressible or (non-solenoidal)

Proof

By the Divergence Theorem (c.f.,[27]), it is enough to show that the J-divergence of (8) satisfies

$J(p,q) \neq 0$ (99)

We have

$J(p,q) = K(p.q) + K(q,p) = \sum_{n=0}^{\infty}(p(n) ln(\frac{p(n)}{q(n)}) + q(n) ln(\frac{q(n)}{p(n)}))$ (100)

It is determined that:

$$P_{KL}(n) = \begin{cases} p_{KL}(0) & n = 0 \\ \frac{p_{KL}(0)\tau_s y^n}{q(0)} & n > 0 \end{cases} \quad (33, KL)$$

Hence, (100) will be given as

$J(p,q) = [(ln\left(\frac{p(0)}{q(0)}\right)^{p(0)} + ln\left(\frac{q(0)}{p(0)}\right)^{q(0)} + ln\left(\frac{p(0)}{q(0)}\right)(\sum_{n=1}^{\infty}(\tau_{sp} y_p^n p(0) - \sum_{n=1}^{\infty} \tau_{sq} y_q^n q(0))$
$+ ln\left(\frac{\tau_{sp}}{\tau_{sq}}\right)(\sum_{n=1}^{\infty}(\tau_{sp} y_p^n p(0) - \sum_{n=1}^{\infty} \tau_{sq} y_q^n q(0)) + ln\left(\frac{y_p}{y_q}\right)(p(0)\sum_{n=1}^{\infty}(n y_p^n -$
$q(0)\sum_{n=1}^{\infty} n y_q^n)]$ (101)

By

$\sum_{k=0}^{\infty}(A + Bd)x^k = \frac{A}{1-x} + \frac{xd}{(1-x)^2}$ (102)

$J(p,q)$ will be given as

$J(p,q) = (ln\left(\frac{p(0)}{q(0)}\right)^{(p(0-q(0))} + ln\left(\frac{p(0)}{q(0)}\right)((\tau_{sp} p(0)\left(\frac{1}{1-y_p} - 1\right) - \tau_{sq} q(0)\left(\frac{1}{1-y_q} - 1\right))$

$+ ln\left(\frac{\tau_{sp}}{\tau_{sq}}\right)(\left(\tau_{sp} p(0)\left(\frac{1}{1-y_p} - 1\right) - \tau_{sq} q(0)\left(\frac{1}{1-y_q} - 1\right)\right) + ln\left(\frac{y_p}{y_q}\right)\left(p(0)\frac{y_p}{(1-y_p)^2} - q(0)\frac{y_q}{(1-y_q)^2}\right))$

(103)

Therefore,

$J(p,q) = (ln\left(\frac{p(0)}{q(0)}\right)^{(p(0-q(0))} + ln\left(\frac{\tau_{sp} p(0)}{\tau_{sq} q(0)}\right)((\tau_{sp} p(0)\left(\frac{1}{1-y_p} - 1\right) - \tau_{sq} q(0)\left(\frac{1}{1-y_q} - 1\right))$

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$+ln\left(\frac{y_p}{y_q}\right)\left(p(0)\frac{y_p}{(1-y_p)^2} - q(0)\frac{y_q}{(1-y_q)^2}\right))$ (104)

Since $y_p, y_q, p(0), q(0), \tau_{sp}, \tau_{sq} \in (0,1)$, $J(p,q) = 0$ if and only if $y_p = y_q, p(0) = q(0), \tau_{sp} = \tau_{sq}$. So, because $p(n)$ and $q(n)$are distinct, the result follows.

8. The Developability of the Stable M/G/1 QM, the Positivity of Its Ricci Curvature Tensor and The Threshold Theorem of Ricci Curvature Tensor

Theorem 8.1 KLDF satisfies:

(i)Developability

(ii) RCT$\neq 0$

Proof

To prove (i), we must prove that:

$K^{(\alpha)} = \frac{R^{(\alpha)}_{1212}}{det(g_{ij})}$

(16)

is zero.

We have

$R^{(\alpha)}_{ijkl} = [(\partial_j \Gamma^{s(\alpha)}_{ik} - \partial_i \Gamma^{s(\alpha)}_{jk})g_{sl} + (\Gamma^{(\alpha)}_{jt,l}\Gamma^{t(\alpha)}_{ik} - \Gamma^{(\alpha)}_{it,l}\Gamma^{t(\alpha)}_{jk})]$ (13)

where $\Gamma^{k(\alpha)}_{ij} = \Gamma^{(\alpha)}_{ij,s} g^{sk}$

The reader can check that:

$R^{(\alpha)}_{1212} = [(\partial_2(\Gamma^{1(\alpha)}_{11} + \Gamma^{2(\alpha)}_{11}) - \partial_1(\Gamma^{1(\alpha)}_{21} + \Gamma^{2(\alpha)}_{21}))(g_{12} + g_{22} + g_{32}) + (\Gamma^{(\alpha)}_{21,2}\Gamma^{1(\alpha)}_{11} + \Gamma^{(\alpha)}_{22,2}\Gamma^{2(\alpha)}_{11} + \Gamma^{(\alpha)}_{23,2}\Gamma^{3(\alpha)}_{11}) - (\Gamma^{(\alpha)}_{11,2}\Gamma^{1(\alpha)}_{21} + \Gamma^{(\alpha)}_{12,2}\Gamma^{2(\alpha)}_{21} + \Gamma^{(\alpha)}_{13,2}\Gamma^{3(\alpha)}_{21})] = 0$

(105)

$\Delta = det(g_{ij}) = \frac{1}{(1-\rho_q)^2(1-\rho_p)^2(\tau_s)^2} \neq 0$. Hence, $K^{(\alpha)} = \frac{R^{(\alpha)}_{1212}}{det(g_{ij})} = 0$, which proves the result.

To prove (ii), we need to show that:

$$R^{(\alpha)}_{ik} = R^{(\alpha)}_{ijkl} g^{jl} \neq 0 \quad (14)$$

(13) implies:

$$R^{(\alpha)}_{1223} = \frac{\alpha-1}{((1-\rho_q)\tau_s)^2} \neq 0 \quad (106)$$

$$R^{(\alpha)}_{2111} = -\frac{(1-\alpha)^2}{((1-\rho_p))^4} \neq 0 \quad (107)$$

Engaging the mathematical formula (13), it is obtained that:

$R^{(\alpha)}_{1111} = R^{(\alpha)}_{1112} = R^{(\alpha)}_{1113} = R^{(\alpha)}_{1211} = R^{(\alpha)}_{1221} = R^{(\alpha)}_{1222} = R^{(\alpha)}_{1311} = R^{(\alpha)}_{1312} = R^{(\alpha)}_{1313} = R^{(\alpha)}_{1321} = R^{(\alpha)}_{1322} = R^{(\alpha)}_{1323} = R^{(\alpha)}_{1333} = R^{(\alpha)}_{2112} = R^{(\alpha)}_{2121} = R^{(\alpha)}_{2323} = R^{(\alpha)}_{2113} = R^{(\alpha)}_{2131} = R^{(\alpha)}_{2333} = R^{(\alpha)}_{2211} = R^{(\alpha)}_{2212} = R^{(\alpha)}_{2311} = R^{(\alpha)}_{1312} = R^{(\alpha)}_{2332} = R^{(\alpha)}_{2313} = R^{(\alpha)}_{2222} = R^{(\alpha)}_{3111} = R^{(\alpha)}_{3112} = R^{(\alpha)}_{3131} = R^{(\alpha)}_{3232} = R^{(\alpha)}_{3113} = R^{(\alpha)}_{3121} = R^{(\alpha)}_{3222} = R^{(\alpha)}_{3211} = R^{(\alpha)}_{3212} = R^{(\alpha)}_{3213} = R^{(\alpha)}_{3311} = R^{(\alpha)}_{3312} = R^{(\alpha)}_{3313} = R^{(\alpha)}_{3321} = R^{(\alpha)}_{3322} = R^{(\alpha)}_{3323} = R^{(\alpha)}_{3331} = R^{(\alpha)}_{3332} = R^{(\alpha)}_{3333} = 0$ (108)

Therefore,

$R^{(\alpha)}_{11} = R^{(\alpha)}_{12} = R^{(\alpha)}_{13} = R^{(\alpha)}_{22} = R^{(\alpha)}_{23} = R^{(\alpha)}_{31} = R^{(\alpha)}_{32} = R^{(\alpha)}_{33} = 0$

(109)

It is obtained that the only non- zero component is:

$R^{(\alpha)}_{21} = \frac{(1-\alpha)^2}{(1-\rho_p)^2}$ (110)

Therefore, (ii) is proved.

In what follows, a novel threshold theorem for the obtained Ricci component of (110) is devised.

Theorem 8.2 The Ricci component $R^{(\alpha)}_{21}$ of (110) is

(i).Forever increasing in ρ_p for all $\rho_p \in (0,1)$.

(ii)Forever increasing in $the\ parameter\ \alpha, \alpha > 1$

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(iii)Forever decreasing in $the\ parameter\ \alpha, \alpha < 1$
Proof

(i)By the preliminary theorem (4.1) , it suffices to prove that $\frac{\partial R_{21}^{(\alpha)}}{\partial \rho_p} > 0$ (8.10). We have

$$\frac{\partial R_{21}^{(\alpha)}}{\partial \rho_p} = \frac{(1-\alpha)^2}{((1-\rho_p))^3} \qquad (111)$$

By $\rho_p \in (0,1)$, it holds that $((1-\rho_p))^3 > 0$. Hence, the result follows.
As for (ii), we have

$$\frac{\partial R_{21}^{(\alpha)}}{\partial \alpha} = -\frac{(1-\alpha)}{((1-\rho_p))^2} \qquad (112)$$

By $\alpha > 1$, it is implied that$(1-\alpha) < 0$. Hence, we get:

$$\frac{\partial R_{21}^{(\alpha)}}{\partial \alpha} > 0 \qquad (113)$$

Hence, the result follows.
It is notable that the Ricci component is not influenced by the service utilization of the secondary distribution,
ρ_q .
Engaging the same procedure proves (iii).
In the following, some illustrative numerical experiments are obtained.
Numerical Experiments for theorem (8.2)
(i) $\boldsymbol{R_{21}^{(\alpha)}}$ *is* Forever increasing in ρ_p for all $\rho_p \in (0,1)$,
$\alpha = 3, \rho_p \in (0,1)$

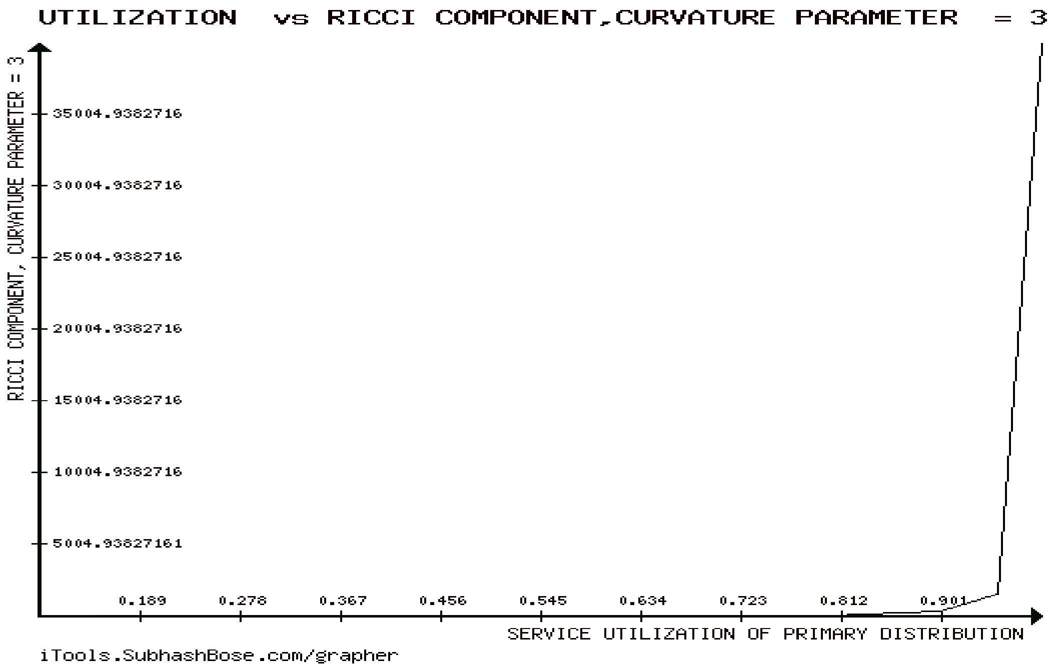

Figure 7.

ii)Forever increasing in $the\ curvature\ parameter\ \alpha, \alpha > 0$
$\alpha > 1, \rho_p = 0.5$

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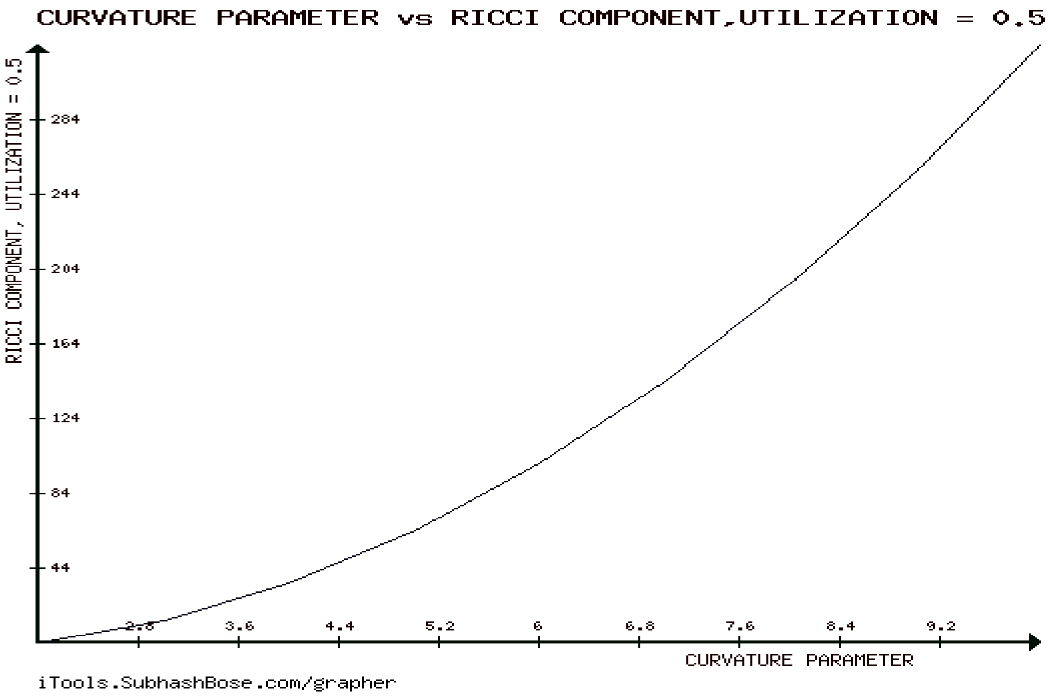

Figure 8.

(iii)Forever decreasing in *the curvature parameter* $\alpha, \alpha < 0$

$\alpha < 1, \rho_p = 0.5$

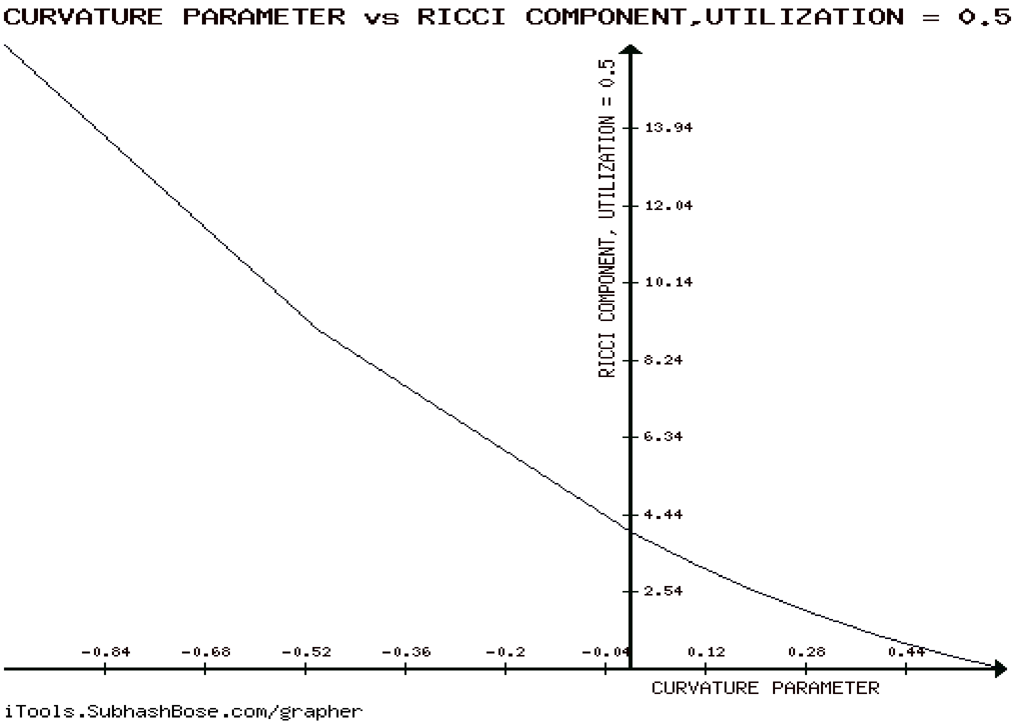

Figure 9.

9. e^A of the underlying manifold

Theorem 10.1 e^A that corresponds to FIM (c.f., (78)) solves $\frac{dx}{dt}$=Ax.

Proof

Recall that:

$$[g_{ij}] = \begin{pmatrix} \frac{-1}{(1-\rho_p)^2} & 0 & 0 \\ 0 & \frac{1}{(1-\rho_q)^2} & 0 \\ 0 & 0 & \frac{-1}{(\tau_s)^2} \end{pmatrix} \quad \text{(c. f., (78))}$$

Rewriting $[g_{ij}]$ in a more simpler form $,[g_{ij}] = \begin{pmatrix} A' & 0 & 0 \\ 0 & \boldsymbol{B'} & 0 \\ 0 & 0 & C' \end{pmatrix}, A' = \frac{-1}{(1-\rho_p)^2}, \boldsymbol{B'} = \frac{1}{(1-\rho_q)^2}, C' = \frac{-1}{(\tau_s)^2}$ (9.1). It follows that:

$$\Phi(\delta) = (\delta) = \det([g_{ij}] - \delta I) = \det \begin{pmatrix} A' - \delta & 0 & 0 \\ 0 & \boldsymbol{B'} - \delta & 0 \\ 0 & 0 & C' - \delta \end{pmatrix} = 0 \ .$$

Therefore, the eigen values are:

$$\delta_{1,2,3} = A', \boldsymbol{B'}, C' \quad (114)$$

It is implied that:

$$D = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix} \quad (115)$$

According to (12), it clearly follows that the matrix $T = T^{-1}$ = unity matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence,

$$e^A = Te^D T^{-1} = \begin{pmatrix} e^{A'} & 0 & 0 \\ 0 & e^{B'} & 0 \\ 0 & 0 & e^{C'} \end{pmatrix} \quad (116)$$

Thid show that e^A solves:

$$\frac{dx}{dt} = \mathrm{Ax} \quad (117)$$

10. Scalar Curvature(Ricci Scalar), $\mathcal{R}$ and Einestein Tensor, $\wp$ and the stress-energy tensor, ϖ of the KLF manifold

This section reports a new discovery of the missing link between information theory, queuing theory and General Relativity, namely the Scalar Curvature(Ricci Scalar), $\mathcal{R}$ and Einestein Tensor, $\wp$ and the stress-energy tensor, ϖ of the underlying Kull- Leibler formalism, KLF for the investigated manifold.

Theorem 10.1 For KLF of the underlying manifold satisfies:

i) $\mathcal{R}$ equals zero.

ii) $\wp$ reads as

$$\wp = \begin{pmatrix} 0 & \frac{(1-\alpha)^2}{(1-\rho)^2} & 0 \\ \frac{(1-\alpha)^2}{(1-\rho)^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (118)$$

iii) The stress-energy tensor, ϖ is given by

$$\wp = \begin{pmatrix} 0 & \frac{c^4 (1-\alpha)^2}{8\pi g (1-\rho)^2} & 0 \\ \frac{c^4 (1-\alpha)^2}{8\pi g (1-\rho)^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (119)$$

iii) In a curved space the area, $A_{curved}(\epsilon)$ approximates to $A_{flat}(\epsilon)$ for some small positive $\epsilon > 0$.

iv) The statement in iii) holds if and only if

$$A_{flat}^{\cdot\cdot}(0) \approx \frac{A_{curved}^{\cdot\cdot}(0) A_{flat}^2(0)}{[A_{flat}(0) A_{curved}(0) - 2(A_{flat}^{\cdot}(0) A_{curved}(0) - A_{curved}^{\cdot}(0) A_{curved}(0))]} \quad (120)$$

Proof

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We have

$\mathcal{R} = R_{ij}^{(\alpha)} g^{ij}, i,j = 1,2,3$ (c.f., (24))

We have

$G_{ij} = R_{ij}^{(\alpha)} - \frac{\mathcal{R}}{2} g_{ij} = \frac{8\pi g \varpi_{ij}}{c^4}, i,j = 1,2,3$ (c. f., (26))

By i), $\mathcal{R} = 0$, which reduces (26) to

$G_{ij} = R_{ij}^{(\alpha)} = \frac{8\pi g \varpi_{ij}}{c^4}, i,j = 1,2,3$ (121)

Hence,

$R_{11}^{(\alpha)} = R_{12}^{(\alpha)} = R_{13}^{(\alpha)} = R_{22}^{(\alpha)} = R_{23}^{(\alpha)} = R_{31}^{(\alpha)} = R_{32}^{(\alpha)} = R_{33}^{(\alpha)} = 0$, $R_{21}^{(\alpha)} = \frac{(1-\alpha)^2}{(1-\rho)^2}$, which implies that

$G_{ij} = R_{ij}^{(\alpha)} = \frac{8\pi g \varpi_{ij}}{c^4} = 0$ in almost all cases expcept only one non − zero component, namely

$G_{21} = R_{21}^{(\alpha)} = \frac{(1-\alpha)^2}{(1-\rho)^2} = \frac{8\pi g \varpi_{21}}{c^4}$ and the remaining components are zero (122)

Clearly, it follows from (122) that both ii) and iii) are satisfied.

iii) holds if and only if:

$\mathcal{R} = lim_{\epsilon\to 0} \frac{6n}{\epsilon^2}[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}] = (6n) lim_{\epsilon\to 0} \frac{1}{\epsilon^2}[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}]$ (25)

Using L'Hopital's rule,

$\mathcal{R} = (6n) lim_{\epsilon\to 0} \frac{1}{\epsilon^2}[1 - \frac{A_{curved}(\epsilon)}{A_{flat}(\epsilon)}]$

$= (\frac{6n}{2}) lim_{\epsilon\to 0} \frac{1}{\epsilon}[\frac{A'_{curved}(\epsilon)A_{curved}(\epsilon) - A'_{flat}(\epsilon)A_{curved}(\epsilon)}{A_{flat}^2(\epsilon)}]$ (Using L'Hopital's rule, (c.f., [56])

$= 3n\,[\frac{A''_{curved}(0)A_{flat}^2(0) - [A_{flat}(0)A_{curved}(0) - 2(A'_{flat}(0)A_{curved}(0) - A'_{curved}(0)A_{curved}(0))]A'_{flat}(0)}{A_{flat}^3(0)}]$

(L'Hopital's rule, (c.f., [56]) (123)

Consequently , iv) holds.

11. The Rényi divergence (RD) , $D_R^\gamma(p||q)$ and the S, AB-divergence, $D_{s,AB}^{\gamma,\eta}(p||q)$ of the KLF manifold

Theorem 11.1 KLF manifold satisfies:

(1)The Rényi divergence (RD) , $D_R^\gamma(p||q)$ (c.f., (2.9)) equals

$$\frac{1}{(\gamma-1)} ln\left(\frac{(p(0))^\gamma (q(0))^{1-\gamma}}{+\left(\frac{p(0)}{q(0)}\right)^2\left(\frac{1+\beta_q}{1+\beta_p}\right)^{\gamma-1}\left(\frac{p(0)}{q(0)}\right)^2\left(\frac{1+\beta_q}{1+\beta_p}\right)^{\gamma-1}}\right) + l\,n\left(E\left[\left(\frac{L_q(L_p-\rho_p)}{L_p(L_q-\rho_q)}\right)^{n(\gamma-1)}\right] - p(0)\right)$$

(124)

(2)The S,AB divergence (RD) , $D_{s,AB}^{\gamma,\eta}(p||q)$ (c.f., (10)) is obtained by:

$$D_{s,AB}^{\gamma,\eta}(p||q) = [\frac{1}{\eta(\eta+\gamma)} ln\left((p(0))^{\gamma+\eta} + E\left(\frac{\rho_p^2(L_p-\rho_p)^{n-1}}{(L_p)^n}\right)^{\gamma+\eta-1} - \frac{\rho_p^2}{(L_p-\rho_p)}\right)$$
$$+ \frac{1}{\gamma(\eta+\gamma)} ln\left((q(0))^{\gamma+\eta} + E\left(\frac{\rho_q^2(L_q-\rho_q)^{n-1}}{(L_q)^n}\right)^{\gamma+\eta-1} - \frac{\rho_q^2}{(L_q-\rho_q)}\right)$$
$$- \frac{1}{\gamma\eta} ln\left((p(0))^\gamma (q(0))^\eta + E\left[\left(\frac{\rho_p^2\rho_q^2((L_p-\rho_p)((L_p-\rho_q))^{n-1}}{(L_pL_q)^n}\right)^{\gamma-1}\right] - \frac{\rho_p^2\rho_q^2}{(L_p-\rho_p)(L_q-\rho_q)}\right)$$

(125)

where $\beta = C_s^2$ for KLF , L_p, L_q are the MQL at the points $p\,,q$ respectively and E defines the expected value.

Proof

(1) The reader can check that for $n \neq 0$,

$$\left(\frac{p(n)}{q(n)}\right)^{\gamma-1}=\left[\left(\frac{p(0)}{q(0)}\right)^{2}\left(\frac{1+\beta_q}{1+\beta_p}\right)^{\gamma-1}\left(\frac{L_q}{L_p}\right)^{n(\gamma-1)}\left(\frac{L_p-\rho_p}{L_q-\rho_q}\right)^{n(\gamma-1)}\right] \quad (126)$$

where β stands for SCV of the underlying manifold, L_p and L_q are the MQL at the points p, q respectively.

We have

$$D_R^{\gamma}(p||q)=\frac{1}{(\gamma-1)}ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma}(q(n))^{1-\gamma}\right) \quad \text{(c.f., (9))}$$

$$=\frac{1}{(\gamma-1)}ln\left(\begin{array}{c}(p(0))^{\gamma}(q(0))^{1-\gamma}\\ +\left(\frac{p(0)}{q(0)}\right)^{2}\left(\frac{1+\beta_q}{1+\beta_p}\right)^{\gamma-1}\left(\frac{p(0)}{q(0)}\right)^{2}\left(\frac{1+\beta_q}{1+\beta_p}\right)^{\gamma-1}\\ +ln\left(E\left[\left(\frac{L_q(L_p-\rho_p)}{L_p(L_q-\rho_q)}\right)^{n(\gamma-1)}\right]-\right.\end{array}\right.$$

$$\left.p(0)\right) \quad \text{(c.f.,(126))}$$

Clearly, p(n) can be re-written for $n>0$ in the form :

$$p(n)=\frac{\rho^{2}(L-\rho)^{n-1}}{L^{n}},\ L=<n>=MQL \quad (127)$$

Now, we have

$$\frac{1}{\eta(\eta+\gamma)}ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma+\eta}\right)=\frac{1}{\eta(\eta+\gamma)}ln\left((p(0))^{\gamma+\eta}+\sum_{n=0}^{\infty}\left(\frac{\rho^{2}(L-\rho)^{n-1}}{L^{n}}\right)^{\gamma+\eta-1}p(n)-\frac{\rho^{2}}{L-\rho}\right)$$

$$=\frac{1}{\eta(\eta+\gamma)}ln\left((p(0))^{\gamma+\eta}+E\left(\frac{\rho_p^{2}(L_p-\rho_p)^{n-1}}{(L_p)^{n}}\right)^{\gamma+\eta-1}-\frac{\rho_p^{2}}{(L_p-\rho_p)}\right) \quad (128)$$

Similarly,

$$\frac{1}{\gamma(\eta+\gamma)}ln\left(\sum_{n=0}^{\infty}(q(n))^{\gamma+\eta}\right)=-\frac{1}{\gamma\eta}ln\left((p(0))^{\gamma}(q(0))^{\eta}+E\left[\left(\frac{\rho_p^{2}\rho_q^{2}((L_p-\rho_p)((L_q-\rho_q))^{n-1}}{(L_pL_q)^{n}}\right)^{\gamma-1}\right]-\frac{\rho_p^{2}\rho_q^{2}}{(L_p-\rho_p)(L_q-\rho_q)}\right) \quad (129)$$

$$D_{s,AB}^{\gamma,\eta}(p||q)=\frac{1}{\eta(\eta+\gamma)}ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma+\eta}\right)+\frac{1}{\gamma(\eta+\gamma)}ln\left(\sum_{n=0}^{\infty}(q(n))^{\gamma+\eta}\right)-\frac{1}{\gamma\eta}ln\left(\sum_{n=0}^{\infty}(p(n))^{\gamma}(q(n))^{\eta}\right) \quad (c.f., (10))$$

for $(\gamma,\eta)\epsilon\mathbb{R}^2$ such that $\gamma\neq 0, \eta\neq 0$ and $\gamma+\eta\neq 0$.

Thus, it follows that:

$$D_{s,AB}^{\gamma,\eta}(p||q)=[\frac{1}{\eta(\eta+\gamma)}ln\left((p(0))^{\gamma+\eta}+E\left(\frac{\rho_p^{2}(L_p-\rho_p)^{n-1}}{(L_p)^{n}}\right)^{\gamma+\eta-1}-\frac{\rho_p^{2}}{(L_p-\rho_p)}\right)$$
$$+\frac{1}{\gamma(\eta+\gamma)}ln\left((q(0))^{\gamma+\eta}+E\left(\frac{\rho_q^{2}(L_q-\rho_q)^{n-1}}{(L_q)^{n}}\right)^{\gamma+\eta-1}-\frac{\rho_q^{2}}{(L_q-\rho_q)}\right)$$
$$-\frac{1}{\gamma\eta}ln\left((p(0))^{\gamma}(q(0))^{\eta}+E\left[\left(\frac{\rho_p^{2}\rho_q^{2}((L_p-\rho_p)((L_p-\rho_q))^{n-1}}{(L_pL_q)^{n}}\right)^{\gamma-1}\right]-\frac{\rho_p^{2}\rho_q^{2}}{(L_p-\rho_p)(L_q-\rho_q)}\right) \quad (130)$$

This proves (2).

12. Numerical Experiments on The Rényi divergence (RD), $D_R^{\gamma}(p||q)$ and the S, AB-divergence, $D_{s,AB}^{\gamma,\eta}(p||q)$ of the KLF of stable M/G/ 1 QM

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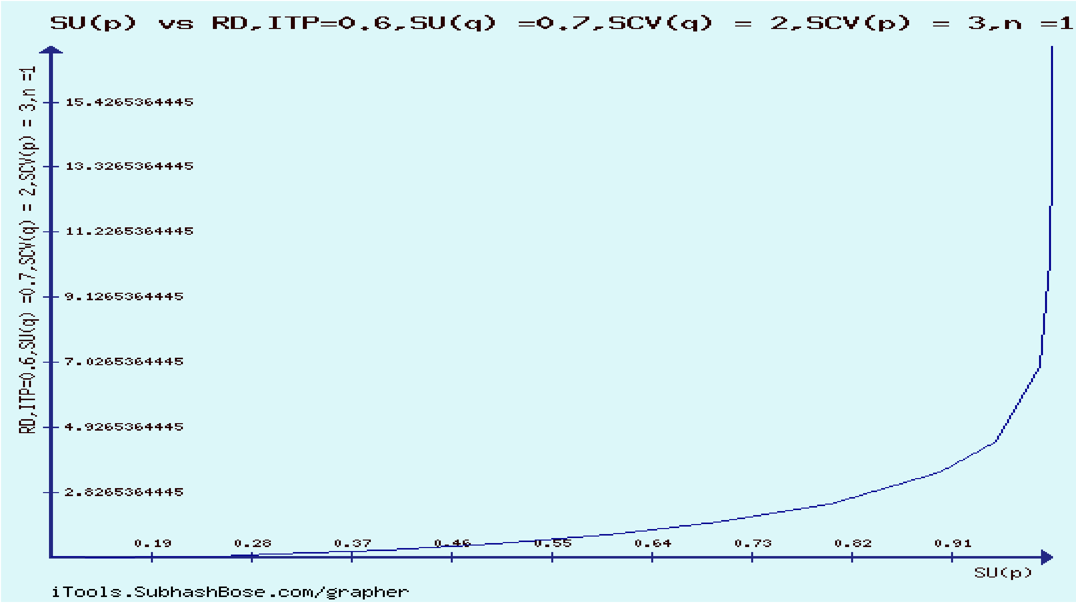

Figure 10.

For non-extensive information theoretic parameter, ITP = γ, figure 10 records the physical phenomenon of increasability of RD by the increase of server utilization and progressively RD approaches infinity as server utilization approaches unity, which drifts the underlying M/G/1 into instability phase. It is slightly observable that at the lower values of server utilization, RD decreases. This is quite clear by observing figure 11. This clearly shows the queueing impact represented by server utilization on RD.

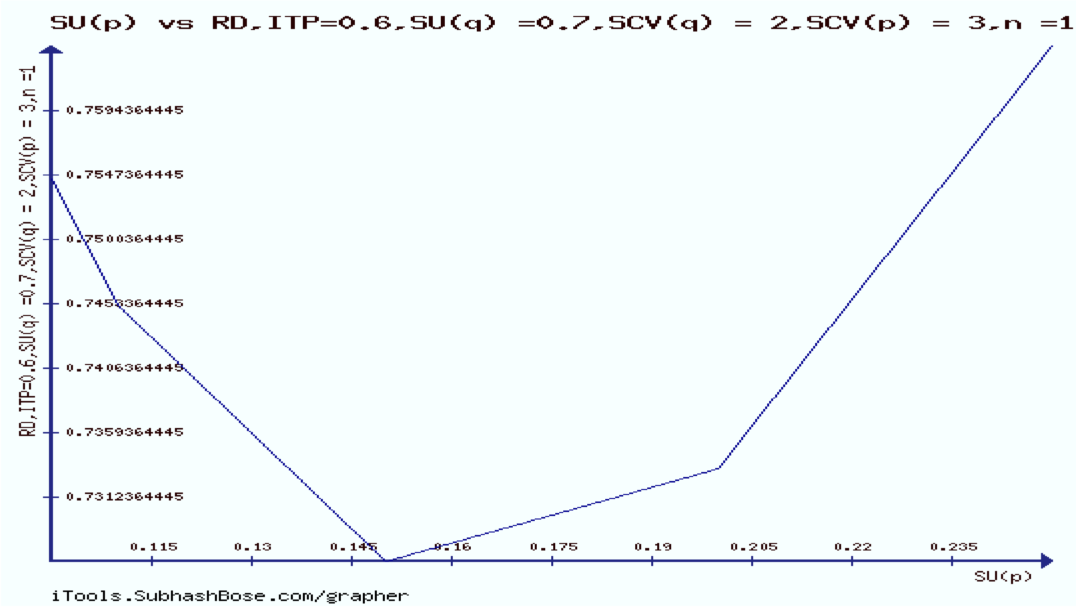

Figure 11.

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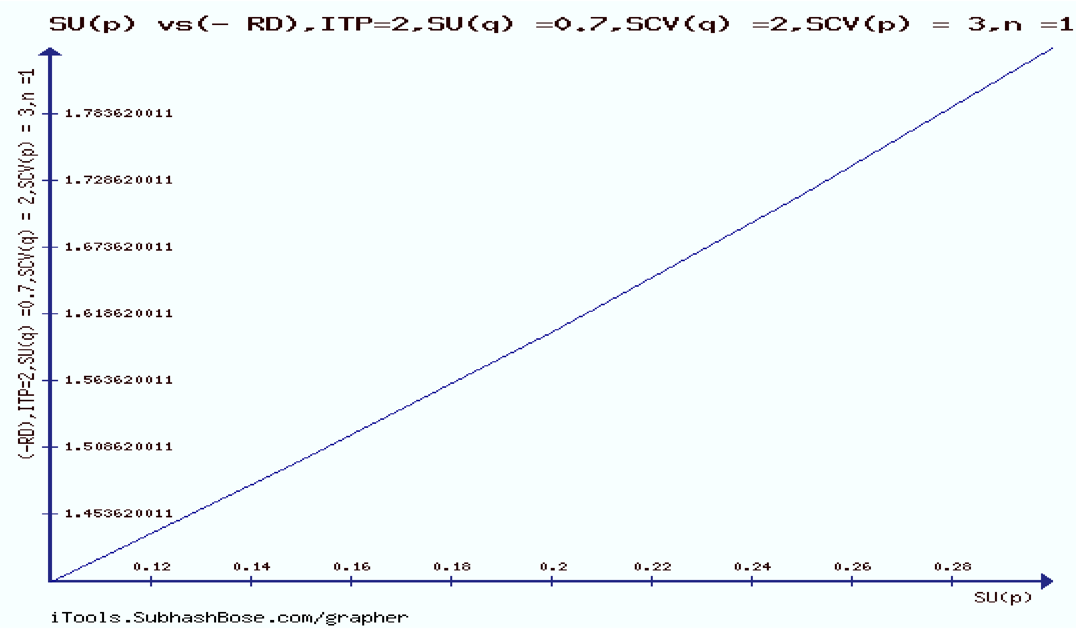

Figure 12.

It is observed by figure 12, that for extensive ITP, the dual information theoretic and queueing theoretic significant impact on RD is clear as RD decreases immensely by the increase of server utilization until it approaches $-\infty$ as server utilization approaches unity(instability phase). Combining all the numerical findings, RD is significantly influenced by both ρ and γ.

The increasability of $D^{0.6,0.4}_{s,AB}(p||q)$ is shown by figure 13 as in below. As ρ progresses to increase beyond 0.25, $D^{0.6,0.4}_{s,AB}(p||q)$ attains complex values.

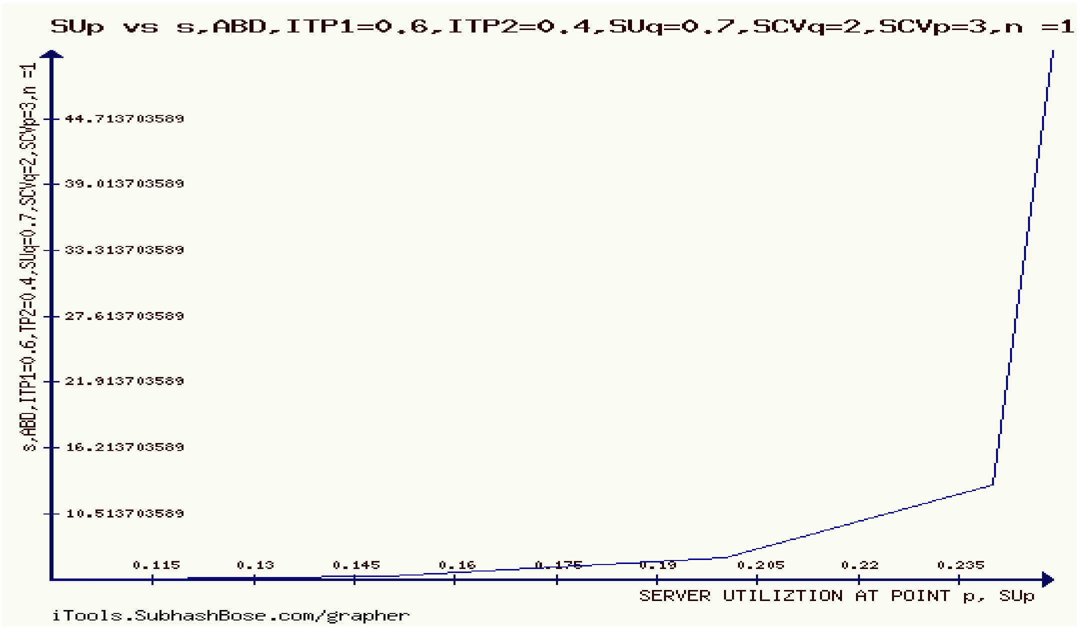

Figure 13. i.

13. Unification of Queueing Systems and KLF of stable M/G/ 1 QM

Define a function φ_α between the stable M/G/1 QM into its IG structure by

$\varphi_\alpha(\rho) = R^{(0)}_{21} = \frac{(1-\alpha)^2}{(1-\rho)^2}$, $\alpha \neq 1$ (c.f., (122)) (131)

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The following theorem is a groundbreaking functional approach which unifies queueing theory with IG.

Theorem 13.1 For a stable M/G/1 QM, the function φ_α of (13.1) generates a family of functionals satisfying the following:

i)φ_α is well-defined.

ii)φ_α is one-to-one.

iii)φ_α is surjective.

iv)φ_α^{-1} exists if and only $\alpha \in (1-\sqrt{\rho}, 1)$ and is given by $\varphi_\alpha^{-1}(\rho) = 1 - \frac{1-\alpha}{\sqrt{\rho}}$, $\rho \in (0,1)$.

Proof

To prove i), let $\varphi_\alpha(\rho_1) = \varphi_\alpha(\rho_2)$ such that $\rho_1 \neq \rho_2$, $\rho_1, \rho_2 \in (0,1)$. Hence, it follows that $\frac{(1-\alpha)^2}{(1-\rho_1)^2} = \frac{(1-\alpha)^2}{(1-\rho_2)^2}$, equivalently $(1-\alpha)^2 = 0$ (equivalently, $\alpha = 1$, which is a contradiction) or $1-\rho_1 = \pm(1-\rho_2)$, which directly implies $\rho_1 = \rho_2$ (contradiction) or $\rho_1 + \rho_2 = 2$(impossible since $\rho_1, \rho_2 \in (0,1)$). This proves that $\rho_1 = \rho_2$. Hence i) is done.

ii)We need to show that:

$$\varphi_\alpha(\rho_1) = \varphi_\alpha(\rho_2) \text{ if and only } \rho_1 = \rho_2 \qquad (132)$$

Let $\varphi_\alpha(\rho_1) = \varphi_\alpha(\rho_2)$, then $\rho_1 = \rho_2$ follows by i). If $\rho_1 = \rho_2$, then it is trivial to show that for all $\alpha \neq 1, \varphi_\alpha(\rho_1) = \varphi_\alpha(\rho_2)$. This proves ii)

iii)It is straightforward to see that for every $\frac{(1-\alpha)^2}{(1-\rho)^2}$, there is a unique ρ assigned to it. Thus, surjectivity follows.

iv) let $\varphi_\alpha(\rho) = \frac{(1-\alpha)^2}{(1-\rho)^2} = w$. Therefore, $1-\rho = \pm\frac{1-\alpha}{\sqrt{w}}$, or $\rho = 1 \mp \frac{1-\alpha}{\sqrt{w}}$. We must ignore the positive sign, to leave us with one choice, $\rho = 1 - \frac{1-\alpha}{\sqrt{w}}$. Since $\rho \in (0,1)$. Then φ_α^{-1} exists if and only if $1 - \frac{1-\alpha}{\sqrt{w}} \in (0,1)$, equivalently,

$0 < 1-\alpha < \sqrt{w}$, or $\alpha \in (1-\sqrt{w}, 1)$. Therefore, $\varphi_\alpha^{-1}(\rho)$ if and only if $\alpha \in (1-\sqrt{\rho}, 1)$.

To interpret the analytic findings of iv), we have obtained

$$\varphi_\alpha^{-1}(\rho) = 1 - \frac{1-\alpha}{\sqrt{\rho}}$$

Let $\rho = 0.1, \alpha \in \left(1-\sqrt{0.1}, 1\right) = (0.683772234,1)$. So, we have

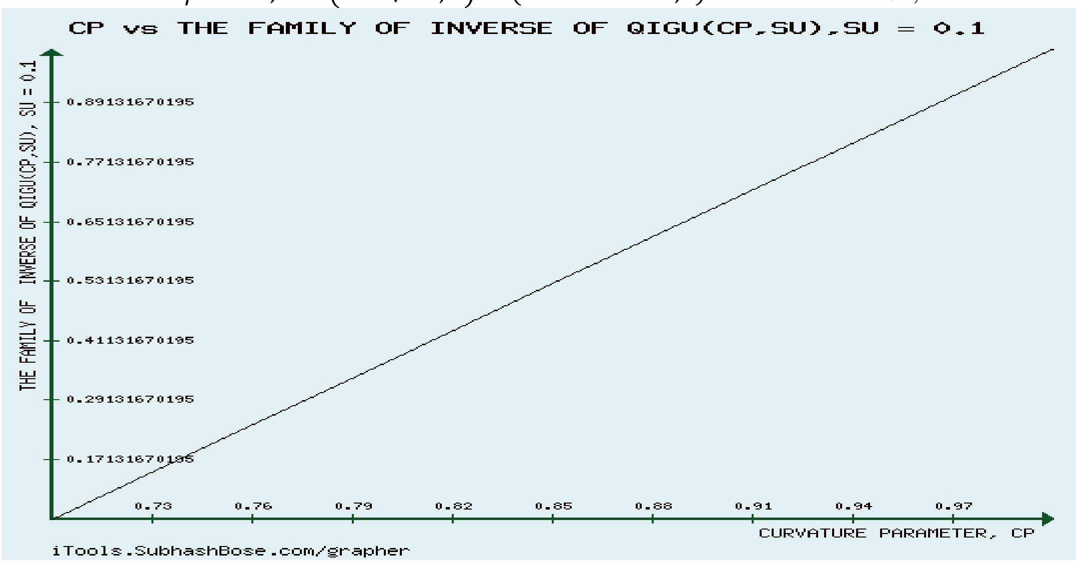

Figure 14.

Figure 14 supports the analytic findings as it shows the significant impact of the curvature parameter on the performance of the family of inverse of the functionals QIGU, queueing-information geometric unifiers.

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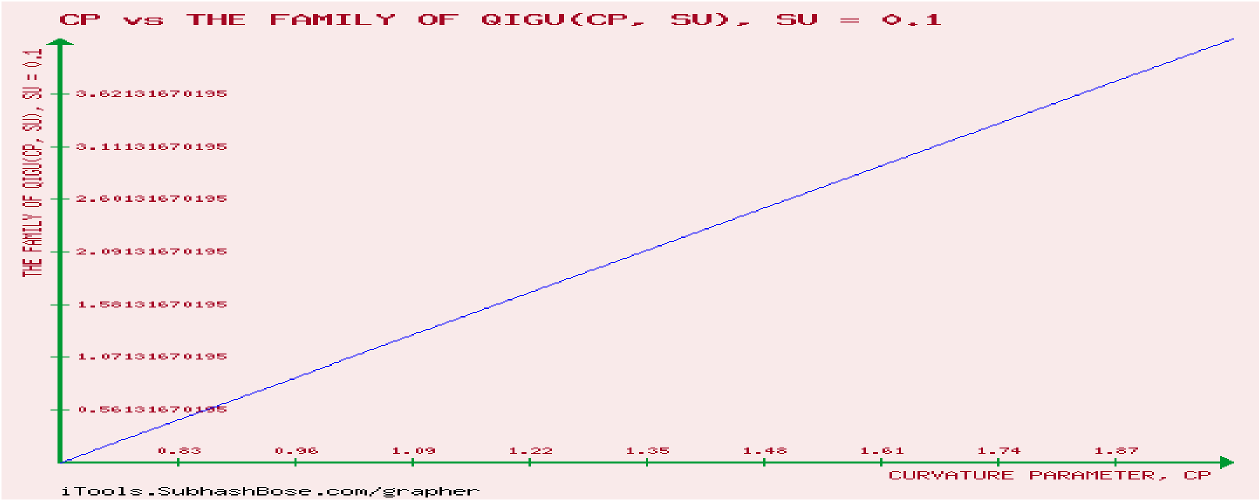

Figure 15.

Let $\rho = 0.1\alpha \notin \left(1 - \sqrt{0.1}, 1\right) = (0.683772234,1)$. So, we have

Figure 15 shows the immense increase of $\varphi_{\alpha}^{-1}(0.1)$ as α increases, until it approaches infinity for sufficiently large α, whereas in figure 16, $\alpha \notin \left(1 - \sqrt{0.1}, 1\right)$, it is observed that $\varphi_{\alpha}^{-1}(0.1)$ is negative and increasing . This is seen from figure 16.

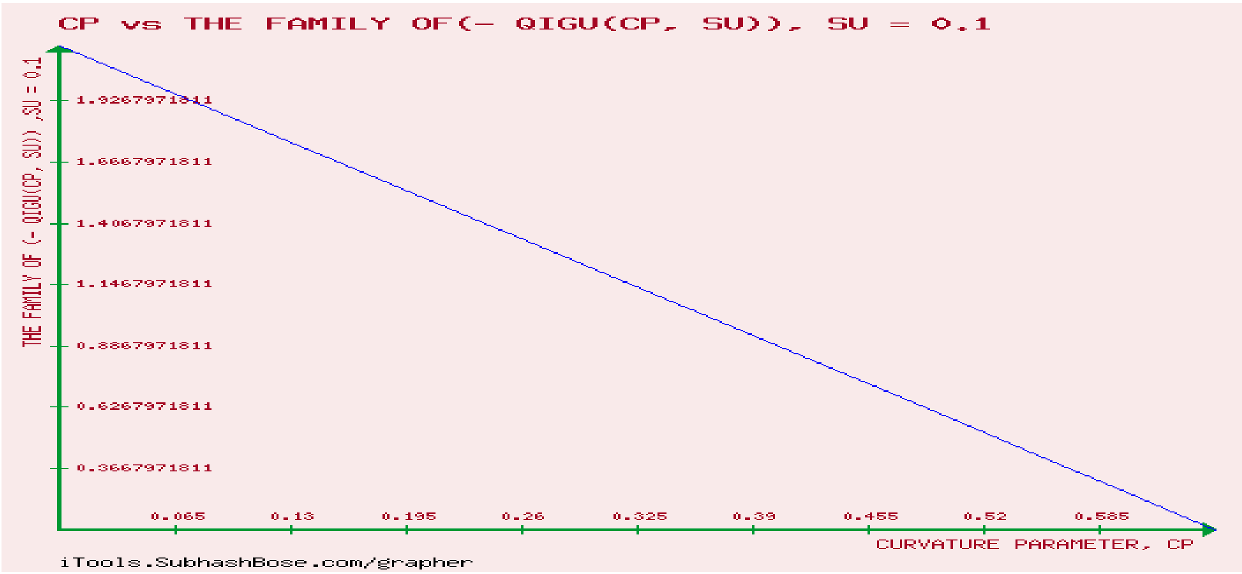

Figure 16.

Let $\rho = 0.1\alpha \notin \left(1 - \sqrt{0.1}, 1\right) = (0.683772234,1)$. So, we have

More to the extreme, for negative values of α, $\varphi_{\alpha}^{-1}(0.1)$ increases by the increase of α, until it approaches $-\infty$ for sufficiently large α. this is recorded by figure 17.

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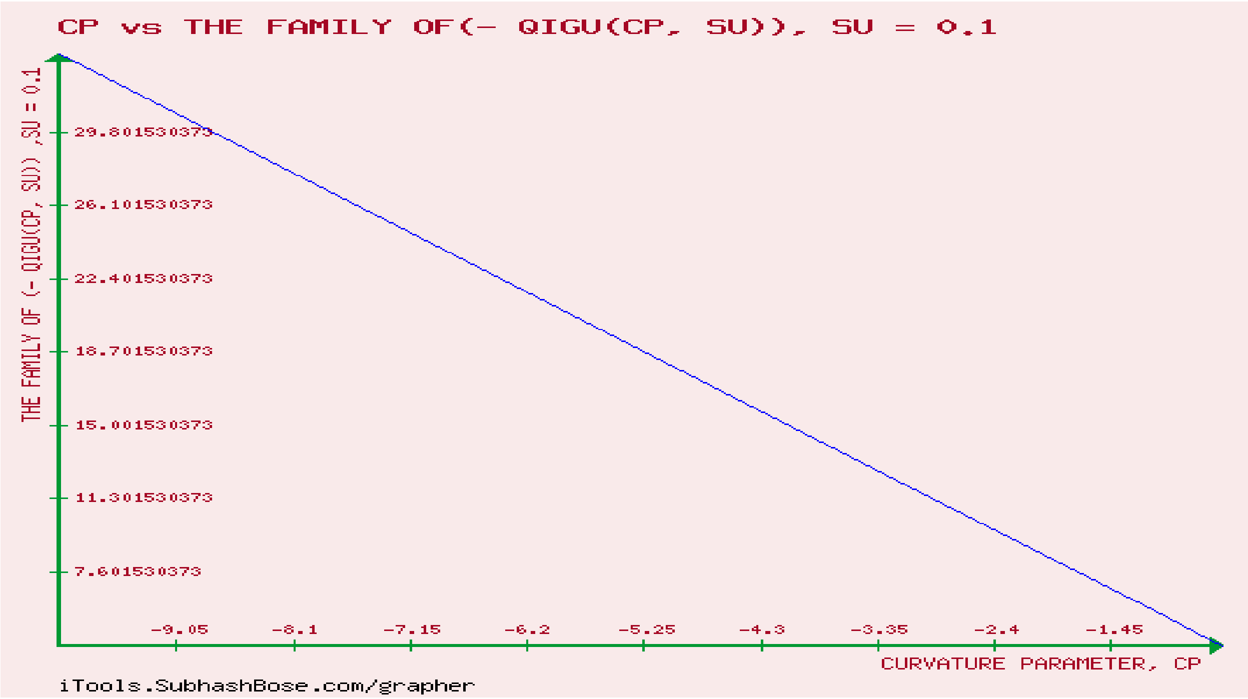

Figure 17.

14. Statistical Queueing functionals (SQFs)of KLF of stable M/G/ 1 QM

There are several SQFs representations. In what follows, we introduce some novel well-defined SQFs and investigate their algebraic structures.

14.1. The first representation

Define the statistical functionals f_1, f_2 from the stable M/G/1 QM into the KLF of it as follows:

$f_1(\rho) = 1 - \rho,\ \rho \in (0,1)$ (133)

$f_2(\rho) = \frac{L-\rho}{L} = 1 - \frac{2}{(1+\frac{\rho\zeta}{1-\rho})}, L = MQL = \frac{\rho}{2}\left(1 + \frac{\rho\zeta}{1-\rho}\right), \zeta = C_{s,KL}^2$ is fixed, $\zeta > 1$ (134)

where $C_{s,KL}^2 = \frac{(2-\tau_{KL})}{\tau_{KL}}$(c.f.,(3.37)) with $\tau_{KL} = \frac{\tau_s}{q(0)}, \tau_s = 2/(1 + C_s^2)$.

Theorem 14.1 For a stable M/G/1 QM, the above defined SQFs, f_1, f_2 (c.f., (133), (134)), it holds that:

i)f_1 is well- defined.

ii) $f_1$is one-to-one.

iii) f_1 is onto

iv)f_1^{-1} exists and is given by:

$f_1^{-1}(\rho) = 1 - \rho$ (135)

v)f_2 is well-defined.

vi) f_2 is one-to-one.

vii) f_2 is onto.

viii)f_2^{-1} exists and is given by:

$f_{2,\zeta}^{-1}(\rho) = \frac{1+\rho}{(1+\zeta)+(1-\zeta)\rho}$ (136)

Proof

i)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_1(\rho_1) = f_1(\rho_2)$. Hence, $1 - \rho_1 = 1 - \rho_2$, or $\rho_1 = \rho_2$(contradiction). Therefore, f_1 is well defined.

ii)Sufficiency:

Let $\rho_1, \rho_2 \in (0,1)$ such that $f_1(\rho_1) = f_1(\rho_2)$. Hence, $\rho_1 = \rho_2$is immediate.

Necessity:

Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 = \rho_2$. Then, it follows that $1-\rho_1 = 1-\rho_2$, orequivalently $f_1(\rho_1) = f_1(\rho_2)$.

iii) Clearly for every $1-\rho$ there is a unique $\rho \in (0,1)$ such that $f_1(\rho) = 1-\rho$. Hence, f_1 is onto.

iv)Let $f_1(\rho) = 1-\rho = y$, $\rho \in (0,1)$. Consequently, $f_1^{-1}(\rho) = 1-\rho$.

v) By (14.2), it follows that

$$f_2(\rho) = 1 - \frac{2}{(1+\frac{\rho\zeta}{1-\rho})} \quad (137)$$

Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_2(\rho_1) = f_2(\rho_2)$. This reduces to $1 - \frac{2}{\left(1+\frac{\rho\zeta}{1-\rho}\right)} = 1 - \frac{2}{(1+\frac{\rho\zeta}{1-\rho})}$, which directly implies by following some algebraic steps that $\rho_1 = \rho_2$ (contradiction), this proves v),

vi)Let $f_2(\rho) = 1 - \frac{2}{(1+\frac{\rho\zeta}{1-\rho})} = z$, $\rho \in (0,1)$. This implies $\rho = 1-z$, $z \in (0,1)$. Consequently, $f_1^{-1}(\rho) = 1-\rho$.

v) By (134), it follows that

$$f_2(\rho) = 1 - \frac{2}{(1+\frac{\rho\zeta}{1-\rho})} \quad (138)$$

Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_2(\rho_1) = f_2(\rho_2)$. Hence, $1 - \frac{2}{\left(1+\frac{\rho_1\zeta}{1-\rho_1}\right)} = 1 - \frac{2}{(1+\frac{\rho_2\zeta}{1-\rho_2})}$, or $\zeta(\rho_1 - \rho_2) = 0$. This implies $\rho_1 \neq \rho_2$(contradiction) or $\zeta = 0$ (impossible since the underlying QM is stable). Hence, v) follows.

Engaging the same methodology as in i) and ii), vi) and vii) will hold.

Let $f_2(\rho) = y$. Thus, $1 - \frac{2}{\left(1+\frac{\rho\zeta}{1-\rho}\right)} = y$, or $\rho = \frac{1+y}{(1+\zeta)+(1-\zeta)y}$. Clearly, this proves viii).

14.2. The second representation

Define the statistical functionals f_1, f_2 from the underlying manifold to its KLDF by:

$f_1(\rho) = \rho^3$, $\rho \in (0,1)$(14.7)

$$f_2(\rho) = \frac{L}{\rho} = \frac{1}{2}\left(1 + \frac{\rho\zeta}{1-\rho}\right), L = MQL = \frac{\rho}{2}\left(1 + \frac{\rho\zeta}{1-\rho}\right), \zeta = C_{s,KL}^2 \text{ is fixed}, \zeta > 1 \quad (139)$$

where $C_{s,KL}^2 = \frac{(2-\tau_{KL})}{\tau_{KL}}$(c.f.,(3.37)) with $\tau_{KL} = \frac{\tau_s}{q(0)}, \tau_s = 2/(1+C_s^2)$.

Theorem 14.2 The defined SQFs, f_1, f_2 (c.f., (14.7), (14.8)) satisfy:

i)f_1 is well defined.

ii) $f_1$is one-to-one.

iii) f_1 is onto.

iv)f_1^{-1} exists and is given by:

$$f_1^{-1}(\rho) = \sqrt[3]{(\rho)} \quad (140)$$

v)f_2 is well defined.

vi) f_2 is one-to-one.

vii) f_2 is onto.

viii)f_2^{-1} exists and is given by:

$$f_{2,\zeta}^{-1}(\rho) = \frac{2\rho-1}{(2\rho-1+\zeta)} \quad (141)$$

i)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_1(\rho_1) = f_1(\rho_2)$. Hence, $\rho_1^3 = \rho_2^3$, or $(\rho_1-\rho_2)[\rho_1^2 + \rho_2^2 + \rho_1\rho_2] = 0$.

Thus, it follows that $\rho_1 = \rho_2$, or $[\rho_1^2 + \rho_2^2 + \rho_1\rho_2] = 0$(impossible , since $\rho_1, \rho_2 \in (0,1)$). Therefore, f_1 is well defined.

It is straightforward to prove ii),iii) and iv).

v)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_2(\rho_1) = f_2(\rho_2)$. This reduces to $\frac{1}{2}\left(1 + \frac{\rho_1\zeta}{1-\rho_1}\right) = \frac{1}{2}\left(1 + \frac{\rho_2\zeta}{1-\rho_2}\right)$, which directly implies $\frac{\rho_1\zeta}{1-\rho_1} = \frac{\rho_2\zeta}{1-\rho_2}$. Hence, $\zeta(\rho_1-\rho_2) = 0$, which implies $\rho_1 = \rho_2$ or $\zeta = 0$(impossible, $\zeta > 1$). This proves v).

It is straightforward to prove vi), vii) .

viii)Let $f_2(\rho) = \frac{1}{2}\left(1 + \frac{\rho\zeta}{1-\rho}\right) = z$, $\rho \in (0,1)$ .This implies $\rho = \frac{2z-1}{(2z-1+\zeta)}$ .Consequently, $f_{2,\zeta}^{-1}(\rho) = \frac{2\rho-1}{(2\rho-1+\zeta)}$ (c.f., (141))

14.3. The third representation

Define the statistical functionals f_1, f_2 from the stable M/G/1 QM into its KLF as follows:

$f_1(\rho) = \rho\,\zeta,\ \rho \in (0,1), \zeta > 1$ (142)

$f_2(\rho) = \frac{L}{L-\rho} = \frac{\rho(\zeta-1)+1}{\rho(\zeta+1)-1}, L = MQL = \frac{\rho}{2}\left(1 + \frac{\rho\zeta}{1-\rho}\right), \zeta = C_{s,KL}^2$ is fixed, $\zeta > 1$ (142)

where $C_{s,KL}^2 = \frac{(2-\tau_{KL})}{\tau_{KL}}$(c.f.,(60)) with $\tau_{KL} = \frac{\tau_s}{q(0)}, \tau_s = 2/(1 + C_s^2)$.

Theorem 14.3 For a stable M/G/1 QM, the above defined SQFs, f_1, f_2 (c.f., (142), (143)), it holds that:

i)f_1 is well defined.
ii) $f_1$is one-to-one.
iii) f_1 is onto.
iv)f_1^{-1} exists and is given by:

$f_1^{-1}(\rho) = \frac{\rho}{\zeta}$ (143)

v)f_2 is well defined.
vi) f_2 is one-to-one.
vii) f_2 is onto.
viii)f_2^{-1} exists and is given by:

$f_{2,\zeta}^{-1}(\rho) = \frac{\rho+1}{\rho(\zeta+1)+(1-\zeta)}$ (144)

i)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_1(\rho_1) = f_1(\rho_2)$. Hence, $\frac{\rho_1}{\zeta} = \frac{\rho_2}{\zeta}$, or $\rho_1 = \rho_2$ or $\frac{1}{\zeta} = 0$(contradiction, since $\zeta > 1$). Therefore, f_1 is well defined.

The easy proofs of ii), iii) and iv) are straightforward.

v)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_2(\rho_1) = f_2(\rho_2)$. This reduces to $\frac{\rho_1(\zeta-1)+1}{\rho_1(\zeta+1)-1} = \frac{\rho_2(\zeta-1)+1}{\rho_2(\zeta+1)-1}$. Hence, it follows that:

$(\rho_1(\zeta-1)+1)(\rho_2(\zeta+1)-1) = (\rho_1(\zeta+1)-1)(\rho_2(\zeta-1)+1)$ (145)

After some calculations, (145) reduces to $\zeta(\rho_1 - \rho_2) = 0$, which implies $\rho_1 = \rho_2$ or $\zeta = 0$(impossible, $\zeta > 1$). This proves v).

The proofs of vi) and vii) are straightforward.

viii)Let $f_2(\rho) = \frac{\rho(\zeta-1)+1}{\rho(\zeta+1)-1} = z$, $\rho \in (0,1)$ .Hence, $\rho = \frac{z+1}{z(\zeta+1)+(1-\zeta)}$ .Consequently, $f_{2,\zeta}^{-1}(\rho) = \frac{\rho+1}{\rho(\zeta+1)+(1-\zeta)}$ (c.f., (144))

14.4. The fourth representation

Define the statistical functionals f_1, f_2 from the underlying manifold to its KLDF by:

$f_1(\rho) = \rho^2,\ \rho \in (0,1), \zeta > 1$ (146)

$f_2(\rho) = MQL = L = \frac{\rho}{2}\left(1 + \frac{\rho\zeta}{1-\rho}\right), \zeta = C_{s,KL}^2$ is fixed, $\zeta > 1$ (147)

where $C_{s,KL}^2 = \frac{(2-\tau_{KL})}{\tau_{KL}}$(c.f.,(60)) with $\tau_{KL} = \frac{\tau_s}{q(0)}, \tau_s = 2/(1 + C_s^2)$.

Theorem 14.4 For a stable M/G/1 QM, the above defined SQFs, f_1, f_2 (c.f., (141), (142)), it holds that:

i)f_1 is well defined.
ii) $f_1$is one-to-one.
iii) f_1 is onto.
iv)f_1^{-1} exists and is given by:

$f_1^{-1}(\rho) = \sqrt{\rho}$ (147)

v)f_2 is well defined.
vi) f_2 is one-to-one.
vii) f_2 is onto

viii)f_2^{-1} exists and is given by:

$f_{2,\zeta}^{-1}(\rho) = \frac{-(2\rho+1)+\sqrt{((2\rho+1)^2+8\rho(\zeta-1))}}{2(\zeta-1)}$ (148)

i)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_1(\rho_1) = f_1(\rho_2)$. Hence, $\rho_1^2 = \rho_2^2$, or $\rho_1 = \rho_2$ or $\rho_1 + \rho_2 = 0$(contradiction, since $\rho_1, \rho_2 \in (0,1)$). Therefore, f_1 is well defined.

The easy proofs of ii), iii) and iv) are straightforward.

v)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_2(\rho_1) = f_2(\rho_2)$. This reduces to $\frac{\rho_1}{2}\left(1+\frac{\rho_1\zeta}{1-\rho_1}\right) = \frac{\rho_2}{2}\left(1+\frac{\rho_2\zeta}{1-\rho_2}\right)$. Hence, it follows that

$(\rho_1-\rho_2)[(1-\rho_1)(1-\rho_2)+\zeta] = 0$ (149)

(149) implies $\rho_1 = \rho_2$ or $[(1-\rho_1)(1-\rho_2)+\zeta] = 0$ (contradiction, since $\rho_1, \rho_2 \in (0,1)$ and $\zeta > 1$). Thus, v) holds.

The proofs of vi) and vii) are easy.

viii)Let $f_2(\rho) = \frac{\rho}{2}\left(1+\frac{\rho\zeta}{1-\rho}\right) = z$.Consequently,

$\rho^2 + \left(\frac{2z+1}{\zeta-1}\right)\rho - \left(\frac{2z}{\zeta-1}\right) = 0$ (150)

Therefore,

$\rho = \frac{-(2z+1)\pm\sqrt{((2z+1)^2+8z(\zeta-1))}}{2(\zeta-1)}$ (151)

Since $\rho \in (0,1)$, it holds by (151) that:

$\rho = \frac{-(2z+1)+\sqrt{((2z+1)^2+8z(\zeta-1))}}{2(\zeta-1)}$ (152)

Clearly, by (14.22), it follows that $f_{2,\zeta}^{-1}(\rho) = \frac{-(2\rho+1)+\sqrt{((2\rho+1)^2+8\rho(\zeta-1))}}{2(\zeta-1)}$ (c.f., (14.18)).

14.5. The fifth representation

Define the statistical functionals f_1, f_2 from the underlying queue manifold into its KLDF by:

$f_1(\rho) = \rho^n$, $\rho \in (0,1), n > 0$ (153)

$f_2(\rho) = L^n, L = \frac{\rho}{2}\left(1+\frac{\rho\zeta}{1-\rho}\right) = MQL, \zeta = C_{s,KL}^2$ is fixed, $\zeta > 1$ (154)

where $C_{s,KL}^2 = \frac{(2-\tau_{KL})}{\tau_{KL}}$(c.f.,(60)) with $\tau_{KL} = \frac{\tau_s}{q(0)}, \tau_s = 2/(1+C_s^2)$.

Theorem 14.5 For a stable M/G/1 QM, the above defined SQFs, f_1, f_2 (c.f., (14.23), (14.24)), it holds that:

i)f_1 is well defined.

ii) $f_1$is one-to-one.

iii) f_1 is onto.

iv)f_1^{-1} exists and is given by:

$f_1^{-1}(\rho) = \rho^{\frac{1}{n}}$ (155)

v)f_2 is well defined.

vi) f_2 is one-to-one.

vii) f_2 is onto.

viii)f_2^{-1} exists and is given by:

$f_{2,\zeta}^{-1}(\rho) = \frac{-\left(2\rho^{\frac{1}{n}}+1\right)+\sqrt{\left(\left(2\rho^{\frac{1}{n}}+1\right)^2+8\rho(\zeta-1)\right)}}{2(\zeta-1)}$ (156)

i)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_1(\rho_1) = f_1(\rho_2)$. Hence, $\rho_1^n = \rho_2^n$, or $\rho_1 = \rho_2$ or $(\rho_1^{n-1} + \rho_1^{n-2}\rho_2 + \cdots + \rho_2^{n-1})$(contradiction, since $\rho_1, \rho_2 \in (0,1)$). Therefore, f_1 is well defined.

The easy proofs of ii), iii) and iv) are straightforward.

v)Let $\rho_1, \rho_2 \in (0,1)$ such that $\rho_1 \neq \rho_2$ and $f_2(\rho_1) = f_2(\rho_2)$. This reduces to $[\frac{\rho_1}{2}\left(1+\frac{\rho_1\zeta}{1-\rho_1}\right)]^n = [\frac{\rho_2}{2}\left(1+\frac{\rho_2\zeta}{1-\rho_2}\right)]^n$. Hence, it follows that:~

$(\rho_1-\rho_2)(\left[\frac{\rho_1}{2}\left(1+\frac{\rho_1\zeta}{1-\rho_1}\right)\right]^{n-1} + \left[\frac{\rho_1}{2}\left(1+\frac{\rho_1\zeta}{1-\rho_1}\right)\right]^{n-2}[\frac{\rho_2}{2}\left(1+\frac{\rho_2\zeta}{1-\rho_2}\right)] + \cdots + \left[\frac{\rho_2}{2}\left(1+\frac{\rho_2\zeta}{1-\rho_2}\right)\right]^{n-1}) = 0$ (157)

(157) implies:

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$\rho_1 = \rho_2$ or ($\left[\frac{\rho_1}{2}\left(1+\frac{\rho_1\zeta}{1-\rho_1}\right)\right]^{n-1} + \left[\frac{\rho_1}{2}\left(1+\frac{\rho_1\zeta}{1-\rho_1}\right)\right]^{n-2}\left[\frac{\rho_2}{2}\left(1+\frac{\rho_2\zeta}{1-\rho_2}\right)\right] + \cdots + \left[\frac{\rho_2}{2}\left(1+\frac{\rho_2\zeta}{1-\rho_2}\right)\right]^{n-1}$) = 0

(Contradiction, since $\rho_1, \rho_2 \in (0,1)$ and $\zeta > 1$). Thus, v) holds.

The proofs of vi) and vii) are straightforward.

viii)Let $f_2(\rho) = [\frac{\rho}{2}\left(1+\frac{\rho\zeta}{1-\rho}\right)]^n = z$.Consequently,

$$\rho^2 + \left(\frac{2k+1}{\zeta-1}\right)\rho - \left(\frac{2k}{\zeta-1}\right) = 0, k = z^{\frac{1}{n}}, \quad (158)$$

Therefore,

$$\rho = \frac{-(2k+1)\pm\sqrt{((2k+1)^2+8k(\zeta-1))}}{2(\zeta-1)} \quad (159)$$

Since $\rho \in (0,1)$, it holds by (151) that:

$$\rho = \frac{-(2k+1)+\sqrt{((2k+1)^2+8k(\zeta-1))}}{2(\zeta-1)} \quad (160)$$

Clearly, by (160), it follows that $f_{2,\zeta}^{-1}(\rho) = \frac{-\left(2\rho^{\frac{1}{n}}+1\right)+\sqrt{(\left(2\rho^{\frac{1}{n}}+1\right)^2+8\rho^{\frac{1}{n}}(\zeta-1))}}{2(\zeta-1)}$ (c.f., (156)).

For $n = 1$, (14.26) reduces to (14.18) as a special case.

15. Closing remarks and next phase of research

FIM, and α-connection for KLDF manifold is presented. Geodesic equations, Kullback divergence, and J-divergence are also developed for this KLDF manifold. According to the findings, the compressibility for the KLDF manifold is proven. Additionally, the underlying manifold possesses a non-zero RCT. Furthermore, the exponential of the FIM is demonstrated to be a differential equation solution, and the work develops information geometric ties to provide queueing-theoretic unification with other mathematical disciplines .

Appendix A: KL Formalism vs. EME Consistency Axioms

1. Uniqueness

It translates to, "If the same problem is solved twice in exactly the same way, the same answer is expected in both cases i.e., the solution should be unique" (c.f., [6]). Let f_{KL}, h_{KI} be two PDFs such that:

$$H^*_{KI}(f_{KL,N}) = H^*_K(h_{KL,N}) \quad (A.1)$$

Hence,

$$\sum_{n=1}^{N}(f_{KL,N})\, ln(\frac{(f_{KL,N})}{q(N)}) = \sum_{n=1}^{N}(h_{KL,n})\, ln(\frac{(h_{KL,N})}{q(N)}) \quad (A.2)$$

By (A.2),it is implied that

$$(f_{KL,N})(ln(f_{KL,N}) - ln\,(q(N))) = (h_{KL,N})(ln(h_{KL,N}) - ln\,(q(N))) \quad (A.3)$$

Let the contradiction be true, namely,$(f_{KL,N}) \neq (h_{KL,N})$. Thus $\exists\, \gamma > 1$ satisfying:

$$(f_{KL,N}) = \gamma(h_{KL,N}) \quad (A.4)$$

Combining (A.3) and (A.4) , we get

$$\gamma(h_{KL,N})(ln\gamma + ln(h_{KL,N}) - ln(q(N)) = (h_{KL,N})(ln(h_{KL,N}) - ln(q(N)) \quad (A.5)$$

Since $(h_{KL,N})$ is non-zero, (A.5) can be transformed into

$$\gamma(ln\gamma) = (1-\gamma)(ln\left(\frac{(h_{KL,N})}{q(N)}\right) \quad (A.6)$$

By default, $q(N) \in (0,1)$

By mathematical analysis, we have the following possibilities:

$q(N) < h_{KL,N}$

It is implied that $\frac{(h_{KL,N})}{q(N)} > 1$, $ln(\frac{(h_{KL,N})}{q(N)}) > 0$. This implies by (A.6), $\gamma > 1$, that $\gamma(ln\gamma) < 0$ (Contradiction)

$q(N) = h_{KL,N}$

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It is implied that $\frac{(h_{KL,N})}{q(N)} = 1$, $ln(\frac{(h_{KL,N})}{q(N)}) = 0$. This implies by (A.6), $\gamma > 1$, that $\gamma(ln\gamma) = 0$ (Contradiction)

$q(N) > h_{KL,N}$

It is implied that $\frac{(h_{KL,N})}{q(N)} < 1$, $ln(\frac{(h_{KL,N})}{q(N)}) < 0$. Hence, KL divergence is negative , which contradicts that fact that KL divergence is non-negative(c.f.[57]).

Therefore., "there cannot be two distinct probability distributions $f_{KL,N}, h_{KL,N} \epsilon \Omega$ having the same KL divergence measure in Ω.Thus, KL formalism satisfies the axiom of uniqueness (c.f., [6]).

2. Invariance

The invariance axiom states that "The same solution should be obtained if the same inference problem is solved twice in two different coordinate systems" (c.f., [58]).Following the analytic methodology proposed in and adopting the notation of Subsection 1, let Ξ be a coordinate transformation from state $\{S_n, n = 1,2,\dots,N\}$ to state $\{M_n, n = 1,2,\dots,N\}$, where Mbe a transformed set of N possible discrete states, namely $M = \{M_n, n = 1,2,\dots,N\}$with $\Gamma(p_{Kl,N}(M_n) = \Xi^{-1}(p_{KL,N}(S_n)$, where J is the Jacobian $J = \frac{\partial(M_n)}{\partial(S_n)}$. Moreover, let $\Gamma\Omega$ be the closed convex set of all probability distributions Γ defined on M such that $\Xi\left(p_{KI,N}(M_n)\right) > 0$ for all $M_n \epsilon M$, n = 1, 2, ...,Nand $\sum_{n=1}^{N} \Xi\left(p_{KL,N}(M_n)\right) = 1$. It can be clearly seen that, transforming variables from $S_n \in S$ into $R_n \in R$, the extended KL divergence (c.f., (2.1)) is transformation invariant namely

$$H^*_{KL}(p_{KL,N}) = H^*_{KL}(\Xi(p_{KL,,N})) \quad (A.7)$$

Thus, the EME formalism satisfies the axiom of invariance since the minimum in $\Xi\Omega$ corresponds to the minimum in Ω" (c.f., [57])._

3. System Independence

It translates to "It should not matter whether one accounts for independent information about independent systems separately in terms of different probabilities or together in terms of the joint probability" (c.f., [6]). The joint probability for any independent systems Q and M is:

$$h_{Kl,N}(x_k, y_n) = Pr(X = x_k, Y = y_n) = f_{Kl,N}(x_k)\, g_{Kl,N}(y_n) \quad (A.8)$$

Thus, KL divergence measure (c.f. (2.1)) can be written as

$$H^*_{KL}(h_{Kl,N}) = \sum_{n=1}^{N}(h_{Kl,N})\ln\left(\frac{(h_{Kl,N})}{q(N)}\right) = \sum_{n=1}^{N} f_{Kl,N}(x_k)\, g_{Kl,N}(y_n)\ln\left(\frac{(f_{Kl,N}(x_k)\, g_{Kl,N}(y_n))}{q(N)}\right) \quad (A.9)$$

Assume that

$$H^*_{KL}(h_{Kl,N}) = H^*_{KL}[f_{KL,N}] + H^*_{Kl}[g(Y_{KL,N})] \quad (A.10)$$

By (A.9) and (A.10), this could be rewritten in the simpler form:

$$fg\, ln\left(\frac{fg}{q(N)}\right) = f\, ln\left(\frac{f}{q(N)}\right) + g\, ln\left(\frac{g}{q(N)}\right) \quad (A.11)$$

Hence, it follows that:

$$fg\left(lnfg - lnq(N)\right) = f\, lnf + g\, lng - (f+g)lnq(N) \quad (A.11)$$

Define $q(N) = 1$, so (A.11) will be:

$$fg\,(lnfg) = f\, lnf + g\, lng \quad (A.12)$$

(A.12) implies,

$$(fg)^{fg} = f^f g^g \quad (A.13)$$

Let $f = g = \frac{1}{2}$ in (A.13)

Hence, it follows that

$$(\tfrac{1}{4})^{\frac{1}{4}} = \frac{1}{2} \quad (A.14)$$

(A.14) is impossible. Thus, system independence is defied because of long-range interactions.

4 Subset Independence(SI)

SI in a physical interpretation reads as "It does not matter whether one treats an independent subset of system states in terms of a separate conditional density or in terms of the full system density" (c.f., [59]).

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In the given context, the notation and concepts related to an aggregate state of a system, denoted as x, and its associated probability distribution $f_{Kl}(x)$. The probability distribution represents the likelihood of the random variable X taking the value x. The text also mentions that the aggregate states ξ_i, where i ranges from 1 to L, can be expressed using this notation.

$$\sum_{S_i^*} f_{KL,i}(x_{ij}) = \xi_i \quad (A.15)$$

We have

$$H_{KL}^*(f_{KL}) = \left(\sum_i \sum_{S_i} \xi_i f_{Kl,i}(x_{ij})\right) \quad (A.16)$$

where $f_{Kl,i}(x)(\epsilon\,\Omega$. Equation (3.10) will read as

$$H_{Kl}^*(f_{KL}) = \left[\sum_i \xi_i \sum_{S_i} f_{KL,i}(x_{ij})\right] \quad)$$

(A.17)

Thus,

$$H_{KL,i}^*(f_{Kl,i}) = \left(\sum_{S_i^*}(f_{Kl,i})(x_{ij})\, ln\left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)\right) \quad (A.18)$$

Hence,

$$H_{KL,i}^*(f_{Kl,i}(x_{ij})) = \left(\sum_{S_i^*} ln\left(\left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)^{(f_{Kl,i})(x_{ij})}\right) = ln\left(\sum_{S_i^*}\left(\left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)^{(f_{Kl,i})(x_{ij})}\right)\right)$$

(A.19)

Hence, apparently by the above proof it holds that

$$\sum_{S_i^*}\left(\left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)^{(f_{Kl,i})(x_{ij})} > \left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)^{(f_{Kl,i})(x_{ij})} > \left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)^{q(N)} = \frac{(f_{Kl,i})(x_{ij})^{q(N)}}{(q(N))^{q(N)}}\right.$$

(A.20)

By (A.20), there exists a positive real number $0 < \sigma < 1$ satisfying:

$$\left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)^{(f_{Kl,i})(x_{ij})} = \frac{(f_{Kl,i})(x_{ij})^{q(N)}}{\sigma(q(N))^{q(N)}} \quad (A.21)$$

Combining (A.19) and (A.21) implies

$$H_{KL,i}^*(f_{Kl,i}(x_{ij})) = \left(\sum_{S_i^*} ln\left(\left(\frac{(f_{Kl,i})(x_{ij})}{q(N)}\right)^{(f_{Kl,i})(x_{ij})}\right) = \left(\frac{(f_{Kl,i})^{q(N)}(x_{ij})}{\sigma(q(N))^{q(N)}}\right) \quad (A.22)$$

Following (A.23),

$$(f_{Kl,i})^{q(N)}(x_{ij}) = \sigma(q(N))^{q(N)} H_{KL,i}^*(f_{Kl,i}(x_{ij})) \quad (A.24)$$

By (A.24),

$$(f_{Kl,i}(x_{ij})) = q(N)\sigma^{\left(\frac{1}{q(N)}\right)}\left(H_{KL,i}^*(f_{Kl,i}(x_{ij}))\right)^{\left(\frac{1}{q(N)}\right)} \quad (A.25)$$

Linking (A.16) with (A.25) yields

$$H_{KL}^*(f_{KL}) = \sum_i \xi_i q(N)\sigma^{\left(\frac{1}{q(N)}\right)}\left(H_{KL,i}^*(f_{Kl,i}(x_{ij}))\right)^{\left(\frac{1}{q(N)}\right)} \quad (A.26)$$

(A.26) implies that KLD satisfies subset independence .

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