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Article

Fractal Dimension of the Generalized Z-Entropy of the Rényian Formalism of Stable *M/G/*1 Queue with Some Potential Applications of Fractal Dimension to Big Data Analytics

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Abstract: In the current work, the stable *M/G/*1 queueing system's Generalised Z-Entropy (GZE) of Rényian formalism is examined in its fractal dimension. Notably, fractal dimension's resulting behaviour that corresponds to the GZE parameters is examined through numerical tests. This research makes a substantial generalization in the literature by fusing fractal geometry and information theory to shed light on how entropy and complexity interact. More fundamentally, the significant role of fractal dimension to advance Big Data Analytics (BDAs) is highlighted. Closing remarks combined with open problems and the next phase of research are provided.

Keywords: fractal dimension (D); generalized Z-entropy; Google Earth satellite (GEs); GNU image manipulation; Big Data Analytics (BDAs)

1. Introduction

The Shannonian entropy [1], namely $H(X)$ reads as:

$$H(X) = \sum_i p(x_i) ln(p(x_i)) \tag{1}$$

p_i serves as the *ith*-event probability.

The probability of the *ith* -event is given in this expression as p_i. This entropy establishes the definition of information in information theory. There are many methods for measuring information, as well as in this scenario, we argue regarding how entropy and D interact with one another. $D[2-7]$, is a measure that assesses how a fractal pattern expands beyond the area it occupies, indicating the complexity of the pattern in spatial dimensions. D[2–7], involves calculations that consider sticks' number (N) required in coastline coverage and the factor of scaling (ε). By analyzing these factors, the fractal dimension provides insights into the intricate nature of patterns and their representation of complexity in spatial dimensions.

$$N \propto \varepsilon^{-D} \tag{2}$$

$$ln\,N = -D = \frac{lnN}{ln\varepsilon} \tag{3}$$

[8] created a map and rigid sticks for an experiment like Richardson's in the book by using GEs pictures combined with GIMP (c.f., Figure 1). The practical application of this technique for measuring the fractal dimension was demonstrated on a part of the Grand Canyon.

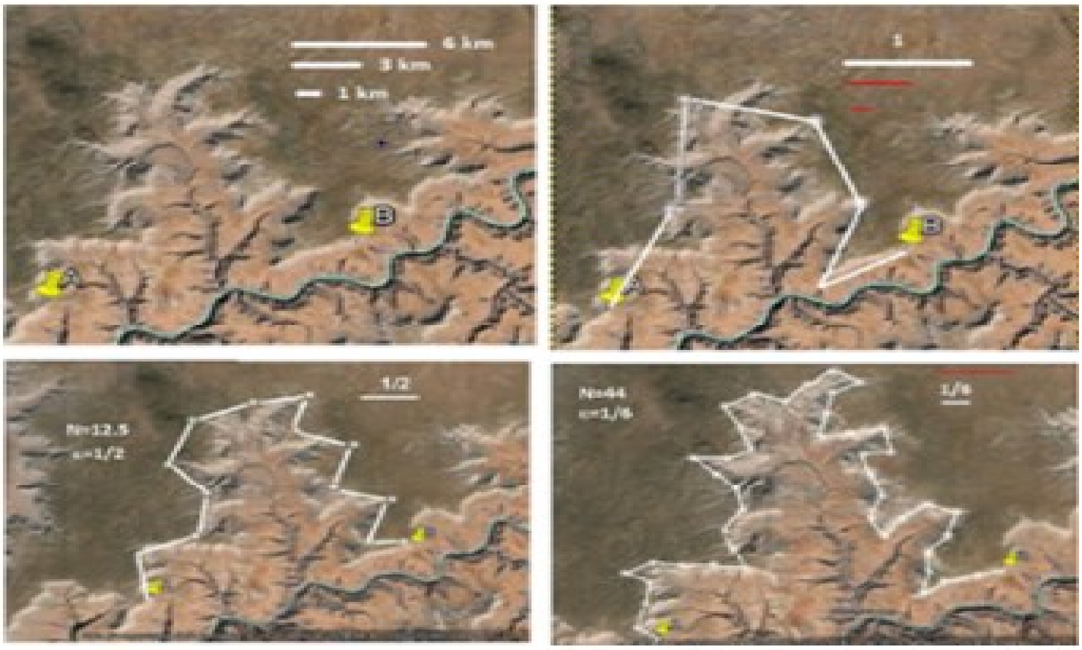

Figure 1. An example of satellite images from Google Earth, specifically showing a portion of the Grand Canyon in Arizona. The mention of "rigid sticks" refers to the creation of visual elements using GIMP, a software for image editing and manipulation[8]. The visualization of how N, D and ε are corelated is illustrated by Figure 2.

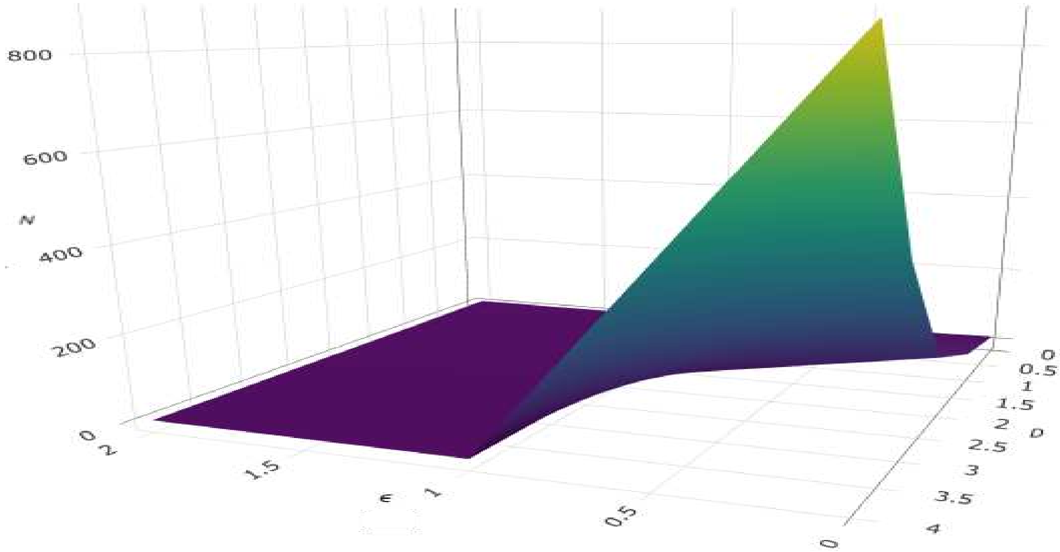

Figure 2. The correlation between N, D and ε.

The road map of this paper is: Section II provides an overview of previous research on deriving fractal dimension using entropy. Section III introduces the definition of the Rényi formalism of $M/G/1$ queue in the stability phase and presents new results combined with a numerical analysis that demonstrates the substantial influence of the $M/G/1$ parameters on the behavior of the fractal dimension. Section IV explores some potential D applications to BDAs, Finally, closing remarks combined with the next phase of research are provided in section V.

2. *D* of Entropies

In [8], the fractal dimension (D)associated with different types of entropy measures was determined, for entropies measures[9–12]. These derivations were conducted under proposing that all outcomes have equal probabilities.

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The Shannonian fractal dimension [8] reads:

$$D_s = lim_{\varepsilon \to 0} \frac{lnN}{ln\frac{1}{\varepsilon}} \tag{4}$$

Rényian dimension of order $q \in (0.5,1)$ reads [10]

$$D_R = lim_{\varepsilon \to 0} \frac{lnN}{ln\frac{1}{\varepsilon}} \tag{5}$$

As for the Tsallisian case[8], of order $q \in (0.5,1)$,

$$D_T = lim_{\varepsilon \to 0} \frac{\frac{1}{1-q}(N^{1-q}-1)}{ln\frac{1}{\varepsilon}} \tag{6}$$

With $q \in (0.5,1)$, $D_T > 0$

The Kaniadakisian fractal dimension [8] for the entropic index, κ reads as:

$$D_K = lim_{\varepsilon \to 0} \frac{\frac{1}{2k}(N^k - N^{-k})}{ln\frac{1}{\varepsilon}} \tag{7}$$

Notably, D_R, D_T *and* D_T are entropic index impacted. The Koch snowflake (KSF) fractal dimension follows for ($N = 4$ *and* $\varepsilon = 1/3$).

3. Rényian Formalism of Stable $M/G/1$ Queueing System

Non-extensivity coins interactions of long range[13].

The Rényi's [9] non-extensive maximum entropy functionals reads:

$$\mathrm{H_{q,R}(p)} = \frac{c}{1-q} ln\left\{\sum_{i=1}^{N}(p_i)^q\right\} \tag{8}$$

respectively, for a constant $c > 0$.

Theorem 1(c.f., [14])

The Rényi's non-extensive maximum entropy solution, $\mathrm{p_{q,R}(n)}$, for $M/G/1$ queue in the stability under normalization, server utilization and Mean Queue Length reads:

$$\mathrm{p_{q,R}(n)} = \begin{cases} \mathrm{p_{q,R}(0)} & \mathrm{n} = 0 \\ \mathrm{p_{q,R}(0)}\tau_s^{\frac{1}{q}} x^n & \mathrm{n} > 0 \end{cases} \tag{9}$$

Such that

$$\mathrm{p_{q,R}(0)} = 1 - \rho \text{ , } \rho \text{ is the server utilization} \tag{10}$$

where τ_s and x to be:

$$\tau_s = 2/(1 + C_{s,1,S}^2) \tag{11}$$

with

$$x = \frac{\rho}{\left(\rho + (1-\rho)\left(\frac{2}{1+\mathrm{C_{s,1,S}^2}}\right)^{\frac{1}{q}}\right)} \tag{12}$$

With

$$\frac{\rho(1-x)}{(1-\rho)x} = \tau_s^{\frac{1}{q}} \tag{13}$$

The Generalized Z-Entropy (GZE) [14] reads as:

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$$H_{q,a,b,\mathbb{Z}}(p) = Z_{a,b} = \frac{1}{(1-\mathrm{q})(\mathrm{a}-\mathrm{b})}\left[\left(\sum_n p_{q,\mathbb{Z}}(n)^q\right)^a - \left(\sum_n p_{q,\mathbb{Z}}(n)^q\right)^b\right] \quad (14)$$

Such that $1 > q > 0.5, a > 0, b \in \mathbb{R}$ *or* $b > 0, a \in \mathbb{R}$ *with* $a \neq b$.

4. New Results

Theorem 2

Engaging [14], (9)-(14), the GZE fractal dimension, $D_{\mathbb{Z}_{a,b}}$ is devised by:

$$D_{\mathbb{Z}_{a,b}} = \frac{1}{(1-q)(a-b)} lim_{\varepsilon\to 0} \frac{1}{ln\frac{1}{\varepsilon}}\left(\left(\rho^q\left(\frac{(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{q-1}\left(1-\left(\frac{\rho}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{(N-1)q}\right)\right)^a - \left(\rho^q\left(\frac{(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{q-1}\left(1-\left(\frac{\rho}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{(N-1)q}\right)\right)^b\right) \quad (15)$$

Proof

By the definition,

$$D_{\mathbb{Z}_{a,b}} = \frac{\mathrm{c}}{(1-\mathrm{q})(\mathrm{a}-\mathrm{b})}\left[\left(\sum_{n=1}^N(\mathrm{p_{q,R}}(0)\tau_s^{\frac{1}{q}}\, x^n)^q\right)^a - \left(\sum_{n=1}^N(\mathrm{p_{q,R}}(0)\tau_s^{\frac{1}{q}}\, x^n)^q\right)^b\right] \text{ (c.f., (9))} \quad (16)$$

$$\sum_{n=1}^N\left(\mathrm{p_{q,R}}(0)\tau_s^{\frac{1}{q}}\, x^n\right)^q = \sum_{n=1}^N(\rho(1-x)\, x^{n-1})^q = (\rho(1-x))^q\left(\frac{1-x^{(N-1)q}}{(1-x)}\right)$$

$$= \rho^q\left(\frac{(1-\rho)\left(\frac{2}{1+C^2_{s,1,S}}\right)^{\frac{1}{q}}}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{s,1,S}}\right)^{\frac{1}{q}}\right)}\right)^{q-1}\left(1-\left(\frac{\rho}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{s,1,S}}\right)^{\frac{1}{q}}\right)}\right)^{(N-1)q}\right)$$

(c.f., (13))

Hence, it follows that:

$$D_{\mathbb{Z}_{q.a.b.N.\varepsilon}} = \frac{1}{(1-q)(a-b)} lim_{\varepsilon\to 0} \frac{1}{ln\frac{1}{\varepsilon}}\left(\left(\rho^q\left(\frac{(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{q-1}\left(1-\left(\frac{\rho}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{(N-1)q}\right)\right)^a - \left(\rho^q\left(\frac{(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{q-1}\left(1-\left(\frac{\rho}{\left(\rho+(1-\rho)\left(\frac{2}{1+C^2_{S.1.S}}\right)^{\frac{1}{q}}\right)}\right)^{(N-1)q}\right)\right)^b\right) \text{ (c.f., (12))} \quad (17)$$

Hence, (15) follows.

Accordingly, let's discuss the following cases:

Case 1: $C_s^2 = 3, \rho = 0.5$

The information-theoretic impact on $D_{\mathbb{Z}_{q,1,-2,4,\frac{1}{3}}}$ is visualized by Figure 3.

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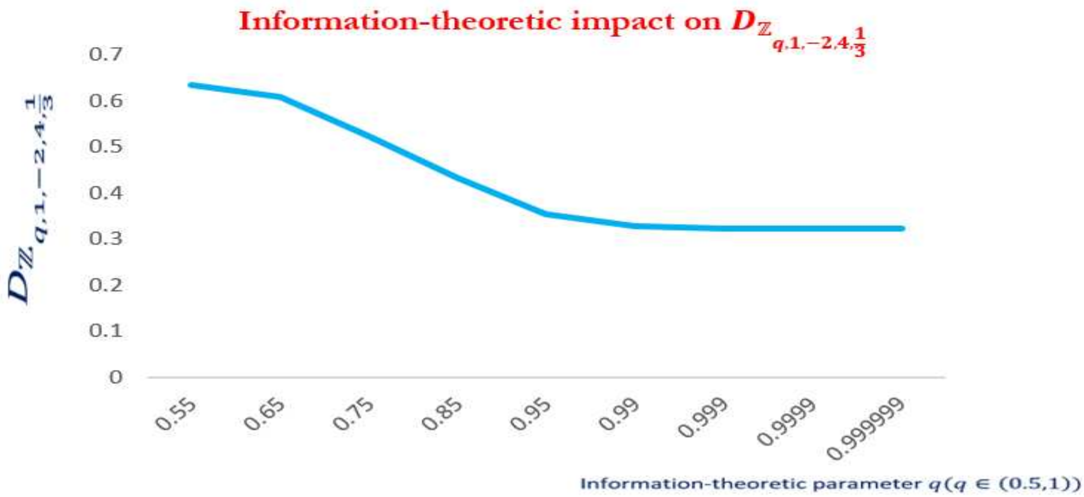

Figure 3. The influence of q on $D_{\mathbb{Z}_{q,1,-2,4,\frac{1}{3}}}$.

Approaching the instability zone, $C_s^2 \to 1$

$$lim_{a\to 0,b\to 0,C_s^2\to 1}\, D_{\mathbb{Z}_{q,a,b,N,\varepsilon}} = \frac{1}{(1-q)} lim_{\varepsilon\to 0} \frac{1}{ln\frac{1}{\varepsilon}}\, ln(\rho^{qN}(1-\rho\ )^{q-1})$$

$$\lim_{C_{s,}^2\to 1} D_{\mathbb{Z}_{q,0,0,4,{}^1/_3}} = \frac{1}{(1-q)ln3}\, ln(\rho^{4N}(1-\rho\ )^{q-1}) = \frac{1}{(1-q)ln3}\, ln(\rho^{4q}(1-\rho\ )^{q-1}) = \frac{4q ln\rho + (q-1)ln\,(1-\rho)}{(1-q)ln3}$$

$$\lim_{C_s^2\to 1} D_{\mathbb{Z}_{0.5,0,0,4,{}^1/_3}} = \frac{4ln\rho - ln\,(1-\rho)}{ln3} = (ln\,(1-\rho)\rho^4)^{ln3}$$

More fundamentally,

$$\lim_{a\to 0} D_{\mathbb{Z}_{q,a,0,4,{}^1/_3}} = \frac{1}{(1-q)ln3}\, ln\left(\rho^q\left(\frac{(1-\rho)(\tau_s)^{\frac{1}{q}}}{\left(\rho+(1-\rho)(\tau_s)^{\frac{1}{q}}\right)}\right)^{q-1}\left(\frac{\rho}{\left(\rho+(1-\rho)(\tau_s)^{\frac{1}{q}}\right)}\right)^{4q}\right)$$

For $\tau_s = 0.5 = \rho$, Clearly, we have $D_{\mathbb{Z}_{1,0,0,4,{}^1/_3}} \to \infty$

Figures 4, 5 and 6 portrays the significant impact of q on $D_{\mathbb{Z}_{q,1,-2,4,\frac{1}{3}}}$ for the prescribed values $\tau_s = 0.5 = \rho$.

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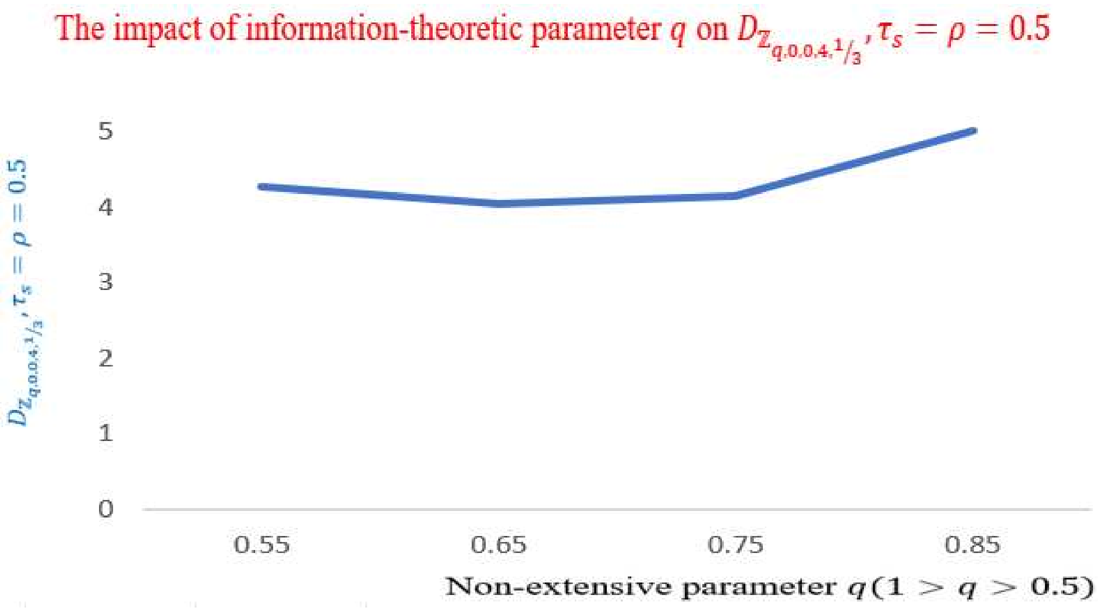

Figure 4.

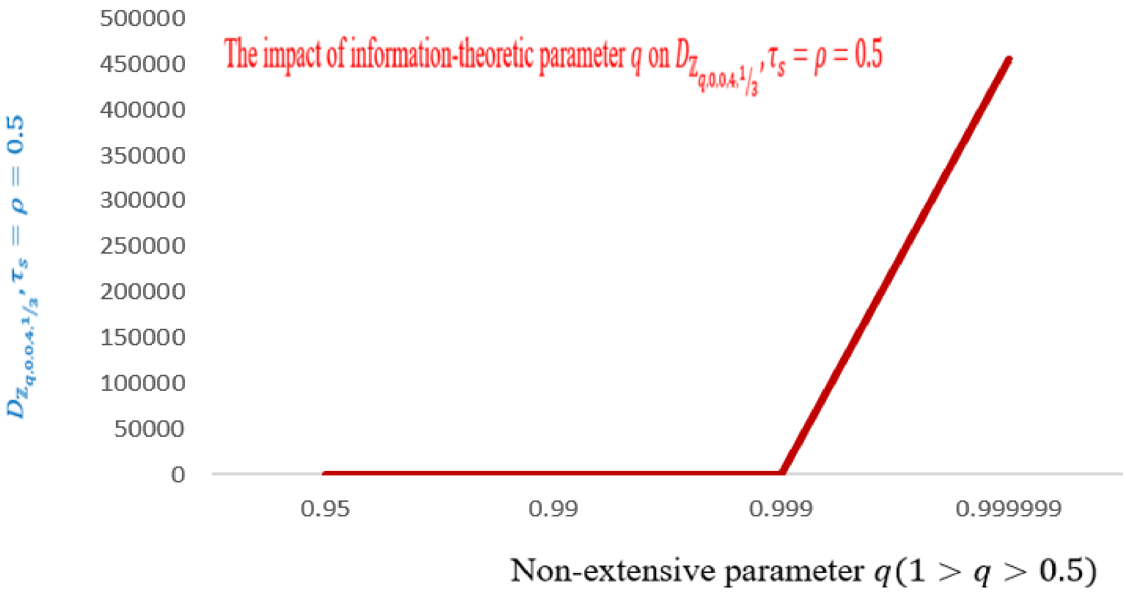

Figure 5.

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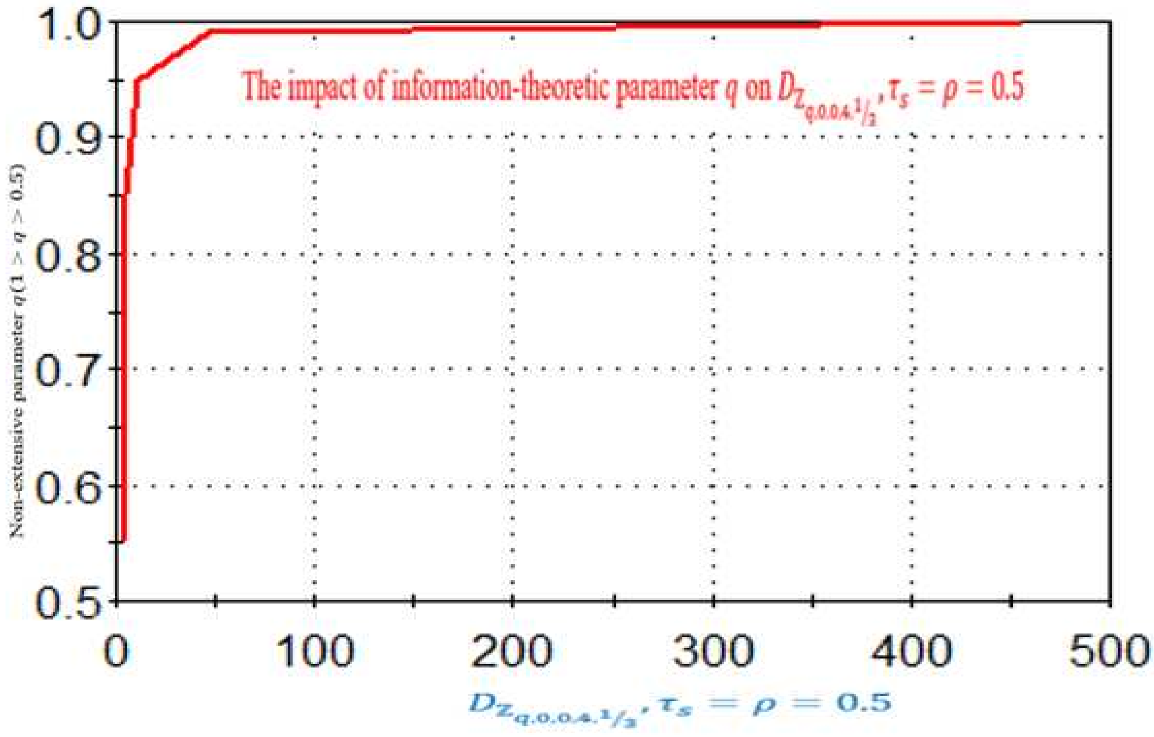

Figure 6.

The impact of information-theoretic parameter q on $D_{\mathbb{Z}_{q,0,0,4,^1/_3}}$ is clear, as $D_{\mathbb{Z}_{q,0,0,4,^1/_3}}$ decreases while q is in the extensivity phase and starts to increase drastically when q is non-extensive.

There is no clearly defined scaling-dimension since the Apollonian gasket[15–17] is only roughly self-similar. However, any "triangular" region enclosed within three circles appeared to be a curved Sierpinski gasket. Remember that scaling it by a factor of 2 requires 3 copies of the Sierpinski gasket as shown by Figures 7 and 8(c.f., [17]). Consequently, we would expect that the Apollonian gasket's fractal dimension will be near to:

$$D = \frac{ln3}{ln2} \approx 1.585 \tag{18}$$

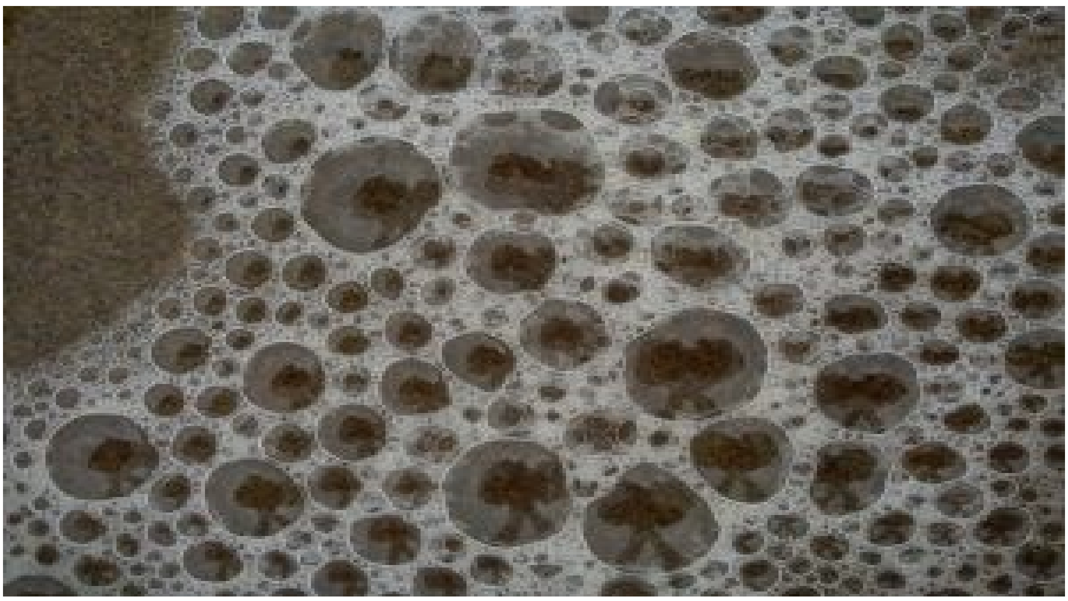

Figure 7. Bubbles are arranged in a fractal way to form foam [17]. .

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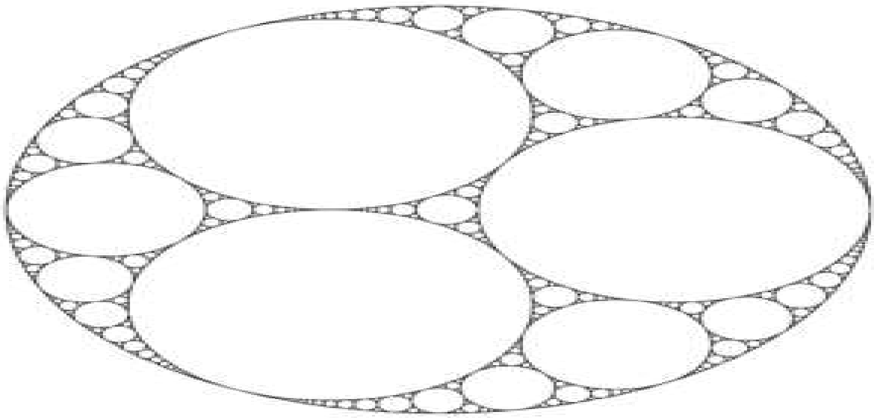

Figure 8. A fractal that can be used to simulate soap bubble foam is the Apollonian gasket[17].

Case 2: $C_s^2 = 1, \rho = 2$(instability)
Sierpiniski Gasket(SG)

$$D_{\mathbb{Z}_{q,1,0,3,\frac{1}{2}}} = \frac{1}{(q-1)ln2}(2^q(-1\ )^q(1-\ (2)^{2q})\ +1)$$

Mathematically speaking, the un-defindedness of SG at many points is based on the fact of attaining complex values at these points, for example:

$$(-1\ )^{0.55} = \left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right)^{0.55} = cos\frac{0.55\pi}{2} + isin\frac{0.55\pi}{2} \qquad \text{(De Moivre Theorem)}$$

So, we are in a situation of a complex valued SG fractal dimension. After some mathematical manipulation, one gets:

$$D_{\mathbb{Z}_{0.55,1,0,3,\frac{1}{2}}}(C_s^2 = 1, \rho = 2)$$

$$= 0.4602098599\left(1.464085696 - \left(cos\frac{0.55\pi}{2} + isin\frac{0.55\pi}{2}\right)\right) \quad = 0.2136291295 + i(0.00693904008)$$

Notably, the fluctuations of the derived values of SG fractal dimension between decreasing and the drastic decreasing along the path of approaching sufficiently large values of q while approaching the extensivity zone, $q > 1$. For $q = 1$, we arrive at invite value for the corresponding SG fractal dimension. Clearly, this shows the significant information-theoretic impact in both non-extensive and extensive phases. This paper provides another revolutionary approach to the traditional definition of both Apollonian and SG dimensions, while mine includes several respective parameters, including queueing and information-theoretic parameters.

5. *D* Applications to PDAs

A data structure called the box locality index (BLI) is used to adapt the box-counting method for fractal dimension calculation [18] for huge data. By encoding the information required for fractal dimension computation, the BLI streamlines the hierarchical structure. Scalable fractal dimension calculation for large data is made possible by the BLI by utilising distributed computing techniques like MapReduce and Spark. This is valuable for a variety of machine learning techniques and data analytics jobs like feature selection and dimensionality reduction.

One methodology that is frequently used to determine the fractal dimension of a dataset is the box-counting method. It involves dividing the dataset's embedding space into a grid of boxes and counting the number of points in each box. By analyzing the relationship between the size of the boxes and the number of points, the fractal dimension can be estimated. This approach is illustrated with three example datasets in Figure 9[18], where the first and third datasets represent one- and two-dimensional objects, respectively, while the second dataset is a well-known fractal called the Sierpinski triangle with a fractal dimension of approximately 1.58. The given one-dimensional dataset generated by a mathematical function. The dataset consists of points (y) that are calculated using the equation $y = 0.1 + 0.8\sin(\pi x) + \sigma$, where x values are uniformly distributed between 0 and 1, and σ represents random noise following a normal distribution with mean 0 and standard deviation 0.01. This dataset is an example of a 1D dataset with a sinusoidal pattern and random noise.

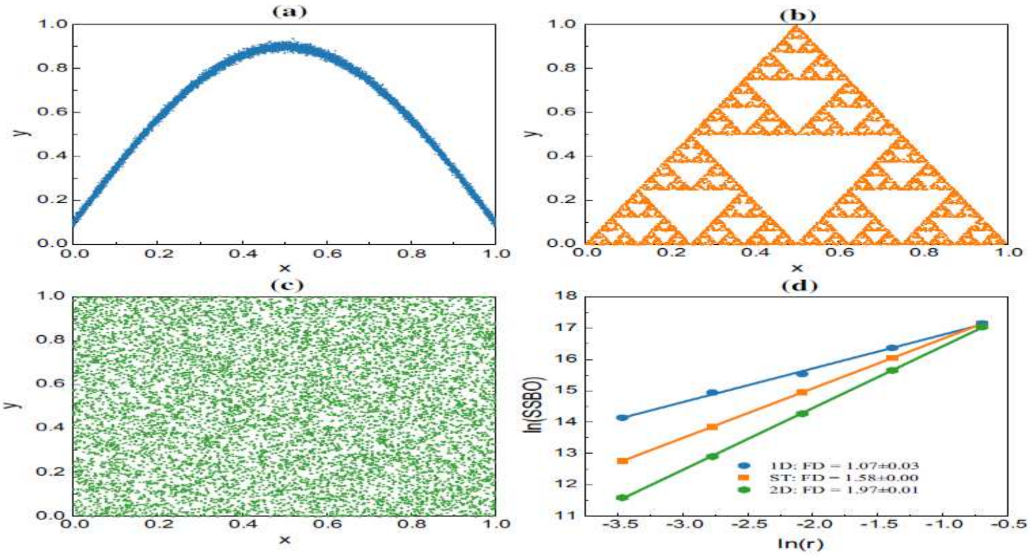

Figure 9. Box-counting plots for three example datasets. The first dataset (1D) is generated by a mathematical function involving sine and random noise. The second dataset is the well-known Sierpinski triangle, and the third dataset (2D) is a uniformly distributed two-dimensional dataset. The box-counting plots help estimate the fractal dimensions of these datasets by analyzing the slopes of the fitted lines.

İn [19], the challenge of online clustering in high-dimensional data and the limitations of existing algorithms in handling this task are thoroughly discussed. Therefore, [19] proposed a novel approach called FractStream to discover core fractal clusters, progressive fractal clusters, and outlier fractal clusters using fractal dimension, basic window technology, and a damped window model. The proposed technique[19] aimed to reduce search complexity, execution time, and memory usage, and its effectiveness and efficiency are demonstrated through experimental studies on various datasets.

An exposition of the construction of a multi-layered nested grid structure for determining the fractal dimension of a dataset is undertaken by [19]. The fractal dimension is calculated by counting the number of data points within each grid of the lowest layer of the grid structure. Additionally, the use of sliding window model for computing cluster partitions on evolving data streams, as discribed by Figure 10 [19].

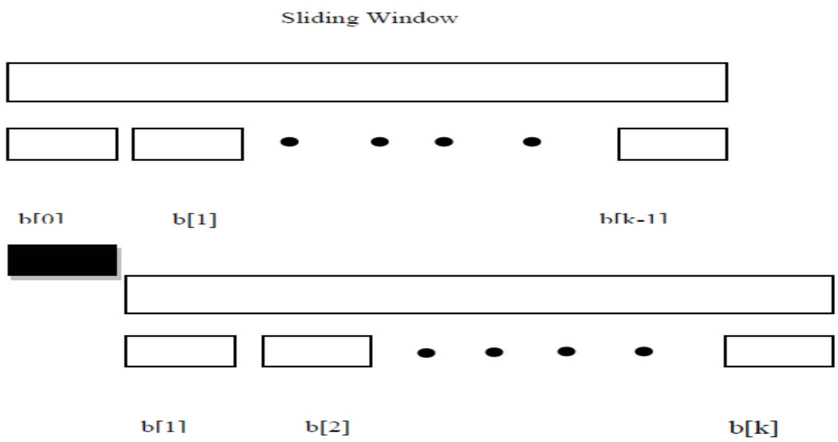

Figure 10. In the context of data analytics and machine learning, sliding window and basic windows are techniques used for analyzing data streams or time series data. A sliding window refers to a fixed-

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size window that moves along the data stream, allowing for continuous analysis of a subset of data. On the other hand, basic windows are non-overlapping windows of fixed size that partition the data stream into distinct segments for analysis. These techniques are commonly employed to extract meaningful patterns and insights from streaming or time-dependent data.

In the given context, a clustering algorithm based on correlation fractal dimension for an evolving data stream was developed[19]. The algorithm involves inserting points into existing clusters based on their relative change in fractal dimension, creating new progressive fractal clusters if certain conditions are met, and creating outlier fractal clusters for points that do not fit into existing clusters. The weight of the clusters is periodically checked, and if a progressive fractal cluster's weight falls below a threshold, it is deleted to make space for new clusters.

The behavior of clusters changes over time when performing online clustering with a window of 1000 data points. This evolution of data can be segmented into intervals, as shown in Figure 11 [19], to analyze the changes in cluster composition.

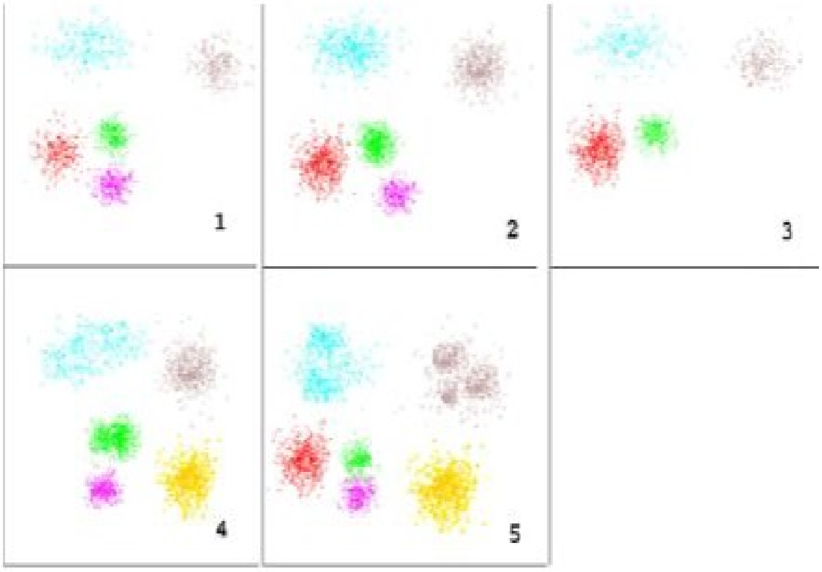

Figure 11. The evolution of clusters over time in the context of online clustering. Initially, there are 5 clusters in a steady state, and as the stream progresses, data points are added to different clusters, resulting in the formation of new clusters and changes in the existing ones. The evaluation of clustering quality is done using the average purity of clusters, which measures the agreement between the cluster labels and the true labels of the data.

In the area of big data applications, disturbances like COVID-19, pollution, or policy changes have a huge effect on economic and financial systems[20]. For expanding the use of big data in financial and economic systems, it is imperative to investigate how these disruptions affect associated time series. The complexity of these time series is analysed using the Generalised Weierstrass-Mandelbrot Function (GWMF) [20], which demonstrates how disturbances in the form of exponential functions can produce multifractal characteristics. Additionally, the model replicates long memory and irregularity, which are evaluated by multifractal analysis and the R/S approach.

Research on how disturbances affect time series produced by the actual part of GWMF, or C(t, μ), and how to replicate multifractal features in time series is scarce[19]. Furthermore, there is little theoretical evidence to support the claim that time series produced by WMF naturally possess the fractal dimension D. Consequently, more statistical examination is required to determine the connection between disturbances and the nonlinear properties of time series produced by C(t, μ), including multifractal analysis and Hurst exponent.

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Disturbance's effect on the statistical characteristics of time series produced by the identified generalized Wavelet Multifractal model. This study demonstrates that disturbances, including feedback terms, can play a complex role in changing the structure of nonlinear time series, shifting from a single fractal dimension to long memory and multifractal features, defying the widely held belief that disturbances have no effect on statistical results. These results emphasize how crucial it is for those working in the financial and economic sectors to comprehend the basic theory of time series, especially when it comes to large data applications[20].

6. Closing Remarks Combined with Open Problems and the Next Phase of Research

This paper explores the relationship between $D_{\mathbb{Z}_{q,a,b,N,\varepsilon}}$ and the information-theoretic queueing parameters. Numerical experiments analyze the behavior of the derived fractal index to evidence that this work represents a significant advancement in unifying information theory and fractal geometry.

An explanation is given to confirm the influential role of fractal dimension in developing and revolutionizing BDAs. The current paper has several emerging open problems.

Open Problem One

Based on the findings of this paper, is it feasible to undertake their approach much further to find the fractal dimension theory of Ismail's Entropy, namely IE(c.f., [21,22]), which is by default the ultimate generalization of numerous in literature?

Open Problem Two

Based on the possibility to unlock open problem one, can we find any mathematical approach to decide the threshold of the involved universal parameters of IE. If so, what will be the expected form of the mathematical relations involved?

Open Problem Three

Can we extend the case to investigate possible applicability of other fractal dimensions in literatures, such as Sierpinski Gasket and Koch Snowflake?

Future research aims to determine the fractal dimensions of other entropies in literature and compare them to further advance the field of Information Theoretic Fractal Geometry (ITFG). Notably, the search is ongoing to possibly find solutions to the above provided open problems.

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