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[Ibtisam Alshammari](#) and [Islam M. Taha](#) *

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Article

On Fuzzy Soft Beta-Continuity and Beta-Irresoluteness: Properties and Applications

Ibtesam Alshammari ¹ and Islam M. Taha ^{2,*}

¹ Department of Mathematics, Faculty of Science, University of Hafr Al Batin, Saudi Arabia

² Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt

* Correspondence: taha@ut.edu.sa or imtaha2010@yahoo.com.

Abstract: In the present study. At first, some properties of r -fuzzy soft β -closed sets are introduced on fuzzy soft topological spaces based on the paper Aygünoğlu et al. (Hacet. J. Math. Stat. 2014, 43, 193-208). Also, we define the closure and interior operators with respect to the classes of r -fuzzy soft β -closed and r -fuzzy soft β -open sets, and investigate some of their properties. In addition, the concept of r -fuzzy soft β -connected sets is introduced and characterized with help of fuzzy soft β -closure operators. Thereafter, some properties of a fuzzy soft β -continuity are studied. Furthermore, we introduce and study the concepts of fuzzy soft almost (weakly) β -continuous functions, which are weaker forms of a fuzzy soft β -continuity. Also, the relationships between these classes of functions are specified with the help of some illustrative examples. Finally, we explore new types of fuzzy soft functions called, fuzzy soft β -irresolute (strongly β -irresolute, β -irresolute open, β -irresolute closed and β -irresolute homeomorphism) functions, and discuss some properties of them. Also, we show that fuzzy soft strongly β -irresolute \Rightarrow fuzzy soft β -irresolute \Rightarrow fuzzy soft β -continuity, but the converse may not be true.

Keywords: fuzzy soft set; fuzzy soft topological space; r -fuzzy soft β -closed (β -open) set; fuzzy soft β -closure (β -interior) operator; fuzzy soft β -continuity; fuzzy soft almost (weakly) β -continuity; fuzzy soft β -irresolute (strongly β -irresolute) function; connectedness

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1. Introduction

Theory of soft set was pioneered by Molodtsov [1], which is a completely novel approach for modeling uncertainty and vagueness. He demonstrated some applications of these theory in solving some practical problems in engineering, economics, and medical science, social science etc. Akdag and Ozkan [2] initiated and studied the concept of soft α -open sets on soft topological spaces. The concept of soft β -open sets was studied by the authors of [3,4], and some properties of soft β -continuity are investigated. Also, the concepts of somewhere dense and Q-sets were defined and studied by the authors of [5,6]. Al-shami et al. [7] initiated the concept of weakly soft semi-open sets and studied its main properties. Also, Al-shami et al. [8] defined and studied the concept of weakly soft β -open sets. Kaur et al. [9] introduced a new approach to studying soft continuous mappings using an induced mapping based on soft sets. Al Ghour and Al-Mufarrij [10] defined new concepts of mappings over soft topological spaces: soft somewhat- r -continuity and soft somewhat- r -openness. Ameen et al. [11] explored more properties of soft somewhere dense continuity.

The notion of fuzzy soft sets was introduced by Maji et al. [12], which combines fuzzy sets [13] and soft sets [1]. Based on fuzzy topologies in the sense of Šostak [14], the notion of fuzzy soft topology is defined and some properties such as fuzzy soft continuity, fuzzy soft interior (closure) set and fuzzy soft subspace is introduced in [15,16]. A new approach to studying separation and regularity axioms via fuzzy soft sets was introduced by the author of [17,18] based on the paper Aygünoğlu et al. [15]. The notion of r -fuzzy soft regularly open sets was defined and studied by Çetkin and Aygün [19]. In addition, the notions of r -fuzzy soft β -open (resp. pre-open) sets were introduced by Taha [20].

The main contribution of this study is arranged as follows:

- In Section 2, we are going to present the notions of fuzzy soft β -closure (β -interior) operators on fuzzy soft topological space based on the article Aygünoğlu et al. [15], and study some properties of them. Also, the concept of r -fuzzy soft β -connected sets is defined and studied.
- In Section 3, we investigate some properties of a fuzzy soft β -continuity. Moreover, we explore and study the notions of fuzzy soft almost (weakly) β -continuous functions, which are weaker forms of fuzzy soft β -continuous functions. Also, we show that fuzzy soft β -continuity \Rightarrow fuzzy soft almost β -continuity \Rightarrow fuzzy soft weakly β -continuity, but the converse may not be true.
- In Section 4, we introduce the notions of fuzzy soft β -irresolute (resp. strongly β -irresolute, β -irresolute open, β -irresolute closed and β -irresolute homeomorphism) functions between two fuzzy soft topological spaces (V, η_F) and (U, τ_E) , and investigate some properties of these functions. Additionally, the relationships between these classes of functions are considered with the help of some examples.
- In the end, we close this study with some conclusions and open a door to suggest some future papers in Section 5.

In this study, nonempty sets will be denoted by V, U etc. F is the set of all parameters for V and $B \subseteq F$. The family of all fuzzy sets on V is denoted by I^V (where $I_o = (0, 1]$, $I = [0, 1]$), and for $t \in I$, $\underline{t}(v) = t$, for all $v \in V$.

The following notions and results will be used in the next sections:

Definition 1. [15, 21, 22] A fuzzy soft set g_B on V is a function from F to I^V such that $g_B(k)$ is a fuzzy set on V , for each $k \in B$ and $g_B(k) = \underline{0}$, if $k \notin B$. The family of all fuzzy soft sets on V is denoted by (\widetilde{V}, F) . In [23], the difference between two fuzzy soft sets g_B and f_A is a fuzzy soft set defined as follows, for each $k \in F$:

$$(g_B \overline{\cap} f_A)(k) = \begin{cases} \underline{0}, & \text{if } g_B(k) \leq f_A(k), \\ g_B(k) \wedge (f_A(k))^c, & \text{otherwise.} \end{cases}$$

Definition 2. [24] A fuzzy soft point k_{v_t} on V is a fuzzy soft set defined as follows:

$$k_{v_t}(e) = \begin{cases} v_t, & \text{if } e = k, \\ \underline{0}, & \text{if } e \in F - \{k\}, \end{cases}$$

where v_t is a fuzzy point on V . k_{v_t} is said to belong to a fuzzy soft set g_B , denoted by $k_{v_t} \tilde{\in} g_B$, if $t \leq g_B(k)(v)$. The family of all fuzzy soft points on V is denoted by $P_t(\widetilde{V})$.

Definition 3. [25] A fuzzy soft point $k_{v_t} \in P_t(\widetilde{V})$ is called a soft quasi-coincident with $g_B \in (\widetilde{V}, F)$, denoted by $k_{v_t} \tilde{q} g_B$, if $t + g_B(k)(v) > 1$. A fuzzy soft set $f_A \in (\widetilde{V}, F)$ is called a soft quasi-coincident with $g_B \in (\widetilde{V}, F)$, denoted by $f_A \tilde{q} g_B$, if there is $k \in F$ and $v \in V$ such that $f_A(k)(v) + g_B(k)(v) > 1$. If f_A is not soft quasi-coincident with g_B , $f_A \tilde{\not{q}} g_B$.

Definition 4. [15] A function $\eta : F \longrightarrow [0, 1]^{(\widetilde{V}, F)}$ is said to be a fuzzy soft topology on V if it satisfies the following, for each $k \in F$:

- (1) $\eta_k(\Phi) = \eta_k(\widetilde{F}) = 1$,
- (2) $\eta_k(g_B \sqcap f_A) \geq \eta_k(g_B) \wedge \eta_k(f_A)$, for each $g_B, f_A \in (\widetilde{V}, F)$,
- (3) $\eta_k(\sqcup_{\delta \in \Delta} (g_B)_\delta) \geq \wedge_{\delta \in \Delta} \eta_k((g_B)_\delta)$, for each $(g_B)_\delta \in (\widetilde{V}, F), \delta \in \Delta$.

Then, (V, η_F) is said to be a fuzzy soft topological space (briefly, FSTS) based on the sense of Šostak [14].

Definition 5. [15] Let (V, η_F) and (U, τ_E) be a FSTSs. A fuzzy soft function $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ is said to be a fuzzy soft continuous if, $\eta_k(\varphi_\psi^{-1}(g_B)) \geq \tau_e(g_B)$ for each $g_B \in (\widetilde{U}, E), k \in F$ and $(e = \psi(k)) \in E$.

Definition 6. [16, 19] In a FSTS (V, η_F) , for each $g_B \in (\widetilde{V}, F), k \in F$ and $r \in I_0$, we define the fuzzy soft operators C_η and $I_\eta : F \times (\widetilde{V}, F) \times I_0 \rightarrow (\widetilde{V}, F)$ as follows:

$$C_\eta(k, g_B, r) = \sqcap \{f_A \in (\widetilde{V}, F) : g_B \sqsubseteq f_A, \eta_k(f_A) \geq r\},$$

$$I_\eta(k, g_B, r) = \sqcup \{f_A \in (\widetilde{V}, F) : f_A \sqsubseteq g_B, \eta_k(f_A) \geq r\}.$$

Definition 7. Let (V, η_F) be a FSTS and $r \in I_0$. A fuzzy soft set $g_B \in (\widetilde{V}, F)$ is said to be an r -fuzzy soft β -open [20] (resp. pre-open [20], semi-open [26] and regularly open [19]) if, $g_B \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, g_B, r), r), r)$ (resp. $g_B \sqsubseteq I_\eta(k, C_\eta(k, g_B, r), r), g_B \sqsubseteq C_\eta(k, I_\eta(k, g_B, r), r)$ and $g_B = I_\eta(k, C_\eta(k, g_B, r), r)$) for each $k \in F$.

Definition 8. [19] Let (V, η_F) be a FSTS and $r \in I_0$. A fuzzy soft set $g_B \in (\widetilde{V}, F)$ is said to be an r -fuzzy soft regularly closed if, $g_B = C_\eta(k, I_\eta(k, g_B, r), r)$ for each $k \in F$.

Definition 9. [20] From the previous definition, we can summarize the relationships among different types of fuzzy soft sets as in the next diagram.

$$\begin{array}{c} \text{regularly open set} \\ \Downarrow \\ \text{pre - open set} \quad \Rightarrow \quad \beta - \text{open set} \end{array}$$

Definition 10. [26] Let (V, η_F) and (U, τ_E) be a FSTSs. A fuzzy soft function $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ is said to be a fuzzy soft almost (resp. weakly) continuous if, for each $k_{v_t} \in P_t(\widetilde{V})$ and each $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r$ containing $\varphi_\psi(k_{v_t})$, there is $f_A \in (\widetilde{V}, F)$ with $\eta_k(f_A) \geq r$ containing k_{v_t} such that $\varphi_\psi(f_A) \sqsubseteq I_\tau(e, C_\tau(e, g_B, r), r)$ (resp. $\varphi_\psi(f_A) \sqsubseteq C_\tau(e, g_B, r)$).

Definition 11. [26] From Definitions 1.5 and 1.9, we have: Fuzzy soft continuity \Rightarrow fuzzy soft almost continuity \Rightarrow fuzzy soft weakly continuity.

The basic results and definitions which we need in the next sections are found in [15,16].

2. Some Properties of r -Fuzzy Soft β -Closed Sets

Here, we introduce the concept of r -fuzzy soft β -closed sets on fuzzy soft topological space based on the sense of Šostak [14], and investigate some properties of them. Also, we define and study the concepts of fuzzy soft β -closure (β -interior) operators. Moreover, the concept of r -fuzzy soft β -connected sets is defined and characterized.

Definition 12. Let (V, η_F) be a FSTS. A fuzzy soft set $g_B \in (\widetilde{V}, F)$ is said to be an r -fuzzy soft β -closed if, $I_\eta(k, C_\eta(k, I_\eta(k, g_B, r), r), r) \sqsubseteq g_B$ for each $k \in F$ and $r \in I_0$.

Proposition 1. Let (V, η_F) be a FSTS, $g_B \in (\widetilde{V}, F)$, $k \in F$ and $r \in I_0$. Then, we have

- (1) g_B is r -fuzzy soft β -closed set iff g_B^c is r -fuzzy soft β -open [20].
- (2) Any intersection of r -fuzzy soft β -closed sets is r -fuzzy soft β -closed.
- (3) Any union of r -fuzzy soft β -open sets is r -fuzzy soft β -open.

Proof. Follows from Definitions 1.7 and 2.1.

□

Proposition 2. Let (V, η_F) be a FSTS, $g_B, f_A \in (\widetilde{V}, F)$, $k \in F$ and $r \in I_0$. If g_B is r -fuzzy soft pre-open set such that $g_B \sqsubseteq f_A \sqsubseteq C_\eta(k, I_\eta(k, g_B, r), r)$, f_A is r -fuzzy soft β -open.

Proof. Since g_B is r -fuzzy soft pre-open and $g_B \sqsubseteq f_A$, then $g_B \sqsubseteq I_\eta(k, C_\eta(k, g_B, r), r) \sqsubseteq I_\eta(k, C_\eta(k, f_A, r), r)$. Since $f_A \sqsubseteq C_\eta(k, I_\eta(k, g_B, r), r)$, then $f_A \sqsubseteq C_\eta(k, I_\eta(k, I_\eta(k, C_\eta(k, f_A, r), r), r), r) = C_\eta(k, I_\eta(k, C_\eta(k, f_A, r), r), r)$, so f_A is r -fuzzy soft β -open.

□

Proposition 3. Let (V, η_F) be a FSTS, $g_B, f_A \in (\widetilde{V}, F)$, $k \in F$ and $r \in I_0$. If g_B is r -fuzzy soft pre-closed set such that $g_B \supseteq f_A \supseteq I_\eta(k, C_\eta(k, g_B, r), r)$, f_A is r -fuzzy soft β -closed.

Proof. Easily proved by a similar way in Proposition 2.2.

□

Definition 13. In a FSTS (V, η_F) , for each $g_B \in (\widetilde{V}, F)$, $k \in F$ and $r \in I_0$, we define a fuzzy soft operator $\beta C_\eta : F \times (\widetilde{V}, F) \times I_0 \rightarrow (\widetilde{V}, F)$ as follows: $\beta C_\eta(k, g_B, r) = \sqcap \{f_A \in (\widetilde{V}, F) : g_B \sqsubseteq f_A, f_A \text{ is } r\text{-fuzzy soft } \beta\text{-closed}\}$.

Theorem 1. In a FSTS (V, η_F) , for each $f_A, g_B \in (\widetilde{V}, F)$, $k \in F$ and $r \in I_0$, the operator $\beta C_\eta : F \times (\widetilde{V}, F) \times I_0 \rightarrow (\widetilde{V}, F)$ satisfies the following properties:

- (1) $\beta C_\eta(k, \Phi, r) = \Phi$.

- (2) $g_B \sqsubseteq \beta C_\eta(k, g_B, r) \sqsubseteq C_\eta(k, g_B, r)$.
- (3) $\beta C_\eta(k, g_B, r) \sqsubseteq \beta C_\eta(k, f_A, r)$, if $g_B \sqsubseteq f_A$.
- (4) $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r) = \beta C_\eta(k, g_B, r)$.
- (5) $\beta C_\eta(k, g_B \sqcup f_A, r) \supseteq \beta C_\eta(k, g_B, r) \sqcup \beta C_\eta(k, f_A, r)$.
- (6) g_B is r -fuzzy soft β -closed iff $\beta C_\eta(k, g_B, r) = g_B$.
- (7) $\beta C_\eta(k, C_\eta(k, g_B, r), r) = C_\eta(k, g_B, r)$.

Proof. (1), (2), (3) and (6) are easily proved from the definition of βC_η .

(4) From (2) and (3), $\beta C_\eta(k, g_B, r) \sqsubseteq \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$. Now we show that $\beta C_\eta(k, g_B, r) \supseteq \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$. Suppose that $\beta C_\eta(k, g_B, r)$ is not contain $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$. Then, there is $v \in V$ and $t \in (0, 1)$ such that $\beta C_\eta(k, g_B, r)(k)(v) < t < \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)(k)(v)$. (A)

Since $\beta C_\eta(k, g_B, r)(k)(v) < t$, by the definition of βC_η , there is h_F is r -fuzzy soft β -closed with $g_B \sqsubseteq h_F$ such that $\beta C_\eta(k, g_B, r)(k)(v) \leq h_F(k)(v) < t$. Since $g_B \sqsubseteq h_F$, we have $\beta C_\eta(k, g_B, r) \sqsubseteq h_F$. Again, by the definition of βC_η , $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r) \sqsubseteq h_F$. Hence $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r)(k)(v) \leq h_F(k)(v) < t$, it is a contradiction for (A). Thus, $\beta C_\eta(k, g_B, r) \supseteq \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$. Then, $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r) = \beta C_\eta(k, g_B, r)$.

(5) Since g_B and $f_A \sqsubseteq g_B \sqcup f_A$, hence by (3), $\beta C_\eta(k, g_B, r) \sqsubseteq \beta C_\eta(k, g_B \sqcup f_A, r)$ and $\beta C_\eta(k, f_A, r) \sqsubseteq \beta C_\eta(k, g_B \sqcup f_A, r)$. Thus, $\beta C_\eta(k, g_B \sqcup f_A, r) \supseteq \beta C_\eta(k, g_B, r) \sqcup \beta C_\eta(k, f_A, r)$.

(7) From (6) and $C_\eta(k, g_B, r)$ is r -fuzzy soft β -closed set, hence $\beta C_\eta(k, C_\eta(k, g_B, r), r) = C_\eta(k, g_B, r)$.
□

Definition 14. Let (V, η_F) be a FSTS, $r \in I_0$ and $g_B, f_A \in (\widetilde{V}, F)$. Then, we have:

(1) Two fuzzy soft sets g_B and f_A are called r -fuzzy soft β -separated iff $g_B \not\tilde{\sqcap} \beta C_\eta(k, f_A, r)$ and $f_A \not\tilde{\sqcap} \beta C_\eta(k, g_B, r)$ for each $k \in F$.

(2) Any fuzzy soft set which cannot be expressed as the union of two r -fuzzy soft β -separated sets is called an r -fuzzy soft β -connected.

Theorem 2. In a FSTS (V, η_F) , we have:

(1) If f_A and $g_B \in (\widetilde{V}, F)$ are r -fuzzy soft β -separated and $h_C, t_D \in (\widetilde{V}, F)$ such that $h_C \sqsubseteq f_A$ and $t_D \sqsubseteq g_B$, then h_C and t_D are r -fuzzy soft β -separated.

(2) If $f_A \not\tilde{\sqcap} g_B$ and either both are r -fuzzy soft β -open or both r -fuzzy soft β -closed, then f_A and g_B are r -fuzzy soft β -separated.

(3) If f_A and g_B are either both r -fuzzy soft β -open or both r -fuzzy soft β -closed, then $f_A \sqcap g_B^c$ and $g_B \sqcap f_A^c$ are r -fuzzy soft β -separated.

Proof. (1) and (2) are obvious.

(3) Let f_A and g_B be an r -fuzzy soft β -open. Since $f_A \sqcap g_B^c \sqsubseteq g_B^c$, $\beta C_\eta(k, f_A \sqcap g_B^c, r) \sqsubseteq g_B^c$ and hence $\beta C_\eta(k, f_A \sqcap g_B^c, r) \tilde{\not\sqsubseteq} g_B$. Then, $\beta C_\eta(k, f_A \sqcap g_B^c, r) \tilde{\not\sqsubseteq} (g_B \sqcap f_A^c)$.

Again, since $g_B \sqcap f_A^c \sqsubseteq f_A^c$, $\beta C_\eta(k, g_B \sqcap f_A^c, r) \sqsubseteq f_A^c$ and hence $\beta C_\eta(k, g_B \sqcap f_A^c, r) \tilde{\not\sqsubseteq} f_A$. Then, $\beta C_\eta(k, g_B \sqcap f_A^c, r) \tilde{\not\sqsubseteq} (f_A \sqcap g_B^c)$. Thus, $f_A \sqcap g_B^c$ and $g_B \sqcap f_A^c$ are r -fuzzy soft β -separated. The other case follows similar lines.

□

Theorem 3. In a FSTS (V, η_F) , then $f_A, g_B \in (\widetilde{V}, \widetilde{F})$ are r -fuzzy soft β -separated iff there exist two r -fuzzy soft β -open sets h_C and t_D such that $f_A \sqsubseteq h_C$, $g_B \sqsubseteq t_D$, $f_A \tilde{\not\sqsubseteq} t_D$ and $g_B \tilde{\not\sqsubseteq} h_C$.

Proof. (\Rightarrow) Let f_A and $g_B \in (\widetilde{V}, \widetilde{F})$ be an r -fuzzy soft β -separated, $f_A \sqsubseteq (\beta C_\eta(k, g_B, r))^c = h_C$ and $g_B \sqsubseteq (\beta C_\eta(k, f_A, r))^c = t_D$, where t_D and h_C are r -fuzzy soft β -open, then $t_D \tilde{\not\sqsubseteq} \beta C_\eta(k, f_A, r)$ and $h_C \tilde{\not\sqsubseteq} \beta C_\eta(k, g_B, r)$. Thus, $g_B \tilde{\not\sqsubseteq} h_C$ and $f_A \tilde{\not\sqsubseteq} t_D$. Hence, we obtain the required result.

(\Leftarrow) Let h_C and t_D be an r -fuzzy soft β -open such that $g_B \sqsubseteq t_D$, $f_A \sqsubseteq h_C$, $g_B \tilde{\not\sqsubseteq} h_C$ and $f_A \tilde{\not\sqsubseteq} t_D$. Then, $g_B \sqsubseteq h_C^c$ and $f_A \sqsubseteq t_D^c$. Hence, $\beta C_\eta(k, g_B, r) \sqsubseteq h_C^c$ and $\beta C_\eta(k, f_A, r) \sqsubseteq t_D^c$. Then, $\beta C_\eta(k, g_B, r) \tilde{\not\sqsubseteq} f_A$ and $\beta C_\eta(k, f_A, r) \tilde{\not\sqsubseteq} g_B$. Thus, g_B and f_A are r -fuzzy soft β -separated. Hence, we obtain the required result.

□

Theorem 4. In a FSTS (V, η_F) , if $g_B \in (\widetilde{V}, \widetilde{F})$ is r -fuzzy soft β -connected such that $g_B \sqsubseteq f_A \sqsubseteq \beta C_\eta(k, g_B, r)$, then f_A is r -fuzzy soft β -connected.

Proof. Suppose that f_A is not r -fuzzy soft β -connected, then there is r -fuzzy soft β -separated sets h_C^* and $t_D^* \in (\widetilde{V}, \widetilde{F})$ such that $f_A = h_C^* \sqcup t_D^*$. Let $h_C = g_B \sqcap h_C^*$ and $t_D = g_B \sqcap t_D^*$, then $g_B = t_D \sqcup h_C$. Since $h_C \sqsubseteq h_C^*$ and $t_D \sqsubseteq t_D^*$, hence by Theorem 2.2(1), h_C and t_D are r -fuzzy soft β -separated, it is a contradiction. Thus, f_A is r -fuzzy soft β -connected, as required.

□

Theorem 5. In a FSTS (V, η_F) , for each $g_B \in (\widetilde{V}, \widetilde{F})$, $k \in F$ and $r \in I_0$, we define a fuzzy soft operator $\beta I_\eta : F \times (\widetilde{V}, \widetilde{F}) \times I_0 \rightarrow (\widetilde{V}, \widetilde{F})$ as follows: $\beta I_\eta(k, g_B, r) = \sqcup \{f_A \in (\widetilde{V}, \widetilde{F}) : f_A \sqsubseteq g_B, f_A \text{ is } r\text{-fuzzy soft } \beta\text{-open}\}$. Then, for each $f_A, g_B \in (\widetilde{V}, \widetilde{F})$, the operator βI_η satisfies the following properties:

- (1) $\beta I_\eta(k, \widetilde{F}, r) = \widetilde{F}$.
- (2) $I_\eta(k, g_B, r) \sqsubseteq \beta I_\eta(k, g_B, r) \sqsubseteq g_B$.
- (3) $\beta I_\eta(k, g_B, r) \sqsubseteq \beta I_\eta(k, f_A, r)$, if $g_B \sqsubseteq f_A$.
- (4) $\beta I_\eta(k, \beta I_\eta(k, g_B, r), r) = \beta I_\eta(k, g_B, r)$.
- (5) $\beta I_\eta(k, g_B, r) \sqcap \beta I_\eta(k, f_A, r) \sqsupseteq \beta I_\eta(k, g_B \sqcap f_A, r)$.
- (6) g_B is r -fuzzy soft β -open iff $\beta I_\eta(k, g_B, r) = g_B$.
- (7) $\beta I_\eta(k, g_B^c, r) = (\beta C_\eta(k, g_B, r))^c$.

Proof. (1), (2), (3) and (6) are easily proved from the definition of βI_η .

(4) and (5) are easily proved by a similar way in Theorem 2.1.

(7) For each $g_B \in (\widetilde{V}, F)$, $k \in F$ and $r \in I_0$, we have $\beta I_\eta(k, g_B^c, r) = \sqcup \{f_A \in (\widetilde{V}, F) : f_A \sqsubseteq g_B^c, f_A \text{ is } r\text{-fuzzy soft } \beta\text{-open}\} = [\sqcap \{f_A^c \in (\widetilde{V}, F) : g_B \sqsubseteq f_A^c, f_A^c \text{ is } r\text{-fuzzy soft } \beta\text{-closed}\}]^c = (\beta C_\eta(k, g_B, r))^c$.
□

3. Weaker Forms of Fuzzy Soft β -Continuity

Here, we investigate some properties of fuzzy soft β -continuity. As a weaker form of fuzzy soft β -continuity, the concepts of fuzzy soft almost (weakly) β -continuous functions are introduced, and some properties are given. Also, we show that fuzzy soft β -continuity \Rightarrow fuzzy soft almost β -continuity \Rightarrow fuzzy soft weakly β -continuity, but the converse may not be true.

Definition 15. Let (V, η_F) and (U, τ_E) be a FSTSs. A fuzzy soft function $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ is said to be a fuzzy soft β -continuous if, $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -closed set for each $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B^c) \geq r, k \in F, (e = \psi(k)) \in E$ and $r \in I_0$.

Theorem 6. Let (V, η_F) and (U, τ_E) be a FSTSs, and $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in (\widetilde{U}, E), k \in F, (e = \psi(k)) \in E$ and $r \in I_0$:

- (1) φ_ψ is fuzzy soft β -continuous.
- (2) For each $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r, \varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -open.
- (3) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.
- (4) $\varphi_\psi^{-1}(I_\tau(e, g_B, r)) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B), r)$.
- (5) $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

Proof. (1) \Leftrightarrow (2) Follows from Proposition 2.1(1) and $\varphi_\psi^{-1}(g_B^c) = (\varphi_\psi^{-1}(g_B))^c$.

(1) \Rightarrow (3) Let $g_B \in (\widetilde{U}, E)$, hence by (1), $\varphi_\psi^{-1}(C_\tau(e, g_B, r))$ is r -fuzzy soft β -closed. Then, we obtain $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

(3) \Leftrightarrow (4) Follows from Theorem 2.5(7).

(3) \Rightarrow (5) Let $g_B \in (\widetilde{U}, E)$, hence by (3), we obtain $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

(5) \Rightarrow (2) Let $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r$, hence by (5), we obtain $(\varphi_\psi^{-1}(g_B))^c = \varphi_\psi^{-1}(g_B^c) \sqsupseteq I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B^c), r), r), r) = (C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r))^c$. Then, $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$, so $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -open.
□

Lemma 1. Every fuzzy soft continuous function [15] is fuzzy soft β -continuous.

Proof. Follows from Definition 1.5 and Theorem 3.1.
□

Remark 1. The converse of Lemma 3.1 is not true, as shown by Example 3.1.

Example 1. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$ and define $g_F, h_F \in (\widetilde{V}, F)$ as follows: $g_F = \{(k_1, \{\frac{v_1}{0.2}, \frac{v_2}{0.3}, \frac{v_3}{0.2}\}), (k_2, \{\frac{v_1}{0.2}, \frac{v_2}{0.3}, \frac{v_3}{0.2}\})\}$, $h_F = \{(k_1, \{\frac{v_1}{0.3}, \frac{v_2}{0.4}, \frac{v_3}{0.8}\}), (k_2, \{\frac{v_1}{0.3}, \frac{v_2}{0.4}, \frac{v_3}{0.8}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \rightarrow [0, 1]^{(\widetilde{V}, F)}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{2}, & \text{if } n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{3}, & \text{if } n_F = h_F, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \rightarrow (V, \tau_F)$ is fuzzy soft β -continuous, but it is not fuzzy soft continuous.

Definition 16. Let (V, η_F) and (U, τ_E) be a FSTSs. A fuzzy soft function $\varphi_\psi : (\widetilde{V}, F) \rightarrow (\widetilde{U}, E)$ is said to be a fuzzy soft almost (resp. weakly) β -continuous if, for each $k_{v_i} \in P_t(\widetilde{V})$ and each $f_A \in (\widetilde{U}, E)$ with $\tau_e(f_A) \geq r$ containing $\varphi_\psi(k_{v_i})$, there is $g_B \in (\widetilde{V}, F)$ is r -fuzzy soft β -open set containing k_{v_i} such that $\varphi_\psi(g_B) \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r)$ (resp. $\varphi_\psi(g_B) \sqsubseteq C_\tau(e, f_A, r)$), $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_o$.

Lemma 2. (1) Every fuzzy soft β -continuous function is fuzzy soft almost β -continuous.

(2) Every fuzzy soft almost β -continuous function is fuzzy soft weakly β -continuous.

Proof. Follows from Definition 3.2 and Theorem 3.1.

□

Remark 2. The converse of Lemma 3.2 is not true, as shown by Examples 3.2 and 3.3.

Example 2. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$ and define $f_F, g_F, h_F \in (\widetilde{V}, F)$ as follows: $f_F = \{(k_1, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.5}\}), (k_2, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.5}\})\}$, $g_F = \{(k_1, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\}), (k_2, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\})\}$, $h_F = \{(k_1, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.6}\}), (k_2, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.6}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \rightarrow [0, 1]^{(\widetilde{V}, F)}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{2}{3}, & \text{if } n_F = f_F, \\ \frac{1}{2}, & \text{if } n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{2}{3}, & \text{if } n_F = f_F, \\ \frac{1}{3}, & \text{if } n_F = h_F, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \rightarrow (V, \tau_F)$ is fuzzy soft almost β -continuous, but it is not fuzzy soft β -continuous.

Example 3. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$ and define $g_F, h_F \in (\widetilde{V}, F)$ as follows: $g_F = \{(k_1, \{\frac{v_1}{0.6}, \frac{v_2}{0.2}, \frac{v_3}{0.4}\}), (k_2, \{\frac{v_1}{0.6}, \frac{v_2}{0.2}, \frac{v_3}{0.4}\})\}$, $h_F = \{(k_1, \{\frac{v_1}{0.3}, \frac{v_2}{0.2}, \frac{v_3}{0.5}\}), (k_2, \{\frac{v_1}{0.3}, \frac{v_2}{0.2}, \frac{v_3}{0.5}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \rightarrow [0, 1]^{(\widetilde{V}, F)}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \tilde{F}\}, \\ \frac{1}{2}, & \text{if } n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \tilde{F}\}, \\ \frac{1}{3}, & \text{if } n_F = h_F, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \longrightarrow (V, \tau_F)$ is fuzzy soft weakly β -continuous, but it is not fuzzy soft almost β -continuous.

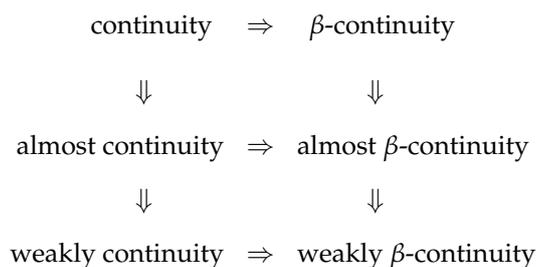
Lemma 3. (1) Every fuzzy soft almost continuous function is fuzzy soft almost β -continuous.

(2) Every fuzzy soft weakly continuous function is fuzzy soft weakly β -continuous.

Proof. Follows from Definitions 1.9 and 3.2.

□

Remark 3. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft continuity as in the next diagram.



Theorem 7. Let (V, η_F) and (U, τ_E) be a FSTSs, and $\varphi_\psi : (\widetilde{V}, \widetilde{F}) \longrightarrow (\widetilde{U}, \widetilde{E})$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in (\widetilde{U}, \widetilde{E})$, $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_0$:

- (1) φ_ψ is fuzzy soft almost β -continuous.
- (2) $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -open, for each g_B is r -fuzzy soft regularly open.
- (3) $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -closed, for each g_B is r -fuzzy soft regularly closed.
- (4) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$, for each g_B is r -fuzzy soft β -open.
- (5) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$, for each g_B is r -fuzzy soft semi-open.
- (6) $\beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r) \supseteq \varphi_\psi^{-1}(g_B)$, for each g_B with $\tau_e(g_B) \geq r$.

Proof. (1) \Rightarrow (2) Let $k_{v_t} \in P_t(\widetilde{V})$ and $g_B \in (\widetilde{U}, \widetilde{E})$ be an r -fuzzy soft regularly open set containing $\varphi_\psi(k_{v_t})$, hence by (1), there is $f_A \in (\widetilde{V}, \widetilde{F})$ is r -fuzzy soft β -open set containing k_{v_t} such that $\varphi_\psi(f_A) \sqsubseteq I_\tau(e, C_\tau(e, g_B, r), r)$.

Thus, $f_A \sqsubseteq \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)) = \varphi_\psi^{-1}(g_B)$ and $k_{v_t} \tilde{\in} f_A \sqsubseteq \varphi_\psi^{-1}(g_B)$. Then, $k_{v_t} \tilde{\in} C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$ and $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$. Therefore, $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -open set.

(2) \Rightarrow (3) Let g_B be an r -fuzzy soft regularly closed set, hence by (2), $\varphi_\psi^{-1}(g_B^c) = (\varphi_\psi^{-1}(g_B))^c$ is r -fuzzy soft β -open set. Then, $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -closed set.

(3) \Rightarrow (4) Let g_B be an r -fuzzy soft β -open set. Since $C_\tau(e, g_B, r)$ is r -fuzzy soft regularly closed set, hence by (3), $\varphi_\psi^{-1}(C_\tau(e, g_B, r))$ is r -fuzzy soft β -closed set. Since $\varphi_\psi^{-1}(g_B) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$, then we have $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

(4) \Rightarrow (5) This is obvious from every r -fuzzy soft semi-open set is r -fuzzy soft β -open set.

(5) \Rightarrow (3) Let g_B be an r -fuzzy soft regularly closed set, hence g_B is r -fuzzy soft semi-open set. Then by (5), $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r)) = \varphi_\psi^{-1}(g_B)$. Therefore, $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -closed set.

(3) \Rightarrow (6) Let $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r$ and $k_{v_t} \tilde{\in} \varphi_\psi^{-1}(g_B)$, then we have $k_{v_t} \tilde{\in} \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r))$. Since $[I_\tau(e, C_\tau(e, g_B, r), r)]^c$ is r -fuzzy soft regularly closed set, hence by (3), $\varphi_\psi^{-1}([I_\tau(e, C_\tau(e, g_B, r), r)]^c)$ is r -fuzzy soft β -closed set. Thus, $\varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r))$ is r -fuzzy soft β -open set and $k_{v_t} \tilde{\in} \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$. Then, $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$.

(6) \Rightarrow (1) Let $k_{v_t} \in P_t(\widetilde{V})$ and $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r$ containing $\varphi_\psi(k_{v_t})$, hence by (6), $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$.

Since $k_{v_t} \tilde{\in} \varphi_\psi^{-1}(g_B)$, then we obtain $k_{v_t} \tilde{\in} \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r) = f_A$ (say). Hence, there is $f_A \in (\widetilde{V}, F)$ is r -fuzzy soft β -open set containing k_{v_t} such that $\varphi_\psi(f_A) \sqsubseteq I_\tau(e, C_\tau(e, g_B, r), r)$. Therefore, φ_ψ is fuzzy soft almost β -continuous.

□

In a similar way, we can prove the following theorem.

Theorem 8. Let (V, η_F) and (U, τ_E) be a FSTSs, and $\varphi_\psi : (\widetilde{V}, F) \rightarrow (\widetilde{U}, E)$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in (\widetilde{U}, E)$, $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_\circ$:

- (1) φ_ψ is fuzzy soft weakly β -continuous.
- (2) $C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(C_\tau(e, g_B, r)), r), r), r) \supseteq \varphi_\psi^{-1}(g_B)$, if $\tau_e(g_B) \geq r$.
- (3) $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, g_B, r)), r), r), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\tau_e(g_B^c) \geq r$.
- (4) $\beta C_\eta(k, \varphi_\psi^{-1}(I_\tau(e, g_B, r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\tau_e(g_B^c) \geq r$.
- (5) $\beta C_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.
- (6) $\beta I_\eta(k, \varphi_\psi^{-1}(C_\tau(e, I_\tau(e, g_B, r), r)), r) \supseteq \varphi_\psi^{-1}(I_\tau(e, g_B, r))$.
- (7) $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(C_\tau(e, g_B, r)), r)$, if $\tau_e(g_B) \geq r$.

Let \mathcal{P} and $\mathcal{Q} : F \times (\widetilde{V}, F) \times I_0 \rightarrow (\widetilde{V}, F)$ be operators on (\widetilde{V}, F) , and \mathcal{R} and $\mathcal{S} : E \times (\widetilde{U}, E) \times I_0 \rightarrow (\widetilde{U}, E)$ be operators on (\widetilde{U}, E) .

Definition 17. [26] Let (V, η_F) and (U, τ_E) be a FSTSs. $\varphi_\psi : (\widetilde{V}, F) \rightarrow (\widetilde{U}, E)$ is said to be a fuzzy soft $(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S})$ -continuous function if, $\mathcal{P}[k, \varphi_\psi^{-1}(\mathcal{S}(e, g_B, r)), r] \sqcap \mathcal{Q}[k, \varphi_\psi^{-1}(\mathcal{R}(e, g_B, r)), r] = \Phi$ for each $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r, k \in F$ and $(e = \psi(k)) \in E$.

In (2014), Aygünoğlu et al. [15] defined the notion of fuzzy soft continuous functions: $\eta_k(\varphi_\psi^{-1}(f_A)) \geq \tau_e(f_A)$, for each $f_A \in (\widetilde{U}, E), k \in F$ and $(e = \psi(k)) \in E$. We can see that Definition 3.3 generalizes the concept of fuzzy soft continuous functions, when we choose \mathcal{P} = identity operator, \mathcal{Q} = interior operator, \mathcal{R} = identity operator and \mathcal{S} = identity operator.

A historical justification of Definition 3.3:

(1) In Section 3, we introduced the notion of fuzzy soft β -continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$, for each $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r$. Here, \mathcal{P} = identity operator, \mathcal{Q} = closure interior closure operator, \mathcal{R} = identity operator and \mathcal{S} = identity operator.

(2) In Section 3, we introduced the notion of fuzzy soft almost β -continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$, for each $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r$. Here, \mathcal{P} = identity operator, \mathcal{Q} = β -interior operator, \mathcal{R} = interior closure operator and \mathcal{S} = identity operator.

(3) In Section 3, we introduced the notion of fuzzy soft weakly β -continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(C_\tau(e, g_B, r)), r)$, for each $g_B \in (\widetilde{U}, E)$ with $\tau_e(g_B) \geq r$. Here, \mathcal{P} = identity operator, \mathcal{Q} = β -interior operator, \mathcal{R} = closure operator and \mathcal{S} = identity operator.

4. Fuzzy Soft β -Irresoluteness

Here, we introduce the concepts of fuzzy soft β -irresolute (resp. strongly β -irresolute, β -irresolute open, β -irresolute closed and β -irresolute homeomorphism) functions between two fuzzy soft topological spaces (V, η_F) and (U, τ_E) , and study some of its features. Also, we show that fuzzy soft strongly β -irresolute \Rightarrow fuzzy soft β -irresolute \Rightarrow fuzzy soft β -continuity, but the converse may not be true.

Definition 18. Let (V, η_F) and (U, τ_E) be a FSTSs. A fuzzy soft function $\varphi_\psi : (\widetilde{V}, F) \rightarrow (\widetilde{U}, E)$ is said to be a fuzzy soft β -irresolute (resp. strongly β -irresolute) if, $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -open (resp. semi-open) set for each $g_B \in (\widetilde{U}, E)$ is r -fuzzy soft β -open set and $r \in I_0$.

Lemma 4. (1) Every fuzzy soft strongly β -irresolute function is fuzzy soft β -irresolute.

(2) Every fuzzy soft β -irresolute function is fuzzy soft β -continuous.

Proof. Follows from Definition 4.1 and Theorem 3.1.

□

Remark 4. The converse of Lemma 4.1 is not true, as shown by Examples 4.1 and 4.2.

Example 4. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$ and define $f_F, g_F \in (\widetilde{V}, F)$ as follows: $f_F = \{(k_1, \{\frac{v_1}{0.3}, \frac{v_2}{0.3}, \frac{v_3}{0.3}\}), (k_2, \{\frac{v_1}{0.3}, \frac{v_2}{0.3}, \frac{v_3}{0.3}\})\}$, $g_F = \{(k_1, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\}), (k_2, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \rightarrow [0, 1]^{(\widetilde{V}, F)}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{2}, & \text{if } n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{2}, & \text{if } n_F = f_F, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \rightarrow (V, \tau_F)$ is fuzzy soft β -irresolute, but it is not fuzzy soft strongly β -irresolute.

Example 5. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$ and define $f_F, g_F \in (\widetilde{V}, F)$ as follows: $f_F = \{(k_1, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.5}\}), (k_2, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.5}\})\}$, $g_F = \{(k_1, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\}), (k_2, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \rightarrow [0, 1]^{(\widetilde{V}, F)}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{3}, & \text{if } n_F = f_F, \\ \frac{1}{2}, & \text{if } n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if } n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{3}, & \text{if } n_F = f_F, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \rightarrow (V, \tau_F)$ is fuzzy soft β -continuous, but it is not fuzzy soft β -irresolute.

Remark 5. From the previous results, we have: Fuzzy soft strongly β -irresolute \Rightarrow fuzzy soft β -irresolute \Rightarrow fuzzy soft β -continuity.

Theorem 9. Let (V, η_F) and (U, τ_E) be a FSTSSs, and $\varphi_\psi : (\widetilde{V}, F) \rightarrow (\widetilde{U}, E)$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in (\widetilde{U}, E)$, $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_0$:

- (1) φ_ψ is fuzzy soft β -irresolute.
- (2) For each g_B is r -fuzzy soft β -closed, $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -closed.
- (3) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.
- (4) $\varphi_\psi^{-1}(\beta I_\tau(e, g_B, r)) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B), r)$.
- (5) $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

Proof. (1) \Leftrightarrow (2) Follows from Proposition 2.1(1) and $\varphi_\psi^{-1}(g_B^c) = (\varphi_\psi^{-1}(g_B))^c$.

(2) \Rightarrow (3) Let $g_B \in (\widetilde{U}, E)$, hence by (2), $\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$ is r -fuzzy soft β -closed. Then, we obtain $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

(3) \Leftrightarrow (4) Follows from Theorem 2.5(7).

(3) \Rightarrow (5) Let $g_B \in (\widetilde{U}, \widetilde{E})$, hence by (3), we obtain $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

(5) \Rightarrow (1) Let $g_B \in (\widetilde{U}, \widetilde{E})$ be an r -fuzzy soft β -open, hence by (5), we obtain $(\varphi_\psi^{-1}(g_B))^c = \varphi_\psi^{-1}(g_B^c) \sqsupseteq I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B^c), r), r), r) = (C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r))^c$. Then, $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$, so $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft β -open. Hence, φ_ψ is fuzzy soft β -irresolute.

□

Theorem 10. Let (V, η_F) and (U, τ_E) be a FSTSSs, and $\varphi_\psi : (\widetilde{V}, \widetilde{F}) \longrightarrow (\widetilde{U}, \widetilde{E})$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in (\widetilde{U}, \widetilde{E})$, $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_0$:

(1) φ_ψ is fuzzy soft strongly β -irresolute.

(2) For each g_B is r -fuzzy soft β -closed, $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft semi-closed.

(3) $I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

Proof. (1) \Leftrightarrow (2) Follows from Proposition 2.1(1) and $\varphi_\psi^{-1}(g_B^c) = (\varphi_\psi^{-1}(g_B))^c$.

(2) \Rightarrow (3) Let $g_B \in (\widetilde{U}, \widetilde{E})$, hence by (2), $\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$ is r -fuzzy soft semi-closed. Then, we obtain $I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

(3) \Rightarrow (1) Let $g_B \in (\widetilde{U}, \widetilde{E})$ be an r -fuzzy soft β -open, hence by (3), we obtain $(\varphi_\psi^{-1}(g_B))^c = \varphi_\psi^{-1}(g_B^c) \sqsupseteq I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B^c), r), r) = (C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r))^c$. Then, $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r)$, so $\varphi_\psi^{-1}(g_B)$ is r -fuzzy soft semi-open. Hence, φ_ψ is fuzzy soft strongly β -irresolute.

□

Proposition 4. Let (V, η_F) , (U, τ_E) and (W, γ_H) be a FSTSSs, and $\varphi_\psi : (\widetilde{V}, \widetilde{F}) \longrightarrow (\widetilde{U}, \widetilde{E})$, $\varphi_{\psi^*}^* : (\widetilde{U}, \widetilde{E}) \longrightarrow (\widetilde{W}, \widetilde{H})$ be two fuzzy soft functions. Then, the composition $\varphi_{\psi^*}^* \circ \varphi_\psi$ is fuzzy soft β -irresolute (resp. strongly β -irresolute and β -continuous) if, φ_ψ is fuzzy soft β -irresolute (resp. strongly β -irresolute and β -irresolute) and $\varphi_{\psi^*}^*$ is fuzzy soft β -irresolute (resp. β -irresolute and β -continuous).

Proof. Follows from Definition 4.1 and Theorem 3.1.

□

Proposition 5. Let (V, η_F) , (U, τ_E) and (W, γ_H) be a FSTSSs, and $\varphi_\psi : (\widetilde{V}, \widetilde{F}) \longrightarrow (\widetilde{U}, \widetilde{E})$, $\varphi_{\psi^*}^* : (\widetilde{U}, \widetilde{E}) \longrightarrow (\widetilde{W}, \widetilde{H})$ be two fuzzy soft functions. Then, the composition $\varphi_{\psi^*}^* \circ \varphi_\psi$ is fuzzy soft almost β -continuous if, φ_ψ is fuzzy soft β -continuous (resp. β -irresolute and β -continuous) and $\varphi_{\psi^*}^*$ is fuzzy soft almost continuous (resp. almost β -continuous and continuous).

Proof. Follows from the above definitions. □

Definition 19. Let (V, η_F) and (U, τ_E) be a FSTSSs and $r \in I_0$. A fuzzy soft function $\varphi_\psi : (\widetilde{V}, \widetilde{F}) \longrightarrow (\widetilde{U}, \widetilde{E})$ is said to be a fuzzy soft β -irresolute open (resp. closed) if, $\varphi_\psi(f_A)$ is r -fuzzy soft β -open (resp. β -closed) set for each $f_A \in (\widetilde{V}, \widetilde{F})$ is r -fuzzy soft β -open (resp. β -closed) set.

Theorem 11. Let (V, η_F) and (U, τ_E) be a FSTSs, and $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ be a fuzzy soft function. The following statements are equivalent for each $f_A \in (\widetilde{V}, F)$, $g_B \in (\widetilde{U}, E)$, $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_0$:

(1) φ_ψ is fuzzy soft β -irresolute open.

(2) $\varphi_\psi(\beta I_\eta(k, f_A, r)) \sqsubseteq \beta I_\tau(e, \varphi_\psi(f_A), r)$.

(3) $\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(\beta I_\tau(e, g_B, r))$.

(4) For each g_B and each f_A is r -fuzzy soft β -closed with $\varphi_\psi^{-1}(g_B) \sqsubseteq f_A$, there is $h_C \in (\widetilde{U}, E)$ is r -fuzzy soft β -closed with $g_B \sqsubseteq h_C$ such that $\varphi_\psi^{-1}(h_C) \sqsubseteq f_A$.

Proof. (1) \Rightarrow (2) Since $\varphi_\psi(\beta I_\eta(k, f_A, r)) \sqsubseteq \varphi_\psi(f_A)$, hence by (1), $\varphi_\psi(\beta I_\eta(k, f_A, r))$ is r -fuzzy soft β -open. Hence, $\varphi_\psi(\beta I_\eta(k, f_A, r)) \sqsubseteq \beta I_\tau(e, \varphi_\psi(f_A), r)$.

(2) \Rightarrow (3) Put $f_A = \varphi_\psi^{-1}(g_B)$, hence by (2), $\varphi_\psi(\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r)) \sqsubseteq \beta I_\tau(e, \varphi_\psi(\varphi_\psi^{-1}(g_B)), r) \sqsubseteq \beta I_\tau(e, g_B, r)$. Then, $\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(\beta I_\tau(e, g_B, r))$.

(3) \Rightarrow (4) Let $g_B \in (\widetilde{U}, E)$ and $f_A \in (\widetilde{V}, F)$ be an r -fuzzy soft β -closed with $\varphi_\psi^{-1}(g_B) \sqsubseteq f_A$. Since $f_A^c \sqsubseteq \varphi_\psi^{-1}(g_B^c)$, $f_A^c = \beta I_\eta(k, f_A^c, r) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B^c), r)$. Hence by (3), $f_A^c \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B^c), r) \sqsubseteq \varphi_\psi^{-1}(\beta I_\tau(e, g_B^c, r))$. Thus, $f_A \supseteq (\varphi_\psi^{-1}(\beta I_\tau(e, g_B^c, r)))^c = \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$. Hence, there is $\beta C_\tau(e, g_B, r) \in (\widetilde{U}, E)$ is r -fuzzy soft β -closed with $g_B \sqsubseteq \beta C_\tau(e, g_B, r)$ such that $\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r)) \sqsubseteq f_A$.

(4) \Rightarrow (1) Let $w_D \in (\widetilde{V}, F)$ be an r -fuzzy soft β -open. Put $g_B = (\varphi_\psi(w_D))^c$ and $f_A = w_D^c$, $\varphi_\psi^{-1}(g_B) = \varphi_\psi^{-1}((\varphi_\psi(w_D))^c) \sqsubseteq f_A$. Hence by (4), there is $h_C \in (\widetilde{U}, E)$ is r -fuzzy soft β -closed with $g_B \sqsubseteq h_C$ such that $\varphi_\psi^{-1}(h_C) \sqsubseteq f_A = w_D^c$. Thus, $\varphi_\psi(w_D) \sqsubseteq \varphi_\psi(\varphi_\psi^{-1}(h_C)) \sqsubseteq h_C^c$. On the other hand, since $g_B \sqsubseteq h_C$, $\varphi_\psi(w_D) = g_B^c \supseteq h_C^c$. Thus, $\varphi_\psi(w_D) = h_C^c$, so $\varphi_\psi(w_D)$ is r -fuzzy soft β -open. Then, φ_ψ is fuzzy soft β -irresolute open. \square

In a similar way, we can prove the following theorem.

Theorem 12. Let (V, η_F) and (U, τ_E) be a FSTSs, and $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ be a bijective fuzzy soft function. The following statements are equivalent for each $f_A \in (\widetilde{V}, F)$, $g_B \in (\widetilde{U}, E)$, $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_0$:

(1) φ_ψ is fuzzy soft β -irresolute closed.

(2) $\beta C_\tau(e, \varphi_\psi(f_A), r) \sqsubseteq \varphi_\psi(\beta C_\eta(k, f_A, r))$.

(3) $\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r)) \sqsubseteq \beta C_\eta(k, \varphi_\psi^{-1}(g_B), r)$.

(4) For each g_B and each f_A is r -fuzzy soft β -open with $\varphi_\psi^{-1}(g_B) \sqsubseteq f_A$, there is $h_C \in (\widetilde{U}, E)$ is r -fuzzy soft β -open with $g_B \sqsubseteq h_C$ such that $\varphi_\psi^{-1}(h_C) \sqsubseteq f_A$.

Proposition 6. Let (V, η_F) and (U, τ_E) be a FSTSs, and $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ be a bijective fuzzy soft function. φ_ψ is fuzzy soft β -irresolute closed iff φ_ψ is fuzzy soft β -irresolute open.

Proof. It is easily proved from;

$$\begin{aligned} \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r)) &\sqsubseteq \beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \\ \iff \varphi_\psi^{-1}(\beta I_\tau(e, g_B^c, r)) &\sqsupseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B^c), r). \end{aligned}$$

□

Definition 20. Let (V, η_F) and (U, τ_E) be a FSTSs. A fuzzy soft function $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ is said to be a fuzzy soft β -irresolute homeomorphism if, φ_ψ is bijective, and both of φ_ψ and φ_ψ^{-1} are β -irresolute.

From the above theorems, we obtain the following corollary.

Corollary 1. Let (V, η_F) and (U, τ_E) be a FSTSs, and $\varphi_\psi : (\widetilde{V}, F) \longrightarrow (\widetilde{U}, E)$ be a bijective fuzzy soft function. The following statements are equivalent for each $f_A \in (\widetilde{V}, F)$, $g_B \in (\widetilde{U}, E)$, $k \in F$, $(e = \psi(k)) \in E$ and $r \in I_0$:

- (1) φ_ψ is fuzzy soft β -irresolute homeomorphism.
- (2) φ_ψ is fuzzy soft β -irresolute and fuzzy soft β -irresolute open.
- (3) φ_ψ is fuzzy soft β -irresolute and fuzzy soft β -irresolute closed.
- (4) $\varphi_\psi(\beta I_\eta(k, f_A, r)) = \beta I_\tau(e, \varphi_\psi(f_A), r)$.
- (5) $\varphi_\psi(\beta C_\eta(k, f_A, r)) = \beta C_\tau(e, \varphi_\psi(f_A), r)$.
- (6) $\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r) = \varphi_\psi^{-1}(\beta I_\tau(e, g_B, r))$.
- (7) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) = \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

5. Conclusions and Future Work

This manuscript is lay out as follows:

(1) In Section 2, fuzzy soft β -closure (β -interior) operators are introduced and studied on fuzzy soft topological space based on the article Aygünöğlü et al. [15]. Moreover, the concept of r -fuzzy soft β -connected sets is defined and characterized.

(2) In Section 3, some properties of a fuzzy soft β -continuity are investigated. As a weaker forms of the notion of fuzzy soft β -continuous functions, the notions of fuzzy soft almost (weakly) β -continuous functions are introduced, and some properties are obtained. Also, we show that fuzzy soft β -continuity \Rightarrow fuzzy soft almost β -continuity \Rightarrow fuzzy soft weakly β -continuity, but the converse may not be true. Furthermore, we have the following results:

- Fuzzy soft $(id_V, C_\eta(I_\eta(C_\eta)), id_U, id_U)$ -continuous function is fuzzy soft β -continuous.
- Fuzzy soft $(id_V, \beta I_\eta, I_\tau(C_\tau), id_U)$ -continuous function is fuzzy soft almost β -continuous.
- Fuzzy soft $(id_V, \beta I_\eta, C_\tau, id_U)$ -continuous function is fuzzy soft weakly β -continuous.

(3) In Section 4, the notions of fuzzy soft β -irresolute (resp. strongly β -irresolute, β -irresolute open, β -irresolute closed and β -irresolute homeomorphism) functions are introduced between two fuzzy

soft topological spaces (V, η_F) and (U, τ_E) , and some properties of these functions are investigated. Additionally, we shown that fuzzy soft strongly β -irresolute \Rightarrow fuzzy soft β -irresolute \Rightarrow fuzzy soft β -continuity, but the converse may not be true.

In upcoming manuscripts, we shall investigate the notions given here in the frames of fuzzy soft r -minimal structures [20]. Also, we will use the fuzzy soft β -closure operator to introduce some new separation axioms on fuzzy soft topological space based on the article Aygünoğlu et al. [15].

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