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Posted Date: 9 September 2024

doi: 10.20944/preprints202401.1070.v2

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Exact Closed-Form M/D/c Queueing Delay Formula

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Abstract: It has been more than a century since Erlang published his paper in 1909. The classical queueing system for the M/D/c queue has been investigated by many authors. The key is to find the roots of an equation using explicit or numerical methods. We present an alternate form of explicit solution(s) of this/these root(s) using the Lambert-W function in this paper. In addition, the unsaturated probabilities of the number of customers in the system is derived explicitly in terms of this/these root(s). Finally, the mean number of customers in the system and the mean queueing delay are obtained from these probabilities. The exact queueing delay formula for the M/D/c queue is expressed in closed-form in terms of the root(s). Numerical results show that they are in excellent agreement with the numerical results published by Seelen, Tijms and Van Hoorn.

Keywords: M/D/c queue; poisson arrivals; deterministic (constant) service time; c multiple servers; root(s) of an equation; mean number of customers; mean queueing delay

1. Introduction

It has been more than a century since Erlang published his pioneering work [1] in 1909. Since then, many authors have studied the M/D/c queue with Poisson arrivals, deterministic (constant) service time and c multiple servers. The M/D/c queue can be applied in wireless networks like the 5G (Fifth Generation) cellular network with fixed-size machine-to-machine packets. Crommelin studied the delay probability of the M/D/c system in [2], while Franx worked out a simple solution for the waiting time distribution of this system. In addition, Jansen and Van Leeuwaarden have derived explicit solutions for the roots of the equation for the M/D/c systems [3]. Tijms presented the mean number of customers for an M/D/c system in terms of the unsaturated probabilities of the number of customers in the system [4]. Furthermore, Knessl has closed-form expressions for the unsaturated probabilities of number of customers in the M/D/2 system [5]. On the other hand, Shortle et al. presented a method of solving the M/D/c system numerically [6]. Seelen, Tijms and Van Hoorn have also published exact numerical results on the mean number of customers in the queue for the M/D/c system [7].

The contributions of this paper are as follows. An alternate form of explicit root(s) of an equation for the M/D/c queue is/are presented in terms of the Lambert-W function. In addition, the unsaturated probabilities of the number of customers in the system are expressed in terms of the root(s). Finally, the mean number of customers in the system and the mean queueing delay in the system are derived in terms of the root(s). The mean queueing delay formula of the M/D/c system is not only exact and of closed-form, it is also a product of the form of the M/D/1 queueing formula and another term to account for the multiple servers. These are the main contributions of this paper.

2. The M/D/c Queueing Delay Formula

We start to find the solution(s) of the roots of the following equation for the roots of the M/D/c system with c servers and the system utilization, denoted by ρ, from [6].

$$z^c = e^{-c\rho(1-z)},\ c = 2, 3, \dots \text{ and } \rho = \lambda b / c, \tag{1}$$

and λ is the arrival rate to the system and b is the deterministic (constant) service time. Let $W(x)$ be the Lambert-W function and it is given by

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$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n. \tag{2}$$

The solution(s) of the roots of equation (1), z_k, $k = 1, 2, \ldots, c - 1$, are as follows. If the value of c is even,

$$z_1 = \frac{-W(\rho e^{-\rho})}{\rho}, \tag{3}$$

$$z_k = \frac{-W((-1)^{k/c}\rho e^{-\rho})}{\rho}, \; k = 2, 4, \ldots, c-2, \text{ and} \tag{4}$$

$$z_{k+1} = \frac{-W(-(-1)^{k/c}\rho e^{-\rho})}{\rho}, \; k+1 = 3, 5, \ldots, c-1. \tag{5}$$

If the value of c is odd,

$$z_k = \frac{-W((-1)^{k/c}\rho e^{-\rho})}{\rho}, \; k = 1, 2, \ldots, c-1. \tag{6}$$

The roots for $c = 1, 2, 3, 4, 5, 6$ are shown in Table 1. One of the advantages of using the Lambert-W function is that it is readily available in the Matlab software for numerical computation.

Table 1. Roots of the M/D/c system with the number of servers, c = 2, 3, 4, 5, 6.

c	z1	z2	z2	z4	z5
2	$\frac{-W(\rho e^{-\rho})}{\rho}$				
3	$\frac{-W((-1)^{1/3}\rho e^{-\rho})}{\rho}$	$\frac{-W((-1)^{2/3}\rho e^{-\rho})}{\rho}$			
4	$\frac{-W(\rho e^{-\rho})}{\rho}$	$\frac{-W((-1)^{1/2}\rho e^{-\rho})}{\rho}$	$\frac{-W(-(-1)^{1/2}\rho e^{-\rho})}{\rho}$		
5	$\frac{-W((-1)^{1/5}\rho e^{-\rho})}{\rho}$	$\frac{-W((-1)^{2/5}\rho e^{-\rho})}{\rho}$	$\frac{-W((-1)^{3/5}\rho e^{-\rho})}{\rho}$	$\frac{-W((-1)^{4/5}\rho e^{-\rho})}{\rho}$	
6	$\frac{-W(\rho e^{-\rho})}{\rho}$	$\frac{-W((-1)^{1/3}\rho e^{-\rho})}{\rho}$	$\frac{-W(-(-1)^{1/3}\rho e^{-\rho})}{\rho}$	$\frac{-W((-1)^{2/3}\rho e^{-\rho})}{\rho}$	$\frac{-W(-(-1)^{2/3}\rho e^{-\rho})}{\rho}$

Let p_k be the unsaturated probabilities of the number of customers in the system, where $k = 0, 1, 2, \ldots, c - 1$. From [6],

$$p_0 = \frac{(-1)^{c-1}(1-\rho)c\prod_{k=1}^{c-1} z_k}{\prod_{k=1}^{c-1}(1-z_k)}, \text{ and} \tag{7}$$

the probabilities of p_k's are solved using the following equation.

$$\begin{bmatrix} a_{11} & \cdots & a_{1,c-1} \\ \vdots & \ddots & \vdots \\ a_{c-1,1} & \cdots & a_{c-1,c-1} \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_{c-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{c-1} \end{bmatrix}, \tag{8}$$

where

$$a_{mn} = z_m^n - z_m^c, \; m, n = 1, 2, \ldots, c-1, \text{ and} \tag{9}$$

$$b_m = (z_m^c - 1\}p_0, \; m = 1, 2, \ldots, c-1. \tag{10}$$

Solving equation (8), we have the unsaturated probabilities of the number of customers in the M/D/c system for c = 1, 2, 3, 4, 5, 6. They are tabulated in Tables 2–6.

Table 2. The unsaturated probability of the number of customers in the M/D/c system with the number of servers, $c = 2$.

k	p_k
1	$\frac{-(1+z_1)}{z_1} p_0$

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Table 3. The unsaturated probabilities of the number of customers in the M/D/c system with the number of servers, $c = 3$.

k	p_k
1	$\frac{-(z_1 + z_2 + z_1 z_2)}{z_1 z_2} p_0$
2	$\frac{(1 + z_1 + z_2)}{z_1 z_2} p_0$

Table 4. The unsaturated probabilities of the number of customers in the M/D/c system with the number of servers, $c = 4$.

k	p_k
1	$\frac{-(z_1 z_2 + z_1 z_3 + z_2 z_3 + z_1 z_2 z_3)}{z_1 z_2 z_3} p_0$
2	$\frac{(z_1 + z_2 + z_3 + z_1 z_2 + z_1 z_3 + z_2 z_3)}{z_1 z_2 z_3} p_0$
3	$\frac{-(1 + z_1 + z_2 + z_3)}{z_1 z_2 z_3} p_0$

Table 5. The unsaturated probabilities of the number of customers in the M/D/c system with the number of servers, $c = 5$.

k	p_k
1	$\frac{-(z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 + z_1 z_2 z_3 z_4)}{z_1 z_2 z_3 z_4} p_0$
2	$\frac{(z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 + z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4)}{z_1 z_2 z_3 z_4} p_0$
3	$\frac{-(z_1 + z_2 + z_3 + z_4 + z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4)}{z_1 z_2 z_3 z_4} p_0$
4	$\frac{(1 + z_1 + z_2 + z_3 + z_4)}{z_1 z_2 z_3 z_4} p_0$

Table 6. The unsaturated probabilities of the number of customers in the M/D/c system with the number of servers, $c = 6$.

k	p_k
1	$\frac{-(z_1 z_2 z_3 z_4 + z_1 z_2 z_3 z_5 + z_1 z_2 z_4 z_5 + z_1 z_3 z_4 z_5 + z_2 z_3 z_4 z_5 + z_1 z_2 z_3 z_4 z_5)}{z_1 z_2 z_3 z_4 z_5} p_0$
2	$\frac{(z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_2 z_5 + z_1 z_3 z_4 + z_1 z_3 z_5 + z_1 z_4 z_5 + z_2 z_3 z_4 + z_2 z_3 z_5 + z_2 z_4 z_5 + z_3 z_4 z_5 + z_1 z_2 z_3 z_4 + z_1 z_2 z_3 z_5 + z_1 z_2 z_4 z_5 + z_1 z_3 z_4 z_5 + z_2 z_3 z_4 z_5)}{z_1 z_2 z_3 z_4 z_5} p_0$
3	$\frac{-(z_1 z_2 + z_1 z_3 + z_1 z_4 + z_1 z_5 + z_2 z_3 + z_2 z_4 + z_2 z_5 + z_3 z_4 + z_3 z_5 + z_4 z_5 + z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_2 z_5 + z_1 z_3 z_4 + z_1 z_3 z_5 + z_1 z_4 z_5 + z_2 z_3 z_4 + z_2 z_3 z_5 + z_2 z_4 z_5 + z_3 z_4 z_5)}{z_1 z_2 z_3 z_4 z_5} p_0$
4	$\frac{(z_1 + z_2 + z_3 + z_4 + z_5 + z_1 z_2 + z_1 z_3 + z_1 z_4 + z_1 z_5 + z_2 z_3 + z_2 z_4 + z_2 z_5 + z_3 z_4 + z_3 z_5 + z_4 z_5))}{z_1 z_2 z_3 z_4 z_5} p_0$
5	$\frac{-(1 + z_1 + z_2 + z_3 + z_4 + z_5)}{z_1 z_2 z_3 z_4 z_5} p_0$

Let us define the followings.

$$g_0 = 1, \text{and} \tag{11}$$

$$g_k = \sum_{i_1=1}^{c-1} \sum_{i_2=i_1+1}^{c-1} \cdots \sum_{i_k=i_{k-1}+1}^{c-1} \prod_{j=1}^{k} z_{i_j}, \; k = 1, 2, \ldots, c-1. \tag{12}$$

From [5], the mean number of customers in the M/D/c queue, N_q, is given by

$$N_q = \frac{U}{2(1-\rho)c}, \text{and} \tag{13}$$

$$U = (cp)^2 - c(c-1) + \sum_{k=0}^{c-1} (c(-1) - k(k-1) p_k. \tag{14}$$

Expressing the probabilities of p_k's in terms of the g_k's in equations (11) and (12), we have

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$$p_k = \frac{(-1)^k (g_{c-k-1} + g_{c-k})}{\prod_{i=1}^{c-1} z_i} p_0,\ k = 1, 2, \ldots, c-1. \tag{15}$$

Furthermore, substituting them into equation (14), we have

$$U = (cp)^2 - c(c-1) + \frac{2c(1-\rho)\sum_{k=1}^{c-1}(-1)^{k-1}(c-k)\, g_{k-1}}{\prod_{i=1}^{c-1}(1-z_i)}. \tag{16}$$

Applying Little's Law, the mean queueing delay of the M/D/c system, denoted by W, is given by

$$W = \frac{N_q}{\lambda}. \tag{17}$$

Simplifying and rearranging some terms, we have

$$W = \frac{\lambda b^2}{2(1-\rho)c} \frac{U}{(c\rho)^2}. \tag{18}$$

Note that the square of the deterministic (constant) service time is equal to the second moment of its service time. Substituting equation (16) into equation (18), we have

$$W = \frac{\lambda b^2}{2(1-\rho)c} \frac{(cp)^2 - c(c-1) + \frac{2c(1-\rho)\sum_{k=1}^{c-1}(-1)^{k-1}(c-k)\, g_{k-1}}{\prod_{i=1}^{c-1}(1-z_i)}}{(c\rho)^2}. \tag{19}$$

Equation (19) is the exact closed-form M/D/c queueing delay formula. They are tabulated in Table 7 for the number of servers, c = 2, 3, 4, 5, 6. Note that c = 2, 3, …, in general.

Table 7. The M/D/c Queueing Delay Formulae with the number of servers, c = 2, 3, 4, 5, 6.

c	M/D/c Queueing Delay Formulae
2	$W = \frac{\lambda b^2}{2(1-\rho)c} \frac{(cp)^2 - c(c-1) + \frac{2c(1-\rho)}{(1-z_1)}}{(c\rho)^2}$
3	$W = \frac{\lambda b^2}{2(1-\rho)c} \frac{(cp)^2 - c(c-1) + \frac{2c(1-\rho)(2-(z_1+z_2))}{(1-z_1)(1-z_2)}}{(c\rho)^2}$
4	$W = \frac{\lambda b^2}{2(1-\rho)c} \frac{(cp)^2 - c(c-1) + \frac{2c(1-\rho)(3-2(z_1+z_2+z_3)+(z_1z_2+z_1z_3+z_2z_3))}{(1-z_1)(1-z_2)(1-z_3)}}{(c\rho)^2}$
5	$W = \frac{\lambda b^2}{2(1-\rho)c} \times \frac{(cp)^2 - c(c-1) + \frac{2c(1-\rho)\left(\begin{matrix} 4-3(z_1+z_2+z_3+z_4)+2(z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4) \\ -(z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4) \end{matrix}\right)}{(1-z_1)(1-z_2)(1-z_3)(1-z_4)}}{(c\rho)^2}$
6	$W = \frac{\lambda b^2}{2(1-\rho)c} \times \frac{(cp)^2 - c(c-1) + \frac{2c(1-\rho)\left(\begin{matrix} 5-4(z_1+z_2+z_3+z_4+z_5) \\ +3(z_1z_2+z_1z_3+z_1z_4+z_1z_5+z_2z_3+z_2z_4+z_2z_5+z_3z_4+z_3z_5+z_4z_5) \\ -2\left(\begin{matrix} z_1z_2z_3+z_1z_2z_4+z_1z_2z_5+z_1z_3z_4+z_1z_3z_5 \\ +z_1z_4z_5+z_2z_3z_4+z_2z_3z_5+z_2z_4z_5+z_3z_4z_5 \end{matrix}\right) \\ +(z_1z_2z_3z_4+z_1z_2z_3z_5+z_1z_2z_4z_5+z_1z_3z_4z_5+z_2z_3z_4z_5) \end{matrix}\right)}{(1-z_1)(1-z_2)(1-z_3)(1-z_4)(1-z_5)}}{(c\rho)^2}$

On further simplification of (19), we have

$$W = \frac{\lambda b^2}{2(1-\rho)c} \frac{(cp)^2 - c(c-1) + 2c(1-\rho)\sum_{k=1}^{c-1}\frac{1}{1-z_i}}{(c\rho)^2}. \tag{20}$$

Note that the service time, b, needs not be set to 1 and there is no need for any rescaling after that.

3. Results

We present the results of the M/D/c system with the number of servers, **c** = 2, 3, 4, 5, 6 and the constant service time, b = 1, to be consistent with [7] in this results section. The mean number of

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customers in the system, N_q, from analysis in Section 2 is compared with known exact numerical results from [7], while the exact mean queueing delay, W, from the analysis in Section 2 is also presented. Note that the constant service time needs not be set to 1 in (20) in general.

Table 8 presents the results for the mean number of customers with our results as compared to available exact numerical results from [7]. Both results match excellently. Only available numerical results from [7] are presented for each system utilization value.

Table 8. The Mean Number of Customers for the M/D/c system with the number of servers, c = 2, 3, 4, 5, 6.

ϱ\c	2		3		4		5		6	
	Ours	**[7]**	**Ours**	**[7]**	**Ours**	**[7]**	**Ours**	**[7]**	**Ours**	**[7]**
0.1	0.0012		0.0003		0.0001		0.0000		0.0000	
0.2	0.0097	0.0097	0.0039	0.0039	0.0016		0.0007		0.0003	
0.3	0.0332	0.0332	0.0181	0.0181	0.0101	0.0101	0.0058	0.0058	0.0033	
0.4	0.0826	0.0826	0.0540	0.0540	0.0363	0.0363	0.0249	0.0249	0.0172	0.0172
0.5	0.1767	0.1767	0.1308	0.1308	0.0993	0.0993	0.0766	0.0766	0.0598	0.0598
0.6	0.3516	0.3516	0.2851	0.2851	0.2361	0.2361	0.1982	0.1982	0.1680	0.1680
0.7	0.6911	0.6911	0.6011	0.6011	0.5312	0.5312	0.4743	0.4743	0.4267	0.4267
0.8	1.4453	1.445	1.3294	1.329	1.2355	1.236	1.1562	1.156	1.0875	1.088
0.9	3.8645	3.864	3.7204	3.720	3.5999	3.600	3.4951	3.495	3.4018	3.402

Table 9 presents our results for the mean queueing delay with our exact closed-form M/D/c queueing delay formula. The results agree with Little's Law and can also be derived from Table 8.

Table 9. The Mean Queueing Delay for the M/D/c system with the number of servers, c = 2, 3, 4, 5, 6.

ϱ\c	2	3	4	5	6
	Ours	**Ours**	**Ours**	**Ours**	**Ours**
0.1	0.0062	0.0009	0.0002	0.0000	0.0000
0.2	0.0242	0.0066	0.0021	0.0007	0.0002
0.3	0.0553	0.0201	0.0085	0.0039	0.0019
0.4	0.1033	0.0450	0.0227	0.0124	0.0072
0.5	0.1767	0.0872	0.0497	0.0307	0.0199
0.6	0.2930	0.1584	0.0984	0.0661	0.0467
0.7	0.4936	0.2862	0.1897	0.1355	0.1016
0.8	0.9033	0.5539	0.3861	0.2891	0.2266
0.9	2.1469	1.3779	1.0000	0.7767	0.6300

4. Discussion

The result in Section 2 provides an exact closed-form M/D/c queueing delay formula. It is in a form that is extending the form of M/D/1 queue expression as well. In addition, this result is dependent on the roots which are expressed in terms of the Lambert-W function. Furthermore, this function is readily available in software like Matlab for ease of numerical computation. The M/D/c queueing system also has applications in wireless networks and other fields.

5. Conclusions

An exact closed-form M/D/c queueing delay formula has been derived. The final form of the result does not require the solution of a set of linear equations anymore. This formula depends on the roots of the solution to an equation for the M/D/c system. Furthermore, the roots are presented in an alternate explicit form in terms of the Lambert-W function for ease of numerical computation. Numerical results show that these results are in excellent agreement with published exact numerical results.

The significance of this queueing delay formula presented in this paper is that it moves us a step closer to an exact closed-form M/G/c queueing system as the exact closed form M/M/c queueing delay formula is also well-known [8].

 doi:10.20944/preprints202401.1070.v2

Author Contributions: The author has sole contribution to this paper.

Data Availability Statement: The numerical data is available in [7].

Acknowledgments: The author would like to thank Prof. Jon W Mark, Prof. Kin Mun Lye and Prof. Kee Chaing Chua for the queueing theory/communication networks courses that they have taught the author.

Conflicts of Interest: The author declares no conflicts of interest.

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