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Description of the Electron in the Electromagnetic Field: The Dirac Type Equation and the Equation for the Wave Function in Spinor Coordinate Space

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Abstract

Physical processes are usually described using four-dimensional vector quantities - coordinate vector, momentum vector, current vector. But at the fundamental level they are characterized by spinors - coordinate spinors, momentum spinors, spinor wave functions. The propagation of fields and their interaction takes place at the spinor level, and since each spinor uniquely corresponds to a certain vector, the results of physical processes appear before us in vector form. For example, the relativistic Schrödinger equation and the Dirac equation are formulated by means of coordinate vectors, momentum vectors and quantum operators corresponding to them. In the Dirac equation a step forward is taken and the wave function is a spinor with complex components, but still coordinates and momentum are vectors. For a closed description of nature using only spinor quantities, it is necessary to have an equation similar to the Dirac equation in which momentum, coordinates and operators are spinors. It is such an equation that is presented in this paper. Using the example of the interaction between an electron and an electromagnetic field, we can see that the spinor equation contains more detailed information about the interaction than the vector equations. This is not new for quantum mechanics, since it describes interactions using complex wave functions, which cannot be observed directly, and only when measured goes to probabilities in the form of squares of the moduli of the wave functions. In the same way spinor quantities are not observable, but they completely determine observable vectors. In Section 2 of the paper, we analyze the quadratic form for an arbitrary four-component complex vector based on Pauli matrices. The form is invariant with respect to Lorentz transformations including any rotations and boosts. The invariance of the form allows us to construct on its basis an equation for a free particle combining the properties of the relativistic wave equation and the Dirac equation. For an electron in the presence of an electromagnetic potential it is shown that taking into account the commutation relations between the momentum and coordinate components allows us to obtain from this equation the known results describing the interactions of the electron spin with the electric and magnetic field. In the presence of a potential the momentum components cease to commute with each other. To neutralize this effect, the Schrödinger equation is supplemented by several equations with mixed partial derivatives on coordinates. In section 3 of the paper this quadratic form is expressed through momentum spinors, which makes it possible to obtain an equation for the spinor wave function in spinor coordinate space by replacing the momentum spinor components by partial derivative operators on the corresponding coordinate spinor component. Section 4 presents a modification of the theory of the path integral, which consists in considering the path integral in the spinor coordinate space. The Lagrangian densities for the scalar field and for the electron field, along with their corresponding propagators, are presented. An equation of motion for the electron is proposed that is relativistically invariant, in contrast to the Dirac equation, which lacks this invariance. This novel equation permitted the construction of an actually invariant procedure for the second quantization of the fermion field in spinor coordinate space. Furthermore, it is demonstrated that the field operators are a combination of plane waves in spinor or vector space, with the coefficients of which being pseudospinors or pseudovectors. Each of these pseudovectors or pseudospinors corresponds to one of the particles presented in the theory of electrodynamics. Furthermore, each plane wave possesses an additional

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coefficient in the form of a creation or annihilation operator. In vector space, these operators commute, whereas in spinor space they anticommutate. The paper presents the spinor and vector representations of the field operators in explicit form, comprising sets of 16 pseudospinors or 4 pseudovectors corresponding to particles represented in electrodynamics. An explicit form of the symmetric traceless tensor with spin two, zero mass and two polarizations is presented, which can serve as a model of the graviton. The results obtained may prompt changes in some aspects of the construction of Feynman diagrams. Among other things, it presents a purely mathematical derivation of Maxwell's inhomogeneous equations without reference to empirical data on the action of electric current, which is usually referred to when deriving equations. Section 4 presents Einstein's inhomogeneous equation, which features the Riemann tensor rather than the Ricci tensor, with the energy-momentum tensor having four indices rather than two.

Keywords: Dirac equation; Schrödinger equation; second quantization; path integral; graviton; Bianchi identity; Einstein's equation

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1. Introduction

Nowadays, the interest to study applications of the Dirac equation to different situations and to find out the conditions of its generalization is not weakening. In particular, in [1] new versions of an extended Dirac equation and the associated Clifford algebra are presented. In [2] a study of the Schrödinger-Dirac covariant equation in the presence of gravity, where the non-commuting gamma matrices become space-time-dependent, is carried out. In [3] an idea is discussed that the visible properties of the electron, including rest mass and magnetic moment, are determined by a massless charge spinning at light speed within a Compton domain. In [4] some aspects of conformal rescaling in detail are explored and the role of the "quantum" potential is discussed as a natural consequence of non-inertial motion and is not exclusive to the quantum domain. Author establishes the fundamental importance of conformal symmetry, in which rescaling of the rest mass plays a vital role. Thus, the basis for a radically new theory of quantum phenomena based on the process of mass-energy flow is proposed. In [5] author has derived the covariant fourth-order/one-function equivalent of the Dirac equation for the general case of an arbitrary set of γ-matrices.

Supporting these search aspirations, in our work we propose a deeper understanding of the Dirac equation with an emphasis on the direct use of the principles of symmetry and invariance to Lorentz transformations. For the first time we present a formulation of the Dirac and Schrödinger equations in spinor coordinate space.

2. Generalized Dirac Type Equation

Let us introduce notations, which will be used further on. The speed of light and the rationalized Planck's constant will be considered as unity.

Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrices constructed from Pauli matrices

$$S_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \quad S_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad S_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad S_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

A vector of matrices

$$\vec{\mathbf{S}}^T \equiv (S_1, S_2, S_3)$$

A set of arbitrary complex numbers and a vector of its three components

$$\mathbf{X}^T \equiv (X_0, X_1, X_2, X_3)$$
$$\vec{\mathbf{X}}^T \equiv (X_1, X_2, X_3)$$

Let us define a 2×2 matrix of Lorentz transformations given by the set of real rotation angles $(\alpha_1, \alpha_2, \alpha_3)$ and boosts $(\beta_1, \beta_2, \beta_3)$

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$$n = exp\left(-\frac{i}{2}\alpha_1\sigma_1\right)exp\left(\frac{1}{2}\beta_1\sigma_1\right)exp\left(-\frac{i}{2}\alpha_2\sigma_2\right)exp\left(\frac{1}{2}\beta_2\sigma_2\right)exp\left(-\frac{i}{2}\alpha_3\sigma_3\right)exp\left(\frac{1}{2}\beta_3\sigma_3\right)$$

and a similar 4×4 transformation matrix

$$N = exp\left(-\frac{i}{2}\alpha_1 S_1\right)exp\left(\frac{1}{2}\beta_1 S_1\right)exp\left(-\frac{i}{2}\alpha_2 S_2\right)exp\left(\frac{1}{2}\beta_2 S_2\right)exp\left(-\frac{i}{2}\alpha_3 S_3\right)exp\left(\frac{1}{2}\beta_3 S_3\right)$$

We also define a 4×4 matrix of Lorentz transformations Λ, where μ and ν take values 0,1,2,3

$$\Lambda^\mu_\nu = \frac{1}{2}Tr[\sigma_\mu n \sigma_\nu n^\dagger]$$

$$\Lambda^\mu_\nu = \frac{1}{4}Tr[S_\mu N S_\nu N^\dagger]$$

$$\Lambda^\mu_\nu = \frac{1}{4}\sum_{\varepsilon=0}^{3}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}(S_\mu)_{\varepsilon\alpha}N_{\alpha\beta}(S_\nu)_{\beta\gamma}\overline{N_{\varepsilon\gamma}}$$

which can also be written explicitly using the 4×4 matrices of rotation generators (R_1, R_2, R_3) and boosts (K_1, K_2, K_3)

$$\Lambda = exp(\alpha_1 R_1)exp(\beta_1 K_1)exp(\alpha_2 R_2)exp(\beta_2 K_2)exp(\alpha_3 R_3)exp(\beta_3 K_3)$$

Let's define a 4×4 matrix

$$M^2 = (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) =$$
$$\left(S_0X_0 - \vec{\mathbf{S}}^T\vec{\mathbf{X}}\right)\left(S_0X_0 + \vec{\mathbf{S}}^T\vec{\mathbf{X}}\right) =$$
$$S_0X_0S_0X_0 - S_1X_1S_1X_1 - S_2X_2S_2X_2 - S_3X_3S_3X_3 +$$
$$S_0X_0(S_1X_1 + S_2X_2 + S_3X_3) - S_1X_1(S_0X_0 + S_2X_2 + S_3X_3) -$$
$$S_2X_2(S_0X_0 + S_1X_1 + S_3X_3) - S_3X_3(S_0X_0 + S_1X_1 + S_2X_2)$$

In fact, we consider a quaternion with complex coefficients, which we multiply by its conjugate quaternion (due to the complexity of the coefficients, these are biquaternions, but we still use quaternionic conjugation, without complex conjugation).

Let us subject the set of complex numbers to the Lorentz transformation

$$\mathbf{X}' = \Lambda\mathbf{X}$$

Let us write a relation whose validity for an arbitrary set of complex numbers can be checked directly

$$(S_0X_0{}' - S_1X_1{}' - S_2X_2{}' - S_3X_3{}')(S_0X_0{}' + S_1X_1{}' + S_2X_2{}' + S_3X_3{}')$$
$$= (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) = M^2$$

The matrix M^2 in the simplest case is diagonal with equal complex elements on the diagonal equal to the square of the length of the vector $\mathbf{X}$ in the metric of Minkowski space, which we denote m^2. Both M^2 and m^2 do not change under any rotations and boosts, in physical applications the invariance of m^2 is usually used, in particular, for the four-component momentum vector this quantity is called the square of mass.

Since the matrices S_μ anticommutate with each other, for a vector $\mathbf{X}$ whose components commute with each other, we have just the simplest case with a diagonal matrix with m^2 on the diagonal. But if the components of vector $\mathbf{X}$ do not commute, the matrix M^2 already has a more complex structure and carries additional physical information compared to m^2. For example, the vector $\mathbf{X}$ may include the electron momentum vector and the electromagnetic potential vector. The four-component potential vector is a function of the four-dimensional coordinates of Minkowski space. The components of the four-component momentum do not commute with the components of the coordinate vector, respectively, and the coordinate function does not commute with the momentum components, and their commutator is expressed through the partial derivative of this function by the corresponding coordinate. If the components of the vector $\mathbf{X}$ do not commute, the matrix M^2 will no longer be invariant with respect to Lorentz transformations.

Assume that the complex numbers we are considering commute with all matrices, and consider that the matrices are pairwise anticommutative and their squares are equal to the unit 4×4 matrix I

$$M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I +$$
$$(S_1X_0X_1 + S_2X_0X_2 + S_3X_0X_3) - (S_1X_1X_0 + S_1S_2X_1X_2 + S_1S_3X_1X_3) -$$
$$(S_2X_2X_0 + S_2S_1X_2X_1 + S_2S_3X_2X_3) - (S_3X_3X_0 + S_3S_1X_3X_1 + S_3S_2X_3X_2) =$$
$$(X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I +$$

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$$
\begin{gathered}
S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) - \\
(S_1S_2X_1X_2 + S_1S_3X_1X_3) - (S_2S_1X_2X_1 + S_2S_3X_2X_3) - (S_3S_1X_3X_1 + S_3S_2X_3X_2) = \\
(X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + \\
S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) - \\
(S_1S_2X_1X_2 + S_2S_1X_2X_1) - (S_2S_3X_2X_3 + S_3S_2X_3X_2) - (S_3S_1X_3X_1 + S_1S_3X_1X_3) = \\
(X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + \\
S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) \\
-\big(S_1S_2X_1X_2 + S_2S_1X_1X_2 + S_2S_1(X_2X_1 - X_1X_2)\big) \\
-\big(S_2S_3X_2X_3 + S_3S_2X_2X_3 + S_3S_2(X_3X_2 - X_2X_3)\big) \\
-\big(S_3S_1X_3X_1 + S_1S_3X_3X_1 + S_1S_3(X_1X_3 - X_3X_1)\big) = \\
(X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + \\
S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) \\
-S_2S_1(X_2X_1 - X_1X_2) - S_3S_2(X_3X_2 - X_2X_3) - S_1S_3(X_1X_3 - X_3X_1)
\end{gathered}
$$

Taking into account the expressions for pairwise products of matrices, we obtain

$$
\begin{gathered}
M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + \\
S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) \\
-S_2S_1(X_2X_1 - X_1X_2) - S_3S_2(X_3X_2 - X_2X_3) - S_1S_3(X_1X_3 - X_3X_1) = \\
(X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + \\
S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) \\
+iS_3(X_2X_1 - X_1X_2) + iS_1(X_3X_2 - X_2X_3) + iS_2(X_1X_3 - X_3X_1) = \\
(X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + \\
S_1(X_0X_1 - X_1X_0) + iS_1(X_3X_2 - X_2X_3) + \\
S_2(X_0X_2 - X_2X_0) + iS_2(X_1X_3 - X_3X_1) + \\
S_3(X_0X_3 - X_3X_0) + iS_3(X_2X_1 - X_1X_2)
\end{gathered}
$$

Consider the case when $\mathbf{X}$ is the sum of the momentum vector and the electromagnetic potential vector, which is a function of coordinates

$$
\begin{gathered}
\mathbf{X} = \mathbf{P} + \mathbf{A} \\
\mathbf{P}^T \equiv (P_0, P_1, P_2, P_3) \\
\mathbf{A}^T \equiv (A_0, A_1, A_2, A_3) \\
\vec{\mathbf{P}}^T \equiv (P_1, P_2, P_3) \\
\vec{\mathbf{A}}^T \equiv (A_1, A_2, A_3)
\end{gathered}
$$

$$
\begin{gathered}
M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] + \\
S_1[(P_0 + A_0)(P_1 + A_1) - (P_1 + A_1)(P_0 + A_0)] + iS_1[(P_3 + A_3)(P_2 + A_2) - (P_2 + A_2)(P_3 + A_3)] + \\
S_2[(P_0 + A_0)(P_2 + A_2) - (P_2 + A_2)(P_0 + A_0)] + iS_2[(P_1 + A_1)(P_3 + A_3) - (P_3 + A_3)(P_1 + A_1)] + \\
S_3[(P_0 + A_0)(P_3 + A_3) - (P_3 + A_3)(P_0 + A_0)] + iS_3[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)]
\end{gathered}
$$

For now, we'll stick with the Heisenberg approach, that is, we will consider the components of the momentum vector P_0, P_1, P_2, P_3 as operators for which there are commutation relations with coordinates or coordinate functions such as A_0, A_1, A_2, A_3. In this approach, the operators do not have to act on any wave function.

Taking into account the commutation relations of the components of the momentum vector and the coordinate vector, the commutator of the momentum component and the coordinate function is expressed through the derivative of this function by the corresponding coordinate, e.g.

$$
\begin{gathered}
[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)] = \\
(P_2P_1 - P_1P_2) + (P_2A_1 - A_1P_2) - (P_1A_2 - A_2P_1) + (A_2A_1 - A_1A_2) = \\
-i\frac{\partial A_1}{\partial x_2} - \left(-i\frac{\partial A_2}{\partial x_1}\right)
\end{gathered}
$$

As a result, we obtain

$$
\begin{gathered}
M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] \\
+S_1\left[-i\frac{\partial A_1}{\partial x_0} + i\frac{\partial A_0}{\partial x_1}\right] + iS_1\left[-i\frac{\partial A_2}{\partial x_3} + i\frac{\partial A_3}{\partial x_2}\right] \\
+S_2\left[-i\frac{\partial A_2}{\partial x_0} + i\frac{\partial A_0}{\partial x_2}\right] + iS_2\left[-i\frac{\partial A_3}{\partial x_1} + i\frac{\partial A_1}{\partial x_3}\right]
\end{gathered}
$$

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$$+S_3\left[-i\frac{\partial A_3}{\partial x_0}+i\frac{\partial A_0}{\partial x_3}\right]+iS_3\left[-i\frac{\partial A_1}{\partial x_2}+i\frac{\partial A_2}{\partial x_1}\right]=$$
$$I[(P_0+A_0)(P_0+A_0)-(P_1+A_1)(P_1+A_1)-(P_2+A_2)(P_2+A_2)-(P_3+A_3)(P_3+A_3)]$$
$$-iS_1\left[\frac{\partial A_1}{\partial x_0}-\frac{\partial A_0}{\partial x_1}\right]+S_1\left[\frac{\partial A_2}{\partial x_3}-\frac{\partial A_3}{\partial x_2}\right]$$
$$-iS_2\left[\frac{\partial A_2}{\partial x_0}-\frac{\partial A_0}{\partial x_2}\right]+S_2\left[\frac{\partial A_3}{\partial x_1}-\frac{\partial A_1}{\partial x_3}\right]$$
$$-iS_3\left[\frac{\partial A_3}{\partial x_0}-\frac{\partial A_0}{\partial x_3}\right]+S_3\left[\frac{\partial A_1}{\partial x_2}-\frac{\partial A_2}{\partial x_1}\right]$$
$$=I[(P_0+A_0)(P_0+A_0)-(P_1+A_1)(P_1+A_1)-(P_2+A_2)(P_2+A_2)-(P_3+A_3)(P_3+A_3)]$$
$$-iS_1F_{01}+S_1F_{32}-iS_2F_{02}+S_2F_{13}-iS_3F_{03}+S_3F_{21}$$
$$=I[(P_0+A_0)(P_0+A_0)-(P_1+A_1)(P_1+A_1)-(P_2+A_2)(P_2+A_2)-(P_3+A_3)(P_3+A_3)]$$
$$-iS_1E_x+S_1B_x-iS_2E_y+S_2B_y-iS_3E_z+S_3B_z$$

where

$$F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$$
$$\partial_\mu\equiv\frac{\partial}{\partial x^\mu}$$
$$F_{\mu\nu}=\begin{pmatrix}0&E_x&E_y&E_z\\-E_x&0&-B_z&B_y\\-E_y&B_z&0&-B_x\\-E_z&-B_y&B_x&0\end{pmatrix}$$

As a result, we have the expression

$$M^2=I[(P_0+A_0)(P_0+A_0)-(P_1+A_1)(P_1+A_1)-(P_2+A_2)(P_2+A_2)-(P_3+A_3)(P_3+A_3)]+\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}$$
$$\vec{\mathbf{B}}^T\equiv(B_x,B_y,B_z)\equiv(B_1,B_2,B_3)$$
$$\vec{\mathbf{E}}^T\equiv(E_x,E_y,E_z)\equiv(E_1,E_2,E_3)$$

The matrix

$$M^2-\left\{\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}=$$
$$I\{(P_0+A_0)(P_0+A_0)-(P_1+A_1)(P_1+A_1)-(P_2+A_2)(P_2+A_2)-(P_3+A_3)(P_3+A_3)\}\equiv$$
$$\equiv Id^2$$

does not change under Lorentz transformations involving any rotations and boosts.

$$Id^2=\big(S_0(P_0+A_0)-S_1(P_1+A_1)-S_2(P_2+A_2)-S_3(P_3+A_3)\big)\big(S_0(P_0+A_0)+S_1(P_1+A_1)+S_2(P_2+A_2)+S_3(P_3+A_3)\big)-\left\{\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}$$
$$=\left(S_0(P_0+A_0)-\vec{\mathbf{S}}^T\left(\vec{\mathbf{P}}+\vec{\mathbf{A}}\right)\right)\left(S_0(P_0+A_0)+\vec{\mathbf{S}}^T\left(\vec{\mathbf{P}}+\vec{\mathbf{A}}\right)\right)-\left\{\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}$$

Taking into account the electron charge we have

$$\mathbf{X}=\mathbf{P}-e\mathbf{A}$$
$$Id^2=\left(S_0(P_0-eA_0)-\vec{\mathbf{S}}^T\left(\vec{\mathbf{P}}+\vec{\mathbf{A}}\right)\right)\left(S_0(P_0-eA_0)+\vec{\mathbf{S}}^T\left(\vec{\mathbf{P}}+\vec{\mathbf{A}}\right)\right)+e\left\{\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}$$

Let us summarize our consideration. There is a correlation

$$Id^2=M^2+e\left\{\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}$$

where

$$M^2\equiv\left(S_0(P_0-eA_0)-\vec{\mathbf{S}}^T\left(\vec{\mathbf{P}}-e\vec{\mathbf{A}}\right)\right)\left(S_0(P_0-eA_0)+\vec{\mathbf{S}}^T\left(\vec{\mathbf{P}}-e\vec{\mathbf{A}}\right)\right)$$
$$Id^2\equiv I\{(P_0-eA_0)^2-(P_1-eA_1)^2-(P_2-eA_2)^2-(P_3-eA_3)^2\}=$$
$$I\left[(P_0-eA_0)(P_0-eA_0)-\left(\vec{\mathbf{P}}-e\vec{\mathbf{A}}\right)^T\left(\vec{\mathbf{P}}-e\vec{\mathbf{A}}\right)\right]=$$
$$I\left\{(P_0-eA_0)^2-\left(\vec{\mathbf{P}}-e\vec{\mathbf{A}}\right)^2\right\}$$

Let's analyze the obtained equality

$$M^2=Id^2-e\left\{\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}$$

Note that the quantity d^2 is invariant to the Lorentz transformations irrespective of whether the momentum and field components commute or not. To solve this equation, we have to make additional simplifications. For example, to arrive at an equation similar to the Dirac equation, we must equate M^2 with the matrix Im^2, where m^2 is the square of the mass of a free electron. Then

$$Im^2=Id^2-e\left\{\vec{\mathbf{S}}^T\vec{\mathbf{B}}-i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}$$

$$Id^2 - Im^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\} = 0$$
$$I\left\{(P_0 - eA_0)^2 - \left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^2\right\} - Im^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\} = 0$$

With this substitution the generalized equation almost coincides with the equation ([6], formula (43.25)), the difference is that there is a plus sign before $e\vec{\mathbf{S}}^T\vec{\mathbf{B}}$, and instead of $i\vec{\mathbf{S}}^T\vec{\mathbf{E}}$ there is $i\vec{\boldsymbol{\alpha}}^T\vec{\mathbf{E}}$, in which the matrices $\boldsymbol{\alpha}$ have the following form

$$\vec{\boldsymbol{\alpha}}^T \equiv (\alpha_1, \alpha_2, \alpha_3)$$
$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

A similar equation is given by Dirac in ([7], Paragraph 76, Equation 24); he does not use the matrices $\vec{\boldsymbol{\alpha}}$, only the matrices $\vec{\mathbf{S}}$, but the signs of the contributions of the magnetic and electric fields are the same.

Along with the original form

$$M^2 = \left(S_0(P_0 - eA_0) - \vec{\mathbf{S}}^T\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)\right)\left(S_0(P_0 - eA_0) + \vec{\mathbf{S}}^T\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)\right) = d^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} - i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

it is possible to consider the form with a different order of the factors. It can be shown that this leads to a change in the sign of the electric field contribution

$$M^2 = \left(S_0(P_0 - eA_0) + \vec{\mathbf{S}}^T\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)\right)\left(S_0(P_0 - eA_0) - \vec{\mathbf{S}}^T\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)\right) = d^2 - e\{\vec{\mathbf{S}}^T\vec{\mathbf{B}} + i\vec{\mathbf{S}}^T\vec{\mathbf{E}}\}$$

Since Id^2, unlike M^2, is invariant to Lorentz transformations, it would be logical to replace it by Im^2. At least both these matrices are diagonal, and in the case of a weak field their diagonal elements are close. Nevertheless, the approach based on the Dirac equation leads to solutions consistent with experiment.

The matrix M^2 in the general case has complex elements and is not diagonal, and in the Dirac equations instead of it is substituted the product of the unit matrix by the square of mass m^2, the physical meaning of such a substitution is not obvious. Apparently it is implied that it is the square of the mass of a free electron. But the square of the length of the sum of the lengths of the electron momentum vectors and the electromagnetic potential vector is not equal to the sum of the squares of the lengths of these vectors, that is, it is not equal to the square of the mass of the electron, even if the square of the length of the potential vector were zero. But, for example, in the case of an electrostatic central field, even the square of the length of one potential vector is not equal to zero. Therefore, it is difficult to find a logical justification for using the mass of a free electron in the Dirac equation in the presence of an electromagnetic field. Due to the noted differences, the solutions of the generalized equation can differ from the solutions arising from the Dirac equation.

In the case when there is a constant magnetic field directed along the z-axis, we can write down

$$A_0 = 0 \qquad A_1 = -\frac{1}{2}B_3x_2 \qquad A_2 = \frac{1}{2}B_3x_1 \quad A_3 = 0$$
$$(S_0P_0)^2 - M^2 - \left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^T\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)I - eS_3B_3 = 0$$
$$(S_0P_0)^2 - M^2 - (P_1 - eA_1)(P_1 - eA_1)I - (P_2 - eA_2)(P_2 - eA_2)I - eS_3B_3 = 0$$
$$(S_0P_0)^2 - M^2 - {P_0}^2I - {P_3}^2I - {P_1}^2 - (eA_1)^2 - {P_2}^2 - (eA_2)^2 + eB_3(x_1P_2 - x_2P_1 + x_1P_2 - x_2P_1) - eS_3B_3 = 0$$
$${P_0}^2I - M^2 - {P_0}^2I - {P_3}^2I - {P_1}^2I - (eA_1)^2I - {P_2}^2I - (eA_2)^2I + eB_3(x_1P_2 - x_2P_1)I - eS_3B_3 = 0$$
$$I\left(-{P_1}^2 - {P_2}^2 - {P_3}^2 - (eA_1)^2 - (eA_2)^2\right) - M^2 - eB_3\begin{pmatrix} L_3+1 & 0 & 0 & 0 \\ 0 & L_3-1 & 0 & 0 \\ 0 & 0 & L_3+1 & 0 \\ 0 & 0 & 0 & L_3-1 \end{pmatrix} = 0$$

Here $(x_1P_2 - x_2P_1) \equiv L_3$. Only when the field is directed along the z-axis, the matrix M^2 is diagonal and real because the third Pauli matrix is diagonal and real. And if the field is weak, M^2 can be approximated by the m^2I matrix. This is probably why it is customary to illustrate the interaction of electron spin with the magnetic field by choosing its direction along the z-axis. In any other direction M^2 is not only non-diagonal, but also complex, so that it is difficult to justify the use of m^2I.

When the influence of the electromagnetic field was taken into account, no specific characteristics of the electron were used. When deriving a similar result using the Dirac equation, it is assumed that since the electron equation is used, the result is specific to the electron. In our case

Pauli matrices and commutation relations are used, apparently these two assumptions or only one of them characterize the properties of the electron, distinguishing it from other particles with non-zero masses.

The proposed equation echoes the Dirac equation, at least from it one can obtain the same formulas for the interaction of spin and electromagnetic field as with the Dirac equation, and in the absence of a field the proposed equation is invariant to the Lorentz transformations. In contrast, to prove the invariance of the Dirac equation even in the absence of a field, the infinitesimal Lorentz transformations are used, but the invariance at finite angles of rotations and boosts is not demonstrated. The proof of invariance of the Dirac equation is based on the claim that a combination of rotations at finite angles can be represented as a combination of infinitesimal rotations. But this is true only for rotations or boosts around one axis, and if there are at least two axes, this statement is not true because of non-commutability of Pauli matrices, which are generators of rotations, so that the exponent of the sum is not equal to the product of exponents if the sum includes generators of rotations or boosts around different axes. By a direct check we can verify that the invariance of the Dirac equation takes place at any combination of rotations, but only under the condition of zero boosts, i.e., only in a rest frame of reference, any boost violates the invariance.

A test case for any theory is the model of the central electrostatic field used in the description of the hydrogen atom, in which the components of the vector potential are zero

$$\left(S_0(P_0 - eA_0) - \vec{\mathbf{S}}^T\vec{\mathbf{P}}\right)\left(S_0(P_0 - eA_0) + \vec{\mathbf{S}}^T\vec{\mathbf{P}}\right) = I\left[(P_0 - eA_0)^2 - {P_1}^2 - {P_2}^2 - {P_3}^2\right] + ie\vec{\mathbf{S}}^T\vec{\mathbf{E}}$$

If again we equate the left part with Im^2, we obtain

$$I\left[(P_0 - eA_0)^2 - {P_1}^2 - {P_2}^2 - {P_3}^2\right] - Im^2 + ie\vec{\mathbf{S}}^T\vec{\mathbf{E}} = 0$$

$$I\left[(P_0 - eA_0)^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 - m^2\right] - ie\left(S_1\frac{\partial A_0}{\partial x_1} + S_2\frac{\partial A_0}{\partial x_2} + S_3\frac{\partial A_0}{\partial x_3}\right) = 0$$

Introducing the notations $\left(A_0 \equiv \varphi(r) = Q/r,\ P_0 \equiv E,\ r = 1/\sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2}\right)$, we obtain

$$I\left[\left(E - \frac{eQ}{r}\right)^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 - m^2\right] - ie\left(S_1\frac{\partial\varphi(r)}{\partial x_1} + S_2\frac{\partial\varphi(r)}{\partial x_2} + S_3\frac{\partial\varphi(r)}{\partial x_3}\right) = 0$$

$$I\left[\left(E - \frac{eQ}{r}\right)^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 - m^2\right] + i\frac{eQ}{r^3}(S_1x_1 + S_2x_2 + S_3x_3) = 0$$

If we substitute operators acting on the wave function instead of momentum components into the equation, we obtain a generalized analog of the relativistic Schrödinger equation, in which the wave function has four components. Using the substitutions

$$P_0 \to i\frac{\partial}{\partial t} \qquad P_1 \to -i\frac{\partial}{\partial x_1} \qquad P_2 \to -i\frac{\partial}{\partial x_2} \qquad P_3 \to -i\frac{\partial}{\partial x_3}$$

the equation for the four-component wave function $\boldsymbol{\psi}(t, x_1, x_2, x_3)$ before all transformations has the form

$$\left(S_0\left(\frac{\partial}{\partial t} - eA_0\right) + \vec{\mathbf{S}}^T(\boldsymbol{\nabla} - e\vec{\mathbf{A}})\right)\left(S_0\left(\frac{\partial}{\partial t} - eA_0\right) - \vec{\mathbf{S}}^T(\boldsymbol{\nabla} - e\vec{\mathbf{A}})\right)\boldsymbol{\psi} + M^2\boldsymbol{\psi} = 0$$

and after transformations

$$\left\{\left(S_0(P_0 - eA_0)\right)^2 - \left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^{\mathbf{2}}I - e\vec{\mathbf{S}}^T\vec{\mathbf{B}} + ie\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}\boldsymbol{\psi} = M^2\boldsymbol{\psi}$$

Once again, note that the matrix M^2 is not diagonal and real.

All the above deductions are also valid when replacing 4×4 matrices S_μ by 2×2 matrices σ_μ, since their commutative and anticommutative properties are the same. The corresponding generalized equation is of the form

$$\left(\sigma_0(P_0 - eA_0)\right)^2 - M^2 - \left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^{\mathbf{2}}I - e\vec{\boldsymbol{\sigma}}^T\vec{\mathbf{B}} + ie\vec{\boldsymbol{\sigma}}^T\vec{\mathbf{E}} = 0$$

where

$$\vec{\boldsymbol{\sigma}}^T \equiv (\sigma_1, \sigma_2, \sigma_3)$$

and the equation for the now two-component wave function looks like

$$\left(\sigma_0\left(\frac{\partial}{\partial t} - eA_0\right) + \vec{\boldsymbol{\sigma}}^T(\boldsymbol{\nabla} - e\vec{\mathbf{A}})\right)\left(\sigma_0\left(\frac{\partial}{\partial t} - eA_0\right) - \vec{\boldsymbol{\sigma}}^T(\boldsymbol{\nabla} - e\vec{\mathbf{A}})\right)\boldsymbol{\psi} + M^2\boldsymbol{\psi} = 0$$

In deriving his equation, Dirac ([7], Paragraph 74) noted that as long as we are dealing with matrices with two rows and columns, we cannot obtain a representation of more than three anticommuting quantities; to represent four anticommuting quantities, he turned to matrices with four rows and columns. In our case, however, three anticommuting matrices are sufficient, so the wave function can also be two-component. Dirac also explains that the presence of four components results in twice as many solutions, half of which have negative energy. In the case of a two-component wave function, however, no negative energy solutions are obtained. Particles with negative energy in this case also exist, but they are described by the same equation in which the signs of all four matrices S or σ are reversed.

One would seem to expect similar results from other representations of the momentum operator, e.g., ([6], formula (24.15))

$$\omega_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \omega_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \omega_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

under the assumption that this representation can describe a particle with spin one. But this expectation is not justified, since the last three matrices do not anticommutate, and therefore the quadratic form constructed on their basis is not invariant under Lorentz transformations.

If one consistently adheres to the Heisenberg approach and does not involve the notion of wave function, it is not very clear how to search for solutions of the presented equations. The Schrödinger approach with finding the eigenvalues of the M^2matrix and their corresponding eigenfunctions can help here.

$$\left\{\left(S_0(P_0 - eA_0)\right)^2 - \left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^2 I - e\vec{\mathbf{S}}^T\vec{\mathbf{B}} + ie\vec{\mathbf{S}}^T\vec{\mathbf{E}}\right\}\boldsymbol{\psi} = M^2\boldsymbol{\psi}$$

In the left-hand side are the operators acting on the wave function, and in the right-hand side is a constant matrix on which the wave function is simply multiplied. This equality must be satisfied for all values of the four-dimensional coordinates (t, x_1, x_2, x_3) at once. Then M^2 is not fixed but can take a set of possible values, finding all these values is the goal of solving the equation.

Thus, we have arrived at an equation containing a matrix M^2 which is non-diagonal, complex and in general depends on the coordinates (t, x_1, x_2, x_3). After the standard procedure of separating the time and space variables, we can go to a stationary equation in which there will be no time dependence, but the dependence the matrix M^2 on the coordinates will remain. It is possible to ignore the dependence of M^2 on the coordinates and its non-diagonality and simply replace this matrix by a unit matrix with a coefficient in the form of the square of the free electron mass. Then the equation will give solutions coinciding with those of the Dirac equation. But this solution can be considered only approximate and the question remains how far we depart from strict adherence to the principle of invariance with respect to Lorentz transformations and how far we deviate from the hypothetical true solution, which is fully consistent with this principle. To find this solution, we need to approach this equation without simplifying assumptions and look for a set of solutions, each of which represents an eigenvalue matrix M^2 of arbitrary form and its corresponding four-component eigenfunction.

Let us return to the question of Lorentz invariance of the expression

$$(S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) = M^2$$

As it was noted, this expression does not change at rotations and boosts in Minkowski space only if the components of (X_0, X_1, X_2, X_3) commute with each other. If they do not commute, the matrix M^2 changes under Lorentz transformations. Two parts can be distinguished in this matrix

$$\begin{gathered} M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + \\ S_1(X_0X_1 - X_1X_0) + iS_1(X_3X_2 - X_2X_3) + \\ S_2(X_0X_2 - X_2X_0) + iS_2(X_1X_3 - X_3X_1) + \\ S_3(X_0X_3 - X_3X_0) + iS_3(X_2X_1 - X_1X_2) \end{gathered}$$

The first row represents the unit matrix multiplied by a value that still does not change under Lorentz transformations. All changes occur in the last three rows. In the particular case of electrodynamics, we have $(e = 1)$

$$\begin{gathered} M^2 = I\big((P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)\big) + \\ +S_1\big((P_0 + A_0)(P_1 + A_1) - (P_1 + A_1)(P_0 + A_0)\big) + iS_1\big((P_3 + A_3)(P_2 + A_2) - (P_2 + A_2)(P_3 + A_3)\big) \\ +S_2\big((P_0 + A_0)(P_2 + A_2) - (P_2 + A_2)(P_0 + A_0)\big) + iS_2\big((P_1 + A_1)(P_3 + A_3) - (P_3 + A_3)(P_1 + A_1)\big) \end{gathered}$$

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$$+S_3\big((P_0+A_0)(P_3+A_3)-(P_3+A_3)(P_0+A_0)\big)+iS_3\big((P_2+A_2)(P_1+A_1)-(P_1+A_1)(P_2+A_2)\big)$$

Here the first line is invariant, but the last three are not. The only way to ensure complete invariance of M^2 is to require these three lines to be zero. Let us again consider the commutation relations, but now we will not assume that the momentum components commute with each other, only the potential components still commute with each other. Now we can write the relations of the form

$$\big((P_0+eA_0)(P_1+eA_1)-(P_1+eA_1)(P_0+eA_0)\big)=$$
$$(P_0P_1-P_1P_0)+e(P_0A_1-A_1P_0)-e(P_1A_0-A_0P_1)+e(A_0A_1-A_1A_0)=$$
$$-i\frac{\partial P_1}{\partial x_0}-ie\frac{\partial A_1}{\partial x_0}-\left(-ie\frac{\partial A_0}{\partial x_1}\right)$$

Such values as $\frac{\partial P_1}{\partial x_0}$ always enter M^2 as a sum with the component of the field, in this case the electric one

$$\frac{\partial P_1}{\partial x_0}+e\left(\frac{\partial A_1}{\partial x_0}-\frac{\partial A_0}{\partial x_1}\right)=\frac{\partial P_1}{\partial x_0}+eE_x$$

If we formally define a new value

$$V_1\equiv\frac{P_1}{m}$$

and suppose that m does not change at rotations and boosts, and we also take into account the presence of charge at the electron, we can require for this and all other similar sums the fulfilment of the condition

$$m\frac{\partial V_1}{\partial x_0}+eE_x=0$$

The value V_1 can be regarded as a component of velocity, and velocity not in the usual sense, as a derivative of the spatial coordinate by time, but simply as a component of momentum divided by the inertial mass m. Then the above equality can be interpreted in the spirit of Newton's law, namely, that the acceleration multiplied by the mass is equal to the force acting on the side of the electric field. If all such equalities are fulfilled, only the first line will remain in the quantity M^2, and it will be invariant under Lorentz transformations. It is possible to go further, and to assume equality of the masses appearing here, namely

$$M^2=Im^2$$

As a result, we obtain a system consisting of the basic equation

$$(P_0+eA_0)(P_0+eA_0)-(P_1+eA_1)(P_1+eA_1)-(P_2+eA_2)(P_2+eA_2)-(P_3+eA_3)(P_3+eA_3)=m^2$$

which can be briefly written as

$$g^{\mu\nu}\big(P_\mu+eA_\mu\big)(P_\nu+eA_\nu)=m^2$$

and additional equations

$$P_\nu P_\mu-P_\mu P_\nu+e\big(\partial_\mu A_\nu-\partial_\nu A_\mu\big)=\partial_\mu P_\nu+e\big(\partial_\mu A_\nu-\partial_\nu A_\mu\big)=0$$

In the general case it is necessary to take into account in the equations an external force, e.g. of mechanical nature, defined at each point of the coordinate space. The equations then have the form

$$\partial_\mu P_\nu+f_{\mu\nu}+e\big(\partial_\mu A_\nu-\partial_\nu A_\mu\big)=0$$

The external force is taken as given, and the acceleration and field are mutually adjusted to nullify the right-hand side. This is true for a charge in an electric field that is fixed stationary, for an electric generator, for an electric motor, and so on.

It is the fulfilment of these equations that causes the mass M^2 and m^2 to acquire the properties we tend to expect of it, namely that the mass is not only invariant under Lorentz transformations, but does not change under accelerations either.

We can introduce tensor notations

$$P_{\mu\nu}+eF_{\mu\nu}=0$$

where in one case we took into account the equality of the derivative commutator, and in the other we simply replaced one of the momentum components with the derivative of the wave function in the coordinate representation, and in both cases we omitted the imaginary unit.

$$P_{\mu\nu}\equiv P_\mu P_\nu-P_\nu P_\mu=-\partial_\mu P_\nu=\partial_\mu P_\nu-\partial_\nu P_\mu$$

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$$P_{\mu\nu} = \begin{pmatrix} 0 & -\partial_0 P_1 & -\partial_0 P_2 & -\partial_0 P_3 \\ \partial_0 P_1 & 0 & -\partial_1 P_2 & -\partial_1 P_3 \\ \partial_0 P_2 & \partial_1 P_2 & 0 & -\partial_2 P_3 \\ \partial_0 P_3 & \partial_1 P_3 & \partial_2 P_3 & 0 \end{pmatrix}$$

$$P_{\mu\nu} = \begin{pmatrix} 0 & -\partial_0 P_1 & -\partial_0 P_2 & -\partial_0 P_3 \\ \partial_0 P_1 & 0 & \partial_1 P_2 - \partial_2 P_1 & \partial_1 P_3 - \partial_3 P_1 \\ \partial_0 P_2 & \partial_2 P_1 - \partial_1 P_2 & 0 & \partial_2 P_3 - \partial_3 P_2 \\ \partial_0 P_3 & \partial_3 P_1 - \partial_1 P_3 & \partial_3 P_2 - \partial_2 P_3 & 0 \end{pmatrix}$$

The resulting system of equations describes not only uniform but also accelerated motion. The presence of an external field leads to a change in momentum, and vice versa, any change in momentum under the influence of an external force perturbs the potential and generates an electromagnetic field. The equations include only the derivatives of the momentum components and lack the values of the momentum components themselves. However, if the particle is moving, the external field at its location changes depending on the nature of the motion, so the instantaneous values of the momentum components are indirectly included in the equations as well.

To accurately calculate the fields at a moving point, it is necessary to use equations that take into account the Lorentz transformations for the given external electromagnetic potential and corresponding transformations for external fields.

In the simplest case, when the speed is constant, the expressions for converting fields at an immovable point into fields at a moving point are as follows.

$$\hat{\mathbf{E}} = \gamma(\mathbf{E} - (\mathbf{V} \times \mathbf{B})) - \Gamma \mathbf{V}(\mathbf{V} \cdot \mathbf{E})$$
$$\hat{\mathbf{B}} = \gamma\big(\mathbf{B} + (\mathbf{V} \times \mathbf{E})\big) - \Gamma \mathbf{V}(\mathbf{V} \cdot \mathbf{B})$$
$$\gamma = \frac{1}{\sqrt{1 - V^2}} \qquad \Gamma = \frac{\gamma - 1}{V^2} \qquad \mathbf{V} = \frac{\vec{\mathbf{P}}}{m}$$

Of course, the very assumption of constant velocity contradicts the essence of Newton's law, in which momentum and velocity change, so Lorentz transformations for fields in general may be more complex. On the other hand, the vector product describing the Lorentz force is usually used in calculations without reservations about the constancy of velocity.

Nevertheless, let us assume that external electric and magnetic fields are specified for a immovable point. From these fields we form electromagnetic tensor. We also need to know the instantaneous momentum vector and find the Lorentz transformation that results in this vector from the momentum vector of a fixed points. We must apply the same transformation to electromagnetic tensor and extract from it the values of the electric and magnetic fields for moving point that we need.

For quantum mechanics we can replace the momentum components in all equations by the derivative operators

$$P_0 \to i\frac{\partial}{\partial x_0} \qquad P_1 \to -i\frac{\partial}{\partial x_1} \qquad P_2 \to -i\frac{\partial}{\partial x_2} \qquad P_3 \to -i\frac{\partial}{\partial x_3}$$

This also applies to equations from the second group, where mixed partial derivatives arise

$$P_\mu P_\nu \to \pm\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}$$

As a result, we obtain for the wave function a system of equations with second order derivatives. The innovation compared to the commonly used Schrödinger equation is the presence in the equations of mixed partial derivatives on all components of the coordinate vector.

The equations proposed here initially take into account the non-commutability of momentum components, their derivation relies only on the unconditional fulfilment (even in coupled systems) of the requirement of invariance to Lorentz transformations for the product of conjugate quaternions with arbitrary coefficients

$$(S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) = M^2$$

Putting all equations together, we write a truly relativistic system of equations for the wave function $\boldsymbol{\psi}(x_0, x_1, x_2, x_3)$

$$\left(\sigma_0\left(\frac{\partial}{\partial x_0} - eA_0\right) + \vec{\boldsymbol{\sigma}}^T\big(\boldsymbol{\nabla} - e\vec{\mathbf{A}}\big)\right)\left(\sigma_0\left(\frac{\partial}{\partial x_0} - eA_0\right) - \vec{\boldsymbol{\sigma}}^T\big(\boldsymbol{\nabla} - e\vec{\mathbf{A}}\big)\right)\boldsymbol{\psi} + M^2\boldsymbol{\psi} = 0$$

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$$S_k\left(\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_k}-\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_0}\right)+ie\left(\frac{\partial}{\partial x_0}A_j+\frac{\partial}{\partial x_k}A_0\right)\right)\boldsymbol{\psi}=0 \qquad k=1..3$$
$$S_1\left(\left(-\frac{\partial}{\partial x_3}\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\right)+ie\left(-\frac{\partial}{\partial x_3}A_2+\frac{\partial}{\partial x_2}A_3\right)\right)\boldsymbol{\psi}=0$$
$$S_2\left(\left(-\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_3}\frac{\partial}{\partial x_1}\right)+ie\left(-\frac{\partial}{\partial x_1}A_3+\frac{\partial}{\partial x_3}A_1\right)\right)\boldsymbol{\psi}=0$$
$$S_3\left(\left(-\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\right)+ie\left(-\frac{\partial}{\partial x_2}A_1+\frac{\partial}{\partial x_1}A_2\right)\right)\boldsymbol{\psi}=0$$

This system is a generalization of the relativistic Schrödinger equation. The essence of the generalization consists not only in taking into account the spin of the electron, which takes place already in the Dirac equation, but also takes into account the non-commutability of the momentum components. It can be assumed that the solutions of this generalized system will give exact values for stationary electron energy levels in the atom, for which no radiative corrections will be needed.

The conditions expressed by additional strings of our equations may be too strong, since they require that each pair of brackets with derivatives is zero. But invariance can also be achieved with a weaker requirement that only their sum as a whole is zero. That is, each pair of brackets can deviate from zero; the main thing is that these deviations are compensated in the total sum. This can work both in classical and quantum mechanics.

If not to substitute the coordinate derivative instead of the momentum component and to remain in the framework of classical physics, the system of equations

$$(P_0+eA_0)^2-(P_1+eA_1)^2-(P_2+eA_2)^2-(P_3+eA_3)^2=m^2$$
$$\partial_\mu P_\nu+e\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)=0$$

describes the motion of a macroscopic charged particle in the presence of an electromagnetic field.

By means of the antisymmetric Levy-Civita symbol we transform antisymmetric tensors into dual tensors

$$\breve{F}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \qquad \breve{P}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_{\rho\sigma}$$

and we use Maxwell's equations written in compact form

$$\partial_\mu F^{\mu\nu}=j^\nu \qquad \partial_\mu\breve{F}^{\mu\nu}=0$$

Let us apply the derivative operator to our proposed equations

$$\partial_\mu P^{\mu\nu}+e\partial_\mu F^{\mu\nu}=0 \qquad \partial_\mu\breve{P}^{\mu\nu}+e\partial_\mu\breve{F}^{\mu\nu}=0$$

then taking into account Maxwell's equations, we obtain

$$\partial_\mu\breve{P}^{\mu\nu}=0$$
$$\partial_\mu P^{\mu\nu}+ej^\nu=0$$
$$\partial_\mu\partial^\mu P^\nu+ej^\nu=0$$

If in the presence of an arbitrary potential there is no particle in the moving point, then our equations are the homogeneous Maxwell's equations for an arbitrarily moving point. If a charge is placed in the point, we obtain inhomogeneous equations for an arbitrarily moving charge.

The proposed equations can be considered as a derivation of Maxwell's equations. There derivation is as follows. The homogeneous part of Maxwell's equations

$$\partial_\mu\breve{F}^{\mu\nu}=0$$

clearly follows from the definition of electromagnetic tensor. The inhomogeneous part

$$e\partial_\mu F^{\mu\nu}=-\partial_\mu P^{\mu\nu}$$

is a direct consequence of the approach we propose. The four-dimensional vector on the right describes the field sources determined by the motion of a charged particle. This is very same current vector that was included in the equation derived by Maxwell from empirical data. We do not need a reference to empirical data, since the generation of electric current by a charged particle with nonzero mass directly follows from the principle of invariance of its mass.

It is noteworthy that the four-dimensional divergence of the right-hand side of the proposed equation

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$$\partial_\nu \partial_\mu P^{\mu\nu} = 0$$

is zero in itself, and not because it is equal to the left-hand side, which has zero divergence. That is, the current determined in this way automatically satisfies the conservation law required of the current in the usual interpretation of Maxwell's equations.

Let us clarify our understanding of the interaction between the electromagnetic field and the momentum of a charged particle. The charged particle creates in the surrounding space lagging potentials and fields, which depend on the nature of its movement - it is motionless, moves evenly or accelerated. In any case, we are talking about the field at points not coinciding with the location of the particle itself. It's another thing when we look at a field where the charged particle is. If there is an external electromagnetic field at this point, the charge interacts with it and moves with acceleration, while the required equality of the derivatives of the momentum and potential at the point where the charge is located is observed, no additional field is created.

If the particle is additionally accelerated or slowed under the influence of external mechanical force, then an additional field, either amplifying or attenuating it, arises to comply with the required equality of derivatives. This additional field under some special conditions can be a source of electromagnetic waves, which is the solution of an inhomogeneous wave equation in the right part of which as a source is just this additional field. But if an additional field at the point of a particle is created at any acceleration of its external force, for example, it takes place in an electrical generator, and then the radiation waves do not generally occur. To generate waves, the field at the point of the particle's presence must be variable, that is, its first derivative must be different from zero, for example, in time, which corresponds to the second derivative of potential. So there must be different from zero also a second derivative of the momentum of the particle, that is, a derivative of acceleration, which is sometimes called a jerk. A charged particle moving in a constant transverse magnetic field has a non-zero derivative of acceleration with respect to time, directed opposite to the velocity of motion, i.e. the necessary second derivative of momentum, and therefore generates an electromagnetic wave. The field in this wave has a different nature than the field of lagging potentials inherent in a particle moving with acceleration. Thus, in the vicinity of a charged particle, there is a superposition of these two types of electromagnetic fields plus the field of external sources. Let us emphasize that the creation of electromotive force in an electric generator is not a field in the vicinity of the particle, but at the point of its location.

Note that the expression

$$P_{\mu\nu} + eF_{\mu\nu} = 0$$

$$\begin{pmatrix} 0 & -\partial_0 P_1 & -\partial_0 P_2 & -\partial_0 P_3 \\ \partial_0 P_1 & 0 & \partial_1 P_2 - \partial_2 P_1 & \partial_1 P_3 - \partial_3 P_1 \\ \partial_0 P_2 & \partial_2 P_1 - \partial_1 P_2 & 0 & \partial_2 P_3 - \partial_3 P_2 \\ \partial_0 P_3 & \partial_3 P_1 - \partial_1 P_3 & \partial_3 P_2 - \partial_2 P_3 & 0 \end{pmatrix} + e \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} = 0$$

illustrates the possibility of controlling parameters of electron, for example, in quantum computing. The first line of the matrix shows that the pulse of the electric field along the x-axis changes the component x of the particle momentum, that is, produces a boost along this axis. After this you can run a boost along the y-axis, then a boost along the x-axis in the opposite direction and also a boost along the y-axis, also in the opposite direction

$$exp\left(\frac{1}{2}\beta_1 S_1\right) exp\left(\frac{1}{2}\beta_2 S_2\right) exp\left(-\frac{1}{2}\beta_1 S_1\right) exp\left(-\frac{1}{2}\beta_2 S_2\right)$$
$$\neq exp\left(\frac{1}{2}\beta_1 S_1\right) exp\left(\frac{1}{2}\beta_2 S_2\right) exp\left(-\frac{1}{2}\beta_2 S_2\right) exp\left(-\frac{1}{2}\beta_1 S_1\right) \neq I$$

Since the boosts on different axes do not commute, this sequence of boosts leads to a rotation, so the particle will rotate at some angle around the z-axis. Incidentally, we note that the commutator of any pair of boosts and/or rotations with different matrices S_i and S_j is equal to a matrix S_k different from them with some real or imaginary coefficient C, for example

$$exp\left(\frac{1}{2}\beta_1 S_1\right) exp\left(-\frac{i}{2}\alpha_2 S_2\right) - exp\left(-\frac{i}{2}\alpha_2 S_2\right) exp\left(\frac{1}{2}\beta_1 S_1\right) = CS_3$$

This is a special case of a general relation for arbitrary complex square matrices A and B, arbitrary complex numbers α, β and some complex number C

$$exp(\alpha A)exp(\beta B) - exp(\beta B)exp(\alpha A) = C(AB - BA)$$

It's interesting whether this relations holds for the case when A and B are operators, not matrices?

Thus, with the help of a sequence of electrical impulses we can turn the electron at arbitrary angles around any axes. But we have three more components of the magnetic field standing in the same positions of the electromagnetic tensor as the derivatives of the momentum.

$$\begin{pmatrix} 0 & -\partial_0 P_1 & -\partial_0 P_2 & -\partial_0 P_3 \\ \partial_0 P_1 & 0 & \partial_1 P_2 - \partial_2 P_1 & \partial_1 P_3 - \partial_3 P_1 \\ \partial_0 P_2 & \partial_2 P_1 - \partial_1 P_2 & 0 & \partial_2 P_3 - \partial_3 P_2 \\ \partial_0 P_3 & \partial_3 P_1 - \partial_1 P_3 & \partial_3 P_2 - \partial_2 P_3 & 0 \end{pmatrix}$$
$$= e\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & \partial_1 A_2 - \partial_2 A_1 & \partial_1 A_3 - \partial_3 A_1 \\ -E_y & \partial_2 A_1 - \partial_1 A_2 & 0 & \partial_2 A_3 - \partial_3 A_2 \\ -E_z & \partial_3 A_1 - \partial_1 A_3 & \partial_3 A_2 - \partial_2 A_3 & 0 \end{pmatrix}$$

The magnetic field on the right-hand side of the equality is the curl of the electromagnetic potential, and the corresponding positions in the tensor on the left can be interpreted as the curl of the momentum, describing the internal rotations of the particle. The presence of a curl in the electromagnetic potential leads to the appearance of a curl in the charged particle momentum. And conversely, due to the presence of a momentum curl of charged particles there is a constant magnetic field.

Here we have Newton's law already for rotations, i.e. the impulse of the magnetic field directly performs the rotation of the electron around the corresponding axis. While the pulse of the magnetic field rotates a particle at a fixed angle, the constant magnetic field rotates it at a constant speed. In a quantum computer, you can use both control of the rotation angle of a particle by an electric or magnetic field, and control of its constant rotation by means of a magnetic field. Because Newton's law works in both directions, by means of electrical or magnetic pulses it is possible not only to initialize the state of the particle, but also to read the parameters of this state after performing manipulations in a quantum computer.

If we recall that in Schrödinger equation, we replace momentum with the derivative of the scalar wave function

$$P_\mu = i\partial_\mu \varphi$$

and, on the other hand, the vector potential is also surely a derivative of some field

$$A_\mu = i\partial_\mu \alpha$$

then the relationship between the momentum rotor and the magnetic field does not seem strange

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\varphi = e(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\alpha$$

It means that the covariant derivatives of fields are not required to commute (if they commuted, there would be no magnetic field), but their commutators are required to compensate each other.

In the Schrödinger equation itself, we use the construction

$$i\partial_\mu \varphi + eA_\mu \varphi = i\partial_\mu \varphi + ie(\partial_\mu \alpha)\varphi$$

It is tempting to generalize it to a more symmetrical form, that is, the derivative of the product of fields

$$ie\alpha(\partial_\mu \varphi) + ie(\partial_\mu \alpha)\varphi = ie\partial_\mu(\alpha\varphi)$$

At the same time, it is unclear why we can usually ignore the electromagnetic field in the first term, perhaps because this field has zero mass?

Let us consider the matrix

$$\Phi = (S_0 X_0 - S_1 X_1 - S_2 X_2 - S_3 X_3)(S_0 Y_0 + S_1 Y_1 + S_2 Y_2 + S_3 Y_3)$$

which includes sets of arbitrary complex numbers

$$\mathbf{X}^T \equiv (X_0, X_1, X_2, X_3)$$
$$\mathbf{Y}^T \equiv (Y_0, Y_1, Y_2, Y_3)$$

Let us subject these sets of complex numbers to the Lorentz transformation

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$$\mathbf{X}' = \Lambda\mathbf{X} \qquad \mathbf{Y}' = \Lambda\mathbf{Y}$$

In this case, the matrix Φ has invariant trace proportional to the scalar product of vectors

$$Tr[(S_0X_0{}' - S_1X_1{}' - S_2X_2{}' - S_3X_3{}')(S_0Y_0{}' + S_1Y_1{}' + S_2Y_2{}' + S_3Y_3{}')] =$$
$$Tr[(S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0Y_0 + S_1Y_1 + S_2Y + S_3Y_3)] = 4\mathbf{X}^T\mathrm{g}\mathbf{Y}$$
$$\mathrm{g} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Having made the same calculations that were made earlier, it is possible to check that

$$\Phi = (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0Y_0 + S_1Y_1 + S_2Y_2 + S_3Y_3) =$$
$$(X_0Y_0 - X_1Y_1 - X_2Y_2 - X_3Y_3)I +$$
$$S_1(X_0Y_1 - X_1Y_0) + iS_1(X_3Y_2 - X_2Y_3) +$$
$$S_2(X_0Y_2 - X_2Y_0) + iS_2(X_1Y_3 - X_3Y_1) +$$
$$S_3(X_0Y_3 - X_3Y_0) + iS_3(X_2Y_1 - X_1Y_2)$$

Let us consider a special case of the phase of a plane wave in Minkowski space in the presence of an electromagnetic potential

$$\Phi =$$
$$(S_0x_0 - S_1x_1 - S_2x_2 - S_3x_3)\big(S_0(P_0 + A_0) + S_1(P_1 + A_1) + S_2(P_2 + A_2) + S_3(P_3 + A_3)\big) =$$
$$\big(x_0(P_0 + A_0) - x_1(P_1 + A_1) - x_2(P_2 + A_2) - x_3(P_3 + A_3)\big)I +$$
$$S_1\big(x_0(P_1 + A_1) - x_1(P_0 + A_0)\big) + iS_1\big(x_3(P_2 + A_2) - x_2(P_3 + A_3)\big) +$$
$$S_2\big(x_0(P_2 + A_2) - x_2(P_0 + A_0)\big) + iS_2\big(x_1(P_3 + A_3) - x_3(P_1 + A_1)\big) +$$
$$S_3\big(x_0(P_3 + A_3) - x_3(P_0 + A_0)\big) + iS_3\big(x_2(P_1 + A_1) - x_1(P_2 + A_2)\big)$$

We again apply the principle of absolute invariance and proceed from the assumption that the trace Φ must be invariant to the Lorentz transformations even in the presence of the field. The first line satisfies this condition, and the trace of the other three must be equated to zero, which leads to the equations for the momentum components. Let us perform the substitutions

$$P_0 \to i\partial_0 \qquad P_1 \to -i\partial_1 \qquad P_2 \to -i\partial_2 \qquad P_3 \to -i\partial_3$$
$$Tr\begin{pmatrix} \displaystyle\sum_{k=1}^{3} S_k\big((x_0\partial_k + x_k\partial_0) + ie(x_0A_k - x_kA_0)\big) \\ +S_1\big((x_3\partial_2 - x_2\partial_3) + ie(x_3A_2 - x_2A_3)\big) \\ +S_2\big((x_1\partial_3 - x_3\partial_1) + ie(x_1A_3 - x_3A_1)\big) \\ +S_3\big((x_2\partial_1 - x_1\partial_2) + ie(x_2A_1 - x_1A_2)\big) \end{pmatrix}\boldsymbol{\psi} = \mathbf{0}$$

We have obtained the equations for the wave function, which, together with the equations

$$\left(\sigma_0\left(\frac{\partial}{\partial x_0} - eA_0\right) + \vec{\boldsymbol{\sigma}}^T(\boldsymbol{\nabla} - e\vec{\mathbf{A}})\right)\left(\sigma_0\left(\frac{\partial}{\partial x_0} - eA_0\right) - \vec{\boldsymbol{\sigma}}^T(\boldsymbol{\nabla} - e\vec{\mathbf{A}})\right)\boldsymbol{\psi} + M^2\boldsymbol{\psi} = 0$$
$$S_k\big((\partial_0\partial_k - \partial_k\partial_0) + ie(\partial_0A_k + \partial_kA_0)\big)\boldsymbol{\psi} = 0 \qquad k = 1..3$$
$$S_1\big((-\partial_3\partial_2 + \partial_2\partial_3) + ie(-\partial_3A_2 + \partial_2A_3)\big)\boldsymbol{\psi} = 0$$
$$S_2\big((-\partial_1\partial_3 + \partial_3\partial_1) + ie(-\partial_1A_3 + \partial_3A_1)\big)\boldsymbol{\psi} = 0$$
$$S_3\big((-\partial_2\partial_1 + \partial_1\partial_2) + ie(-\partial_2A_1 + \partial_1A_2)\big)\boldsymbol{\psi} = 0$$

form a complete system that describes a charged quantum particle in the presence of an electromagnetic field.

For the system with mixed partial derivatives we have applied the most stringent requirements possible, equating to zero each of the expressions with matrices S_k. But invariance can also be achieved with less stringent requirements

$$S_1\begin{pmatrix} \big((i\partial_0 + A_0)(-i\partial_1 + A_1) - (-i\partial_1 + A_1)(i\partial_0 + A_0)\big) \\ +i\big((-i\partial_3 + A_3)(-i\partial_2 + A_2) - (-i\partial_2 + A_2)(-i\partial_3 + A_3)\big) \end{pmatrix}\boldsymbol{\psi} = 0$$
$$S_2\begin{pmatrix} \big((i\partial_0 + A_0)(-i\partial_2 + A_2) - (-i\partial_2 + A_2)(i\partial_0 + A_0)\big) \\ +i\big((-i\partial_1 + A_1)(-i\partial_3 + A_3) - (-i\partial_3 + A_3)(-i\partial_1 + A_1)\big) \end{pmatrix}\boldsymbol{\psi} = 0$$
$$S_3\begin{pmatrix} \big((i\partial_0 + A_0)(-i\partial_3 + A_3) - (-i\partial_3 + A_3)(i\partial_0 + A_0)\big) \\ +i\big((-i\partial_2 + A_2)(-i\partial_1 + A_1) - (-i\partial_1 + A_1)(-i\partial_2 + A_2)\big) \end{pmatrix}\boldsymbol{\psi} = 0$$

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Let us formulate the essence of the proposed approach. For each point of Minkowski space we have equations arising from the requirement of invariance of mass to Lorentz transformations and to changes of momentum (simply speaking, to accelerations). They reduce to the equality to zero of the set of differences between the derivatives of momentum and the derivatives of some potential field with possible addition of an external force. In the conventional sense, these are the equations of motion. If we want to obtain a complete picture of the behavior of the system in the form of a trajectory in coordinate space, we must choose such a trajectory at each point of which the equations of motion are satisfied. To do this, we have to integrate the mentioned differences over some measure and choose the trajectory with the minimum of the integral. This way we get the only classical trajectory. It is imperative to emphasize that the equations of motion were not obtained from the principle of least action; rather, they were derived from the requirement of invariance of mass to Lorentz transformations and its invariance to accelerations.

When transitioning from classical interpretation to quantum mechanics, the equations of motion are applied in a different way. In this context, let us clarify the relationship between the Lagrangian, the equations of motion, and the translation operator. This operator transforms the quantum state and its corresponding wave function in coordinate representation from one point to another infinitely close point, acting either in configuration space or in Minkowski real space

$$e^{i(\widehat{P_0}dX_0-\widehat{P_1}dX_1-\widehat{P_2}dX_2-\widehat{P_3}dX_3)} \qquad e^{i(\hat{H}dt-\hat{P}dq)}$$

Here, momentum is an operator, and coordinate is a number – one of the eigenvalues of the coordinate operator, since we assume that the state is described by a wave function in coordinate representation. To move a finite distance between two points, you can choose a trajectory connecting these points, and apply the translator translation to each step. The resulting operator will be the product of the operators, i.e., the product of exponents of the specified type

$$\Pi e^{i(\widehat{P_0}dX_0-\widehat{P_1}dX_1-\widehat{P_2}dX_2-\widehat{P_3}dX_3)} \qquad \Pi e^{i(\hat{H}dt-\hat{P}dq)}$$

To achieve our goal – the transition to Lagrangian – we will sacrifice rigor and replace the product of the exponent with the exponent of the sum, turning a blind eye to the possible non-commutativity of the exponents with each other

$$e^{i\int(\widehat{P_0}dX_0-\widehat{P_1}dX_1-\widehat{P_2}dX_2-\widehat{P_3}dX_3)} \qquad e^{i\int(\hat{H}dt-\hat{P}dq)}$$

Next, we will replace the momentum operators with their eigenvalues, interpreting the resulting quantity as the probability amplitude of the of transition along the selected trajectory. This substitution may also be incorrect due to the probable non-commutativity of the momentum component operators with each other. After that, we find the sum of the probability amplitudes for all possible trajectories

$$\int e^{i\int(P_0dX_0-P_1dX_1-P_2dX_2-P_3dX_3)} \qquad \int e^{i\int(Hdt-Pdq)}$$

The total phase along the trajectory is influenced by the fact that at each point of the trajectory the momentum components are not independent; they are subject to constraints imposed by the equation of motion. To account for this constraints, we use the ratio between the Hamiltonian and the Lagrangian

$$H = P\frac{dq}{dt} - L$$

$$Hdt - Pdq = -Ldt$$

$$\int e^{i\int(Hdt-Pdq)} = \int e^{-i\int Ldt}$$

Using the two controversial assumptions mentioned above, we arrived at the well-known scheme using the Lagrangian and the corresponding action. The transition to the Lagrangian simplifies the calculation of the phase, reducing integration over multidimensional coordinate space to integration over time. Moreover, since, for a given Lagrangian, the equation of motion is determined by the Euler-Lagrange equation, it is assumed that the constraints on the momentum components contained in the equation of motion are automatically taken into account in the Lagrangian. However, if the equations of motion are considered to be a more fundamental

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description of nature than the Lagrangian, then doubts remain as to whether all the constraints defined by these equations are taken into account in it.

Thus, if we assume that nature at a fundamental level is described by the translation operator over an infinitesimally small distance and by the equations of motion, and these two laws are absolute truths, then the integral over trajectories method using the Lagrangian is only an approximate truth, which can be arrived at only at the cost of several controversial assumptions.

Let us formulate the phase and mass invariance requirement in a more general form using the metric tensor of the flat Minkowski space

$$\Phi = (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3)$$
$$= (S_0, S_1, S_2, S_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}(S_0, S_1, S_2, S_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} =$$
$$= (S_\mu \mathrm{g}^{\mu\nu} X_\nu)(S_\rho I^{\rho\sigma} P_\sigma)$$
$$\mathrm{Tr}\left[(S_\mu \mathrm{g}^{\mu\nu} X_\nu)(S_\rho I^{\rho\sigma} P_\sigma)\right] = 4\mathbf{X}^{\mathrm{T}}\mathrm{g}\mathbf{P}$$
$$M^2 = (S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3)(S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3) =$$
$$= (S_0, S_1, S_2, S_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}(S_0, S_1, S_2, S_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} =$$
$$= (S_\mu \mathrm{g}^{\mu\nu} P_\nu)(S_\rho I^{\rho\sigma} P_\sigma) = (\mathbf{P}^{\mathrm{T}}\mathrm{g}\mathbf{P})I$$
$$\mathrm{Tr}\left[(S_\mu \mathrm{g}^{\mu\nu} P_\nu)(S_\rho I^{\rho\sigma} P_\sigma)\right] = 4\mathbf{P}^{\mathrm{T}}\mathrm{g}\mathbf{P}$$

Let us note an important difference. In the absence of an external field the matrix M^2 is diagonal and invariant to Lorentz transformations, while the matrix Φ is not diagonal even in the absence of an external field and it changes under Lorentz transformations. However, its trace is invariant and equal to four invariant phases. So we can formulate the general principle as a requirement that at addition of an external field the trace of matrices in both cases remains invariant to Lorentz transformations.

If we replace the metric tensor of the Minkowski space by the metric tensor of the space curved by the action of gravitation of the general theory of relativity, will the phase and mass remain invariant to the Lorentz transformations? Since the curvature of space is caused only by the action of external masses, we can assume that the invariance principle is absolute. Then we have at our disposal the equations imposing restrictions on the metric of curved space.

The matrix may be rewritten in another form

$$(M^2)_{\mu\nu} = \sum_{\rho=0}^{3} \mathrm{g}^{\alpha\beta} I^{\gamma\delta} (S_\alpha)_{\mu\rho} (S_\gamma)_{\rho\nu} P_\beta P_\delta = \sum_{\rho=0}^{3} \mathrm{g}^{\alpha\beta} I^{\gamma\delta} (S_\alpha)_{\mu\rho} (S_\gamma)_{\rho\nu} T_{\beta\delta}$$

where

$$T_{\beta\delta} \equiv P_\beta P_\delta$$

has the meaning of the energy-momentum tensor. However, in this form it is not very clear how to add the external field carefully. Earlier we added a vector field to the momentum vector, but perhaps it makes sense to add at once some tensor field to the momentum-energy tensor. The tensor of this field may have a more general form than the direct product of the sums of the electromagnetic potential vector and the momentum vector. In this context the question arises whether it is possible in this way to take into account the influence of the gravitational field, which is described by a tensor, while requiring the conservation of mass invariance.

Let's go back to our equation

$$P_\nu P_\mu - P_\mu P_\nu + e(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu P_\nu + e(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$$

and look at it from a different point of view, taking into account the correspondence of the momentum component and the derivatives according to the coordinates

$$P_0 \to i\frac{\partial}{\partial x_0} = i\partial^0 \quad P_1 \to -i\frac{\partial}{\partial x_1} = -i\partial^1 \quad P_2 \to -i\frac{\partial}{\partial x_2} = -i\partial^2 \quad P_3 \to -i\frac{\partial}{\partial x_3} = -i\partial^3$$

or, when using covariant derivatives instead of contravariant derivatives

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$$P_0 \to i\partial_0 \qquad P_1 \to i\partial_1 \qquad P_2 \to i\partial_2 \qquad P_3 \to i\partial_3$$

Then the previously obtained condition of absolute mass invariance will have the form

$$\partial_\mu\partial_\nu - \partial_\nu\partial_\mu + e(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$$

Add the vector field A_μ to the covariant derivative

$$\nabla_\mu \equiv \partial_\mu + eA_\mu$$

and find the commutator component of a covariant derivative acting on the scalar field φ

$$\nabla_\mu\nabla_\nu\varphi = eA_\mu\partial_\nu\varphi + eA_\mu eA_\nu\varphi + \partial_\mu\partial_\nu\varphi + e\partial_\mu A_\nu\varphi + eA_\nu\partial_\mu\varphi$$
$$\nabla_\nu\nabla_\mu\varphi = eA_\nu\partial_\mu\varphi + eA_\nu eA_\mu\varphi + \partial_\nu\partial_\mu\varphi + e\partial_\nu A_\mu\varphi + eA_\mu\partial_\nu\varphi$$
$$[\nabla_\mu, \nabla_\nu]\varphi = \nabla_\mu\nabla_\nu\varphi - \nabla_\nu\nabla_\mu\varphi = \partial_\mu\partial_\nu\varphi - \partial_\nu\partial_\mu\varphi + e\partial_\mu A_\nu\varphi - e\partial_\nu A_\mu\varphi$$

The commutator coincides exactly with the left part of our equation, providing absolute mass invariance. Thus, in order for mass to be invariant, it is necessary that the covariant derivative containing the vector field and acting on the scalar field has a zero commutator of its components. This condition determines the equation of motion of a charged particle in an electromagnetic field, and the commutator of the derivative includes an electromagnetic tensor containing components of the electromagnetic field voltage. This principle of commutability of a covariant derivative works both in the classical interpretation for a charged particle in an electromagnetic field

$$P_\nu P_\mu - P_\mu P_\nu + e(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu P_\nu + e(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$$

and is the same in quantum interpretation when the scalar field φ is a scalar wave function, and the operator equation of motion is an additional condition for the Schrödinger equation.

By the way, the basic equation given earlier

$$g^{\mu\nu}(P_\mu + eA_\mu)(P_\nu + eA_\nu) - m^2 = 0$$

can also be written using the covariant derivative

$$(g^{\mu\nu}\nabla_\mu\nabla_\mu - m^2)\varphi = 0$$

The requirement of the commutativity of the covariant derivative is now added to the basic equation

$$[\nabla_\mu, \nabla_\nu]\varphi = 0$$

It is sometimes assumed that the components of the ordinary derivative and the corresponding components of the momentum are switched between each other. If this were true, then the components of the momentum, for example, could not change over time, that is, the particle could not accelerate, because the derivative of the momentum in time is equal to the momentum commutator with the energy operator. In fact, the components of a generalized derivative that is obtained after adding a compensating field are commutated. It is the addition of this field that provides commutability of the derivative and mass invariance.

Extend the formulated principle to the gravitational field, for which we define a covariant derivative including affine connection $\Gamma_{\mu\beta}{}^\sigma$ and act on the vector field Φ^σ

$$\nabla_\mu\Phi^\sigma \equiv \partial_\mu\Phi^\sigma + \Gamma_{\mu\beta}{}^\sigma\Phi^\beta$$

Use familiar expressions to find the commutator of a covariant derivative

$$\nabla_\mu\nabla_\nu\Phi^\sigma = \partial_\mu\partial_\nu\Phi^\sigma + \partial_\mu\Gamma_{\nu\gamma}{}^\sigma\Phi^\gamma + \Gamma_{\nu\gamma}{}^\sigma\partial_\mu\Phi^\gamma + \Gamma_{\mu\gamma}{}^\sigma\partial_\nu\Phi^\gamma + \Gamma_{\mu\gamma}{}^\sigma\Gamma_{\nu\beta}{}^\gamma\Phi^\beta - \Gamma_{\mu\nu}{}^\gamma\partial_\gamma\Phi^\sigma - \Gamma_{\mu\nu}{}^\gamma\Gamma_{\gamma\beta}{}^\sigma\Phi^\beta$$
$$\nabla_\nu\nabla_\mu\Phi^\sigma = \partial_\nu\partial_\mu\Phi^\sigma + \partial_\nu\Gamma_{\mu\gamma}{}^\sigma\Phi^\gamma + \Gamma_{\mu\gamma}{}^\sigma\partial_\nu\Phi^\gamma + \Gamma_{\nu\gamma}{}^\sigma\partial_\mu\Phi^\gamma + \Gamma_{\nu\gamma}{}^\sigma\Gamma_{\mu\beta}{}^\gamma\Phi^\beta - \Gamma_{\nu\mu}{}^\gamma\partial_\gamma\Phi^\sigma - \Gamma_{\nu\mu}{}^\gamma\Gamma_{\gamma\beta}{}^\sigma\Phi^\beta$$
$$[\nabla_\mu, \nabla_\nu]\Phi^\sigma = (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Phi^\sigma +$$
$$\Phi^\gamma(\partial_\mu\Gamma_{\nu\gamma}{}^\sigma - \partial_\nu\Gamma_{\mu\gamma}{}^\sigma + \Gamma_{\mu\beta}{}^\sigma\Gamma_{\nu\gamma}{}^\beta - \Gamma_{\nu\beta}{}^\sigma\Gamma_{\mu\gamma}{}^\beta) - 2\Gamma_{[\mu\nu]}{}^\gamma(\partial_\gamma\Phi^\sigma + \Gamma_{\gamma\beta}{}^\sigma\Phi^\beta)$$
$$[\nabla_\mu, \nabla_\nu]\Phi^\sigma = (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Phi^\sigma + R^\sigma{}_{\gamma\mu\nu}\Phi^\gamma + T_{\mu\nu}{}^\gamma\nabla_\gamma\Phi^\sigma$$

Here used designations: torsion tensor

$$-T_{\mu\nu}{}^\gamma = 2\Gamma_{[\mu\nu]}{}^\gamma = \Gamma_{\mu\nu}{}^\gamma - \Gamma_{\nu\mu}{}^\gamma$$

and Riemann tensor

$$R^\sigma{}_{\gamma\mu\nu} = \partial_\mu\Gamma_{\nu\gamma}{}^\sigma - \partial_\nu\Gamma_{\mu\gamma}{}^\sigma + \Gamma_{\mu\beta}{}^\sigma\Gamma_{\nu\gamma}{}^\beta - \Gamma_{\nu\beta}{}^\sigma\Gamma_{\mu\gamma}{}^\beta$$

which plays the role of a gravitational field strength.

According to the proposed principle, components of a covariant derivative must commute, so

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Phi^\sigma + R^\sigma{}_{\gamma\mu\nu}\Phi^\gamma + T_{\mu\nu}{}^\gamma(\partial_\gamma\Phi^\sigma + \Gamma_{\gamma\beta}{}^\sigma\Phi^\beta) = 0$$

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In the interpretation of quantum mechanics we have obtained an operator equation for the vector wave function Φ^{σ} in coordinate representation. In the classical interpretation, it is necessary to replace the components of the operator derivative with the components of momentum

$$i\partial_{\mu}\Phi^{\sigma} \to P_{\mu}^{\sigma} \qquad -\partial_{\mu}\partial_{\nu}\Phi^{\sigma} \to P_{\mu}^{\sigma}P_{\nu}^{\sigma}$$

$$-\left(P_{\mu}^{\sigma}P_{\nu}^{\sigma} - P_{\nu}^{\sigma}P_{\mu}^{\sigma}\right) + R^{\sigma}{}_{\gamma\mu\nu}\Phi^{\gamma} + T_{\mu\nu}{}^{\gamma}\left(-iP_{\gamma}^{\sigma} + \Gamma_{\gamma\beta}{}^{\sigma}\Phi^{\beta}\right) = 0$$

Again, let us consider that the commutator of the coordinate function and the operator of the momentum is equal to a derivative of this function according to the corresponding coordinate

$$P_{\mu}^{\sigma}P_{\nu}^{\sigma} - P_{\nu}^{\sigma}P_{\mu}^{\sigma} = -\partial_{\mu}P_{\nu}^{\sigma}$$

$$\partial_{\mu}P_{\nu}^{\sigma} + R^{\sigma}{}_{\gamma\mu\nu}\Phi^{\gamma} + T_{\mu\nu}{}^{\gamma}\left(-iP_{\gamma}^{\sigma} + \Gamma_{\gamma\beta}{}^{\sigma}\Phi^{\beta}\right) = 0$$

$$\partial_{\mu}P_{\nu}^{\sigma} = -R^{\sigma}{}_{\gamma\mu\nu}\Phi^{\gamma} - T_{\mu\nu}{}^{\gamma}\left(-iP_{\gamma}^{\sigma} + \Gamma_{\gamma\beta}{}^{\sigma}\Phi^{\beta}\right)$$

In the left part there is a derivative momentum, in particular, for the time coordinate it gives acceleration, and in the right part the force of gravity that causes this acceleration. This expression is an analogue of Newton's law and it describes interaction of a particle with the gravitational field, the tension of which is described by Riemann's tensor. Unlike the case of the electromagnetic field, in the right part there is also a component of the momentum that is Newton's law contains non-linearity. The formula works in the opposite direction. If the particle is exposed to external influences, such as electromagnetic nature, then it accelerates, so at this point an additional gravitational field is created that adds to the external one. If the second derivative of the momentum is not zero, the source appears in the right part of some wave equation. Then gravitational waves are generated.

The correspondence between the quantum mechanical and classical interpretations can be established in a slightly different way. Let us take the original equation

$$\left(\partial_{\mu}\partial_{\nu} - \partial_{\nu}\partial_{\mu}\right)\Phi^{\sigma} + R^{\sigma}{}_{\gamma\mu\nu}\Phi^{\gamma} + T_{\mu\nu}{}^{\gamma}\left(\partial_{\gamma}\Phi^{\sigma} + \Gamma_{\gamma\beta}{}^{\sigma}\Phi^{\beta}\right) = 0$$

and use a different substitution

$$i\partial_{\mu}\Phi^{\sigma} \to P_{\mu}^{\sigma}$$

$$-i\left(\partial_{\mu}P_{\nu}^{\sigma} - \partial_{\nu}P_{\mu}^{\sigma}\right) + R^{\sigma}{}_{\gamma\mu\nu}\Phi^{\gamma} + T_{\mu\nu}{}^{\gamma}\left(-iP_{\gamma}^{\sigma} + \Gamma_{\gamma\beta}{}^{\sigma}\Phi^{\beta}\right) = 0$$

Now the equation includes an antisymmetric tensor $\partial_{\mu}P_{\nu}^{\sigma} - \partial_{\nu}P_{\mu}^{\sigma}$ containing, among other things, the curl of the momentum components. The equation also includes components of the vector field Φ^{γ}, which in the case of electrodynamics, where the corresponding field is scalar, we simply remove from the equation. Now it is more difficult to do this, since there is contraction of the field components with the Riemann tensor.

Similarly, as the equation of motion of a charged particle is an addition to the basic equation that includes mass, so the equation of motion of particles in a gravitational field is an addition to some equation that also includes mass. Suppose that in the case of gravity this equation has the analogous form, including a diagonal matrix with a square mass on the diagonal

$$\left(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} - M^{2}\right)\Phi^{\sigma} = 0$$

The invariance of this mass to changes in the momentum is provided precisely by meeting the commutability requirement of the generalized covariant derivative

$$\left[\nabla_{\mu}, \nabla_{\nu}\right]\Phi^{\sigma} = 0$$

Strong and weak interaction theories also use covariant derivatives with corresponding compensating fields. It is logical to extend to them the principle of the vanishing commutator of covariant derivatives. Then we will get the equations of motion for these theories not on the basis of the principle of least action, but on the basis of the requirement of absolute mass invariance.

For example, in Yang-Mills theory, the covariant derivative for a field Ψ, which can be viewed as a vector in a certain vector space and on which gauge fields act, is written as follows

$$\nabla_{\mu}\mathbf{\Psi} \equiv \partial_{\mu}\mathbf{\Psi} - igA_{\mu}{}^{\sigma}T^{\sigma}\mathbf{\Psi}$$

where g is the coupling constant, $A_{\mu}{}^{\sigma}$ is the component of the gauge field, T^{σ} are the generators of the gauge group. Then, in accordance with the principle of mass invariance (in some cases, the invariant mass may be zero), the field obeys two equations

$$\left(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} - M^{2}\right)\mathbf{\Psi} = 0$$

$$\left[\nabla_{\mu}, \nabla_{\nu}\right]\mathbf{\Psi} = 0$$

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The first of these is the equation of motion, and the second is Newton's law. The principle of least action, the Lagrangian, and the Euler equation only allow us to obtain the equation of motion, while both of these equations follow immediately from the principle of mass invariance.

We believe that natural processes are described by an equation in the form of a product of conjugate quaternions, in which mass is invariant to Lorentz transformations.

$$\left((S_0\nabla_0 - S_1\nabla_1 - S_2\nabla_2 - S_3\nabla_3)(S_0\nabla_0 + S_1\nabla_1 + S_2\nabla_2 + S_3\nabla_3) - M^2\right)\mathbf{\Psi} = 0$$

Due to the anti-commutativity of Pauli matrices, we can move on to the equation

$$\left(\mathrm{g}^{\mu\nu}\nabla_\mu\nabla_\nu - M^2\right)\mathbf{\Psi} = 0$$

This transition leaves the mass invariant even under accelerated motion, but only on condition that the components of the covariant derivative commute. Why should we start with the initial quaternionic equation rather than immediately with the Klein-Gordon equation? One argument is that it is precisely through the transition process between them that we automatically obtain the electromagnetic field tensor and the Riemann tensor, as well as the derivation of Newton's law

3. Quantum Theory in Spinor Coordinates Space

Let us consider the set of arbitrary complex numbers, for simplicity we will call it a vector

$$\mathbf{X}^{\mathbf{T}} \equiv (X_0, X_1, X_2, X_3)$$

and let us consider arbitrary four-component complex spinors

$$\mathbf{x1}^{\mathbf{T}} \equiv (x1_0, x1_1, x1_2, x1_3)$$

$$\mathbf{x2}^{\mathbf{T}} \equiv (x2_0, x2_1, x2_2, x2_3)$$

Among all possible vectors, let us select a set of such vectors for which there is a representation of components through arbitrary complex spinors using gamma matrices in the Weyl basis

$$\gamma_0^V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_1^V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2^V = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3^V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$X_\mu = \frac{1}{2}\mathbf{x1}^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{x2}$$

and there is another way to calculate them

$$X_\mu = \frac{1}{2}Tr[\mathbf{x1}\mathbf{x2}^\dagger(\gamma_0^V\gamma_\mu^V)]$$

Let us narrow this space down to Minkowski space, for which we will assume that both spinors are identical, and then the vector X constructed from them has real components

$$X_\mu = \frac{1}{2}\mathbf{x}^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{x}$$

and we will assume that this is the coordinate vector of an electron, constructed from the coordinate spinor **x**. Thus, an electron can be characterized by its location in either spinor or vector coordinate space, with the spinor coordinates being arbitrary and the vector coordinates being calculated from them. It is important to note that not just any four real numbers are a vector in Minkowski space, but only those obtained using a bilinear form from arbitrary spinor coordinates.

$$X_0 = \frac{1}{2}(\overline{x_0}x_0 + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3)$$

$$X_1 = \frac{1}{2}(-\overline{x_0}x_1 - \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)$$

$$X_2 = \frac{1}{2}(i\overline{x_0}x_1 - i\overline{x_1}x_0 - i\overline{x_2}x_3 + i\overline{x_3}x_2)$$

$$X_3 = \frac{1}{2}(-\overline{x_0}x_0 + \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)$$

As we can see, the components of the vector in Minkowski space are interdependent, from this dependence automatically follow the relations of the special theory of relativity between space and time. For the same reason, the coordinates of Minkowski space cannot serve as independent variables in the equations. And since we do not doubt the truth of the theory of relativity, we cannot doubt

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the reality of spinor space, which by means of the simplest arithmetic operations generates our space and time.

The quantity

$$X_0 = \frac{1}{2}(\overline{x_0}x_0 + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3) \equiv t$$

represents time in four-dimensional vector space. An interesting fact is that time is always a positive quantity. As an assumption it can be noted that since we observe that time value goes forward, i.e. the value of t grows, and it is possible only due to scaling of all components of spinor space, such scaling leads to increase of distance between any two points of Minkowski space. As a result, with the passage of time the Minkowski space should expand, herewith at first relatively quickly, and then more and more slowly.

Let us introduce another arbitrary spinor

$$\mathbf{p}^{\mathbf{T}} \equiv (p_0, p_1, p_2, p_3)$$

from which a vector with real coordinates is formed

$$\mathbf{P}^{\mathbf{T}} \equiv (P_0, P_1, P_2, P_3)$$

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{p}$$

$$P_\mu = \frac{1}{2}Tr[\mathbf{p}\mathbf{p}^\dagger(\gamma_0^V\gamma_\mu^V)]$$

We will assume that this is the momentum vector of an electron, constructed from the complex momentum spinor **p**.

Consider the complex quantity

$$\mathbf{p}^\dagger\gamma_0^V\mathbf{x} = (\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3})\begin{pmatrix} x_2 \\ x_3 \\ x_0 \\ x_1 \end{pmatrix} =$$

$$= \overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 \equiv (\mathbf{p}, \mathbf{x})$$

In order to achieve consistency between the Lorentz transformation acting on the spinor and the Lorentz transformation that correctly transforms not only the vector obtained from it, but also the tensor of any rank, it is necessary to change the transformation low of the spinor by changing the sign of the angle at the boost in the first matrix

$$n1 =$$

$$exp\left(-\frac{i}{2}\alpha_1\sigma_1\right)exp\left(-\frac{i}{2}\alpha_2\sigma_2\right)exp\left(-\frac{i}{2}\alpha_3\sigma_3\right)exp\left(-\frac{1}{2}\beta_1\sigma_1\right)exp\left(-\frac{1}{2}\beta_2\sigma_2\right)exp\left(-\frac{1}{2}\beta_3\sigma_3\right)$$

$$n2 =$$

$$exp\left(-\frac{i}{2}\alpha_1\sigma_1\right)exp\left(-\frac{i}{2}\alpha_2\sigma_2\right)exp\left(-\frac{i}{2}\alpha_3\sigma_3\right)exp\left(\frac{1}{2}\beta_1\sigma_1\right)exp\left(\frac{1}{2}\beta_2\sigma_2\right)exp\left(\frac{1}{2}\beta_3\sigma_3\right)$$

$$N = \begin{pmatrix} n1 & 0 \\ 0 & n2 \end{pmatrix}$$

In this case, the transformed spinor yields a vector that can be obtained in parallel from the original vector by applying the corresponding Lorentz transformation matrix in Minkowski vector space

$$\Lambda = exp(\alpha_1R_1)exp(\alpha_2R_2)exp(\alpha_3R_3)exp(\beta_1K_1)exp(\beta_2K_2)exp(\beta_3K_3)$$

$$\Lambda g\Lambda^T = g$$

The quantity $\mathbf{p}^{\boldsymbol{T}}\gamma_0^V\mathbf{x}$ is invariant under the Lorentz transformation defined in this way, applied simultaneously to the momentum and coordinate spinors, which automatically transforms both corresponding vectors as well

$$\mathbf{p}' = N\mathbf{p}$$

$$P'_\mu = \frac{1}{2}Tr[\mathbf{p}'\mathbf{p}'^\dagger(\gamma_0^V\gamma_\mu^V)]$$

$$P'_\mu = \frac{1}{2}\mathbf{p}'^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{p}'$$

$$\mathbf{P}' = \Lambda\mathbf{P}$$

$$\mathbf{x}' = N\mathbf{x}$$

$$X'_\mu = \frac{1}{2}Tr[\mathbf{x}'\mathbf{x}'^\dagger(\gamma_0^V\gamma_\mu^V)]$$

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$$X'_\mu = \frac{1}{2}\mathbf{x}'^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{x}'$$
$$\mathbf{X}' = \Lambda\mathbf{X}$$
$$\mathbf{p}'^T\gamma_0^V\mathbf{x}' = \mathbf{p}^T\gamma_0^V\mathbf{x}$$

Accordingly, the exponent

$$exp\left(-i(\mathbf{p}^\dagger\gamma_0^V\mathbf{x} + \overline{\mathbf{p}^\dagger\gamma_0^V\mathbf{x}})\right) =$$
$$= exp(-i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1}))$$

is also invariant and characterizes the propagation process of a plane wave in spinor space with phase invariant to Lorentz transformations.

This plane wave is the solution to the following wave equations

$$\left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial \overline{x_0}} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial \overline{x_1}}\right) exp(-i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1})) =$$
$$= -(\overline{p_0}p_2 + \overline{p_1}p_3)\, exp(-i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1}))$$
$$= -m\, exp(-i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1}))$$
$$\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial \overline{x_2}} + \frac{\partial}{\partial x_1}\frac{\partial}{\partial \overline{x_3}}\right) exp(-i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1})) =$$
$$= -(\overline{p_2}p_0 + \overline{p_3}p_1)\, exp(-i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1}))$$
$$= -\overline{m}\, exp(-i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1}))$$

where

$$\overline{p_0}p_2 + \overline{p_1}p_3 \equiv m$$
$$\overline{p_2}p_0 + \overline{p_3}p_1 \equiv \overline{m}$$

These definitions make sense if the equality is true

$$\overline{\overline{p_0}p_2 + \overline{p_1}p_3} = \overline{p_2}p_0 + \overline{p_3}p_1$$
$$p_0\overline{p_2} + p_1\overline{p_3} = \overline{p_2}p_0 + \overline{p_3}p_1$$

that is, if the corresponding components of the spinor commute, which is known to be the case for a free particle. The complex quantity m is invariant under the action on the momentum spinor $\mathbf{p}$ with the transformation N.

Let us clarify that by the derivative on a complex variable from a complex function we here understand the derivative from an arbitrary stepped complex function using the formula that is valid at least for any integer degrees, complex conjugate quantities are considered as independent variables

$$\frac{\partial z^k}{\partial z} = kz^{k-1} \qquad \frac{\partial \bar{z}^k}{\partial \bar{z}} = k\bar{z}^{k-1}$$

In particular, this is true for the exponential function, which is an infinite power series.

It is not by chance that we denote the invariant quantity by the symbol m, because if we form the momentum vector from the momentum spinor $\mathbf{p}$ included in the expression for the plane wave

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{p}$$

then for the square of its length the following equality will be satisfied

$$\mathbf{P}^T\mathrm{g}\mathbf{P} = {P_0}^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 = m^2 = m\overline{m} = (\overline{p_0}p_2 + \overline{p_1}p_3)(\overline{p_2}p_0 + \overline{p_3}p_1)$$
$$\Lambda\mathrm{g}\Lambda^T = \mathrm{g}$$
$$\mathrm{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

that is the square of the modulus m has the sense of the square of the mass of a free particle, which is described by a plane wave in spinor space as well as by a plane wave in vector space.

The mass determined in this way can be complex or real with positive and negative values, including zero. Its modulus coincides with the ordinary mass, defined as the length of the momentum vector in Minkowski space.

For the momentum spinor of a bosonic type particle having in the rest frame the following form

$$\mathbf{p}^T = (p_0, p_1, \overline{p_1}, -\overline{p_0})$$

mass is real and equal to zero

$$m = \overline{p_0}p_2 + \overline{p_1}p_3 = \overline{p_0}\overline{p_1} - \overline{p_1}\overline{p_0} = 0$$

For the fermionic type momentum spinor having in the rest frame the following form

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$$\mathbf{p}^T = (p_0, p_1, p_0, p_1)$$

it is not zero

$$m = \overline{p_0}p_2 + \overline{p_1}p_3 = \overline{p_0}p_0 + \overline{p_1}p_1 > 0$$

Another version of the fermion

$$\mathbf{p}^T = (p_0, p_1, -p_0, -p_1)$$

also leads to a nonzero mass, but with a negative value

$$m = \overline{p_0}p_2 + \overline{p_1}p_3 = -\overline{p_0}p_0 - \overline{p_1}p_1 < 0$$

This particle with negative mass can be treated as an antiparticle, and in the rest frame its energy is equal to its mass modulo and it is positive

$$P_0 = \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) = \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_0}p_0 + \overline{p_1}p_1) = \overline{p_0}p_0 + \overline{p_1}p_1$$

For simplicity it is possible to consider, for example, the mass of the electron as negative and that of the positron as positive.

We will further represent the field wave function as a four-component spinor function of four-component spinor coordinates

$$\boldsymbol{\psi}(x_0, x_1, x_2, x_3) = \begin{pmatrix} \psi_0(x_0, x_1, x_2, x_3) \\ \psi_1(x_0, x_1, x_2, x_3) \\ \psi_2(x_0, x_1, x_2, x_3) \\ \psi_3(x_0, x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \varphi(x_0, x_1, x_2, x_3)$$

where the coefficients u_μ are complex quantities independent of coordinates. In fact, as shown at the end of the paper, the wave function is a linear combination of such right-hand sides with operator coefficients.

We operate within the framework of classical concepts of quantum mechanics, simply using coordinate and momentum representations in spinor space instead of coordinate and momentum representations in vector space to describe the state of a physical system. Both types of representations are equally valid, and there is no need to express the wave function in one representation through the wave function in the other; both wave functions equally describe the same physical state. Moreover, since vector coordinates and momenta can be easily expressed in terms of their spinor analogues, we would give priority to spinor representations as more fundamental.

Let us summarize the relations between quantum-mechanical quantities for the spinor space

$$\mathbf{x}^\mathbf{T} \equiv (x_0, x_1, x_2, x_3) \qquad \hat{\mathbf{x}}^\mathbf{T} \equiv (\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$$
$$\mathbf{p}^\mathbf{T} \equiv (p_0, p_1, p_2, p_3) \qquad \hat{\mathbf{p}}^\mathbf{T} \equiv (\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3)$$
$$(\mathbf{p}, \mathbf{x}) = \overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1$$
$$\overline{(\mathbf{p}, \mathbf{x})} = p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{1}$$

The complete orthonormalized system of eigenvectors of the momentum operator

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$
$$\hat{p}_\alpha|\mathbf{p}\rangle = p_\alpha|\mathbf{p}\rangle$$
$$\langle\mathbf{p}|\mathbf{p}'\rangle = (2\pi)^4\delta(\mathbf{p} - \mathbf{p}')$$
$$\int \frac{d^4p}{(2\pi)^4} |\mathbf{p}\rangle_{(\mathbf{x})} \langle\mathbf{p}|_{(\mathbf{x}')} = \mathbb{1}_{(\mathbf{x})(\mathbf{x}')}$$
$$\hat{\mathbf{p}}_{(\mathbf{x})(\mathbf{x}')} = \int \frac{d^4p}{(2\pi)^4} |\mathbf{p}\rangle_{(\mathbf{x})}\mathbf{p}\langle\mathbf{p}|_{(\mathbf{x}')}$$
$$\hat{p}_{\alpha(\mathbf{x})(\mathbf{x}')} = \int \frac{d^4p}{(2\pi)^4} |\mathbf{p}\rangle_{(\mathbf{x})}p_\alpha\langle\mathbf{p}|_{(\mathbf{x}')}$$
$$|\boldsymbol{\varphi}\rangle = \int \frac{d^4p}{(2\pi)^4} \boldsymbol{\varphi}(\mathbf{p})|\mathbf{p}\rangle$$
$$\boldsymbol{\varphi}(\mathbf{p}) = \langle\mathbf{p}|\boldsymbol{\varphi}\rangle$$

The complete orthonormalized system of eigenvectors of the coordinate operator

$$\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$$
$$\hat{x}_\alpha|\mathbf{x}\rangle = x_\alpha|\mathbf{x}\rangle$$
$$\langle\mathbf{x}|\mathbf{x}'\rangle = \delta(\mathbf{x} - \mathbf{x}')$$
$$\int d^4x\, |\mathbf{x}\rangle_{(\mathbf{p})} \langle\mathbf{x}|_{(\mathbf{p}')} = \mathbb{1}_{(\mathbf{p})(\mathbf{p}')}$$

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$$\hat{\mathbf{x}}_{(\mathbf{p})(\mathbf{p}')} = \int d^4x\, |\mathbf{x}\rangle_{(\mathbf{p})}\mathbf{x}\langle\mathbf{x}|_{(\mathbf{p}')}$$
$$\hat{x}_{\alpha\,(\mathbf{p})(\mathbf{p}')} = \int d^4x\, |\mathbf{x}\rangle_{(\mathbf{p})}x_\alpha\langle\mathbf{x}|_{(\mathbf{p}')}$$
$$|\boldsymbol{\varphi}\rangle = \int d^4x\;\boldsymbol{\varphi}(\mathbf{x})|\mathbf{x}\rangle$$
$$\boldsymbol{\varphi}(\mathbf{x}) = \langle\mathbf{x}|\boldsymbol{\varphi}\rangle$$

The relation between wave function in momentum and coordinate representations and the relation between eigenvectors of the coordinate operator and the momentum operator

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$
$$|\boldsymbol{\varphi}\rangle = \int d^4x\;\boldsymbol{\varphi}(\mathbf{x})|\mathbf{x}\rangle = \int d^4x\left(\int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}\right)|\mathbf{x}\rangle$$
$$= \int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})\left(\int d^4x\;e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}|\mathbf{x}\rangle\right)$$
$$|\boldsymbol{\varphi}\rangle = \int \frac{d^4p}{(2\pi)^4}\;\boldsymbol{\varphi}(\mathbf{p})|\mathbf{p}\rangle$$
$$|\mathbf{p}\rangle = \int d^4x\;e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}|\mathbf{x}\rangle$$
$$\langle\mathbf{x}|\mathbf{p}\rangle = e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$
$$\hat{p}_\alpha|\boldsymbol{\varphi}\rangle = \int d^4x\;\frac{\partial\boldsymbol{\varphi}(\mathbf{x})}{\partial x_\alpha}|\mathbf{x}\rangle$$
$$\hat{x}_\alpha|\boldsymbol{\varphi}\rangle = \int d^4x\;x_\alpha\boldsymbol{\varphi}(\mathbf{x})\,|\mathbf{x}\rangle$$

The wave function in coordinate or momentum spinor representation has the classical interpretation adopted in quantum mechanics; the square of its modulus represents the probability density of a measurable quantity taking a particular eigenvalue.

4. Path Integral and Second Quantization in Spinor Coordinate Space

Based on the above, we can modify the theory of the path integral. We will consider it in the notations in which it is presented in [9]. For a free scalar field with sources J(X) the path integral has the form

$$Z(J) = \int D\varphi(\mathbf{X})\;exp(i\mathcal{S}(\varphi(\mathbf{X}))) = \int D\varphi(\mathbf{X})\;exp\left(i\int d^4X\{\mathcal{L}(\varphi(\mathbf{X})) + J(\mathbf{X})\varphi(\mathbf{X})\}\right)$$
$$= \int D\varphi(\mathbf{X})\;exp\left(i\int d^4X\left\{\frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial X_0}\right)^2 - \left(\frac{\partial\varphi}{\partial X_1}\right)^2 - \left(\frac{\partial\varphi}{\partial X_2}\right)^2 - \left(\frac{\partial\varphi}{\partial X_3}\right)^2 - m^2\varphi(\mathbf{X})^2\right)\right.\right.$$
$$\left.\left. + J(\mathbf{X})\varphi(\mathbf{X})\right\}\right)$$

It includes the action of

$$\mathcal{S}(\varphi(\mathbf{X}) = \int d^4X\{\mathcal{L}(\varphi(\mathbf{X})) + J(\mathbf{X})\varphi(\mathbf{X})\}$$

and the Lagrangian density for the free field

$$\mathcal{L}(\varphi(\mathbf{X}) = \frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial X_0}\right)^2 - \left(\frac{\partial\varphi}{\partial X_1}\right)^2 - \left(\frac{\partial\varphi}{\partial X_2}\right)^2 - \left(\frac{\partial\varphi}{\partial X_3}\right)^2 - m^2\varphi(\mathbf{X})^2\right)$$

For convenience and clarity, the following notations are introduced

$$(\partial\varphi)^2 = \partial_\mu\varphi\partial^\mu\varphi = \eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = (\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2 - (\partial_3\varphi)^2$$
$$= \left(\frac{\partial\varphi}{\partial X_0}\right)^2 - \left(\frac{\partial\varphi}{\partial X_1}\right)^2 - \left(\frac{\partial\varphi}{\partial X_2}\right)^2 - \left(\frac{\partial\varphi}{\partial X_3}\right)^2$$
$$\partial_\mu \equiv \frac{\partial}{\partial X_\mu}$$

For the general case the Lagrangian density has the form

$$\mathcal{L}(\varphi(\mathbf{X})) = \frac{1}{2}(\partial\varphi(\mathbf{X}))^2 - V(\varphi(\mathbf{X}))$$

where $V(\varphi(\mathbf{X}))$-polynomial over the field $\varphi(\mathbf{X})$.

Substituting the Lagrangian density into the Euler equation

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)} - \frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

the field equation of motion is obtained.

The free field theory is developed for a special kind of polynomial

$$V\big(\varphi(X)\big) = \frac{1}{2}m^2\varphi^2$$

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]$$

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)} = \frac{1}{2}\frac{\delta(\partial\varphi)^2}{\delta(\partial_\mu\varphi)} = \frac{1}{2}\frac{\delta[(\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2 - (\partial_3\varphi)^2]}{\delta(\partial_\mu\varphi)} = \pm\frac{1}{2}\frac{\delta\big(\partial_\mu\varphi\big)^2}{\delta(\partial_\mu\varphi)} = \pm\partial_\mu\varphi$$

$$\frac{\delta\mathcal{L}}{\delta\varphi} = \frac{1}{2}\left[-m^2\frac{\delta\varphi^2}{\delta\varphi}\right] = -m^2\varphi$$

In summary, Euler's equation defines the equation of motion

$$\partial_0(\partial_0\varphi) - \partial_0(\partial_0\varphi) - \partial_0(\partial_0\varphi) - \partial_0(\partial_0\varphi) + m^2\varphi = 0$$

$$\partial_0^2\varphi - \partial_1^2\varphi - \partial_2^2\varphi - \partial_3^2\varphi + m^2\varphi = 0$$

$$\partial^2\varphi \equiv \partial_0^2\varphi - \partial_1^2\varphi - \partial_2^2\varphi - \partial_3^2\varphi$$

$$\partial^2\varphi + m^2\varphi = 0$$

$$(\partial^2 + m^2)\varphi = 0$$

The notations used here are

$$\partial^2\varphi \equiv \partial_0^2\varphi - \partial_1^2\varphi - \partial_2^2\varphi - \partial_3^2\varphi$$

$$\partial^2 \equiv \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$$

Thus, there is a correspondence of the Lagrangian density and the equation of motion for the free field

$$\mathcal{L}\big(\varphi(\mathbf{X})\big) = \frac{1}{2}\Big[\big(\partial_0\varphi(\mathbf{X})\big)^2 - \big(\partial_1\varphi(\mathbf{X})\big)^2 - \big(\partial_2\varphi(\mathbf{X})\big)^2 - \big(\partial_3\varphi(\mathbf{X})\big)^2 - m^2\varphi(\mathbf{X})^2\Big]$$

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]$$

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2 - (\partial_3\varphi)^2 - m^2\varphi^2]$$

$$\partial_0^2\varphi(\mathbf{X}) - \partial_1^2\varphi(\mathbf{X}) - \partial_2^2\varphi(\mathbf{X}) - \partial_3^2\varphi(\mathbf{X}) + m^2\varphi(\mathbf{X}) = 0$$

Our proposal is to replace the Lagrangian density in vector coordinate space by the Lagrangian density in spinor coordinate space. For this purpose, we use the equation of motion in spinor coordinate space and we want to find the Lagrangian density for which the Euler equation defines this equation of motion

$$\left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial \overline{x_0}} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial \overline{x_1}}\right)\varphi(\mathbf{x}) = -m\varphi(\mathbf{x})$$

$$\left(\partial_2\overline{\partial_0} + \partial_3\overline{\partial_1}\right)\varphi(\mathbf{x}) + m\varphi(\mathbf{x}) = 0$$

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi(\mathbf{x}))} - \frac{\delta\mathcal{L}}{\delta\varphi(\mathbf{x})} = 0$$

For the sake of clarity, we use the same notation for the spinor coordinate derivative as for the vector coordinate derivative; the context allows us to distinguish between them

$$\partial_\mu \equiv \frac{\partial}{\partial x_\mu} \qquad \overline{\partial_\mu} \equiv \frac{\partial}{\partial \overline{x_\mu}}$$

Let us write the Lagrangian density plus sources in the form

$$\mathcal{L}\big(\varphi(\mathbf{x})\big) = \frac{1}{2}\left[\partial_2\varphi(\mathbf{x})\overline{\partial_0}\varphi(\mathbf{x}) + \partial_3\varphi(\mathbf{x})\overline{\partial_1}\varphi(\mathbf{x})\right] - V\big(\varphi(\mathbf{x})\big) + j(\mathbf{x})\varphi(\mathbf{x})$$

And let's substitute the Lagrangian density into the Euler equation

$$\partial_0\frac{\delta\mathcal{L}}{\delta(\partial_0)} + \partial_1\frac{\delta\mathcal{L}}{\delta(\partial_1)} + \partial_2\frac{\delta\mathcal{L}}{\delta(\partial_2)} + \partial_3\frac{\delta\mathcal{L}}{\delta(\partial_3)} +$$

$$\overline{\partial_0}\frac{\delta\mathcal{L}}{\delta\big(\overline{\partial_0}\big)} + \overline{\partial_1}\frac{\delta\mathcal{L}}{\delta\big(\overline{\partial_1}\big)} + \overline{\partial_2}\frac{\delta\mathcal{L}}{\delta\big(\overline{\partial_2}\big)} + \overline{\partial_3}\frac{\delta\mathcal{L}}{\delta\big(\overline{\partial_3}\big)} - \frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

$$\frac{1}{2}\left[\partial_2\left(\overline{\partial_0}\varphi(\mathbf{x})\right) + \partial_3\left(\overline{\partial_1}\varphi(\mathbf{x})\right) + \overline{\partial_0}\big(\partial_2\varphi(\mathbf{x})\big) + \overline{\partial_1}\big(\partial_3\varphi(\mathbf{x})\big)\right] - \frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

For the case of a free field the derivative operators commute, so we can write

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$$(\partial_2\overline{\partial_0} + \partial_3\overline{\partial_1})\varphi(\mathbf{x}) - \left(\frac{\delta\mathcal{L}}{\delta\varphi}\right) = 0$$
$$(\partial_2\overline{\partial_0} + \partial_3\overline{\partial_1})\varphi(\mathbf{x}) - \left(\frac{\delta V(\varphi)}{\delta\varphi}\right) = 0$$

It is pleasant that the Euler equation in invariant form works also in this situation, so that we obtain the desired form of the equation of motion in the spinor coordinate space. It is important that the proposed Lagrangian density has a relativistically invariant form, even in the general case, and not only at commuting derivatives.

The polynomial has the form
$$V(\varphi) = \frac{1}{2}m\varphi(\mathbf{x})^2 + \frac{g}{3!}\varphi(\mathbf{x})^3 + \frac{\lambda}{4!}\varphi(\mathbf{x})^4 + \cdots$$
In the case of a free field we restrict ourselves to the first term of the polynomial
$$V(\varphi) = \frac{1}{2}\,m\varphi(\mathbf{x})^2$$
Then the Lagrangian density and the equation of motion for the scalar field in spinor coordinate space have the form
$$\mathcal{L}\big(\varphi(\mathbf{x})\big) = \frac{1}{2}\big[\partial_2\varphi(\mathbf{x})\overline{\partial_0}\varphi(\mathbf{x}) + \partial_3\varphi(\mathbf{x})\overline{\partial_1}\varphi(\mathbf{x})\big] - \frac{1}{2}\,m\varphi(\mathbf{x})^2$$
$$(\partial_2\overline{\partial_0} + \partial_3\overline{\partial_1})\varphi(\mathbf{x}) + m\varphi(\mathbf{x}) = 0$$
Now we have to find the path integral, which, along with the Lagrangian density, includes the sources
$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\{\mathcal{L}(\varphi(\mathbf{x})) + j(\mathbf{x})\varphi(\mathbf{x})\}\right)$$
$$= \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{\frac{1}{2}[\partial_1\varphi(\mathbf{x})\partial_2\varphi(\mathbf{x}) - \partial_0\varphi(\mathbf{x})\partial_3\varphi(\mathbf{x}))] - \frac{1}{2}m\varphi(\mathbf{x})^2 + j(\mathbf{x})\varphi(\mathbf{x})\right\}\right)$$
The components of spinors are complex, and we have already noted that the derivatives on complex variables are applied to the degree functions, which, most likely, can describe physical fields, respectively, the finding of an indefinite integral for the function of a complex variable can be treated similarly, i.e. as an indefinite integral from the degree function.

It is possible to recover Planck's constant, which provides a transition to the classical limit
$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(\frac{i}{\hbar}\int d^4x\mathcal{L}(\varphi(\mathbf{x})\right)$$
One of the steps in computing the path integral in [9] is to find the free propagator from equation
$$-(\partial^2 + m^2)D(\mathbf{X}-\mathbf{Y}) = \delta(\mathbf{X}-\mathbf{Y})$$
the solution of which has the form
$$D(\mathbf{X}-\mathbf{Y}) = \int \frac{d^4P}{(2\pi)^4}\frac{e^{i\mathbf{P}(\mathbf{X}-\mathbf{Y})}}{P^2 - m^2 + i\varepsilon}$$
herewith
$$\delta(\mathbf{X}-\mathbf{Y}) = \int \frac{d^4P}{(2\pi)^4}e^{i\mathbf{P}(\mathbf{X}-\mathbf{Y})}$$
In our case, we want to find
$$Z(j) =$$
$$= \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{\frac{1}{2}[\partial_1\varphi(\mathbf{x})\partial_2\varphi(\mathbf{x}) - \partial_0\varphi(\mathbf{x})\partial_3\varphi(\mathbf{x}))] - \frac{1}{2}m\varphi(\mathbf{x})^2 + j(\mathbf{x})\varphi(\mathbf{x})\right\}\right)$$
After integration by parts by analogy with ([9], Chapter 1.3) we obtain for the special case of a free field
$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{-\frac{1}{2}\varphi(\mathbf{x})\big[(\partial_2\overline{\partial_0} + \partial_3\overline{\partial_1}) + m\big]\varphi(\mathbf{x}) + j(\mathbf{x})\varphi(\mathbf{x})\right\}\right)$$
In the process of calculation, it is necessary to find the solution of the equation
$$-\big(\partial_2\overline{\partial_0} + \partial_3\overline{\partial_1} + m\big)D(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$
For this purpose, we pass to the momentum space by means of the integral transformation
$$\varphi(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\,\varphi(\mathbf{p})e^{i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})})}$$

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The assumed propagator has the form

$$D(\mathbf{x}-\mathbf{y}) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{i\left(\overline{p_0}(x_2-y_2)+\overline{p_1}(x_3-y_3)+\overline{p_2}(x_0-y_0)+\overline{p_3}(x_1-y_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(\overline{p_0}p_2+\overline{p_1}p_3)-m}$$

which is verified by substitution into Eq. Here it is assumed that the representation of the delta function

$$\delta(\mathbf{x}-\mathbf{y}) = \int \frac{d^4p}{(2\pi)^4} e^{i\left(\overline{p_0}(x_2-y_2)+\overline{p_1}(x_3-y_3)+\overline{p_2}(x_0-y_0)+\overline{p_3}(x_1-y_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}$$

We include a conjugate phase to the exponent

$$\overline{(\mathbf{p},\mathbf{x})} = p_0\overline{x_2} + p_1\overline{x_3} + p_2\overline{x_0} + p_3\overline{x_1}$$

Which, on the one hand, provides convergence of the integral, and on the other hand, it does not affect the result of calculating those derivatives from the lists ∂_μ and $\overline{\partial_\mu}$ that are included in this particular equation.

We note at once that there is no simple correspondence between the so defined phase of a plane wave in spinor space and the phase of a plane wave in vector space, but both parts of the inequality are invariant under Lorentz transformations.

One can see the difference between the propagators, since in one case m^2 is real and positive, while in spinor space m is complex in general. We can use the approximate equality

$$\frac{1}{(\overline{p_0}p_2+\overline{p_1}p_3)-m} = \frac{(\overline{p_2}p_0+\overline{p_3}p_1)+\overline{m}}{\left((\overline{p_0}p_2+\overline{p_1}p_3)-m\ \right)\left((\overline{p_2}p_0+\overline{p_3}p_1)+\overline{m}\ \right)} =$$

$$= \frac{(\overline{p_2}p_0+\overline{p_3}p_1)+\overline{m}}{(\overline{p_0}p_2+\overline{p_1}p_3)(\overline{p_2}p_0+\overline{p_3}p_1)+(\overline{p_0}p_2+\overline{p_1}p_3)\overline{m}-m(\overline{p_2}p_0+\overline{p_3}p_1)-m^2} \cong$$

$$\cong \frac{(\overline{p_2}p_0+\overline{p_3}p_1)+\overline{m}}{(\overline{p_0}p_2+\overline{p_1}p_3)(\overline{p_2}p_0+\overline{p_3}p_1)-m^2} = \frac{(\overline{p_2}p_0+\overline{p_3}p_1)+\overline{m}}{P^2-m^2}$$

in which it is taken into account that

$$P^2 = {P_0}^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 = m^2 = m\overline{m} = (\overline{p_0}p_2+\overline{p_1}p_3)(\overline{p_2}p_0+\overline{p_3}p_1)$$

where

$$P^2 \equiv P_0^2 - P_1^2 - P_2^2 - P_3^2$$

Now the propagator has the form

$$D(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{(\overline{p_2}p_0+\overline{p_3}p_1)+\overline{m}}{P^2-m^2} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

What are the advantages of the transition from path integral in vector space to path integral in spinor space? A possible answer is that there are new conditions for working with divergent integrals. Now integration is performed over spinor space, so that in the numerator there is a four-dimensional differential element d^4p instead of element d^4P in the case of vector space. The spinor element has the order of magnitude P^2 instead of P^4 for the vector element, whish decreases the order of magnitude of the numerator, while the order of magnitude of the denominator does not change.

If the spinor coordinate space is indeed more fundamental, and the vector coordinate space is an offspring of it, then we ma y benefit from this transition in any case.

Now let us move from the scalar field to the field of an electron, that is, the field of a particle with half-integer spin. We will use gamma matrices in the Weyl basis

$$\gamma_0^V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_1^V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2^V = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3^V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let us consider the Dirac equation

$$\left(i\gamma_0^V \frac{\partial}{\partial X_0} - i\gamma_0^V \frac{\partial}{\partial X_0} - i\gamma_0^V \frac{\partial}{\partial X_0} - i\gamma_0^V \frac{\partial}{\partial X_0} - m\right)\boldsymbol{\varphi}(\mathbf{X}) = 0$$

Taking into account the substitution

$$P_0 \to i\frac{\partial}{\partial X_0} \qquad P_1 \to -i\frac{\partial}{\partial X_1} \qquad P_2 \to -i\frac{\partial}{\partial X_2} \qquad P_3 \to -i\frac{\partial}{\partial X_3}$$

we can record

$$(\gamma_0^V P_0 + \gamma_1^V P_1 + \gamma_2^V P_2 + \gamma_3^V P_3 - m)\boldsymbol{\varphi}(\mathbf{X}) = 0$$

Let us substitute the expressions of the vector components through the components of the momentum spinor

$$P_0 - P_3 = \overline{p_0}p_0 + \overline{p_3}p_3$$
$$P_0 + P_3 = \overline{p_1}p_1 + \overline{p_2}p_2$$
$$-P_1 + iP_2 = \overline{p_1}p_0 - \overline{p_3}p_2$$
$$-P_1 - iP_2 = \overline{p_0}p_1 - \overline{p_2}p_3$$
$$\gamma_0^V P_0 + \gamma_1^V P_1 + \gamma_2^V P_2 + \gamma_3^V P_3 =$$
$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} P_0 + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} P_1 +$$
$$+ \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} P_2 + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} P_3 =$$
$$= \begin{pmatrix} 0 & 0 & P_0 + P_3 & P_1 - iP_2 \\ 0 & 0 & P_1 + iP_2 & P_0 - P_3 \\ P_0 - P_3 & -P_1 + iP_2 & 0 & 0 \\ -P_1 - iP_2 & P_0 + P_3 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 0 & \overline{p_1}p_1 + \overline{p_2}p_2 & -\overline{p_1}p_0 + \overline{p_3}p_2 \\ 0 & 0 & -\overline{p_0}p_1 + \overline{p_2}p_3 & \overline{p_0}p_0 + \overline{p_3}p_3 \\ \overline{p_0}p_0 + \overline{p_3}p_3 & \overline{p_1}p_0 - \overline{p_3}p_2 & 0 & 0 \\ \overline{p_0}p_1 - \overline{p_2}p_3 & \overline{p_1}p_1 + \overline{p_2}p_2 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 0 & \overline{p_1}p_1 & -\overline{p_1}p_0 \\ 0 & 0 & -\overline{p_0}p_1 & \overline{p_0}p_0 \\ \overline{p_0}p_0 & \overline{p_1}p_0 & 0 & 0 \\ \overline{p_0}p_1 & \overline{p_1}p_1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \overline{p_2}p_2 & \overline{p_3}p_2 \\ 0 & 0 & \overline{p_2}p_3 & \overline{p_3}p_3 \\ \overline{p_3}p_3 & -\overline{p_3}p_2 & 0 & 0 \\ -\overline{p_2}p_3 & \overline{p_2}p_2 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 0 & \overline{p_1}p_1 & -\overline{p_1}p_0 \\ 0 & 0 & -\overline{p_0}p_1 & \overline{p_0}p_0 \\ p_0\overline{p_0} - [p_0\overline{p_0} - \overline{p_0}p_0] & p_0\overline{p_1} - [p_0\overline{p_1} - \overline{p_1}p_0] & 0 & 0 \\ p_1\overline{p_0} - [p_1\overline{p_0} - \overline{p_0}p_1] & p_1\overline{p_1} - [p_1\overline{p_1} - \overline{p_1}p_1] & 0 & 0 \end{pmatrix} +$$
$$+ \begin{pmatrix} 0 & 0 & p_2\overline{p_2} - [p_2\overline{p_2} - \overline{p_2}p_2] & p_2\overline{p_3} - [p_2\overline{p_3} - \overline{p_3}p_2] \\ 0 & 0 & p_3\overline{p_2} - [p_3\overline{p_2} - \overline{p_2}p_3] & p_3\overline{p_3} - [p_3\overline{p_3} - \overline{p_3}p_3] \\ \overline{p_3}p_3 & -\overline{p_3}p_2 & 0 & 0 \\ -\overline{p_2}p_3 & \overline{p_2}p_2 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 0 & \overline{p_1}p_1 & -\overline{p_1}p_0 \\ 0 & 0 & -\overline{p_0}p_1 & \overline{p_0}p_0 \\ p_0\overline{p_0} & p_0\overline{p_1} & 0 & 0 \\ p_1\overline{p_0} & p_1\overline{p_1} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & p_2\overline{p_2} & p_2\overline{p_3} \\ 0 & 0 & p_3\overline{p_2} & p_3\overline{p_3} \\ \overline{p_3}p_3 & -\overline{p_3}p_2 & 0 & 0 \\ -\overline{p_2}p_3 & \overline{p_2}p_2 & 0 & 0 \end{pmatrix} +$$
$$- \begin{pmatrix} 0 & 0 & [p_2\overline{p_2} - \overline{p_2}p_2] & [p_2\overline{p_3} - \overline{p_3}p_2] \\ 0 & 0 & [p_3\overline{p_2} - \overline{p_2}p_3] & [p_3\overline{p_3} - \overline{p_3}p_3] \\ [p_0\overline{p_0} - \overline{p_0}p_0] & [p_0\overline{p_1} - \overline{p_1}p_0] & 0 & 0 \\ [p_1\overline{p_0} - \overline{p_0}p_1] & [p_1\overline{p_1} - \overline{p_1}p_1] & 0 & 0 \end{pmatrix} =$$
$$\equiv S^V(\mathbf{p}) - K^V(\mathbf{p})$$

Let us represent the matrix $S^V(\mathbf{p})$ as a sum of direct products of spinors

$$S^V(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ p_0 \\ p_1 \end{pmatrix} (\overline{p_0}, \overline{p_1}, 0, 0) + \begin{pmatrix} \overline{p_1} \\ -\overline{p_0} \\ 0 \\ 0 \end{pmatrix} (0, 0, p_1, -p_0) + \begin{pmatrix} 0 \\ 0 \\ \overline{p_3} \\ -\overline{p_2} \end{pmatrix} (p_3, -p_2, 0, 0) + \begin{pmatrix} p_2 \\ p_3 \\ 0 \\ 0 \end{pmatrix} (0, 0, \overline{p_2}, \overline{p_3})$$

For a free field the components of the momentum spinor commute, therefore

$$\gamma_0^V P_0 + \gamma_1^V P_1 + \gamma_2^V P_2 + \gamma_3^V P_3 = S^V(\mathbf{p})$$

Complex mass

$$m = (\overline{p_0}p_2 + \overline{p_1}p_3)$$
$$\overline{m} = (\overline{p_2}p_0 + \overline{p_3}p_1)$$

does not change at rotations and boosts for an arbitrary complex spinor. Moreover, by a direct check it is possible to check that for an arbitrary spinor

$$S^V(\mathbf{p})S^V(\mathbf{p}) = m^2 I$$

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For a free field, when all components of the momentum spinor commute, we can write the relativistic equation of motion of the fermionic field

$$S^V(\mathbf{p})S^V(\mathbf{p})\boldsymbol{\varphi}(\mathbf{x}) = m^2 I\boldsymbol{\varphi}(\mathbf{x})$$

where the matrix of derivatives S^V is obtained from the matrix $S^V(\mathbf{p})$ by substitutions

$$p_1 \to i\overline{\partial_3} \qquad p_0 \to i\overline{\partial_2} \qquad p_3 \to i\overline{\partial_1} \qquad p_2 \to i\overline{\partial_0}$$
$$\overline{p_1} \to i\partial_3 \qquad \overline{p_0} \to i\partial_2 \qquad \overline{p_3} \to i\partial_1 \qquad \overline{p_2} \to i\partial_0$$
$$\overline{\partial_\mu}\varphi(\mathbf{x}) \equiv \frac{\partial\varphi(\mathbf{x})}{\partial\overline{x_\mu}}$$
$$-S^V = \begin{pmatrix} 0 \\ 0 \\ \overline{\partial_2} \\ \overline{\partial_3} \end{pmatrix}(\partial_2, \partial_3, 0,0) + \begin{pmatrix} \partial_3 \\ -\partial_2 \\ 0 \\ 0 \end{pmatrix}\left(0,0, \overline{\partial_3}, -\overline{\partial_2}\right) + \begin{pmatrix} 0 \\ 0 \\ \partial_1 \\ -\partial_0 \end{pmatrix}\left(\overline{\partial_1}, -\overline{\partial_0}, 0,0\right) + \begin{pmatrix} \overline{\partial_0} \\ \overline{\partial_1} \\ 0 \\ 0 \end{pmatrix}(0,0, \partial_0, \partial_1)$$

Using the transformations performed, we obtained the Dirac equation for the wave function in the spinor coordinate representation, as opposed to the traditional form for the wave function in the vector coordinate representation

$$(\gamma_0^V P_0 + \gamma_1^V P_1 + \gamma_2^V P_2 + \gamma_3^V P_3 - m)\boldsymbol{\varphi}(\mathbf{X}) = 0$$
$$\left(i\gamma_0^V \frac{\partial}{\partial X_0} - i\gamma_0^V \frac{\partial}{\partial X_0} - i\gamma_0^V \frac{\partial}{\partial X_0} - i\gamma_0^V \frac{\partial}{\partial X_0} - m\right)\boldsymbol{\varphi}(\mathbf{X}) = 0$$
$$(S^V(\mathbf{p}) - mI)\boldsymbol{\varphi}(\mathbf{x}) = 0$$
$$(S^V + mI)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

We again want to find the path integral

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\{\mathcal{L}(\varphi(\mathbf{x})) + j(\mathbf{x})\varphi(\mathbf{x})\}\right)$$

For this, we need the Lagrangian density, from which, by means of the Euler equation, the equation of motion is derived.

It is proposed to use the density of the Lagrangian

$$\mathcal{L} = \frac{1}{2}\boldsymbol{\varphi}(\mathbf{x})^T S^V \boldsymbol{\varphi}(\mathbf{x}) - \frac{1}{2}m\boldsymbol{\varphi}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x})$$

Let us substitute the Lagrangian density into the Euler equation and obtain the equation of motion. Since the Lagrangian density includes, along with the derivatives of ∂_μ, the derivatives of $\overline{\partial_\mu}$, it is logical to use a different definition of Euler's equation

$$\partial_0 \frac{\delta\mathcal{L}}{\delta(\partial_0)} + \overline{\partial_0}\frac{\delta\mathcal{L}}{\delta(\overline{\partial_0})} + \partial_1\frac{\delta\mathcal{L}}{\delta(\partial_1)} + \overline{\partial_1}\frac{\delta\mathcal{L}}{\delta(\overline{\partial_1})} +$$
$$+\partial_2\frac{\delta\mathcal{L}}{\delta(\partial_2)} + \overline{\partial_2}\frac{\delta\mathcal{L}}{\delta(\overline{\partial_2})} + \partial_3\frac{\delta\mathcal{L}}{\delta(\partial_3)} + \overline{\partial_3}\frac{\delta\mathcal{L}}{\delta(\overline{\partial_3})} - \frac{\delta\mathcal{L}}{\delta\varphi} = 0$$

Then for the free field case when the derivative operators commute with each other, we obtain the equation of motion

$$S^V\boldsymbol{\varphi}(\mathbf{x}) + m\boldsymbol{\varphi}(\mathbf{x}) = 0$$

If to follow the invariance principle strictly, we should start from the product of two matrices, i.e. to use the Lagrangian density

$$\mathcal{L} = \frac{1}{2}[\boldsymbol{\varphi}(\mathbf{x})^T S^V S^V \boldsymbol{\varphi}(\mathbf{x}) - m^2\boldsymbol{\varphi}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x})]$$

only such a product remains unchanged under Lorentz transformations; a single matrix does not possess this property. It is in this sense that the Dirac equation cannot be considered invariant.

Nevertheless, further we will search for the path integral in the simplest case with the originally proposed Lagrangian density and in addition assume commutativity of all derivative operators

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{\frac{1}{2}\boldsymbol{\varphi}(\mathbf{x})^T S^V\boldsymbol{\varphi}(\mathbf{x}) - \frac{1}{2}m\boldsymbol{\varphi}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x}) + \mathbf{j}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x})\right\}\right)$$

After integration by parts, we presumably obtain

$$Z(j) = \int D\varphi(\mathbf{x})\, exp\left(i\int d^4x\left\{-\frac{1}{2}\boldsymbol{\varphi}(\mathbf{x})^T[S^V + mI]\boldsymbol{\varphi}(\mathbf{x}) + j(\mathbf{x})\boldsymbol{\varphi}(\mathbf{x})\right\}\right)$$

Then it is necessary to find the solution of the equation

$$-(S^V + mI)\mathbf{D}(\mathbf{x}) = I\delta(\mathbf{x})$$

For this purpose, we pass to the momentum space by means of the integral transformation

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$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p}) e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

We get the equation

$$(S^V(\mathbf{p}) - mI)D^V(\mathbf{p}) = I$$

with the decision

$$D^V(\mathbf{p}) = \frac{S^V(\mathbf{p}) + \overline{m}I}{P^2 - \overline{m}m}$$

Indeed

$$\frac{(S^V(\mathbf{p}) - mI)(S^V(\mathbf{p}) + \overline{m}I)}{P^2 - \overline{m}m} = \frac{(P^2 - \overline{m}m)I}{P^2 - \overline{m}m} = I$$

Here we use the equality, which is valid for an arbitrary complex spinor $\mathbf{p}$

$$(S^V(\mathbf{p}) - mI)(S^V(\mathbf{p}) + \overline{m}I) = P^2I - (m - \overline{m})S^V(\mathbf{p}) - \overline{m}mI = (P^2 - m^2)I$$

$$P^2 = P_0^2 - P_1^2 - P_2^2 - P_3^2$$

It is based on the ratio

$$\overline{m}m = P_0^2 - P_1^2 - P_2^2 - P_3^2$$

it is also taken into account that we consider fermions whose mass is real. As a result, the propagator has the form

$$D^V(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{S^V(\mathbf{p}) + \overline{m}I}{P^2 - \overline{m}m} e^{i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})})}$$

here we assume the validity of the relation

$$\delta(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, e^{i(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})})}$$

Let's return to the question about the use of completely relativistically invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2}[\boldsymbol{\varphi}(\mathbf{x})^T S^V S^V \boldsymbol{\varphi}(\mathbf{x}) - m^2 \boldsymbol{\varphi}(\mathbf{x})^T \boldsymbol{\varphi}(\mathbf{x})]$$

We would like to find a matrix such that, under Lorentz transformations, not only its square is invariant, but the matrix itself remains unchanged. Developing the idea of invariance, we pass to the set of reference spinors with wider filling, but continuing to form matrices possessing the invariance property

$$\mathbf{u1} = \begin{pmatrix} \overline{p_3} \\ -\overline{p_2} \\ p_0 \\ p_1 \end{pmatrix} \quad \mathbf{u2} = \begin{pmatrix} \overline{p_2} \\ \overline{p_3} \\ p_1 \\ -p_0 \end{pmatrix} \quad \mathbf{u3} = \begin{pmatrix} -p_0 \\ -p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix} \quad \mathbf{u4} = \begin{pmatrix} p_1 \\ -p_0 \\ \overline{p_2} \\ \overline{p_3} \end{pmatrix}$$

$$\mathbf{v1} = \begin{pmatrix} p_0 \\ p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix} \quad \mathbf{v2} = \begin{pmatrix} p_1 \\ -p_0 \\ -\overline{p_2} \\ -\overline{p_3} \end{pmatrix} \quad \mathbf{v3} = \begin{pmatrix} -\overline{p_3} \\ \overline{p_2} \\ p_0 \\ p_1 \end{pmatrix} \quad \mathbf{v4} = \begin{pmatrix} \overline{p_2} \\ \overline{p_3} \\ -p_1 \\ p_0 \end{pmatrix}$$

Let's define the matrix

$$S^R(\mathbf{p}) = \mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}) + \mathbf{v1}(\mathbf{p})\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{v3}(\mathbf{p})\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p}) =$$

$$= \begin{pmatrix} \overline{p_3} \\ -\overline{p_2} \\ p_0 \\ p_1 \end{pmatrix}(p_1, -p_0, \overline{p_2}, \overline{p_3}) - \begin{pmatrix} -p_0 \\ -p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix}(\overline{p_2}, \overline{p_3}, p_1, -p_0) +$$

$$+ \begin{pmatrix} p_0 \\ p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix}(\overline{p_2}, \overline{p_3}, -p_1, p_0) - \begin{pmatrix} -\overline{p_3} \\ \overline{p_2} \\ p_0 \\ p_1 \end{pmatrix}(-p_1, p_0, \overline{p_2}, \overline{p_3}) =$$

$$= 2\begin{pmatrix} \overline{p_2}p_0 + \overline{p_3}p_1 & 0 & 0 & 0 \\ 0 & \overline{p_2}p_0 + \overline{p_3}p_1 & 0 & 0 \\ 0 & 0 & \overline{p_2}p_0 + \overline{p_3}p_1 & 0 \\ 0 & 0 & 0 & \overline{p_2}p_0 + \overline{p_3}p_1 \end{pmatrix} =$$

$$= 2\begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & \overline{m} & 0 \\ 0 & 0 & 0 & \overline{m} \end{pmatrix} = 2\overline{m}I$$

and matrix

$$S_R(\mathbf{p}) = S^R(\mathbf{p})^T = \mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{v2}(\mathbf{p})\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p})$$

This matrix $\overline{S^R(\mathbf{p})}$ does not change at rotations and boosts, so it can be stated that the equation of motion in the form of

$$\left(\overline{S^R} + 2mI\right)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

in which the matrix of derivatives S^R is obtained from the matrix $S^R(\mathbf{p})$ by substitution

$$\begin{matrix} p_1 \to i\overline{\partial_3} & p_0 \to i\overline{\partial_2} & p_3 \to i\overline{\partial_1} & p_2 \to i\overline{\partial_0} \\ \overline{p_1} \to i\partial_3 & \overline{p_0} \to i\partial_2 & \overline{p_3} \to i\partial_1 & \overline{p_2} \to i\partial_0 \end{matrix}$$

$$-S^R = \begin{pmatrix} \partial_1 \\ -\partial_0 \\ \overline{\partial_2} \\ \overline{\partial_3} \end{pmatrix}\left(\overline{\partial_3}, -\overline{\partial_2}, \partial_0, \partial_1\right) - \begin{pmatrix} \overline{-\partial_2} \\ -\partial_1 \\ -\partial_1 \\ \partial_0 \end{pmatrix}\left(\partial_0, \partial_1, \partial_1, \overline{-\partial_2}\right) +$$

$$+\begin{pmatrix} \overline{\partial_2} \\ \overline{\partial_3} \\ -\partial_1 \\ \partial_0 \end{pmatrix}\left(\overline{p_2}, \partial_1, -\overline{\partial_3}, \overline{\partial_2}\right) - \begin{pmatrix} -\partial_1 \\ \overline{p_2} \\ \overline{\partial_2} \\ \overline{\partial_3} \end{pmatrix}\left(\overline{-\partial_3}, \overline{\partial_2}, \partial_0, \partial_1\right)$$

is truly relativistically invariant equation. Respectively we can use the invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\boldsymbol{\varphi}(\mathbf{x})^T\overline{S^R}\boldsymbol{\varphi}(\mathbf{x}) - 2mI\boldsymbol{\varphi}(\mathbf{x})^T\boldsymbol{\varphi}(\mathbf{x})\right]$$

to which corresponds the relativistically invariant propagator of the fermion having a real mass, which is negative for the electron and positive for the positron

$$D^R(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{\overline{S^R}(\mathbf{p}) - 2mI}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

Taking into account the ratios

$$\frac{1}{\overline{S^R}(\mathbf{p}) - 2mI} = \frac{S^R(\mathbf{p}) + 2\overline{m}I}{\left(\overline{S^R}(\mathbf{p}) - 2mI\right)\left(S^R(\mathbf{p}) + 2\overline{m}I\ \right)} =$$

$$= \frac{S^R(\mathbf{p}) + 2\overline{m}I}{(\overline{p_0}p_2 + \overline{p_1}p_3)(\overline{p_2}p_0 + \overline{p_3}p_1)I + 2(\overline{p_0}p_2 + \overline{p_1}p_3)\overline{m}I - 2m(\overline{p_2}p_0 + \overline{p_3}p_1)I - 4m^2I} \cong$$

$$\cong \frac{S^R(\mathbf{p}) + 2\overline{m}I}{(\overline{p_0}p_2 + \overline{p_1}p_3)(\overline{p_2}p_0 + \overline{p_3}p_1) - 4m^2I} = \frac{S^R(\mathbf{p}) + 2\overline{m}I}{(P^2 - 4m^2)I}$$

the propagator can be written as

$$D^R(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\frac{S^R(\mathbf{p}) + 2\overline{m}I}{(P^2 - 4m^2)I}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

Let us compare the propagator in spinor space with the propagator of the fermion given in ([9], formula II.2.22 and formula II.5.18)

$$D(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4}\frac{e^{-i\mathbf{PX}}}{\gamma^\mu P_\mu - mI} = \int \frac{d^4P}{(2\pi)^4}\frac{\gamma^\mu P_\mu + mI}{P^2 - m^2}e^{-i\mathbf{PX}}$$

In [9] this formula is obtained by applying the second quantization procedure or using Grassmann integrals. The results are similar, but the integration here is performed in the vector momentum space. The Dirac equation and the corresponding Lagrangian density are not relativistically invariant. Besides, here the mass is considered always real and positive, but then it is not clear how electron and positron differ from the point of view of this formula.

Nevertheless, this fact and the fact of no invariance of the Dirac equation itself do not cancel the value of the second quantization procedure and the final form of the fermion propagator, which allows to make accurate predictions of the experimental results.

We hope that the proposed Lagrangian density for the spinor coordinate space can find application in the calculation of the path integral, but already in the spinor space. Whether such a calculation in spinor space has an advantage over the calculation of the path integral in vector space can be shown by their real comparison.

Let us decompose the fermion field into plane waves with operator coefficients

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

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$$+\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}$$

Let's impose the anticommutation conditions on the operator coefficients

$$\begin{array}{ll}
b_1(\mathbf{p})b_1^*(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & b_1^*(\mathbf{p}')b_1(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}') = \delta(\mathbf{p}'-\mathbf{p}) \\
d_1(\mathbf{p})d_1^*(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & d_1^*(\mathbf{p})d_1(\mathbf{p}') + d_1(\mathbf{p}')d_1^*(\mathbf{p}) = \delta(\mathbf{p}'-\mathbf{p}) \\
d_2(\mathbf{p})d_2^*(\mathbf{p}') + d_2^*(\mathbf{p}')d_2(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & d_2^*(\mathbf{p})d_2(\mathbf{p}') + d_2(\mathbf{p}')d_2^*(\mathbf{p}) = \delta(\mathbf{p}'-\mathbf{p}) \\
b_2(\mathbf{p})b_2^*(\mathbf{p}') + b_2^*(\mathbf{p}')b_2(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & b_2^*(\mathbf{p})b_2(\mathbf{p}') + b_2(\mathbf{p}')b_2^*(\mathbf{p}) = \delta(\mathbf{p}'-\mathbf{p}) \\
d_3(\mathbf{p})d_3^*(\mathbf{p}') + d_3^*(\mathbf{p}')d_3(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & d_3^*(\mathbf{p})d_3(\mathbf{p}') + d_3(\mathbf{p}')d_3^*(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') \\
b_3(\mathbf{p})b_3^*(\mathbf{p}') + b_3^*(\mathbf{p}')b_3(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & b_3^*(\mathbf{p})b_3(\mathbf{p}') + b_3(\mathbf{p}')b_3^*(\mathbf{p}) = \delta(\mathbf{p}'-\mathbf{p}) \\
b_4(\mathbf{p})b_4^*(\mathbf{p}') + b_4^*(\mathbf{p}')b_4(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & b_4^*(\mathbf{p})b_4(\mathbf{p}') + b_4(\mathbf{p}')b_4^*(\mathbf{p}) = \delta(\mathbf{p}'-\mathbf{p}) \\
d_4(\mathbf{p})d_4^*(\mathbf{p}') + d_4^*(\mathbf{p}')d_4(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{p}') & d_4^*(\mathbf{p})d_4(\mathbf{p}') + d_4(\mathbf{p}')d_4^*(\mathbf{p}) = \delta(\mathbf{p}'-\mathbf{p})
\end{array}$$

We consider the rest anticommutators to be equal to zero. Then we can write the expression for the anticommutator of the field

$$\{\varphi_i(\mathbf{x}),\varphi_j(\mathbf{x}')\} = \varphi_i(\mathbf{x})\varphi_j(\mathbf{x}') + \varphi_j(\mathbf{x}')\varphi_i(\mathbf{x}) = \left(\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^T(\mathbf{x}') + \left(\boldsymbol{\varphi}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T\right)_{ij}$$

$$\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^T(\mathbf{x}') + \left(\boldsymbol{\varphi}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T =$$

$$\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$

$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix}$$

$$e^{i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}$$

$$+$$

$$\begin{pmatrix} \begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}(\mathbf{p}') + ib_2(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p}') + b_1(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}(\mathbf{p}') + ib_3(\mathbf{p}')\overline{\mathbf{v2}}(\mathbf{p}') + b_4(\mathbf{p}')\overline{\mathbf{v4}}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + ib_2^*(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}^T(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}^T(\mathbf{p}) \\ +d_4^*(\mathbf{p})\mathbf{v1}^+(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v3}^+(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}^T(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}^T(\mathbf{p}) \end{bmatrix} \\ e^{i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{-i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})} \end{pmatrix}^T$$

$$+$$

$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}$$

$$\begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') + ib_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') + ib_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + b_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix}$$

$$e^{-i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}e^{i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}$$

$$+$$

$$\begin{pmatrix} \begin{bmatrix} b_1^*(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p}') + ib_2^*(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\overline{\mathbf{v1}}(\mathbf{p}') + ib_3^*(\mathbf{p}')\overline{\mathbf{v3}}(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}^T(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}^T(\mathbf{p}) + ib_2(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + b_1(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}^T(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}^T(\mathbf{p}) + ib_3(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) + b_4(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) \end{bmatrix} \\ e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})} \end{pmatrix}^T$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^T(\mathbf{p}') + d_1(\mathbf{p}')d_1^*(\mathbf{p})\left(\mathbf{u1}(\mathbf{p}')\mathbf{u4}^T(\mathbf{p})\right)^T \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^T(\mathbf{p}') - d_2(\mathbf{p}')d_2^*(\mathbf{p})\left(\mathbf{u3}(\mathbf{p}')\mathbf{u2}^T(\mathbf{p})\right)^T + \cdots \\ e^{i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \end{bmatrix} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})\overline{b_1^*(\mathbf{p}')\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') + b_1(\mathbf{p}'b_1^*(\mathbf{p}))\left(\overline{\mathbf{u4}}(\mathbf{p}')\mathbf{u1}^+(\mathbf{p})\right)^T \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') - b_2(\mathbf{p}')b_2^*(\mathbf{p})\left(\overline{\mathbf{u2}}(\mathbf{p}')\mathbf{u3}^+(\mathbf{p})\right)^T + \cdots \\ e^{i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{-i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})} \end{bmatrix} \end{bmatrix}$$

$$
+\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}
\begin{bmatrix}
\begin{bmatrix}
b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u4}^+(\mathbf{p})\right)^{\boldsymbol{T}} \\
-b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}') - b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u2}^+(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots
\end{bmatrix} \\
e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\
+ \\
\begin{bmatrix}
d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u4}(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} \\
-d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}') - d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u2}(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots
\end{bmatrix} \\
e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}
\end{bmatrix}
$$

$$
=\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}
\begin{bmatrix}
\begin{bmatrix}
d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') + d_1(\mathbf{p}')d_1^*(\mathbf{p})\left(\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}')\right) \\
-d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') - d_2(\mathbf{p}')d_2^*(\mathbf{p})\left(\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}')\right) + \cdots
\end{bmatrix} \\
e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\
+ \\
\begin{bmatrix}
b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') + \ b_1\left(\mathbf{p}'b_1^*(\mathbf{p})\right)\left(\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}')\right) \\
-b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') - b_2(\mathbf{p}')b_2^*(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}')\right) + \cdots
\end{bmatrix} \\
e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}
\end{bmatrix}
$$

$$
+\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}
\begin{bmatrix}
\begin{bmatrix}
b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}') + b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}')\right) \\
-b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}') - b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}')\right) + \cdots
\end{bmatrix} \\
e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\
+ \\
\begin{bmatrix}
d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}')\right) \\
-d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}') - d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}')\right) + \cdots
\end{bmatrix} \\
e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}
\end{bmatrix}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}
\begin{bmatrix}
\begin{bmatrix}
\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}) + \cdots \\
-\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}) + \cdots
\end{bmatrix} \\
e^{i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\
+ \\
\begin{bmatrix}
\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) + \cdots \\
-\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + \cdots
\end{bmatrix} \\
e^{-i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
\end{bmatrix}
$$

$$
+\int \frac{d^4p}{(2\pi)^4}
\begin{bmatrix}
\begin{bmatrix}
\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) + \cdots \\
-\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + \cdots
\end{bmatrix} \\
e^{-i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\
+ \\
\begin{bmatrix}
\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) + \cdots \\
-\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + \cdots
\end{bmatrix} \\
e^{i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
\end{bmatrix}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}
\begin{bmatrix}
\begin{bmatrix}
\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}) + \cdots + \\
\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + \cdots +
\end{bmatrix} \\
e^{i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\
+ \\
\begin{bmatrix}
\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) - \overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + \cdots + \\
\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) - \overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + \cdots +
\end{bmatrix} \\
e^{-i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
\end{bmatrix}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}
\begin{bmatrix}
\begin{bmatrix}
\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}) + \mathbf{v1}(\mathbf{p})\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{v3}(\mathbf{p})\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p}) + \\
\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}) - \mathbf{v2}(\mathbf{p})\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p})
\end{bmatrix} \\
e^{i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\
+ \\
\begin{bmatrix}
\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}) - \overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}) + \overline{\mathbf{v4}}(\mathbf{p})\mathbf{v1}^+(\mathbf{p}) - \overline{\mathbf{v2}}(\mathbf{p})\mathbf{v3}^+(\mathbf{p}) + \\
\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) - \overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + \overline{\mathbf{v1}}(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) - \overline{\mathbf{v3}}(\mathbf{p})\mathbf{v2}^+(\mathbf{p})
\end{bmatrix} \\
e^{-i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
\end{bmatrix}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}\left(S^R(\mathbf{p}) + S_R(\mathbf{p})\right)e^{i\left(\overline{p_0}(x_2-x'_2)+\overline{p_1}(x_3-x'_3)+\overline{p_2}(x_0-x'_0)+\overline{p_3}(x_1-x'_1)+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} +
$$

$$\int \frac{d^4p}{(2\pi)^4}\left(\overline{S_R}(\mathbf{p})+\overline{S^R}(\mathbf{p})\right)e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}$$

$$=$$

$$\int \frac{d^4p}{(2\pi)^4}4\begin{pmatrix}\overline{m} & 0 & 0 & 0\\ 0 & \overline{m} & 0 & 0\\ 0 & 0 & \overline{m} & 0\\ 0 & 0 & 0 & \overline{m}\end{pmatrix}e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}+$$

$$\int \frac{d^4p}{(2\pi)^4}4\begin{pmatrix}m & 0 & 0 & 0\\ 0 & m & 0 & 0\\ 0 & 0 & m & 0\\ 0 & 0 & 0 & m\end{pmatrix}e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}$$

$$=4\overline{m}I\delta(\mathbf{x}'-\mathbf{x})+4mI\delta(\mathbf{x}-\mathbf{x}')$$

We will consider this relation as a proof of the anti-symmetry of the fermion wave function under the stipulated anticommutation relations. It is important that all the above deductions are valid in any frame of reference, while the proof of anticommutativity of the fermion field in [9] is carried out for the rest frame.

Let us calculate the total energy of the fermion field

$$E=P_0=\int d^4x\,\boldsymbol{\varphi}^+(\mathbf{x})\boldsymbol{\varphi}(\mathbf{x})$$

$$=\int d^4x\iint\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix}\begin{bmatrix}d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}')-id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}')-ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}')+b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}')\\+d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}')-id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}')-ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}')+b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}')\end{bmatrix}e^{-i\left(\overline{p_0'}x_2+\overline{p_1'}x_3+\overline{p_2'}x_0+\overline{p_3'}x_1+\overline{(\mathbf{p}',\mathbf{x})}\right)}\\+\begin{bmatrix}b_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}')-ib_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}')-id_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}')+d_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}')\\+b_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}')-ib_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}')-id_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}')+d_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}')\end{bmatrix}e^{i\left(\overline{p_0'}x_2+\overline{p_1'}x_3+\overline{p_2'}x_0+\overline{p_3'}x_1+\overline{(\mathbf{p}',\mathbf{x})}\right)}\end{bmatrix}$$

$$\begin{bmatrix}\begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}^T(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}^T(\mathbf{p})+ib_2(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+b_1(\mathbf{p})\mathbf{u4}^+(\mathbf{p})\\+d_4(\mathbf{p})\mathbf{v1}^T(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}^T(\mathbf{p})+ib_3(\mathbf{p})\mathbf{v2}^+(\mathbf{p})+b_4(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\end{bmatrix}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}\\+\begin{bmatrix}b_1^*(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+ib_2^*(\mathbf{p})\mathbf{u3}^+(\mathbf{p})+id_2^*(\mathbf{p})\mathbf{u2}^T(\mathbf{p})+d_1^*(\mathbf{p})\mathbf{u4}^T(\mathbf{p})\\+b_4^*(\mathbf{p})\mathbf{v1}^+(\mathbf{p})+ib_3^*(\mathbf{p})\mathbf{v3}^+(\mathbf{p})+id_3^*(\mathbf{p})\mathbf{v2}^T(\mathbf{p})+d_4^*(\mathbf{p})\mathbf{v4}^T(\mathbf{p})\end{bmatrix}e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}\end{bmatrix}$$

$$=\int d^4x\iint\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix}\begin{bmatrix}d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}')-id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}')-ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}')+b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}')\\+d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}')-id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}')-ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}')+b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}')\end{bmatrix}\\ \begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}(\mathbf{p})+ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p})+b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\\+d_4(\mathbf{p})\mathbf{v1}(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}(\mathbf{p})+ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p})+b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p})\end{bmatrix}\\ e^{-i\left(\overline{p_0'}x_2+\overline{p_1'}x_3+\overline{p_2'}x_0+\overline{p_3'}x_1+\overline{(\mathbf{p}',\mathbf{x})}\right)}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}\\+\begin{bmatrix}b_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}')-ib_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}')-id_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}')+d_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}')\\+b_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}')-ib_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}')-id_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}')+d_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}')\end{bmatrix}\\ \begin{bmatrix}b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p})+ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p})+id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p})+d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p})\\+b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p})+ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p})+id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p})+d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p})\end{bmatrix}\\ e^{i\left(\overline{p_0'}x_2+\overline{p_1'}x_3+\overline{p_2'}x_0+\overline{p_3'}x_1+\overline{(\mathbf{p}',\mathbf{x})}\right)}e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}\end{bmatrix}$$

$$=\iint\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$

$$\begin{bmatrix}\begin{bmatrix}d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}')-id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}')-ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}')+b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}')\\+d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}')-id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}')-ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}')+b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}')\end{bmatrix}\\ \begin{bmatrix}d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})+id_2(\mathbf{p})\mathbf{u3}(\mathbf{p})+ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p})+b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\\+d_4(\mathbf{p})\mathbf{v1}(\mathbf{p})+id_3(\mathbf{p})\mathbf{v3}(\mathbf{p})+ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p})+b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p})\end{bmatrix}\\ \delta(\mathbf{p}'-\mathbf{p})\\+\begin{bmatrix}b_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}')-ib_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}')-id_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}')+d_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}')\\+b_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}')-ib_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}')-id_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}')+d_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}')\end{bmatrix}\\ \begin{bmatrix}b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p})+ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p})+id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p})+d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p})\\+b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p})+ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p})+id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p})+d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p})\end{bmatrix}\\ \delta(\mathbf{p}-\mathbf{p}')\end{bmatrix}$$

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$$= \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} d_1^*(\mathbf{p})d_1(\mathbf{p})\mathbf{u1}^+(\mathbf{p})\mathbf{u1}(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p})\mathbf{u4}^+(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_1(\mathbf{p})b_1^*(\mathbf{p})\mathbf{u1}^T(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p})\mathbf{u4}^T(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +b_2(\mathbf{p})b_2^*(\mathbf{p})\mathbf{u3}^T(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p})\mathbf{u2}^T(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) \\ +d_2^*(\mathbf{p})d_2(\mathbf{p})\mathbf{u3}^+(\mathbf{p})\mathbf{u3}(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p})\mathbf{u2}^+(\mathbf{p})\mathbf{u2}(\mathbf{p}) \\ +d_4^*(\mathbf{p})d_4(\mathbf{p})\mathbf{v1}^+(\mathbf{p})\mathbf{v1}(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\mathbf{v4}(\mathbf{p}) \\ +b_4(\mathbf{p})b_4^*(\mathbf{p})\mathbf{v1}^T(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p})\mathbf{v4}^T(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \\ +b_3(\mathbf{p})b_3^*(\mathbf{p})\mathbf{v3}^T(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p})\mathbf{v2}^T(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) \\ +d_3^*(\mathbf{p})d_3(\mathbf{p})\mathbf{v3}^+(\mathbf{p})\mathbf{v3}(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p})\mathbf{v2}^+(\mathbf{p})\mathbf{v2}(\mathbf{p}) \end{bmatrix}$$

$$= \int \frac{d^4p}{(2\pi)^4} e_0(\mathbf{p}) \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p}) \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p}) + d_2^*(\mathbf{p})d_2(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) \\ +b_4(\mathbf{p})b_4^*(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) \\ +b_3(\mathbf{p})b_3^*(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) \end{bmatrix}$$

$$= 8\int \frac{d^4p}{(2\pi)^4} e_0(\mathbf{p})\delta(\mathbf{0}) = 8\int d^4x \int \frac{d^4p}{(2\pi)^4} e_0(\mathbf{p})$$

here

$$e_0(\mathbf{p}) = \overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3$$

Each summand in brackets represents the operator of the number of particles with a certain reference spinor. The operator's action consists of consecutive application of the annihilation operator and the operator of the creation of a particle. On initial examination, it would appear that the energy associated with zero-point fluctuations in the vacuum has been overlooked. However, an examination of the final expression reveals that the field always possesses a constant energy, regardless of the particles that contribute to it. This constant energy of the field can be interpreted as the energy of zero-point fluctuations of the vacuum.

The following relations were taken into account in the derivation

$$\begin{array}{ll} b_1(\mathbf{p})b_1^*(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p}) = \delta(\mathbf{0}) & b_1^*(\mathbf{p}')b_1(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}) = \delta(\mathbf{0}) \\ d_1(\mathbf{p})d_1^*(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) = \delta(\mathbf{0}) & d_1^*(\mathbf{p})d_1(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p}) = \delta(\mathbf{0}) \\ d_2(\mathbf{p})d_2^*(\mathbf{p}) + d_2^*(\mathbf{p}')d_2(\mathbf{p}) = \delta(\mathbf{0}) & d_2^*(\mathbf{p})d_2(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) = \delta(\mathbf{0}) \\ b_2(\mathbf{p})b_2^*(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p}) = \delta(\mathbf{0}) & b_2^*(\mathbf{p})b_2(\mathbf{p}) + b_2(\mathbf{p})b_2^*(\mathbf{p}) = \delta(\mathbf{0}) \\ d_3(\mathbf{p})d_3^*(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) = \delta(\mathbf{0}) & d_3^*(\mathbf{p})d_3(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) = \delta(\mathbf{0}) \\ b_3(\mathbf{p})b_3^*(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) = \delta(\mathbf{0}) & b_3^*(\mathbf{p})b_3(\mathbf{p}) + b_3(\mathbf{p})b_3^*(\mathbf{p}) = \delta(\mathbf{0}) \\ b_4(\mathbf{p})b_4^*(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) = \delta(\mathbf{0}) & b_4^*(\mathbf{p})b_4(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p}) = \delta(\mathbf{0}) \\ d_4(\mathbf{p})d_4^*(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) = \delta(\mathbf{0}) & d_4^*(\mathbf{p})d_4(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) = \delta(\mathbf{0}) \end{array}$$

$$\delta(\mathbf{0}) = \int d^4x$$

Let us draw an analogy between our approach and the relations given in ([11], Volume 1, Chapter 3, Section 3.3.1). There it is noted that the creation and annihilation operators of the fermionic field must satisfy such commutation relations that the equality expressing translational invariance is satisfied

$$\boldsymbol{\varphi}(\mathbf{X}+\mathbf{A}) = e^{i\mathbf{P}^T\mathbf{A}}\,\boldsymbol{\varphi}(\mathbf{X})e^{-i\mathbf{P}^T\mathbf{A}}$$

which in differential form is written as

$$\partial_\mu\boldsymbol{\varphi}(\mathbf{X}) = i[P_\mu, \boldsymbol{\varphi}(\mathbf{X})]$$

The coordinates here are the components of the Minkowski vector space. On the basis of these relations the anticommutation relations between the creation and annihilation operators are derived.

In the spinor coordinate space, we can express the translational invariance of the field operator by the relations

$$\boldsymbol{\varphi}(\mathbf{x}+\mathbf{a}) = e^{i(p_0a_1 - p_1a_0 + p_2a_3 - p_3a_2 + \overline{(\mathbf{p},\mathbf{a})})}\,\boldsymbol{\varphi}(\mathbf{x})e^{-i(p_0a_1 - p_1a_0 + p_2a_3 - p_3a_2 + \overline{(\mathbf{p},\mathbf{a})})}$$

$$\begin{array}{ll} \partial_0\boldsymbol{\varphi}(\mathbf{x}) = i[\overline{p_2}, \boldsymbol{\varphi}(\mathbf{x})] & \partial_1\boldsymbol{\varphi}(\mathbf{x}) = i[\overline{p_3}, \boldsymbol{\varphi}(\mathbf{x})] \\ \partial_2\boldsymbol{\varphi}(\mathbf{x}) = i[\overline{p_0}, \boldsymbol{\varphi}(\mathbf{x})] & \partial_3\boldsymbol{\varphi}(\mathbf{x}) = i[\overline{p_1}, \boldsymbol{\varphi}(\mathbf{x})] \\ [p_1, x_0] = -i & [p_0, x_1] = -i \\ [p_3, x_2] = -i & [p_2, x_3] = -i \end{array}$$

It is interesting to find out in what relation these translational operators are - one operator acts in vector space, the other in spinor space. Both operators act on the same state, but in one case the state is described by a wave function in vector coordinate representation, and in the other case in

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spinor coordinate representation. The translation mechanism of the operators is essentially the same, but it is not possible to replace the action of one translation operator by some combination of actions of the other. Because of this, the question arises as to which of these operators better describes nature. Our point of view is that the translation operator in spinor space is primary, and the operator in vector space just successfully copies it, without being exact, but being some approximation. It attracted the attention of physicists first because vector space is more accessible for investigation. When integrating over a four-dimensional vector space in some cases there is a divergence, then use renormalization. When integrating over four-dimensional spinor space, the differential element has two orders of magnitude of the vector momentum component smaller, while the denominator in the integrand remains of the same order as when integrating over vector space. This difference possibly affects the convergence.

Let us calculate the total mass of the fermion field

$$M = \int d^4x\, \boldsymbol{\varphi}^T(\mathbf{x})\, \boldsymbol{\varphi}(\mathbf{x}) =$$

$$\int d^4x \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') + ib_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') + ib_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + b_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix}$$
$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}$$
$$e^{i\left(\overline{p_0'}x_2+\overline{p_1'}x_3+\overline{p_2'}x_0+\overline{p_3'}x_1+\overline{(\mathbf{p}',\mathbf{x})}\right)}e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$+\int d^4x \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$
$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$
$$e^{-i\left(\overline{p_0'}x_2+\overline{p_1'}x_3+\overline{p_2'}x_0+\overline{p_3'}x_1+\overline{(\mathbf{p}',\mathbf{x})}\right)}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p}')\mathbf{u1}^T(\mathbf{p}') + id_2(\mathbf{p}')\mathbf{u3}^T(\mathbf{p}') + ib_2(\mathbf{p}')\mathbf{u2}^+(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}^+(\mathbf{p}') \\ +d_4(\mathbf{p}')\mathbf{v1}^T(\mathbf{p}') + id_3(\mathbf{p}')\mathbf{v3}^T(\mathbf{p}') + ib_3(\mathbf{p}')\mathbf{v2}^+(\mathbf{p}') + b_4(\mathbf{p}')\mathbf{v4}^+(\mathbf{p}') \end{bmatrix}$$
$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}\delta(\mathbf{p}-\mathbf{p}') +$$
$$+\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$
$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}\delta(\mathbf{p}'-\mathbf{p}) =$$

$$= \int \frac{d^4p}{(2\pi)^4}\begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p})\mathbf{u1}^T(\mathbf{p})\mathbf{u4}(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p})\mathbf{u4}^+(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) \\ -d_2(\mathbf{p})d_2^*(\mathbf{p})\mathbf{u3}^T(\mathbf{p})\mathbf{u2}(\mathbf{p}) - b_2(\mathbf{p})b_2^*(\mathbf{p})\mathbf{u2}^+(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) \\ +d_4(\mathbf{p})d_4^*(\mathbf{p})\mathbf{v1}^T(\mathbf{p})\mathbf{v4}(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p})\mathbf{v4}^+(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) \\ -d_3(\mathbf{p})d_3^*(\mathbf{p})\mathbf{v3}^T(\mathbf{p})\mathbf{v2}(\mathbf{p}) - b_3(\mathbf{p})b_3^*(\mathbf{p})\mathbf{v2}^+(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) \end{bmatrix} +$$
$$+\int \frac{d^4p}{(2\pi)^4}\begin{bmatrix} b_1^*(\mathbf{p})\, b_1(\mathbf{p})\mathbf{u1}^+(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p})\mathbf{u4}^T(\mathbf{p})\mathbf{u1}(\mathbf{p}) \\ -b_2^*(\mathbf{p})b_2(\mathbf{p})\mathbf{u3}^+(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) - d_2^*(\mathbf{p})d_2(\mathbf{p})\mathbf{u2}(\mathbf{p})\mathbf{u3}(\mathbf{p}) \\ +b_4^*(\mathbf{p})b_4(\mathbf{p})\mathbf{v1}^+(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p})\mathbf{v4}^T(\mathbf{p})\mathbf{v1}(\mathbf{p}) \\ -b_3^*(\mathbf{p})b_3(\mathbf{p})\mathbf{v3}^+(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) - d_3^*(\mathbf{p})d_3(\mathbf{p})\mathbf{v2}^T(\mathbf{p})\mathbf{v3}(\mathbf{p}) \end{bmatrix} =$$
$$= \int \frac{d^4p}{(2\pi)^4}\,(m+\overline{m})\begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p}) \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) + b_3(\mathbf{p})b_3^*(\mathbf{p}) \\ +b_1^*(\mathbf{p})\, b_1(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) \\ +b_2^*(\mathbf{p})b_2(\mathbf{p}) + d_2^*(\mathbf{p})d_2(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) \end{bmatrix} =$$
$$= \int \frac{d^4p}{(2\pi)^4}16(m+\overline{m})\,\delta(\mathbf{0}) = \int d^4x \int \frac{d^4p}{(2\pi)^4}\,16(m+\overline{m})$$

The ratios used in the derivation are

$$\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\mathbf{u4}(\mathbf{p}) = \overline{p_3}p_1 + \overline{p_2}p_0 + p_0\overline{p_2} + p_1\overline{p_3} = \overline{m} + m$$
$$-\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\mathbf{u2}(\mathbf{p}) = -(-p_0\overline{p_2} - p_1\overline{p_3} - \overline{p_3}p_1 - \overline{p_2}p_0) = \overline{m} + m$$
$$\mathbf{u4}^{+}(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) = \overline{m} + m$$
$$-\mathbf{u2}^{+}(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) = \overline{m} + m$$
$$\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p})\mathbf{v4}(\mathbf{p}) = \overline{m} + m$$
$$\mathbf{v4}^{+}(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) = \overline{m} + m$$
$$-\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p})\mathbf{v2}(\mathbf{p}) = \overline{m} + m$$
$$-\mathbf{v2}^{+}(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) = \overline{m} + m$$
$$\mathbf{u1}^{+}(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) = \overline{m} + m$$
$$\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p})\mathbf{u1}(\mathbf{p}) = \overline{m} + m$$
$$-\mathbf{u3}^{+}(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) = \overline{m} + m$$
$$-\mathbf{u2}(\mathbf{p})\mathbf{u3}(\mathbf{p}) = \overline{m} + m$$
$$\mathbf{v1}^{+}(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) = \overline{m} + m$$
$$\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p})\mathbf{v1}(\mathbf{p}) = \overline{m} + m$$
$$-\mathbf{v3}^{+}(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) = \overline{m} + m$$
$$-\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p})\mathbf{v3}(\mathbf{p}) = \overline{m} + m$$
$$b_1(\mathbf{p})b_1^*(\mathbf{p}) + b_1^*(\mathbf{p})b_1(\mathbf{p}) = b_1^*(\mathbf{p})b_1(\mathbf{p}) + b_1(\mathbf{p})b_1^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_1(\mathbf{p})d_1^*(\mathbf{p}) + d_1^*(\mathbf{p})d_1(\mathbf{p}) = d_1^*(\mathbf{p})d_1(\mathbf{p}) + d_1(\mathbf{p})d_1^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_2(\mathbf{p})d_2^*(\mathbf{p}) + d_2^*(\mathbf{p})d_2(\mathbf{p}) = b_2^*(\mathbf{p})b_2(\mathbf{p}) + b_2(\mathbf{p})b_2^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_2(\mathbf{p})b_2^*(\mathbf{p}) + b_2^*(\mathbf{p})b_2(\mathbf{p}) = d_2^*(\mathbf{p})d_2(\mathbf{p}) + d_2(\mathbf{p})d_2^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_3(\mathbf{p})d_3^*(\mathbf{p}) + d_3^*(\mathbf{p})d_3(\mathbf{p}) = b_3^*(\mathbf{p})b_3(\mathbf{p}) + b_3(\mathbf{p})b_3^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_3(\mathbf{p})b_3^*(\mathbf{p}) + b_3^*(\mathbf{p})b_3(\mathbf{p}) = d_3^*(\mathbf{p})d_3(\mathbf{p}) + d_3(\mathbf{p})d_3^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$b_4(\mathbf{p})b_4^*(\mathbf{p}) + b_4^*(\mathbf{p})b_4(\mathbf{p}) = b_4^*(\mathbf{p})b_4(\mathbf{p}) + b_4(\mathbf{p})b_4^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$d_4(\mathbf{p})d_4^*(\mathbf{p}) + d_4^*(\mathbf{p})d_4(\mathbf{p}) = d_4^*(\mathbf{p})d_4(\mathbf{p}) + d_4(\mathbf{p})d_4^*(\mathbf{p}) = \delta(\mathbf{0})$$
$$\delta(\mathbf{0}) = \int d^4x$$

Let us give an interpretation of the operator coefficients for this approach

$$\mathbf{u1} = \begin{pmatrix} \overline{p_3} \\ -\overline{p_2} \\ p_0 \\ p_1 \end{pmatrix} \quad \mathbf{u2} = \begin{pmatrix} \overline{p_2} \\ \overline{p_3} \\ p_1 \\ -p_0 \end{pmatrix} \quad \mathbf{u3} = \begin{pmatrix} -p_0 \\ -p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix} \quad \mathbf{u4} = \begin{pmatrix} p_1 \\ -p_0 \\ \overline{p_2} \\ \overline{p_3} \end{pmatrix}$$
$$\mathbf{v1} = \begin{pmatrix} p_0 \\ p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix} \quad \mathbf{v2} = \begin{pmatrix} p_1 \\ -p_0 \\ -\overline{p_2} \\ -\overline{p_3} \end{pmatrix} \quad \mathbf{v3} = \begin{pmatrix} -\overline{p_3} \\ \overline{p_2} \\ p_0 \\ p_1 \end{pmatrix} \quad \mathbf{v4} = \begin{pmatrix} \overline{p_2} \\ \overline{p_3} \\ -p_1 \\ p_0 \end{pmatrix}$$

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + \ b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$
$$+ \begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2 + \overline{(\mathbf{p},\mathbf{x})})}$$

$d_1^*(\mathbf{p})$ creates and $d_1(\mathbf{p})$ destroys a particle $\mathbf{u1}(\mathbf{p})$, $d_1^*(\mathbf{p})d_1(\mathbf{p})$ is the operator of the number of such particles

$b_1(\mathbf{p})$ creates and $b_1^*(\mathbf{p})$ destroys a particle $\overline{\mathbf{u1}}(\mathbf{p})$, $b_1(\mathbf{p})b_1^*(\mathbf{p})$ is the operator of the number of such particles

$d_1(\mathbf{p})$ creates and $d_1^*(\mathbf{p})$ destroys a particle $\mathbf{u4}(\mathbf{p})$, $d_1(\mathbf{p})d_1^*(\mathbf{p})$ is the operator of the number of such particles

$b_1^*(\mathbf{p})$ creates and $b_1(\mathbf{p})$ destroys a particle $\overline{\mathbf{u4}}(\mathbf{p})$, $b_1^*(\mathbf{p})b_1(\mathbf{p})$ is the operator of the number of such particles

The properties of all particles and operators are summarized in a table

creates	destroys	spinor	vector

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$d_1^*(\mathbf{p})$	$d_1(\mathbf{p})$	$\mathbf{u1}(\mathbf{p}) = \begin{pmatrix} \overline{p_3} \\ -\overline{p_2} \\ p_0 \\ p_1 \end{pmatrix}$	$\mathbf{U1} = \begin{pmatrix} P_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$
$d_1(\mathbf{p})$	$d_1^*(\mathbf{p})$	$\mathbf{u4}(\mathbf{p}) = \begin{pmatrix} p_1 \\ -p_0 \\ \overline{p_2} \\ \overline{p_3} \end{pmatrix}$	$\mathbf{U4} = \begin{pmatrix} P_0 \\ Q_1 \\ -Q_2 \\ Q_3 \end{pmatrix}$
$b_1(\mathbf{p})$	$b_1^*(\mathbf{p})$	$\overline{\mathbf{u1}}(\mathbf{p}) = \begin{pmatrix} p_3 \\ -p_2 \\ \overline{p_0} \\ \overline{p_1} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ Q_1 \\ -Q_2 \\ Q_3 \end{pmatrix}$
$b_1^*(\mathbf{p})$	$b_1(\mathbf{p})$	$\overline{\mathbf{u4}}(\mathbf{p}) = \begin{pmatrix} \overline{p_1} \\ -\overline{p_0} \\ p_2 \\ p_3 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$
$d_4^*(\mathbf{p})$	$d_4(\mathbf{p})$	$\mathbf{v1}(\mathbf{p}) = \begin{pmatrix} p_0 \\ p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix}$	$\mathbf{V1} = \begin{pmatrix} P_0 \\ -Q_1 \\ -Q_2 \\ -Q_3 \end{pmatrix}$
$d_4(\mathbf{p})$	$d_4^*(\mathbf{p})$	$\mathbf{v4}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ \overline{p_3} \\ -p_1 \\ p_0 \end{pmatrix}$	$\mathbf{V4} = \begin{pmatrix} P_0 \\ -Q_1 \\ Q_2 \\ -Q_3 \end{pmatrix}$
$b_4(\mathbf{p})$	$b_4^*(\mathbf{p})$	$\overline{\mathbf{v1}}(\mathbf{p}) = \begin{pmatrix} \overline{p_0} \\ \overline{p_1} \\ -p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -Q_1 \\ Q_2 \\ -Q_3 \end{pmatrix}$
$b_4^*(\mathbf{p})$	$b_4(\mathbf{p})$	$\overline{\mathbf{v4}}(\mathbf{p}) = \begin{pmatrix} p_2 \\ p_3 \\ -\overline{p_1} \\ \overline{p_0} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -Q_1 \\ -Q_2 \\ -Q_3 \end{pmatrix}$
$d_2^*(\mathbf{p})$	$d_2(\mathbf{p})$	$\mathbf{u3}(\mathbf{p}) = \begin{pmatrix} -p_0 \\ -p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -Q_1 \\ -Q_2 \\ -Q_3 \end{pmatrix}$
$d_2(\mathbf{p})$	$d_2^*(\mathbf{p})$	$\mathbf{u2}(\mathbf{p}) = \begin{pmatrix} \overline{p_2} \\ \overline{p_3} \\ p_1 \\ -p_0 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -Q_1 \\ Q_2 \\ -Q_3 \end{pmatrix}$
$b_2(\mathbf{p})$	$b_2^*(\mathbf{p})$	$\overline{\mathbf{u3}}(\mathbf{p}) = \begin{pmatrix} -\overline{p_0} \\ -\overline{p_1} \\ -p_3 \\ p_2 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -Q_1 \\ Q_2 \\ -Q_3 \end{pmatrix}$
$b_2^*(\mathbf{p})$	$b_2(\mathbf{p})$	$\overline{\mathbf{u2}}(\mathbf{p}) = \begin{pmatrix} p_2 \\ p_3 \\ \overline{p_1} \\ -\overline{p_0} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ -Q_1 \\ -Q_2 \\ -Q_3 \end{pmatrix}$
$d_3^*(\mathbf{p})$	$d_3(\mathbf{p})$	$\mathbf{v3}(\mathbf{p}) = \begin{pmatrix} -\overline{p_3} \\ \overline{p_2} \\ p_0 \\ p_1 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$

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$d_3(\mathbf{p})$	$d_3^*(\mathbf{p})$	$\mathbf{v2}(\mathbf{p}) = \begin{pmatrix} p_1 \\ -p_0 \\ -\overline{p_2} \\ -\overline{p_3} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ Q_1 \\ -Q_2 \\ Q_3 \end{pmatrix}$
$b_3(\mathbf{p})$	$b_3^*(\mathbf{p})$	$\overline{\mathbf{v3}}(\mathbf{p}) = \begin{pmatrix} -p_3 \\ p_2 \\ \overline{p_0} \\ \overline{p_1} \end{pmatrix}$	$\begin{pmatrix} P_0 \\ Q_1 \\ -Q_2 \\ Q_3 \end{pmatrix}$
$b_3^*(\mathbf{p})$	$b_3(\mathbf{p})$	$\overline{\mathbf{v2}}(\mathbf{p}) = \begin{pmatrix} \overline{p_1} \\ -\overline{p_0} \\ -p_2 \\ -p_3 \end{pmatrix}$	$\begin{pmatrix} P_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$

Here the column "vector" shows the vector obtained from the corresponding spinor by the formula of the form

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{p}$$
$$U1_\mu = \frac{1}{2}\mathbf{u1}^\dagger(\gamma_0^V\gamma_\mu^V)\mathbf{u1}$$
$$\mathbf{U1} = \begin{pmatrix} P_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

Although we have used the term vector for quantities like **U1**, they are not really vectors in the sense that if a Lorentz transformation is applied to a coordinate spinor and hence a coordinate vector, the true vector must undergo the same transformation. For a momentum vector this is the case, but for quantities like **U1** it is not right.

Their time component coincides with the component of the momentum vector, but the spatial components differ from the corresponding components of the momentum vector.

By the words $d_1(\mathbf{p})$ destroys the particle $\mathbf{u1}(\mathbf{p})$ it should be understood that this operator transforms this particle into the particle $\mathbf{u4}(\mathbf{p})$, and the operator $d_1^*(\mathbf{p})$ performs the reverse transformation of $\mathbf{u4}(\mathbf{p})$ into $\mathbf{u1}(\mathbf{p})$. The action of the operator $d_1(\mathbf{p})$ on any other particle gives zero.

Let us see what result we get if we apply another definition of anticommutativity of the fermionic field.

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$
$$+\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$
$$\{\varphi_i(\mathbf{x}),\overline{\varphi_J}(\mathbf{x}')\} = \varphi_i(\mathbf{x})\overline{\varphi_J}(\mathbf{x}') + \overline{\varphi_J}(\mathbf{x}')\varphi_i(\mathbf{x}) = \left(\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^+(\mathbf{x}') + \left(\overline{\boldsymbol{\varphi}}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T\right)_{ij}$$

$$\boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}^+(\mathbf{x}') + \left(\overline{\boldsymbol{\varphi}}(\mathbf{x}')\boldsymbol{\varphi}^T(\mathbf{x})\right)^T =$$
$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$
$$\begin{bmatrix} d_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') - id_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') - ib_2^*(\mathbf{p}')\mathbf{u2}^T(\mathbf{p}') + b_1^*(\mathbf{p}')\mathbf{u4}^T(\mathbf{p}') \\ +d_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') - id_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') - ib_3^*(\mathbf{p}')\mathbf{v2}^T(\mathbf{p}') + b_4^*(\mathbf{p}')\mathbf{v4}^T(\mathbf{p}') \end{bmatrix}$$
$$e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}$$
$$+$$

$$\begin{pmatrix} \begin{bmatrix} d_1^*(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p}') - id_2^*(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p}') - ib_2^*(\mathbf{p}')\mathbf{u2}(\mathbf{p}') + b_1(\mathbf{p}')\mathbf{u4}(\mathbf{p}') \\ +d_4^*(\mathbf{p}')\overline{\mathbf{v1}}(\mathbf{p}') - id_3^*(\mathbf{p}')\overline{\mathbf{v3}}(\mathbf{p}') - ib_3^*(\mathbf{p}')\mathbf{v2}(\mathbf{p}') + b_4^*(\mathbf{p}')\mathbf{v4}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + ib_2(\mathbf{p}')\mathbf{u2}^{+}(\mathbf{p}') + b_1^*(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p}) + ib_3(\mathbf{p})\mathbf{v2}^{+}(\mathbf{p}) + b_4(\mathbf{p})\mathbf{v4}^{+}(\mathbf{p}) \end{bmatrix} \\ e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{pmatrix}^T$$

$$+$$

$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix}$$
$$\begin{bmatrix} b_1(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') - ib_2(\mathbf{p}')\mathbf{u3}^{T}(\mathbf{p}') - id_2(\mathbf{p}')\mathbf{u2}^{+}(\mathbf{p}') + d_1(\mathbf{p}')\mathbf{u4}^{+}(\mathbf{p}') \\ +b_4(\mathbf{p}')\mathbf{v1}^{T}(\mathbf{p}') - ib_3(\mathbf{p}')\mathbf{v3}^{T}(\mathbf{p}') - id_3(\mathbf{p}')\mathbf{v2}^{+}(\mathbf{p}') + d_4(\mathbf{p}')\mathbf{v4}^{+}(\mathbf{p}') \end{bmatrix}$$
$$e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)}$$

$$+$$

$$\begin{pmatrix} \begin{bmatrix} b_1(\mathbf{p}')\mathbf{u1}(\mathbf{p}') - ib_2(\mathbf{p}')\mathbf{u3}(\mathbf{p}') - id_2(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p}') + d_1(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p}') \\ +b_4(\mathbf{p}')\mathbf{v1}(\mathbf{p}') - ib_3(\mathbf{p}')\mathbf{v3}(\mathbf{p}') - id_3(\mathbf{p}')\overline{\mathbf{v2}}(\mathbf{p}') + d_4(\mathbf{p}')\overline{\mathbf{v4}}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}) + ib_2^*(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\mathbf{v1}^{+}(\mathbf{p}) + ibib_{3_3}^*(\mathbf{p})\mathbf{v3}^{+}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}^{T}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p}) \end{bmatrix} \\ e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{pmatrix}^{\boldsymbol{T}}$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}') + \left(d_1^*(\mathbf{p}')d_1(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \\ +d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}') + \left(d_2^*(\mathbf{p}')d_2(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^{T}(\mathbf{p}') + \left(b_1^*(\mathbf{p}')b_1(\mathbf{p})\mathbf{u4}(\mathbf{p}')\mathbf{u4}^{+}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^{T}(\mathbf{p}') + \left(b_2^*(\mathbf{p}')b_2(\mathbf{p}')\mathbf{u2}(\mathbf{p}')\mathbf{u2}^{+}(\mathbf{p}')\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}$$

$$+ \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') + \left(b_1(\mathbf{p}')b_1^*(\mathbf{p})\mathbf{u1}(\mathbf{p}')\mathbf{u1}^{+}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \\ +b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^{T}(\mathbf{p}') + \left(b_2(\mathbf{p}')b_2^*(\mathbf{p})\mathbf{u3}(\mathbf{p}')\mathbf{u3}^{+}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}') + \left(d_1(\mathbf{p}')d_1^*(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}')\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \\ +d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}') + \left(d_2(\mathbf{p}')d_2^*(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}')\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} + \cdots \end{bmatrix} \\ e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}') + \left(d_1^*(\mathbf{p}')d_1(\mathbf{p})\mathbf{u1}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}')\right) + \cdots \\ +d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}') + \left(d_2^*(\mathbf{p}')d_2(\mathbf{p})\mathbf{u3}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^{T}(\mathbf{p}') + \left(b_1^*(\mathbf{p}')b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^{T}(\mathbf{p}')\right) + \cdots \\ +b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^{T}(\mathbf{p}') + \left(b_2^*(\mathbf{p}')b_2(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^{T}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}$$

$$+ \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} b_1^*(\mathbf{p})b_1(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}') + \left(b_1(\mathbf{p}')b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}')(\mathbf{p}')\right) + \cdots \\ +b_2^*(\mathbf{p})b_2(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^{T}(\mathbf{p}') + \left(b_2(\mathbf{p}')b_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^{T}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} e^{i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} d_1^*(\mathbf{p})d_1(\mathbf{p}')\mathbf{u4}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}') + \left(d_1(\mathbf{p}')d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}')\right) + \cdots \\ +d_2^*(\mathbf{p})d_2(\mathbf{p}')\mathbf{u2}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}') + \left(d_2(\mathbf{p}')d_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}')\right) + \cdots \end{bmatrix} \\ e^{-i\left(\overline{p'_0}x'_2+\overline{p'_1}x'_3+\overline{p'_2}x'_0+\overline{p'_3}x'_1+\overline{(\mathbf{p}',\mathbf{x}')}\right)} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}$$

$$
= \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}) + \cdots \\ +\mathbf{u3}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^{T}(\mathbf{p}) + \cdots \\ +\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^{T}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}
$$

$$
+ \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} \overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^{T}(\mathbf{p}) + \cdots \\ +\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^{T}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \mathbf{u4}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}) + \cdots \\ +\mathbf{u2}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}
$$

$$
= \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}) + \mathbf{u3}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}) + \\ \mathbf{u4}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}) + \mathbf{u2}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^{T}(\mathbf{p}) + \overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^{T}(\mathbf{p}) + \\ \overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^{T}(\mathbf{p}) + \overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^{T}(\mathbf{p}) + \cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}
$$

$$
= \int \frac{d^4p}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}) + \mathbf{u2}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}) + \mathbf{u3}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}) + \mathbf{u4}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}) + \\ \mathbf{v1}(\mathbf{p})\mathbf{v1}^{+}(\mathbf{p}) + \mathbf{v2}(\mathbf{p})\mathbf{v2}^{+}(\mathbf{p}) + \mathbf{v3}(\mathbf{p})\mathbf{v3}^{+}(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v4}^{+}(\mathbf{p}) \end{bmatrix} \\ e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \overline{\mathbf{u1}}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}) + \overline{\mathbf{u2}}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}) + \overline{\mathbf{u3}}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}) + \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p}) + \\ \overline{\mathbf{v1}}(\mathbf{p})\mathbf{v1}^{+}(\mathbf{p}) + \overline{\mathbf{v2}}(\mathbf{p})\mathbf{v2}^{+}(\mathbf{p}) + \overline{\mathbf{v3}}(\mathbf{p})\mathbf{v3}^{+}(\mathbf{p}) + \overline{\mathbf{v4}}(\mathbf{p})\mathbf{v4}^{+}(\mathbf{p}) \end{bmatrix} \\ e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}
$$

$$
= \int \frac{d^4p}{(2\pi)^4} \left(T^R(\mathbf{p}) + T_R(\mathbf{p})\right) e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} +
$$

$$
+ \int \frac{d^4p}{(2\pi)^4} \left(\overline{T_R}(\mathbf{p}) + \overline{T^R}(\mathbf{p})\right) e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} =
$$

$$
= \int \frac{d^4p}{(2\pi)^4} 4 \begin{pmatrix} e(\mathbf{p}) & 0 & 0 & 0 \\ 0 & e(\mathbf{p}) & 0 & 0 \\ 0 & 0 & e(\mathbf{p}) & 0 \\ 0 & 0 & 0 & e(\mathbf{p}) \end{pmatrix} e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
$$

$$
+ \int \frac{d^4p}{(2\pi)^4} 4 \begin{pmatrix} e(\mathbf{p}) & 0 & 0 & 0 \\ 0 & e(\mathbf{p}) & 0 & 0 \\ 0 & 0 & e(\mathbf{p}) & 0 \\ 0 & 0 & 0 & e(\mathbf{p}) \end{pmatrix} e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
$$

$$
= 4e(\mathbf{p})I\delta(\mathbf{x}'-\mathbf{x}) + 4e(\mathbf{p})I\delta(\mathbf{x}-\mathbf{x}')
$$

where

$$
T^R(\mathbf{p}) = \mathbf{u1}(\mathbf{p})\mathbf{u1}^{+}(\mathbf{p}) + \mathbf{u2}(\mathbf{p})\mathbf{u2}^{+}(\mathbf{p}) + \mathbf{u3}(\mathbf{p})\mathbf{u3}^{+}(\mathbf{p}) + \mathbf{u4}(\mathbf{p})\mathbf{u4}^{+}(\mathbf{p})
$$

$$
T_R(\mathbf{p}) = \mathbf{v1}(\mathbf{p})\mathbf{v1}^{+}(\mathbf{p}) + \mathbf{v2}(\mathbf{p})\mathbf{v2}^{+}(\mathbf{p}) + \mathbf{v3}(\mathbf{p})\mathbf{v3}^{+}(\mathbf{p}) + \mathbf{v4}(\mathbf{p})\mathbf{v4}^{+}(\mathbf{p})
$$

$$
T^R(\mathbf{p}) + T_R(\mathbf{p}) + \overline{T^R}(\mathbf{p}) + \overline{T_R}(\mathbf{p}) =
$$

$$
4(p_0\overline{p_0} + p_1\overline{p_1} + p_2\overline{p_2} + p_3\overline{p_3}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4e(\mathbf{p})I
$$

The last operation of taking the value $(p_0\overline{p_0} + p_1\overline{p_1} + p_2\overline{p_2} + p_3\overline{p_3})$ out from under the sign of the integral seems doubtful because of its dependence on the momentum over which the integration is performed. If one closes one's eyes to this, as is generally accepted in the literature, in particular in [9], this relation is taken to be interpreted as a proof of the anti-symmetry of the fermion wave function under the stipulated anticommutation relations. The only situation where this is unquestionably true is when considering in a rest system where boosts are excluded, energy is equal to mass, and invariant to rotations.

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It is noteworthy that the antisymmetric treatment, whether or not complex conjugation is considered, yields a diagonal matrix that is invariant in one case but not in the other. It is encouraging to observe that the set of reference spinors remain consistent.

It is crucial to note that the proposed invariant approach cannot be realized within the Minkowski vector space. To achieve this, it is necessary to transition to the spinor space. This reiterates the secondary role of the Minkowski space in comparison to the spinor space.

Dirac's equation can be expressed in both spinor and vector spaces, a fact that led Dirac to discover it. In contrast, the invariant equation can be written in spinor space but not in vector space, which explains why it was unknown.

Let us define four vectors

$$\mathbf{U1}=\begin{pmatrix}P_0\\Q_1\\Q_2\\Q_3\end{pmatrix}\qquad \mathbf{U4}=\begin{pmatrix}P_0\\Q_1\\-Q_2\\Q_3\end{pmatrix}\qquad \mathbf{V1}=\begin{pmatrix}P_0\\-Q_1\\-Q_2\\-Q_3\end{pmatrix}\qquad \mathbf{V4}=\begin{pmatrix}P_0\\-Q_1\\Q_2\\-Q_3\end{pmatrix}$$

Why we have chosen these 4 vectors out of 8 possible combinations of signs of three spatial components? Because they are represented in the previously given table of variants of spinor particles.

An alternative view of the selected four vectors is possible. The initial 16 spinors can be interpreted in the spirit of quantum mechanics as vectors in a Hilbert space. They describe pure states and form a complete basis, since the sum of their density matrices (projectors) is equal to the diagonal matrix with the following value on its diagonal

$$4(\overline{p_0}p_0+\overline{p_1}p_1+\overline{p_2}p_2+\overline{p_3}p_3)$$

16 spinors are necessary precisely to ensure completeness. This basis is not orthogonal.

It is interesting that if one does not use complex conjugation when forming the tensor product of spinors, then the sum of 16 such products will also be a diagonal matrix with the following value on the diagonal

$$4Re(p_0p_0+p_1p_1+p_2p_2+p_3p_3)$$

The three spatial components of the four Minkowski space vectors presented above are nothing more than a three-dimensional polarization vector, defined as the average of the spin operator over one of the 16 initial states. The polarization vectors form two pairs in which their directions are opposite. If we take two states with opposite polarization vectors, we can treat them as two electrons with opposite spins and use the tensor product to form a singlet state from these electrons. For example, we can use the state **u1**, which corresponds to the polarization vector **V1**, and $\overline{\mathbf{u2}}$, which corresponds to the opposite vector **U1**

$$|\mathbf{s}\rangle=|\mathbf{u1}\rangle\otimes|\overline{\mathbf{u2}}\rangle-|\overline{\mathbf{u2}}\rangle\otimes|\mathbf{u1}\rangle=\begin{pmatrix}\overline{p_3}\\-\overline{p_2}\\p_0\\p_1\end{pmatrix}\otimes\begin{pmatrix}p_2\\p_3\\\overline{p_1}\\-\overline{p_0}\end{pmatrix}-\begin{pmatrix}p_2\\p_3\\\overline{p_1}\\-\overline{p_0}\end{pmatrix}\otimes\begin{pmatrix}\overline{p_3}\\-\overline{p_2}\\p_0\\p_1\end{pmatrix}$$

This pure state is characterized by an antisymmetric wave function, each of the electrons here is in a mixed state, end they are entangled with each other. Let us define the extended spin operator as four 8×8 matrices

$$\Gamma_\mu=\sigma_0\otimes(\gamma_0^V\gamma_\mu^V)=\begin{pmatrix}\gamma_0^V\gamma_\mu^V & 0\\0 & \gamma_0^V\gamma_\mu^V\end{pmatrix}$$

and let us find its average value in the singlet state

$$P_\mu=\langle\mathbf{s}|\Gamma_\mu|\mathbf{s}\rangle=Tr(|\mathbf{s}\rangle\langle\mathbf{s}|\Gamma_\mu)=\sum_{\alpha=0}^{7}\sum_{\beta=0}^{7}\left[\frac{1}{4}\overline{s_\alpha}s_\beta(\Gamma_\mu)_{\alpha\beta}\right]$$

As a result, we obtain a four-dimensional polarization vector that has the form

$$\mathbf{P}=\begin{pmatrix}P_0\\0\\0\\0\end{pmatrix}$$

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This vector does not change with any rotations of the initial spinors, and during boosts its form remaining the same, but the energy value changes. The polarization vector of the three triplet states of this spin pair has exactly the same properties with the same energy value

$$|\mathbf{u1}\rangle \otimes |\overline{\mathbf{u2}}\rangle + |\overline{\mathbf{u2}}\rangle \otimes |\mathbf{u1}\rangle$$
$$|\mathbf{u1}\rangle \otimes |\mathbf{u1}\rangle - |\overline{\mathbf{u2}}\rangle \otimes |\overline{\mathbf{u2}}\rangle$$
$$|\mathbf{u1}\rangle \otimes |\mathbf{u1}\rangle + |\overline{\mathbf{u2}}\rangle \otimes |\overline{\mathbf{u2}}\rangle$$

Spinor **u1** forms a set of four states with the same properties with only one spinor from the remaining 14, which is the spinor $\overline{\mathbf{v4}}$ corresponding to the polarization vector **U1**

$$|\mathbf{u1}\rangle \otimes |\overline{\mathbf{v4}}\rangle - |\overline{\mathbf{v4}}\rangle \otimes |\mathbf{u1}\rangle = \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix} \otimes \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix} - \begin{pmatrix} \overline{p_2} \\ -\overline{p_3} \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix} \otimes \begin{pmatrix} -p_3 \\ -p_2 \\ p_1 \\ p_0 \end{pmatrix}$$

In general, for any of the 16 spinors, there are two other spinors with which such singlet and triplet states are formed. These states are similar to Bell states, but Bell states only include zeros and ones, and the states themselves are an idealized representation of the properties of electrons, the result of a thought experiment. In our case, the states are associated with electron momentum and describe a pair of real relativistic particles under the action of arbitrary Lorentz transformations. The hypothesis that the model of combining two electrons using the tensor product of states adequately describes nature is confirmed at least by the fact that these four states are found in this form in the helium atom.

When we consider the 16 states expressed through the components of the momentum spinor, the question arises as to how to interpret them if the electron is in an atom. It should be noted that in this case, the components of the electron's momentum are replaced by derivatives of the wave function in the spinor coordinate representation with respect to the corresponding spinor coordinate. In this case, each spatial mode is assigned 16 pseudo-spinors. This interpretation provides a physical basis for understanding the nature of the spin of an electron in an atom.

The polarization vectors for all other pairs do not have such a simple form and change under rotations and boosts. But among the spinors there are still pairs with interesting properties. For example, the state **u4**, which corresponds to the polarization vector **U4**, and **v1**, which corresponds to the vector **V1**. Their singlet and triplet states correspond to the following polarization vectors

$$|\mathbf{u4}\rangle \otimes |\mathbf{v1}\rangle - |\mathbf{v1}\rangle \otimes |\mathbf{u4}\rangle = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} \otimes \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \otimes \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} \qquad \begin{pmatrix} P_0 \\ 0 \\ P_2 \\ 0 \end{pmatrix}$$

$$|\mathbf{u4}\rangle \otimes |\mathbf{v1}\rangle + |\mathbf{v1}\rangle \otimes |\mathbf{u4}\rangle = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} \otimes \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} + \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \otimes \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} \qquad \begin{pmatrix} Q_0 \\ 0 \\ Q_2 \\ 0 \end{pmatrix}$$

$$|\mathbf{u4}\rangle \otimes |\mathbf{u4}\rangle - |\mathbf{v1}\rangle \otimes |\mathbf{v1}\rangle = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} \otimes \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} + \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \otimes \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \qquad \begin{pmatrix} Q_0 \\ 0 \\ Q_2 \\ 0 \end{pmatrix}$$

$$|\mathbf{u4}\rangle \otimes |\mathbf{u4}\rangle + |\mathbf{v1}\rangle \otimes |\mathbf{v1}\rangle = \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} \otimes \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} + \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \otimes \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix} \qquad \begin{pmatrix} P_0 \\ 0 \\ P_2 \\ 0 \end{pmatrix}$$

All components of these polarization vectors change during rotations and boosts, but the structure of the vectors and their coincidence in the singlet and one of the triplet states remain unchanged.

Since electrons tend to form pairs (in atoms or in superconductors), it is logical to consider the tensor product of two Hilbert spaces. This product is itself a Hilbert space, and 256 tensor products of the original 16 spinors form a complete basis of states in this space. The sum of all projection operators is equal to the diagonal matrix with a diagonal element

$$(4(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3))^2$$

This interpretation leads us to believe that physical fields and their corresponding particles are described at a basic level by the 16 spinor states presented. The momentum vectors available to our perception are the result of a kind of measurement we make of the average values of various operators, in particular, the spin operator. The picture of the world that we see is the result of a quantum measurement procedure applied to the state space. Which is more productive: describing nature using initial quantum states, or settling for the average values of a certain operators? At the same time, one may ask whether the real electron is a pure state in the form of a superposition of 16 basis states, or whether it is a mixed state in the form of a weighted sum of 16 projectors onto these states.

Let us look for a representation of the electromagnetic field operator in vector space without first referring to spinor space. Let us define four vectors expressed through the components of the momentum vector

$$\mathbf{U1}=\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}\qquad \mathbf{U4}=\begin{pmatrix}P_0\\P_1\\-P_2\\P_3\end{pmatrix}\qquad \mathbf{V1}=\begin{pmatrix}P_0\\-P_1\\-P_2\\-P_3\end{pmatrix}\qquad \mathbf{V4}=\begin{pmatrix}P_0\\-P_1\\P_2\\-P_3\end{pmatrix}$$

Here, instead of the spatial components Q_μ obtained from the basis vectors using the matrix $\gamma_0^V\gamma_\mu^V$, we have returned to the spatial components P_μ obtained using the matrix S_μ. This makes the relationship between the components of the momentum vector and the components of the pseudovectors that make up the field more obvious. We temporarily forget about the origin of these pseudovectors; their origin from pseudospinors is not used in any way.

For the selected vectors the following relations are valid

$$\mathbf{V1}*\mathbf{U1}^{T}-\mathbf{U1}*\mathbf{V1}^{T}+\mathbf{V4}*\mathbf{V1}^{T}-\mathbf{V1}*\mathbf{V4}^{T}+$$
$$\mathbf{U4}*\mathbf{V4}^{T}-\mathbf{V4}*\mathbf{U4}^{T}+\mathbf{U1}*\mathbf{U4}^{T}-\mathbf{U4}*\mathbf{U1}^{T}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\mathbf{V1}*\mathbf{U1}^{T}+\mathbf{U1}*\mathbf{V1}^{T}+\mathbf{V4}*\mathbf{V1}^{T}+\mathbf{V1}*\mathbf{V4}^{T}+$$
$$+\mathbf{U4}*\mathbf{V4}^{T}+\mathbf{V4}*\mathbf{U4}^{T}+\mathbf{U1}*\mathbf{U4}^{T}+\mathbf{U4}*\mathbf{U1}^{T}=$$
$$=\begin{pmatrix}8P_0^2&0&0&0\\0&0&0&0\\0&0&-8P_2^2&0\\0&0&0&0\end{pmatrix}$$

$$\mathbf{V1}*\mathbf{U1}^{T}+\mathbf{V4}*\mathbf{V1}^{T}+\mathbf{U4}*\mathbf{V4}^{T}+\mathbf{U1}*\mathbf{U4}^{T}=\begin{pmatrix}4P_0^2&0&0&0\\0&0&0&0\\0&0&-4P_2^2&0\\0&0&0&0\end{pmatrix}$$

$$\mathbf{U1}*\mathbf{V1}^{T}+\mathbf{V1}*\mathbf{V4}^{T}+\mathbf{V4}*\mathbf{U4}^{T}+\mathbf{U4}*\mathbf{U1}^{T}=\begin{pmatrix}4P_0^2&0&0&0\\0&0&0&0\\0&0&-4P_2^2&0\\0&0&0&0\end{pmatrix}$$

$$(\mathbf{U1}*\mathbf{U1}^{T}+\mathbf{U4}*\mathbf{U4}^{T}+\mathbf{V1}*\mathbf{V1}^{T}+\mathbf{V4}*\mathbf{V4}^{T})+$$
$$+(\mathbf{U1}*\mathbf{V1}^{T}+\mathbf{V1}*\mathbf{U1}^{T}+\mathbf{V4}*\mathbf{U4}^{T}+\mathbf{U4}*\mathbf{V4}^{T})=$$
$$=\begin{pmatrix}8P_0^2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$8P_0^2=\mathbf{U1}^{T}*\mathbf{U1}+\mathbf{U4}^{T}*\mathbf{U4}+\mathbf{V1}^{T}*\mathbf{V1}+\mathbf{V4}^{T}*\mathbf{V4}+4M^2$$
$$(\mathbf{U1}^{T}*\mathbf{U1}+\mathbf{U4}^{T}*\mathbf{U4}+\mathbf{V1}^{T}*\mathbf{V1}+\mathbf{V4}^{T}*\mathbf{V4})+$$
$$+(\mathbf{U1}^{T}*\mathbf{V1}+\mathbf{V1}^{T}*\mathbf{U1}+\mathbf{V4}^{T}*\mathbf{U4}+\mathbf{U4}^{T}*\mathbf{V4})=8P_0^2$$
$$\mathbf{U1}^{T}*\mathbf{V1}+\mathbf{V1}^{T}*\mathbf{U1}+\mathbf{V4}^{T}*\mathbf{U4}+\mathbf{U4}^{T}*\mathbf{V4}=4M^2$$
$$\mathbf{U1}^{T}*\mathbf{U1}+\mathbf{U4}^{T}*\mathbf{U4}+\mathbf{V1}^{T}*\mathbf{V1}+\mathbf{V4}^{T}*\mathbf{V4}=8P_0^2-4M^2$$
$$\mathbf{U1}^{T}*\mathbf{V1}=\mathbf{V1}^{T}*\mathbf{U1}=\mathbf{V4}^{T}*\mathbf{U4}=\mathbf{U4}^{T}*\mathbf{V4}=M^2$$
$$\mathbf{U1}^{T}*\mathbf{U1}=\mathbf{U4}^{T}*\mathbf{U4}=\mathbf{V1}^{T}*\mathbf{V1}=\mathbf{V4}^{T}*\mathbf{V4}=\mathbf{P}^{T}*\mathbf{P}=2P_0^2-M^2$$
$$\mathbf{U1}^{T}\mathrm{g}\mathbf{U1}=\mathbf{U4}^{T}\mathrm{g}\mathbf{U4}=\mathbf{V1}^{T}\mathrm{g}\mathbf{V1}=\mathbf{V4}^{T}\mathrm{g}\mathbf{V4}=M^2$$
$$M^2=\mathbf{P}^{T}\mathrm{g}\mathbf{P}\equiv(\mathbf{P},\mathbf{P})$$

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$$g \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(\mathbf{U1}-\mathbf{U4}) = \begin{pmatrix} 0 \\ 0 \\ 2P_2 \\ 0 \end{pmatrix} \quad (\mathbf{V1}-\mathbf{V4}) = \begin{pmatrix} 0 \\ 0 \\ -2P_2 \\ 0 \end{pmatrix}$$

$$(\mathbf{U1}+\mathbf{V1}) = \begin{pmatrix} 2P_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (\mathbf{U4}+\mathbf{V4}) = \begin{pmatrix} 2P_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let us decompose the field into plane waves with operator coefficients and let's find the commutation relations for them. We will use the next notation for the scalar product of vectors

$$(\mathbf{P},\mathbf{X}) \equiv \mathbf{P}^T g \mathbf{X}$$

$$\boldsymbol{\varphi}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X})}$$
$$+$$
$$\begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

$$\boldsymbol{\varphi}(\mathbf{X}') = \int \frac{d^4P'}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}') + b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}') \\ +d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}') + b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}') \end{bmatrix} e^{i(\mathbf{P}',\mathbf{X}')}$$
$$+$$
$$\begin{bmatrix} b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}') + d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}') \\ +d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}') \end{bmatrix} e^{-i(\mathbf{P}',\mathbf{X}')}$$

$$\left[\varphi_i(\mathbf{X}), \varphi_j(\mathbf{X}')\right] = \varphi_i(\mathbf{X})\varphi_j(\mathbf{X}') - \varphi_j(\mathbf{X}')\varphi_i(\mathbf{X}) = \left(\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{X}') - \left(\boldsymbol{\varphi}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T\right)_{ij}$$

$$\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{X}') - \left(\boldsymbol{\varphi}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T =$$
$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\begin{bmatrix} \left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T \\ +\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1^*(\mathbf{P})\mathbf{v4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left(d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T \end{bmatrix}$$
$$+ \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\begin{bmatrix} \left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T \\ +\left(b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left(b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left(b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^T \end{bmatrix}$$
$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - d_1(\mathbf{P}')d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ + b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - b_1(\mathbf{P}')b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ +b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - b_4(\mathbf{P}')b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - d_4(\mathbf{P}')(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

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$$+\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} b_4^*(\mathbf{P})b_4(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - b_4^*(\mathbf{P}')b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\,e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +d_4^*(\mathbf{P})d_4(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - d_4^*(\mathbf{P}')d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +d_1^*(\mathbf{P})d_1(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - d_1^*(\mathbf{P}')d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +b_1^*(\mathbf{P})b_1(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - b_1^*(\mathbf{P}')b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} (d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P}))\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P}))\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P}))\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$+\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} (b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}))\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}))\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}))\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

Let us apply the following commutation relations

$$d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$d_1(\mathbf{P}')d_1^*(\mathbf{P}) - d_1^*(\mathbf{P})d_1(\mathbf{P}') = \delta(\mathbf{P}'-\mathbf{P})$$
$$d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$
$$b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$
$$d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$
$$b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \delta(\mathbf{P}-\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix} + \begin{bmatrix} -\delta(\mathbf{P}'-\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i((\mathbf{P},\mathbf{X}'))} \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \end{bmatrix} + \begin{bmatrix} -\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - \begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - \int \frac{d^4P}{(2\pi)^4}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

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$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^4} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^4} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X}-\mathbf{X}') - \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X}'-\mathbf{X}) = 0$$

Here it is taken into account that

$$\mathbf{V1} * \mathbf{U1}^{\boldsymbol{T}} + \mathbf{V4} * \mathbf{V1}^{\boldsymbol{T}} + \mathbf{U4} * \mathbf{V4}^{\boldsymbol{T}} + \mathbf{U1} * \mathbf{U4}^{\boldsymbol{T}} = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{U1} * \mathbf{V1}^{\boldsymbol{T}} + \mathbf{V1} * \mathbf{V4}^{\boldsymbol{T}} + \mathbf{V4} * \mathbf{U4}^{\boldsymbol{T}} + \mathbf{U4} * \mathbf{U1}^{\boldsymbol{T}} = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We will consider this relation as a proof of the symmetry of the wave function under the stipulated commutation relations.

Let us find the commutation relations for the wave function and its time derivative, which in this case play the role of canonical momentum

$$[\varphi_i(\mathbf{X}), \dot{\varphi}_j(\mathbf{X}')] = \varphi_i(\mathbf{X})\dot{\varphi}_j(\mathbf{X}') - \dot{\varphi}_j(\mathbf{X}')\varphi_i(\mathbf{X}) = \left(\boldsymbol{\varphi}(\mathbf{X})\dot{\boldsymbol{\varphi}}^T(\mathbf{X}') - \left(\dot{\boldsymbol{\varphi}}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T\right)_{ij}$$

where

$$\dot{\varphi}_j(\mathbf{X}) \equiv \frac{\partial \varphi_i(\mathbf{X})}{\partial X_0}$$

$$\boldsymbol{\varphi}(\mathbf{X})\dot{\boldsymbol{\varphi}}^T(\mathbf{X}') - \left(\dot{\boldsymbol{\varphi}}(\mathbf{X}')\boldsymbol{\varphi}^T(\mathbf{X})\right)^T =$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l} \left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{T} \\ +\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)\left((-iP_0')d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((iP_0')d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{T} \end{array}\right]$$

$$+ \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l} \left(b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')b_4(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')b_4^*(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')d_4(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')d_4^*(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \\ +\left(d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')d_1(\mathbf{P}')\mathbf{V1}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')d_1^*(\mathbf{P}')\mathbf{U1}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{T} \\ +\left(b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}\right)\left((iP_0')b_1(\mathbf{P}')\mathbf{U4}(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}\right)^{\boldsymbol{T}} - \left(\left((-iP_0')b_1^*(\mathbf{P}')\mathbf{V4}(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}\right)\left(b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}\right)^{\boldsymbol{T}}\right)^{T} \end{array}\right]$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\left[\begin{array}{l} (-iP_0')d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')d_1(\mathbf{P}')d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ + (-iP_0')b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')b_1(\mathbf{P}')b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ +(-iP_0')b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')b_4(\mathbf{P}')b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \\ +(-iP_0')d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} - (iP_0')d_4(\mathbf{P}')d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P}',\mathbf{X}')}e^{-i(\mathbf{P},\mathbf{X})} \end{array}\right]$$

$$+ \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

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$$\begin{bmatrix} (iP_0')b_4^*(\mathbf{P})b_4(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')b_4^*(\mathbf{P}')b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\,e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +(iP_0')d_4^*(\mathbf{P})d_4(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')d_4^*(\mathbf{P}')d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +(iP_0')d_1^*(\mathbf{P})d_1(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')d_1^*(\mathbf{P}')d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \\ +(iP_0')b_1^*(\mathbf{P})b_1(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{p}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} - (-iP_0')b_1^*(\mathbf{P}')b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} (-iP_0')(d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P}))\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(-iP_0')(\,b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P}))\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(-iP_0')(b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(-iP_0')(d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P}))\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$+ \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} (iP_0')(b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}))\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(iP_0')(d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}))\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(iP_0')(d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}))\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ +(iP_0')(b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}))\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

The commutation relations remain the same

$$d_1(\mathbf{P})d_1^*(\mathbf{P}') - d_1^*(\mathbf{P}')d_1(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$b_1(\mathbf{P})b_1^*(\mathbf{P}') - b_1^*(\mathbf{P}')b_1(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$b_4(\mathbf{P})b_4^*(\mathbf{P}') - b_4^*(\mathbf{P}')b_4(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$d_4(\mathbf{P})d_4^*(\mathbf{P}') - d_4^*(\mathbf{P}')d_4(\mathbf{P}) = \delta(\mathbf{P}-\mathbf{P}')$$
$$d_1^*(\mathbf{P})d_1(\mathbf{P}') - d_1(\mathbf{P}')d_1^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$
$$b_1^*(\mathbf{P})b_1(\mathbf{P}') - b_1(\mathbf{P}')b_1^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$
$$d_4^*(\mathbf{P})d_4(\mathbf{P}') - d_4(\mathbf{P}')d_4^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$
$$b_4^*(\mathbf{P})b_4(\mathbf{P}') - b_4(\mathbf{P}')b_4^*(\mathbf{P}) = -\delta(\mathbf{P}'-\mathbf{P})$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$(-iP_0')\begin{bmatrix} \delta(\mathbf{P}-\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +\delta(\mathbf{P}-\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix} + (iP_0')\begin{bmatrix} -\delta(\mathbf{P}'-\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \\ -\delta(\mathbf{P}'-\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$(-iP_0)\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i((\mathbf{P},\mathbf{X}'))} \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \end{bmatrix} + (iP_0)\begin{bmatrix} -\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \\ -\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{-i(\mathbf{P},\mathbf{X})}e^{i(\mathbf{P},\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}$$

$$(-iP_0)\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - (iP_0)\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})} =$$

$$(-iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^2} e^{i(\mathbf{P},\mathbf{X}-\mathbf{X}')} - (iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^2} e^{i(\mathbf{P},\mathbf{X}'-\mathbf{X})}$$

$$= (-iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X}-\mathbf{X}') - (iP_0)\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X}'-\mathbf{X})$$

$$= -iP_0\begin{pmatrix} 8P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\delta(\mathbf{X}-\mathbf{X}')$$

As one would expect, the field has only two degrees of freedom. This relation is valid for any reference frame, but the values of the momentum components in each of them are different.

Let us calculate the square of the field energy

$$E^2 = \int d^4X\, \boldsymbol{\varphi}^+(\mathbf{X})\boldsymbol{\varphi}(\mathbf{X}) =$$

$$= \int d^4X \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$

$$\begin{bmatrix} \begin{bmatrix} d_1^*(\mathbf{P}')\mathbf{V1}^{T}(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{U4}^{T}(\mathbf{P}') \\ +d_4^*(\mathbf{P}')\mathbf{U1}^{T}(\mathbf{P}') + b_4^*(\mathbf{P}')\mathbf{V4}^{T}(\mathbf{P}') \end{bmatrix} e^{-i(\mathbf{P}',\mathbf{X})} \\ + \begin{bmatrix} b_1(\mathbf{P}')\mathbf{V1}^{T}(\mathbf{P}') + d_1(\mathbf{P}')\mathbf{U4}^{T}(\mathbf{P}') \\ +b_4(\mathbf{P}')\mathbf{U1}^{T}(\mathbf{P}') + d_4(\mathbf{P}')\mathbf{V4}^{T}(\mathbf{P}') \end{bmatrix} e^{i(\mathbf{P}',\mathbf{X})} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}^{T}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}^{T}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}^{T}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}^{T}(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X})} \\ + \begin{bmatrix} b_1^*(\mathbf{P})\mathbf{V1}^{T}(\mathbf{P}) + d_1^*(\mathbf{P})\mathbf{U4}^{T}(\mathbf{P}) \\ +b_4^*(\mathbf{P})\mathbf{U1}^{T}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{V4}^{T}(\mathbf{P}) \end{bmatrix} e^{-(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$= \int d^4X \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1^*(\mathbf{P}')\mathbf{V1}^{T}(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{U4}^{T}(\mathbf{P}') \\ +d_4^*(\mathbf{P}')\mathbf{U1}^{T}(\mathbf{P}') + b_4^*(\mathbf{P}')\mathbf{V4}^{T}(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ e^{-i(\mathbf{P}',\mathbf{X})}e^{i(\mathbf{P},\mathbf{X})} \\ + \begin{bmatrix} b_1(\mathbf{P}')\mathbf{V1}^{T}(\mathbf{P}') + d_1(\mathbf{P}')\mathbf{U4}^{T}(\mathbf{P}') \\ +b_4(\mathbf{P}')\mathbf{U1}^{T}(\mathbf{P}') + d_4(\mathbf{P}')\mathbf{V4}^{T}(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_1^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +b_4^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ e^{i(\mathbf{P}',\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X})} \end{bmatrix}$$

$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4} \begin{bmatrix} \begin{bmatrix} d_1^*(\mathbf{P}')\mathbf{V1}^{+}(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{U4}^{T}(\mathbf{P}') \\ +d_4^*(\mathbf{P}')\mathbf{U1}^{+}(\mathbf{P}') + b_4^*(\mathbf{P}')\mathbf{V4}^{T}(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ \delta(\mathbf{P}-\mathbf{P}') \\ + \begin{bmatrix} b_1(\mathbf{P}')\mathbf{V1}^{T}(\mathbf{p}') + d_1(\mathbf{P}')\mathbf{U4}^{+}(\mathbf{P}') \\ +b_4(\mathbf{P}')\mathbf{U1}^{T}(\mathbf{p}') + d_4(\mathbf{P}')\mathbf{V4}^{+}(\mathbf{P}') \end{bmatrix} \\ \begin{bmatrix} b_1^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_1^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +b_4^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} \\ \delta(\mathbf{P}'-\mathbf{P}) \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} d_1^*(\mathbf{P})d_1(\mathbf{P})\mathbf{V1}^{T}(\mathbf{p})\mathbf{V1}(\mathbf{P}) + d_1(\mathbf{P})d_1^*(\mathbf{P})\mathbf{U4}^{T}(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +b_1(\mathbf{P})b_1^*(\mathbf{P})\mathbf{V1}^{T}(\mathbf{p})\mathbf{V1}(\mathbf{P}) + b_1^*(\mathbf{P})b_1(\mathbf{P})\mathbf{U4}^{T}(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4^*(\mathbf{P})d_4(\mathbf{P})\mathbf{U1}^{T}(\mathbf{p})\mathbf{U1}(\mathbf{P}) + d_4(\mathbf{P})d_4^*(\mathbf{P})\mathbf{V4}^{T}(\mathbf{P})\mathbf{V4}(\mathbf{P}) \\ +b_4(\mathbf{P})b_4^*(\mathbf{P})\mathbf{U1}^{T}(\mathbf{p})\mathbf{U1}(\mathbf{P}) + b_4^*(\mathbf{P})b_4(\mathbf{P})\mathbf{V4}^{T}(\mathbf{P})\mathbf{V}(\mathbf{P}) \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4} \begin{bmatrix} b_1(\mathbf{P})b_1^*(\mathbf{P}) + b_1^*(\mathbf{P})b_1(\mathbf{P}) + d_1^*(\mathbf{P})d_1(\mathbf{P}) + d_1(\mathbf{P})d_1^*(\mathbf{P}) \\ +b_4(\mathbf{P})b_4^*(\mathbf{P}) + b_4^*(\mathbf{P})b_4(\mathbf{P}) + d_4^*(\mathbf{P})d_4(\mathbf{P}) + d_4(\mathbf{P})d_4^*(\mathbf{P}) \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}\, \mathbf{P}^{\mathbf{T}}\mathbf{P} \begin{bmatrix} (b_1^*(\mathbf{P})b_1(\mathbf{P}) + \delta(\mathbf{0})) + b_1^*(\mathbf{P})b_1(\mathbf{P}) + d_1^*(\mathbf{P})d_1(\mathbf{P}) + (d_1^*(\mathbf{P})d_1(\mathbf{P}) + \delta(\mathbf{0})) \\ +\left(b_4^*(\mathbf{P})b_4(\mathbf{P}) + \delta(\mathbf{0})\right) + b_4^*(\mathbf{P})b_4(\mathbf{P}) + d_4^*(\mathbf{P})d_4(\mathbf{P}) + (d_4^*(\mathbf{P})d_4(\mathbf{P}) + \delta(\mathbf{0})) \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}\, 2\mathbf{P}^{\mathbf{T}}\mathbf{P} \begin{bmatrix} b_1^*(\mathbf{P})b_1(\mathbf{P}) + d_1^*(\mathbf{P})d_1(\mathbf{P}) \\ +b_4^*(\mathbf{P})b_4(\mathbf{P}) + d_4^*(\mathbf{P})d_4(\mathbf{P}) \end{bmatrix} + \int \frac{d^4P}{(2\pi)^4}\, 4\mathbf{P}^{\mathbf{T}}\mathbf{P}\delta(\mathbf{0})$$

here

$$\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{V1}^{\mathbf{T}}(\mathbf{P})\mathbf{V1}(\mathbf{P}) = \mathbf{U4}^{\mathbf{T}}(\mathbf{P})\mathbf{U4}(\mathbf{P}) = \mathbf{U1}^{\mathbf{T}}(\mathbf{P})\mathbf{U1}(\mathbf{P}) = \mathbf{V4}^{\mathbf{T}}(\mathbf{P})\mathbf{V4}(\mathbf{P}) =$$
$$= 2P_0^2 - M^2 = 2P_0^2 - \mathbf{P}^{\mathbf{T}}\mathrm{g}\mathbf{P} = 2P_0^2 - (\mathbf{P},\mathbf{P}) = 2P_0^2 - P^2$$

If we consider the photon field, the mass is zero, so that only the energy of the field remains in the formula. Each summand in brackets under the integral represents the operator of number of particles with a certain reference vector, its action consists in the consecutive application of the

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annihilation operator and the particle creation operator. The last summand describes the energy of zero-point fluctuations of vacuum. When there is no particle, we have the equality

$$E^2 = \int d^4X\, \boldsymbol{\varphi}^+(\mathbf{X})\boldsymbol{\varphi}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4}\, 4\mathbf{P}^{\mathrm{T}}\mathbf{P}\delta(\mathbf{0})$$

In this connection it is logical to use the normalization for the wave operator

$$\frac{\boldsymbol{\varphi}(\mathbf{X})}{2\mathbf{P}^{\mathrm{T}}\mathbf{P}}$$

If the mass is not zero, then we can relate $\mathbf{U1}(\mathbf{P})$ and $\mathbf{V1}(\mathbf{P})$ to the current of electrons with different spins and, respectively, relate $\mathbf{U4}(\mathbf{P})$ and $\mathbf{V4}(\mathbf{P})$ to the current of positrons with different spins.

As we have seen, neither electron current vectors nor electromagnetic field vectors are true vectors. When transforming the coordinate system, the same transformation acts on the components of the momentum vector, from these transformed components in each frame of reference the pseudovectors of the field are formed. But we know that the interaction between current and electromagnetic field is described by an additional term in the Lagrangian density of the electrodynamics theory. This term is the scalar product of the current and the electromagnetic potential and it is necessary for this product to be a scalar. But to form a scalar using a metric tensor, two true vectors are needed, and these are not available. There remains only one way to provide the scalar, it is necessary that signs of components in pseudovectors of current and field coincide, then they will compensate each other, and in fact we will get the scalar product of two vectors, and hence we will get a scalar.

Thus, there is a direct interrelation between the spinor description of the field and its vector description. 16 pseudospinors pass into 4 pseudovectors, moreover, the modulus of the complex mass in spinor space is equal to the mass in vector space. At all this by the value of the phase of a plane wave in spinor space by any direct way it is not possible to calculate the phase of a plane wave in vector space. Hence the assumption arises that operators in spinor space describe nature exactly, while operators in vector space provide only an approximate description. This may partly explain the problems with divergence when integrating in vector space.

To describe the evolution of the field state, we consider the vacuum averaged expression having the sense of the propagator. Before we do so, let us explain the meaning of operators included in the field decomposition

$$\boldsymbol{\varphi}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + \ b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

$$+$$

$$\begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

For example, $d_1(\mathbf{P})$ is an operator of annihilation of a particle with pseudovector $\mathbf{V1}(\mathbf{P})$, similarly, other operators without asterisks annihilate particles with pseudovector which stands in expansion with these operators. Accordingly, the operator $d_1^*(\mathbf{P})d_1(\mathbf{P})$is the operator of the number of particles with pseudovector $\mathbf{V1}(\mathbf{P})$.

Let us define a vacuum state of the field with zero filling numbers of particles of each of four varieties by specifying its properties with respect to the action of annihilation operators

$$d_1(\mathbf{P})|\Psi_0\rangle = 0 \qquad d_4(\mathbf{P})|\Psi_0\rangle = 0 \qquad b_1(\mathbf{P})|\Psi_0\rangle = 0 \qquad b_4(\mathbf{P})|\Psi_0\rangle = 0$$

$$\langle\Psi_0|d_1^*(\mathbf{P}) = 0 \qquad \langle\Psi_0|\, d_4^*(\mathbf{P}) = 0 \qquad \langle\Psi_0|b_1^*(\mathbf{P}) = 0 \qquad \langle\Psi_0|b_4^*(\mathbf{P}) = 0$$

It follows from these relations that

$$\langle\Psi_0|d_1(\mathbf{P})d_1^*(\mathbf{P}')|\Psi_0\rangle = \langle\Psi_0|[\,d_1(\mathbf{P}), d_1^*(\mathbf{P}')]|\Psi_0\rangle = \langle\Psi_0|\delta(\mathbf{P}-\mathbf{P}')|\Psi_0\rangle$$

Let us construct the amplitude of the field component, which is born at the point with coordinates $\mathbf{X} = \mathbf{0}$ and annihilated at the point with coordinates $\mathbf{X}$

$$\langle\Psi_0|\varphi_i(\mathbf{X})\varphi_j(\mathbf{0})|\Psi_0\rangle = (\langle\Psi_0|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle)_{ij}$$

$$\langle\Psi_0|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle =$$

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$$\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}\langle\Psi_0\left|\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})+b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})+b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\end{bmatrix}\begin{bmatrix} b_4^*(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}')+d_4^*(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}')\\ +d_1^*(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}')+b_1^*(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}')\end{bmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$
$$=\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}\langle\Psi_0\left|\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')+b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')\\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')+b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\end{bmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$
$$=\int \frac{d^4P}{(2\pi)^4}\langle\Psi_0\left|\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})+\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})\\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})+\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})\end{bmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$
$$=\int \frac{d^4P}{(2\pi)^4}\langle\Psi_0\left|\begin{pmatrix} 4P_0^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -4P_2^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\right|\Psi_0\rangle\, e^{i(\mathbf{P},\mathbf{X})}$$

For the reasons given above, let us apply the normalization of the field operator

$$\frac{\boldsymbol{\varphi}(\mathbf{X})}{2\mathbf{P}^{\mathbf{T}}\mathbf{P}}$$

As a result, we get

$$\frac{1}{4\mathbf{P}^{\mathbf{T}}\mathbf{P}}\langle\Psi_0|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle =$$
$$\int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_0|\Psi_0\rangle}{4(2P_0^2-M^2)}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -4P_2^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} e^{i(\mathbf{P},\mathbf{X})} =$$
$$=\int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_0|\Psi_0\rangle}{2P_0^2-M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -P_2^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

If the mass is zero, this expression is the matrix element between states of the photon.

Note that the matrix entering the matrix element has no inverse, so we do not try to find the equation of motion or Lagrangian density, they are not necessary in this case, since we have an explicit expression for the field operator. We do not have to worry about following the principles of Lorentzian covariance, gauge invariance, or following ideas of symmetry. Instead, we rely only on the fulfilment of canonical commutation relations for the field operator. The field operator is written identically in any frame of reference, and to pass to another frame it is enough to know how the momentum vector is transformed, which is transformed by exactly the same law as the coordinate vector, which ensures the invariance of the phase of the plane wave. In other words, the field is not a vector but a set of pseudovectors (pseudospinors in spinor space), only momentum and coordinate are vectors (spinor).

We can make our reasoning more intuitively clear if we define the creation and annihilation operators of the field particle

$$\boldsymbol{B}(\mathbf{X})=\int \frac{d^4P}{(2\pi)^4}\begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P})+d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P})\\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P})+b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})\end{bmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$
$$\boldsymbol{A}(\mathbf{X})=\int \frac{d^4P}{(2\pi)^4}\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})+b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})+b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\end{bmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

Let us find the commutation relations between the components of these operators

$$\left[A_i(\mathbf{X}),B_j(\mathbf{X}')\right]=A_i(\mathbf{X})B_j(\mathbf{X}')-B_j(\mathbf{X}')A_i(\mathbf{X})=\left(\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{X}')-\left(\boldsymbol{B}(\mathbf{X}')\boldsymbol{A}^T(\mathbf{X})\right)^T\right)_{ij}$$
$$\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{X}')-\left(\boldsymbol{B}(\mathbf{X}')\boldsymbol{A}^T(\mathbf{X})\right)^T =$$
$$=\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}-d_1^*(\mathbf{P}')d_1(\mathbf{P})\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})}\\ +b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}-b_1^*(\mathbf{P}')b_1(\mathbf{P})\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})}\\ +b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}-b_4^*(\mathbf{P}')b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\,e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})}\\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}-d_4^*(\mathbf{P}')d_4(\mathbf{P})\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{-i(\mathbf{P}',\mathbf{X}')}e^{i(\mathbf{P},\mathbf{X})}\end{bmatrix}$$
$$=\iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}\begin{bmatrix} \left(d_1(\mathbf{P})d_1^*(\mathbf{P}')-d_1^*(\mathbf{P}')d_1(\mathbf{P})\right)\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}\\ +\left(b_1(\mathbf{P})b_1^*(\mathbf{P}')-b_1^*(\mathbf{P}')b_1(\mathbf{P})\right)\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}\\ +\left(b_4(\mathbf{P})b_4^*(\mathbf{P}')-b_4^*(\mathbf{P}')b_4(\mathbf{P})\right)\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}\\ +\left(d_4(\mathbf{P})d_4^*(\mathbf{P}')-d_4^*(\mathbf{P}')d_4(\mathbf{P})\right)\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')}\end{bmatrix}$$

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$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}\begin{bmatrix} (\delta(\mathbf{P}-\mathbf{P}'))\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(\delta(\mathbf{P}-\mathbf{P}'))\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(\delta(\mathbf{P}-\mathbf{P}'))\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \\ +(\delta(\mathbf{P}-\mathbf{P}'))\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}')e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P}',\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P})e^{i(\mathbf{P},\mathbf{X})}e^{-i(\mathbf{P},\mathbf{X}')} \end{bmatrix}$$

$$= \int \frac{d^4P}{(2\pi)^4}\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \end{bmatrix} e^{-i(\mathbf{P},\mathbf{X}-\mathbf{X}')}$$

$$= \int \frac{d^4P}{(2\pi)^4}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X}-\mathbf{X}')}$$

$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int \frac{d^4P}{(2\pi)^4} e^{-i(\mathbf{P},\mathbf{X}-\mathbf{X}')}$$

$$= \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(\mathbf{X}-\mathbf{X}')$$

$$A_i(\mathbf{X})B_j(\mathbf{X}') - B_j(\mathbf{X}')A_i(\mathbf{X}) = \delta(\mathbf{X}-\mathbf{X}')\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

As we see, the commutation relations are satisfied for the creation and annihilation operators. Let us define the total particle number operator in the form

$$N_{ji}(\mathbf{X}) = B_j(\mathbf{X})A_i(\mathbf{X})$$

$$N_{ji} = \int d^4X\, B_j(\mathbf{X})A_i(\mathbf{X})$$

Let's find the commutator

$$[N_{ji}, B_j(\mathbf{X})] = \int d^4\mathrm{X}' \{B_j(\mathbf{X}')A_i(\mathbf{X}')B_j(\mathbf{X}) - B_j(\mathbf{X})B_j(\mathbf{X}')A_i(\mathbf{X}')\} =$$

$$\int d^4\mathrm{X}' \{B_j(\mathbf{X}')A_i(\mathbf{X}')B_j(\mathbf{X}) - B_j(\mathbf{X}')B_j(\mathbf{X})A_i(\mathbf{X}')\} =$$

$$\int d^4\mathrm{X}' \left\{B_j(\mathbf{X}')\left(A_i(\mathbf{X}')B_j(\mathbf{X}) - B_j(\mathbf{X})A_i(\mathbf{X}')\right)\right\} =$$

$$\int d^4\mathrm{X}' \{B_j(\mathbf{X}')\delta(\mathbf{X}'-\mathbf{X})\}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} = B_j(\mathbf{X})\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

Let's define the vacuum state using the relations

$$d_1(\mathbf{P})|\Psi_0\rangle = 0 \quad b_1(\mathbf{P})|\Psi_0\rangle = 0 \quad d_4(\mathbf{P})|\Psi_0\rangle = 0 \quad b_4(\mathbf{P})|\Psi_0\rangle = 0$$

which implies

$$A_i(\mathbf{X})|\Psi_0\rangle = 0$$

$$N_{ji}|\Psi_0\rangle = \int d^4X\, B_j(\mathbf{X})A_i(\mathbf{X})|\Psi_0\rangle = 0$$

Let's act on vacuum by the creation operator and for the obtained state we find eigenvalues of the particle number operator

$$\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X}) = [N_{ji}, B_j(\mathbf{X})] = N_{ji}B_j(\mathbf{X}) - B_j(\mathbf{X})N_{ji}$$

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$$\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X})|\Psi_0\rangle = N_{ji}B_j(\mathbf{X})|\Psi_0\rangle - B_j(\mathbf{X})N_{ji}|\Psi_0\rangle$$

$$N_{ji}\big(B_j(\mathbf{X})|\Psi_0\rangle\big) = \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} \big(B_j(\mathbf{X})|\Psi_0\rangle\big)$$

If we apply normalization

$$\frac{A(\mathbf{X})}{2\mathbf{P}^{\mathrm{T}}\mathbf{P}} \quad \frac{B(\mathbf{X})}{2\mathbf{P}^{\mathrm{T}}\mathbf{P}}$$

then the eigenvalues will have the form

$$N_{ji}\big(B_j(\mathbf{X})|\Psi_0\rangle\big) = \frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} \big(B_j(\mathbf{X})|\Psi_0\rangle\big)$$

Note that in the case of the photon field, the matrix, taking into account the normalization, contains elements whose modulus is less than or equal to ½, since at zero mass $P_2^2 \le P_0^2$.

The fact that for the creation and annihilation operator's commutation relations are fulfilled, allows to conclude that quanta of the field obey Bose statistics, therefore a single action of the creation operator increases the number of particles in the field by one, and the action of the annihilation operator decreases this number by one. Hence, by means of these operators it is possible to write the matrix element not only for the case when the initial and final states are vacuum, but also for the initial state with an arbitrary number of particles

$$\frac{\langle\Psi_n|\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle}{4\mathbf{P}^{\mathrm{T}}\mathbf{P}} = \int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_n|\Psi_n\rangle}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

For illustration let us consider a one-particle state

$$|\Psi_1\rangle \equiv B_j(\mathbf{X})|\Psi_0\rangle$$

$$N_{ji}|\Psi_1\rangle = \frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_1\rangle$$

and act on it with the creation operator. Again, let's take into account

$$\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X}) = \big[N_{ji}, B_j(\mathbf{X})\big] = N_{ji}B_j(\mathbf{X}) - B_j(\mathbf{X})N_{ji}$$

$$\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X})|\Psi_1\rangle = N_{ji}B_j(\mathbf{X})|\Psi_1\rangle - B_j(\mathbf{X})N_{ji}|\Psi_1\rangle$$

$$= N_{ji}B_j(\mathbf{X})|\Psi_1\rangle - B_j(\mathbf{X})\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_1\rangle$$

The result is

$$N_{ji}B_j(\mathbf{X})|\Psi_1\rangle = 2\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} B_j(\mathbf{X})|\Psi_1\rangle$$

The eigenvalue of the particle number operator has increased, instead of a one-particle state we have a two-particle state

$$|\Psi_2\rangle \equiv B_j(\mathbf{X})|\Psi_1\rangle$$

$$N_{ji}|\Psi_2\rangle = 2\frac{1}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_2\rangle$$

Further application of the creation operator increases the number of particles to any value. Now let us find a commutator for the annihilation operator, without taking into account the normalization for the moment

$$[N_{ji}, A_i(\mathbf{X})] = \int d^4\mathrm{X}' \{B_j(\mathbf{X}')A_i(\mathbf{X}')A_i(\mathbf{X}) - A_i(\mathbf{X})B_j(\mathbf{X}')A_i(\mathbf{X}')\} =$$

$$\int d^4\mathrm{X}' \{B_j(\mathbf{X}')A_i(\mathbf{X})A_i(\mathbf{X}') - A_i(\mathbf{X})B_j(\mathbf{X}')A_i(\mathbf{X}')\} =$$

$$\int d^4\mathrm{X}' \left\{\left(B_j(\mathbf{X}')A_i(\mathbf{X}) - A_i(\mathbf{X})B_j(\mathbf{X}')\right)A_i(\mathbf{X}')\right\} =$$

$$\int d^4\mathrm{X}' \{-\delta(\mathbf{X}' - \mathbf{X})A_i(\mathbf{X}')\}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} = -A_i(\mathbf{X})\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

The ratios have been taken into account here

$$A_i(\mathbf{X})B_j(\mathbf{X}') - B_j(\mathbf{X}')A_i(\mathbf{X}) = \delta(\mathbf{X} - \mathbf{X}')\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

$$B_j(\mathbf{X}')A_i(\mathbf{X}) - A_i(\mathbf{X})B_j(\mathbf{X}') = -\delta(\mathbf{X} - \mathbf{X}')\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}$$

Let's act by the annihilation operator

$$-\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} A_i(\mathbf{X}) = [N_{ji}, A_i(\mathbf{X})] = N_{ji}A_i(\mathbf{X}) - A_i(\mathbf{X})N_{ji}$$

on the two-particle state and for the obtained state

$$-\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} A_i(\mathbf{X})|\Psi_2\rangle = N_{ji}A_i(\mathbf{X})|\Psi_2\rangle - A_i(\mathbf{X})N_{ji}|\Psi_2\rangle$$

we find the eigenvalues of the particle number operator

$$N_{ji}(A_i(\mathbf{X})|\Psi_2\rangle) =$$

$$= -\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} A_i(\mathbf{X})|\Psi_2\rangle + 2\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} (A_i(\mathbf{X})|\Psi_2\rangle)$$

$$N_{ji}(A_i(\mathbf{X})|\Psi_2\rangle) = 1\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} A_i(\mathbf{X})|\Psi_2\rangle$$

Here, the fact has been used that without taking into account the normalization we have

$$N_{ji}|\Psi_2\rangle = 2\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_2\rangle$$

Thus, the annihilation operator reduces the number of particles and puts the field into a single-particle state.

Separate application of the creation and annihilation operators more corresponds to the ideology of second quantization than their use only as a sum, i.e. only as a field operator

$$\boldsymbol{\varphi}(\mathbf{X}) = \boldsymbol{A}(\mathbf{X}) + \boldsymbol{B}(\mathbf{X})$$

In particular, since

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$$\langle\Psi_0|\boldsymbol{\varphi}(\mathbf{X})\boldsymbol{\varphi}^T(\mathbf{0})|\Psi_0\rangle = \langle\Psi_0|\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{0})|\Psi_0\rangle$$

then the matrix element really acquires the sense of the amplitude of the probability that the particle is born at the origin and annihilated at the point with coordinates $\mathbf{X}$.

Moreover, now the matrix element can be not bound to the vacuum state, but can be applied to the field state with arbitrary number of particles n>0. The application of the sum of operators to some state makes sense only in the case when all operators except one give as a result zero. Therefore, at the usual approach we have to work only with the vacuum state so that at calculation of the matrix element the annihilation operator gives zero. In our approach this restriction is removed, the operators are not summed, but only multiplied, and they can be applied to a state with any number of particles. For this purpose, let us take into account the following relations

$$\langle\Psi(\mathbf{P},d_1)_n|d_1(\mathbf{P})d_1^*(\mathbf{P}')|\Psi(\mathbf{P},d_1)_n\rangle = \langle\Psi(\mathbf{P},d_1)_n|d_1^*(\mathbf{P})d_1(\mathbf{P}')|\Psi(\mathbf{P},d_1)_n\rangle$$
$$= \langle\Psi(\mathbf{P},d_1)_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},d_1)_n\rangle = \langle\Psi(\mathbf{P})_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$
$$\langle\Psi(\mathbf{P},b_1)_n|b_1(\mathbf{P})b_1^*(\mathbf{P}')|\Psi(\mathbf{P},b_1)_n\rangle = \langle\Psi(\mathbf{P},b_1)_n|b_1^*(\mathbf{P})b_1(\mathbf{P}')|\Psi(\mathbf{P},b_1)_n\rangle$$
$$= \langle\Psi(\mathbf{P},b_1)_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},b_1)_n\rangle = \langle\Psi(\mathbf{P})_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$
$$\langle\Psi(\mathbf{P},d_4)_n|d_4(\mathbf{P})d_4^*(\mathbf{P}')|\Psi(\mathbf{P},d_4)_n\rangle = \langle\Psi(\mathbf{P},d_4)_n|d_4^*(\mathbf{P})d_4(\mathbf{P}')|\Psi(\mathbf{P},d_4)_n\rangle$$
$$= \langle\Psi(\mathbf{P},d_4)_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},d_4)_n\rangle = \langle\Psi(\mathbf{P})_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$
$$\langle\Psi(\mathbf{P},b_4)_n|b_4(\mathbf{P})b_4^*(\mathbf{P}')|\Psi(\mathbf{P},b_4)_n\rangle = \langle\Psi(\mathbf{P},b_4)_n|b_4^*(\mathbf{P})b_4(\mathbf{P}')|\Psi(\mathbf{P},b_4)_n\rangle$$
$$= \langle\Psi(\mathbf{P},b_4)_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P},b_4)_n\rangle = \langle\Psi(\mathbf{P})_n|\delta(\mathbf{P}-\mathbf{P}')|\Psi(\mathbf{P})_n\rangle$$

$$\langle\Psi_n|\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{0})|\Psi_n\rangle = \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\langle\Psi(\mathbf{P})_n\left|\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P})\end{bmatrix}\begin{bmatrix} b_4^*(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + d_4^*(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_1^*(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + b_1^*(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}')\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{i(\mathbf{P},\mathbf{X})}$$
$$= \iint \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\langle\Psi(\mathbf{P})_n\left|\begin{bmatrix} d_1(\mathbf{P})d_1^*(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}') + b_1(\mathbf{P})b_1^*(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}') \\ +d_4(\mathbf{P})d_4^*(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}') + b_4(\mathbf{P})b_4^*(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}')\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{i(\mathbf{P},\mathbf{X})}$$
$$= \int \frac{d^4P}{(2\pi)^4}\langle\Psi(\mathbf{P})_n\left|\begin{bmatrix} \mathbf{V1}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) + \mathbf{U4}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \\ +\mathbf{U1}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) + \mathbf{V4}(\mathbf{P})\mathbf{V1}^T(\mathbf{P})\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{i(\mathbf{P},\mathbf{X})}$$
$$= \int \frac{d^4P}{(2\pi)^4}\langle\Psi(\mathbf{P})_n\left|\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{i(\mathbf{P},\mathbf{X})}$$
$$= \int \frac{d^4P}{(2\pi)^4}\langle\Psi(\mathbf{P})_n|\Psi(\mathbf{P})_n\rangle\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$
$$= \langle\Psi_n|\Psi_n\rangle\int \frac{d^4P}{(2\pi)^4}\begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

The assumption used here is that the scalar products $\langle\Psi(\mathbf{P})_n|\Psi(\mathbf{P})_n\rangle = \langle\Psi_n|\Psi_n\rangle$ are the same for any values of momentum. Taking into account the normalization

$$\langle\Psi_n|\boldsymbol{A}(\mathbf{X})\boldsymbol{B}^T(\mathbf{0})|\Psi_n\rangle = \int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_n|\Psi_n\rangle}{2P_0^2 - M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} e^{i(\mathbf{P},\mathbf{X})}$$

At non-zero number of particles we can change the order of operators and first apply the annihilation operator

$$\langle\Psi_n|\boldsymbol{B}(\mathbf{X})\boldsymbol{A}^T(\mathbf{0})|\Psi_n\rangle = \int\int \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\langle\Psi(\mathbf{P})_n\left|\begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P})\end{bmatrix}\begin{bmatrix} d_1(\mathbf{P}')\mathbf{V1}^T(\mathbf{P}') + b_1(\mathbf{P}')\mathbf{U4}^T(\mathbf{P}') \\ +d_4(\mathbf{P}')\mathbf{U1}^T(\mathbf{P}') + b_4(\mathbf{P}')\mathbf{V4}^T(\mathbf{P}')\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{-i(\mathbf{P},\mathbf{X})}$$
$$= \int\int \frac{d^4P}{(2\pi)^4}\frac{d^4P'}{(2\pi)^4}$$
$$\langle\Psi(\mathbf{P})_n\left|\begin{bmatrix} d_1^*(\mathbf{P})d_1(\mathbf{P}')\mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}') + b_1^*(\mathbf{P})b_1(\mathbf{P}')\mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}') \\ +d_4^*(\mathbf{P})d_4(\mathbf{P}')\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}') + b_4^*(\mathbf{P})b_4(\mathbf{P}')\mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}')\end{bmatrix}\right|\Psi(\mathbf{P})_n\rangle e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n \left| \begin{bmatrix} \mathbf{U1}(\mathbf{P})\mathbf{V1}^T(\mathbf{P}) + \mathbf{V4}(\mathbf{P})\mathbf{U4}^T(\mathbf{P}) \\ +\mathbf{U4}(\mathbf{P})\mathbf{U1}^T(\mathbf{P}) + \mathbf{V1}(\mathbf{P})\mathbf{V4}^T(\mathbf{P}) \end{bmatrix} \right| \Psi(\mathbf{P})_n \rangle \, e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n \left| \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| \Psi(\mathbf{P})_n \rangle \, e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \int \frac{d^4P}{(2\pi)^4} \langle \Psi(\mathbf{P})_n | \Psi(\mathbf{P})_n \rangle \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

$$= \langle \Psi_n | \Psi_n \rangle \int \frac{d^4P}{(2\pi)^4} \begin{pmatrix} 4P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

After normalization we obtain

$$\langle \Psi_n | \boldsymbol{B}(\mathbf{X}) \boldsymbol{A}^T(\mathbf{0}) | \Psi_n \rangle = \int \frac{d^4P}{(2\pi)^4} \frac{\langle \Psi_n | \Psi_n \rangle}{2P_0^2 - M^2} \begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i(\mathbf{P},\mathbf{X})}$$

Let's return to the previously used definition of the vacuum state by means of relations

$$d_1(\mathbf{P})|\Psi_0\rangle = 0 \quad b_1(\mathbf{P})|\Psi_0\rangle = 0 \quad d_4(\mathbf{P})|\Psi_0\rangle = 0 \quad b_4(\mathbf{P})|\Psi_0\rangle = 0$$

$$d_1^*(\mathbf{P})d_1(\mathbf{P})|\Psi_0\rangle = 0 \quad b_1^*(\mathbf{P})b_1(\mathbf{P})|\Psi_0\rangle = 0 \quad d_4^*(\mathbf{P})d_4(\mathbf{P})|\Psi_0\rangle = 0 \quad b_4^*(\mathbf{P})b_4(\mathbf{P})|\Psi_0\rangle = 0$$

which implies

$$A_i(\mathbf{X})|\Psi_0\rangle = 0$$

$$N_{ji}|\Psi_0\rangle = \int d^4X \, B_j(\mathbf{X}) A_i(\mathbf{X}) |\Psi_0\rangle = 0$$

As we have seen, the action of the creation operator transforms the zero-particle state into a one-particle state

$$N_{ji}\big(B_j(\mathbf{X})|\Psi_0\rangle\big) = N_{ji}|\Psi_1\rangle = \frac{1}{2P_0^2 - M^2} \begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} |\Psi_1\rangle$$

At that, none of the operators of the number of particles with a particular value of momentum

$$d_1^*(\mathbf{P})d_1(\mathbf{P})|\Psi_1\rangle \quad b_1^*(\mathbf{P})b_1(\mathbf{P})|\Psi_1\rangle \quad d_4^*(\mathbf{P})d_4(\mathbf{P})|\Psi_1\rangle \quad b_4^*(\mathbf{P})b_4(\mathbf{P})|\Psi_1\rangle$$

has no definite meaning, since the particle is only one. In this connection it makes sense not to define the vacuum in such a detailed way, it is enough to define that the vacuum state is characterized by only one condition

$$N_{ji}|\Psi_0\rangle = \int d^4X \, B_j(\mathbf{X}) A_i(\mathbf{X}) |\Psi_0\rangle = 0$$

At this approach the field energy is not equal to the sum of energies of partial oscillations, accordingly the question about the energy of zero-point oscillations of each oscillator constituting the field is removed. We get rid of the problem of infinite energy of the sum of zero-point vibrations of an infinite number of oscillators.

We can use the creation and annihilation operators instead of the field operator, and we can apply them to an arbitrary state, not just the vacuum state. So, we don't need to calculate the vacuum mean and apply Wick's theorem.

We would like the matrix element to have properties of the Green's function, i.e., to satisfy the equation

$$-\left(\frac{\partial^2}{\partial {X_0}^2} - \frac{\partial^2}{\partial {X_1}^2} - \frac{\partial^2}{\partial {X_2}^2} - \frac{\partial^2}{\partial {X_3}^2} + m^2\right) D(\mathbf{X}) = \delta(\mathbf{X})$$

The solution of this equation has the form

$$D(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4} \frac{e^{i(\mathbf{P},\mathbf{X})}}{P_0^2 - P_1^2 - P_2^2 - P_3^2 - M^2} = \int \frac{d^4P}{(2\pi)^4} \frac{e^{i(\mathbf{P},\mathbf{X})}}{P^2 - M^2}$$

Therefore, we complement the denominator of the integrand, for what we relate the creation and annihilation operators

$$\widetilde{\boldsymbol{B}}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4}\begin{bmatrix} b_4^*(\mathbf{P})\mathbf{V1}(\mathbf{P}) + d_4^*(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_1^*(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_1^*(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix}\frac{e^{-i(\mathbf{P},\mathbf{X})}}{\sqrt{P^2-M^2}}$$

$$\widetilde{\boldsymbol{A}}(\mathbf{X}) = \int \frac{d^4P}{(2\pi)^4}\begin{bmatrix} d_1(\mathbf{P})\mathbf{V1}(\mathbf{P}) + b_1(\mathbf{P})\mathbf{U4}(\mathbf{P}) \\ +d_4(\mathbf{P})\mathbf{U1}(\mathbf{P}) + b_4(\mathbf{P})\mathbf{V4}(\mathbf{P}) \end{bmatrix}\frac{e^{i(\mathbf{P},\mathbf{X})}}{\sqrt{P^2-M^2}}$$

$$\langle\Psi_n|\widetilde{\boldsymbol{B}}(\mathbf{X})\widetilde{\boldsymbol{A}}^T(\mathbf{0})|\Psi_n\rangle = \int \frac{d^4P}{(2\pi)^4}\frac{\langle\Psi_n|\Psi_n\rangle}{2P_0^2-M^2}\begin{pmatrix} P_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -P_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\frac{e^{-i(\mathbf{P},\mathbf{X})}}{P^2-M^2}$$

After such normalization, one doubts the expediency of the normalization introduced earlier, namely, the inclusion of the multiplier in the formula

$$\frac{\langle\Psi_n|\Psi_n\rangle}{2P_0^2-M^2}$$

By analogy with the introduced creation and annihilation operators for fields in vector space, let us describe the corresponding operators for fields in spinor space. As an initial one we use the previously described field operator for the fermionic field

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$
$$+\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

Let us define the creation and annihilation operators

$$\boldsymbol{b}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$\boldsymbol{a}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

Let's find anticommutation relations between components of these operators

$$\{a_i(\mathbf{x}), b_j(\mathbf{x}')\} = a_i(\mathbf{x})b_j(\mathbf{x}') + b_j(\mathbf{x}')a_i(\mathbf{x}) = \left(\boldsymbol{a}(\mathbf{x})\boldsymbol{b}(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T\right)_{ij}$$

$$\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^T(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}')\boldsymbol{a}^T(\mathbf{x})\right)^T =$$
$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix}$$
$$\begin{bmatrix} b_1^*(\mathbf{p}')\mathbf{u1}^+(\mathbf{p}') + ib_2^*(\mathbf{p}')\mathbf{u3}^+(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\mathbf{v1}^+(\mathbf{p}') + ib_3^*(\mathbf{p}')\mathbf{v3}^+(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p}') \end{bmatrix}$$
$$e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}$$
$$+$$
$$\left(\begin{matrix}\begin{bmatrix} b_1^*(\mathbf{p}')\overline{\mathbf{u1}}(\mathbf{p}') + ib_2^*(\mathbf{p}')\overline{\mathbf{u3}}(\mathbf{p}') + id_2^*(\mathbf{p}')\mathbf{u2}(\mathbf{p}') + d_1^*(\mathbf{p}')\mathbf{u4}(\mathbf{p}') \\ +b_4^*(\mathbf{p}')\overline{\mathbf{v1}}(\mathbf{p}') + ib_3^*(\mathbf{p}')\overline{\mathbf{v3}}(\mathbf{p}') + id_3^*(\mathbf{p}')\mathbf{v2}(\mathbf{p}') + d_4^*(\mathbf{p}')\mathbf{v4}(\mathbf{p}') \end{bmatrix} \\ \begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p}) + ib_2(\mathbf{p})\mathbf{u2}^+(\mathbf{p}) + b_1(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p}) + ib_3(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) + b_4(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) \end{bmatrix} \\ e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}\end{matrix}\right)^T$$

$$= \iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}') \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}') + \cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}') \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}') + \cdots \end{bmatrix} \\ e^{i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}$$

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$$
+\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u4}^+(\mathbf{p})\right)^{\boldsymbol{T}} \\ -b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u2}^+(\mathbf{p})\right)^{\boldsymbol{T}}+\cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u4}(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u2}(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}}+\cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}
$$

$$
=\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}')+d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u4}(\mathbf{p}')\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}} \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}')-d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u2}(\mathbf{p}')\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})\right)^{\boldsymbol{T}}+\cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}')+b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u1}}(\mathbf{p}')\mathbf{u4}^+(\mathbf{p})\right)^{\boldsymbol{T}} \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}')-b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u3}}(\mathbf{p}')\mathbf{u2}^+(\mathbf{p})\right)^{\boldsymbol{T}}+\cdots \end{bmatrix} \\ e^{i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}
$$

$$
=\iint \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} d_1(\mathbf{p})d_1^*(\mathbf{p}')\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}')+d_1^*(\mathbf{p}')d_1(\mathbf{p})\left(\mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p}')\right) \\ -d_2(\mathbf{p})d_2^*(\mathbf{p}')\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}')-d_2^*(\mathbf{p}')d_2(\mathbf{p})\left(\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p}')\right)+\cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}e^{-i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} b_1(\mathbf{p})b_1^*(\mathbf{p}')\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}')+b_1^*(\mathbf{p}')b_1(\mathbf{p})\left(\overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p}')\right) \\ -b_2(\mathbf{p})b_2^*(\mathbf{p}')\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}')-b_2^*(\mathbf{p}')b_2(\mathbf{p})\left(\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p}')\right)+\cdots \end{bmatrix} \\ e^{i\left(\overline{p_0'}x_2'+\overline{p_1'}x_3'+\overline{p_2'}x_0'+\overline{p_3'}x_1'+\overline{(\mathbf{p}',\mathbf{x}')}\right)}e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)} \end{bmatrix}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p})+\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p}) \\ -\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p})-\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})+\cdots \end{bmatrix} \\ e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p}) \\ -\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})-\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})+\cdots \end{bmatrix} \\ e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}\begin{bmatrix} \begin{bmatrix} \mathbf{u1}(\mathbf{p})\mathbf{u4}^{\boldsymbol{T}}(\mathbf{p})+\mathbf{u4}(\mathbf{p})\mathbf{u1}^{\boldsymbol{T}}(\mathbf{p})+\mathbf{v1}(\mathbf{p})\mathbf{v4}^{\boldsymbol{T}}(\mathbf{p})+\mathbf{v4}(\mathbf{p})\mathbf{v1}^{\boldsymbol{T}}(\mathbf{p}) \\ -\mathbf{u3}(\mathbf{p})\mathbf{u2}^{\boldsymbol{T}}(\mathbf{p})-\mathbf{u2}(\mathbf{p})\mathbf{u3}^{\boldsymbol{T}}(\mathbf{p})-\mathbf{v3}(\mathbf{p})\mathbf{v2}^{\boldsymbol{T}}(\mathbf{p})-\mathbf{v2}(\mathbf{p})\mathbf{v3}^{\boldsymbol{T}}(\mathbf{p}) \end{bmatrix} \\ e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \\ + \\ \begin{bmatrix} \overline{\mathbf{u4}}(\mathbf{p})\mathbf{u1}^+(\mathbf{p})+\overline{\mathbf{u1}}(\mathbf{p})\mathbf{u4}^+(\mathbf{p})+\overline{\mathbf{v4}}(\mathbf{p})\mathbf{v1}^+(\mathbf{p})+\overline{\mathbf{v1}}(\mathbf{p})\mathbf{v4}^+(\mathbf{p}) \\ -\overline{\mathbf{u2}}(\mathbf{p})\mathbf{u3}^+(\mathbf{p})-\overline{\mathbf{u3}}(\mathbf{p})\mathbf{u2}^+(\mathbf{p})-\overline{\mathbf{v2}}(\mathbf{p})\mathbf{v3}^+(\mathbf{p})-\overline{\mathbf{v3}}(\mathbf{p})\mathbf{v2}^+(\mathbf{p}) \end{bmatrix} \\ e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)} \end{bmatrix}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}\left(S^R(\mathbf{p})+S_R(\mathbf{p})\right)e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
$$

$$
+\int \frac{d^4p}{(2\pi)^4}\left(\overline{S_R}(\mathbf{p})+\overline{S^R}(\mathbf{p})\right)e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
$$

$$
=\int \frac{d^4p}{(2\pi)^4}\,4\begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix}e^{i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
$$

$$
+\int \frac{d^4p}{(2\pi)^4}\,4\begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & \overline{m} & 0 \\ 0 & 0 & 0 & \overline{m} \end{pmatrix}e^{-i\left(\overline{p_0}(x_2-x_2')+\overline{p_1}(x_3-x_3')+\overline{p_2}(x_0-x_0')+\overline{p_3}(x_1-x_1')+\overline{(\mathbf{p},\mathbf{x}-\mathbf{x}')}\right)}
$$

$$
=4mI\delta(\mathbf{x}'-\mathbf{x})+4\overline{m}I\delta(\mathbf{x}-\mathbf{x}')
$$

$$
\left\{a_i(\mathbf{x}'),b_j(\mathbf{x})\right\}=a_i(\mathbf{x}')b_j(\mathbf{x})+b_j(\mathbf{x})a_i(\mathbf{x}')=4Re(m)\delta(\mathbf{x}'-\mathbf{x})\delta_{ij}
$$

$$
\left\{b_j(\mathbf{x}'),a_i(\mathbf{x})\right\}=b_j(\mathbf{x}')a_i(\mathbf{x})+a_i(\mathbf{x})b_j(\mathbf{x}')=4Re(m)\delta(\mathbf{x}-\mathbf{x}')\delta_{ij}
$$

Besides these relations, the following `anti-commutation relations take place between the components of the annihilation and creation operators

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$$\{b_i(\mathbf{x}), b_j(\mathbf{x}')\} = b_i(\mathbf{x})b_j(\mathbf{x}') + b_j(\mathbf{x}')b_i(\mathbf{x}) = 0$$
$$\{a_i(\mathbf{x}), a_j(\mathbf{x}')\} = a_i(\mathbf{x})a_j(\mathbf{x}') + a_j(\mathbf{x}')a_i(\mathbf{x}) = 0$$

Let's define operators of the total number of particles in the form

$$N_{ji}(\mathbf{x}) = b_j(\mathbf{x})a_i(\mathbf{x}) \qquad N_{ji} = \int d^4x\, b_j(\mathbf{x})a_i(\mathbf{x})$$

Let's find the commutators

$$[N_{ji}, b_j(\mathbf{x})] = \int d^4\mathrm{x}' \,[b_j(\mathbf{x}')a_i(\mathbf{x}')b_j(\mathbf{x}) - b_j(\mathbf{x})b_j(\mathbf{x}')a_i(\mathbf{x}')] =$$
$$\int d^4\mathrm{x}' \,[b_j(\mathbf{x}')a_i(\mathbf{x}')b_j(\mathbf{x}) + b_j(\mathbf{x}')b_j(\mathbf{x})a_i(\mathbf{x}')] =$$
$$\int d^4\mathrm{x}' \left[b_j(\mathbf{x}')\left(a_i(\mathbf{x}')b_j(\mathbf{x}) + b_j(\mathbf{x})a_i(\mathbf{x}')\right)\right] =$$
$$4Re(m)\int d^4\mathrm{x}'\, b_j(\mathbf{x}')\delta(\mathbf{x}' - \mathbf{x})\delta_{ij} = 4Re(m)\delta_{ij}b_j(\mathbf{x}) = [N_{ji}, b_j(\mathbf{x})]$$
$$[N_{ji}, a_i(\mathbf{x})] = \int d^4\mathrm{x}' \,[b_j(\mathbf{x}')a_i(\mathbf{x}')a_i(\mathbf{x}) - a_i(\mathbf{x})b_j(\mathbf{x}')a_i(\mathbf{x}')] =$$
$$\int d^4\mathrm{x}' \,[-b_j(\mathbf{x}')a_i(\mathbf{x})a_i(\mathbf{x}') - a_i(\mathbf{x})b_j(\mathbf{x}')a_i(\mathbf{x}')] =$$
$$-\int d^4\mathrm{x}' \left[\left(b_j(\mathbf{x}')a_i(\mathbf{x}) + a_i(\mathbf{x})b_j(\mathbf{x}')\right)a_i(\mathbf{x}')\right] =$$
$$-4Re(m)\int d^4\mathrm{x}'\, \delta(\mathbf{x}' - \mathbf{x})\delta_{ij}a_i(\mathbf{x}') = -4Re(m)\delta_{ij}a_i(\mathbf{x}) = [N_{ji}, a_i(\mathbf{x})]$$

Instead of defining the vacuum state through its properties under the action of annihilation operators

$$d_1(\mathbf{p})|\Psi_0\rangle = d_2(\mathbf{p})|\Psi_0\rangle = b_2(\mathbf{p})|\Psi_0\rangle = b_1(\mathbf{p})|\Psi_0\rangle = 0$$
$$d_4(\mathbf{p})|\Psi_0\rangle = d_3(\mathbf{p})|\Psi_0\rangle = b_3(\mathbf{p})|\Psi_0\rangle = b_3(\mathbf{p})|\Psi_0\rangle = 0$$

which would entail the ratios

$$\boldsymbol{a}(\mathbf{x})|\Psi_0\rangle = 0 \qquad N_{ji}|\Psi_0\rangle = 0$$

we will not require from operators all these properties, but we will be limited by a weaker and simpler definition of vacuum, namely, absence of particles in vacuum

$$N_{ji}|\Psi_0\rangle = 0$$

Let's use the found commutator

$$4Re(m)\delta_{ij}b_j(\mathbf{x}) = N_{ji}b_j(\mathbf{x}) - b_j(\mathbf{x})N_{ji}$$
$$4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_0\rangle = N_{ji}b_j(\mathbf{x})|\Psi_0\rangle - b_j(\mathbf{x})N_{ji}|\Psi_0\rangle$$
$$N_{ji}b_j(\mathbf{x})|\Psi_0\rangle = 4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_0\rangle$$
$$|\Psi_1\rangle \equiv b_j(\mathbf{x})|\Psi_0\rangle$$
$$N_{ji}|\Psi_1\rangle = 4Re(m)\delta_{ij}|\Psi_1\rangle$$

On the obtained one-particle state let's act on the obtained one-particle state by the creation operator again

$$4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_1\rangle = N_{ji}b_j(\mathbf{x})|\Psi_1\rangle - b_j(\mathbf{x})N_{ji}|\Psi_1\rangle$$
$$4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_1\rangle = N_{ji}b_j(\mathbf{x})|\Psi_1\rangle - 4Re(m)\delta_{ij}b_j(\mathbf{x})|\Psi_1\rangle$$
$$N_{ji}b_j(x)|\Psi_1\rangle = 2(4Re(m)\delta_{ij})b_j(x)|\Psi_1\rangle$$
$$|\Psi_2\rangle \equiv b_j(\mathbf{x})|\Psi_1\rangle$$
$$N_{ji}|\Psi_2\rangle = 2(4Re(m)\delta_{ij})|\Psi_2\rangle$$

We have obtained a state with two particles and we can thus increase the number of particles to infinity. All particles are identical and indistinguishable from each other, each of them is in all allowed states, of which the free field has infinitely many. Electrons in an atom have fewer allowed states, but still any electron occupies all of them equally with the others. This theory describes both electron and positron, the difference between them being only in the sign of the mass, it being convenient to consider that the electron has a negative mass and the positron a positive one.

Similarly, we use the commutator of the annihilation operator

$$-4Re(m)\delta_{ij}a_i(\mathbf{x}) = N_{ji}a_i(\mathbf{x}) - a_i(\mathbf{x})N_{ji}$$
$$-4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_2\rangle = N_{ji}a_i(\mathbf{x})|\Psi_2\rangle - a_i(\mathbf{x})N_{ji}|\Psi_2\rangle$$
$$-4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_2\rangle = N_{ji}a_i(\mathbf{x})|\Psi_2\rangle - 2(4Re(m)\delta_{ij})a_i(\boldsymbol{x})|\Psi_2\rangle$$
$$N_{ji}a_i(\mathbf{x})|\Psi_2\rangle = 4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_2\rangle$$

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Thus, the action of the annihilation operator has transformed the two-particle state into a one-particle state. Using the same calculations, we obtain the result of the annihilation operator action on the one-particle state

$$N_{ji}a_i(\mathbf{x})|\Psi_1\rangle = 0 * a_i(\mathbf{x})|\Psi_1\rangle$$

And in the same way we define the result of its action on the null state

$$N_{ji}a_i(\mathbf{x})|\Psi_0\rangle = -4Re(m)\delta_{ij}a_i(\mathbf{x})|\Psi_0\rangle = 4Re(-m)\delta_{ij}a_i(\mathbf{x})|\Psi_0\rangle$$

We obtain a state with the number of particles minus one, but we see that in fact it is a state with one particle whose mass is negative. Thus, the positron annihilation operator is also the electron creation operator. It destroys positrons until they run out, after which it starts creating electrons. The creation operator, on the contrary, destroys electrons, and when they run out, starts creating positrons. Thus, since there are many electrons in our universe, this operator cannot be creating positrons because it cannot destroy all electrons due to their number. Moreover, the operator of annihilation of positrons because of the absence of the latter, only creates more and more electrons.

If the mass is zero, then in any state the number of particles is zero, i.e., for example, the electromagnetic field in spinor space, where it should be fermionic, simply has no particles. The absence of particles does not contradict the presence of the field, which is represented by the same 16 spinors, this field obeys Fermi statistics, and it has no charge and can be treated as a Majorana fermion. This field interacts with electrons in spinor space, and the result of the interaction manifests itself in vector space.

With the help of the creation and annihilation operators we can write the matrix element for the situation when the initial and final states are states with arbitrary number of particles

$$\langle\Psi_n|\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle =$$
$$\int\frac{d^4p}{(2\pi)^4}\langle\Psi_n|\Psi_n\rangle 4Re(m)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}e^{i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}$$
$$\langle\Psi_n|\boldsymbol{b}(\mathbf{x})\boldsymbol{a}^{\mathrm{T}}(\mathbf{0})|\Psi_n\rangle =$$
$$\int\frac{d^4p}{(2\pi)^4}\langle\Psi_n|\Psi_n\rangle 4Re(m)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}e^{-i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}$$

An indirect argument in favor of the fact that the sign of a fermion's charge is determined by the sign of its mass is as follows. In quantum electrodynamics, the amplitude of the scattering of fermions on each other is used to determine the interaction potential between two fermions. This amplitude is proportional to the product of the fermion's masses. The corresponding product is positive for two electrons or two positrons and negative for different particles. Therefore, depending on the sign of the product of the masses, fermions are either attracted or repelled.

We would like the spinor matrix element to have properties of the Green's function, i.e. to satisfy the equations which for this case are given below and which can be combined into one equation

$$\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial\overline{x_2}}+\frac{\partial}{\partial x_1}\frac{\partial}{\partial\overline{x_3}}+\overline{m}\right)D(\mathbf{x}) = \delta(\mathbf{x})$$
$$\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial\overline{x_2}}+\frac{\partial}{\partial x_1}\frac{\partial}{\partial\overline{x_3}}+\overline{m}\right)D(\mathbf{x}) = \delta(\mathbf{x})$$
$$\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial\overline{x_2}}+\frac{\partial}{\partial x_1}\frac{\partial}{\partial\overline{x_3}}+\overline{m}\right)\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial\overline{x_2}}+\frac{\partial}{\partial x_1}\frac{\partial}{\partial\overline{x_3}}+\overline{m}\right)D(\mathbf{x}) = \delta(\mathbf{x})$$

where the delta function can be represented as

$$\delta(\mathbf{x}) = \int\frac{d^4p}{(2\pi)^4}\,e^{i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1)})}$$

The solution of the combined equation has the form

$$D(\mathbf{x}) = \int\frac{d^4p}{(2\pi)^4}\frac{e^{i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}}{(\overline{p_0}p_2+\overline{p_1}p_3-m)(\overline{p_2}p_0+\overline{p_3}p_1-\overline{m})}$$

Therefore, we will supplement the denominator of the integrand, for which we will normalize the creation and annihilation operators

$$\widetilde{\boldsymbol{b}}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} \frac{e^{-i\left(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})}\right)}}{(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$
$$\widetilde{\boldsymbol{a}}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} \frac{e^{i\left(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})}\right)}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})}$$
$$\langle \Psi_n | \widetilde{\boldsymbol{a}}(\mathbf{x}) \widetilde{\boldsymbol{b}}^{\mathrm{T}}(\mathbf{y}) | \Psi_n \rangle =$$
$$= \int \frac{d^4p}{(2\pi)^4} \langle \Psi_n | \Psi_n \rangle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{4Re(m) e^{i\left((\mathbf{p},\mathbf{x}-\mathbf{y}) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$
$$\langle \Psi_n | \widetilde{\boldsymbol{b}}(\mathbf{x}) \widetilde{\boldsymbol{a}}^{\mathrm{T}}(\mathbf{y}) | \Psi_n \rangle =$$
$$= \int \frac{d^4p}{(2\pi)^4} \langle \Psi_n | \Psi_n \rangle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{4Re(m) e^{-i\left((\mathbf{p},\mathbf{x}-\mathbf{y}) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$

The electron and positron have different mass sign, so their matrix elements will be different.

We can repeat the above calculations, keeping the annihilation operator, but defining the creation operator differently

$$\boldsymbol{a}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} e^{i\left(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})}\right)}$$
$$\boldsymbol{b}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) - id_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) - ib_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + b_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +d_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) - id_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) - ib_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + b_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} e^{-i\left(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})}\right)}$$

As a result, we obtain the anticommutator

$$\left\{a_i(\mathbf{x}), b_j(\mathbf{x}')\right\} = a_i(\mathbf{x}) b_j(\mathbf{x}') + b_j(\mathbf{x}') a_i(\mathbf{x}) = \left(\boldsymbol{a}(\mathbf{x}) \boldsymbol{b}(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}') \boldsymbol{a}^T(\mathbf{x})\right)^T\right)_{ij}$$
$$\boldsymbol{a}(\mathbf{x}) \boldsymbol{b}^T(\mathbf{x}') + \left(\boldsymbol{b}(\mathbf{x}') \boldsymbol{a}^T(\mathbf{x})\right)^T = 4P_0 I \delta(\mathbf{x}' - \mathbf{x}) + 4P_0 I \delta(\mathbf{x} - \mathbf{x}') = 8P_0 \delta(\mathbf{x} - \mathbf{x}')$$
$$P_0 = p_0\overline{p_0} + p_1\overline{p_1} + p_2\overline{p_2} + p_3\overline{p_3}$$

As before, using the creation and annihilation operators, we construct the matrix element for a state with an arbitrary number of particles

$$\widetilde{\boldsymbol{a}}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} \frac{e^{i\left(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})}\right)}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})}$$
$$\widetilde{\boldsymbol{b}}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) - id_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) - ib_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + b_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +d_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) - id_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) - ib_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + b_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} \frac{e^{-i\left(\overline{p_0}x_2 + \overline{p_1}x_3 + \overline{p_2}x_0 + \overline{p_3}x_1 + \overline{(\mathbf{p},\mathbf{x})}\right)}}{(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$
$$\langle \Psi_n | \widetilde{\boldsymbol{a}}(\mathbf{x}) \widetilde{\boldsymbol{b}}^{\mathrm{T}}(\mathbf{y}) | \Psi_n \rangle =$$
$$= \int \frac{d^4p}{(2\pi)^4} \langle \Psi_n | \Psi_n \rangle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{8P_0 e^{i\left((\mathbf{p},\mathbf{x}-\mathbf{y}) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$
$$\langle \Psi_n | \widetilde{\boldsymbol{b}}(\mathbf{x}) \widetilde{\boldsymbol{a}}^{\mathrm{T}}(\mathbf{y}) | \Psi_n \rangle =$$
$$= \int \frac{d^4p}{(2\pi)^4} \langle \Psi_n | \Psi_n \rangle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{8P_0 e^{-i\left((\mathbf{p},\mathbf{x}-\mathbf{y}) + \overline{(\mathbf{p},\mathbf{x}-\mathbf{y})}\right)}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$

Now instead of mass the matrix element includes energy, therefore such theory is applicable also to the field with zero mass, i.e. it can serve as a model not only for the electron, but also for the electromagnetic field in spinor space. The only problem is that if earlier the action of the

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annihilation operator on the zero-point state gave a particle with negative mass, now this action gives a particle with negative energy, which makes the interpretation of such theory more difficult.

Note that in this revision the creation and annihilation operators are conjugate to each other

$$\widetilde{\boldsymbol{a}}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + \ b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} \frac{e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})}$$

$$\widetilde{\boldsymbol{b}}(\mathbf{x}) = \overline{\widetilde{\boldsymbol{a}}(\mathbf{x})} = \int \frac{d^4p}{(2\pi)^4}$$

$$\begin{bmatrix} d_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) - id_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) - ib_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + b_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +d_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) - id_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) - ib_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + b_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} \frac{e^{-i\left(\overline{(\mathbf{p},\mathbf{x})}+(\mathbf{p},\mathbf{x})\right)}}{(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$

The considered free field matrix elements describe the situation when there is a point source with coordinate $\mathbf{x}$ and a point sink with coordinate $\mathbf{y}$. In the general case in the spinor space the distribution of source-stocks $\boldsymbol{J}(\mathbf{x})$ can be given and the value of

$$W(J) = -\frac{1}{2}\iint d^4x\, d^4y\, J_i(\mathbf{x})D_{ij}(\mathbf{x}-\mathbf{y})J_j(\mathbf{y})$$

which is used for finding the integral over the trajectories and which can be written using the Fourier transform for the spinor space

$$J_i(\mathbf{p}) \equiv \int \frac{d^4x}{(2\pi)^4} J_i(\mathbf{x}) e^{-i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$W(J) = -\frac{1}{2}\iint \frac{d^4p}{(2\pi)^4}\overline{J_i(\mathbf{p})}\,\delta_{ij}\frac{8(p_0\overline{p_0} + p_1\overline{p_1} + p_2\overline{p_2} + p_3\overline{p_3})}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}J_j(\mathbf{p})$$

In quantum field theory it is customary to calculate a similar quantity

$$W(J) = -\frac{1}{2}\iint d^4X\, d^4Y\, J_i(\mathbf{X})D_{ij}(\mathbf{X}-\mathbf{Y})J_j(\mathbf{Y})$$

in which the coordinates, momenta and the Fourier transform connecting them belong to the vector space. In our opinion, the transition to spinor space, more fundamental than vector space, which is a superstructure over spinor space, can eliminate divergences in calculating integrals in the framework of the formalism of the integral over trajectories. In momentum space the similarity is even more obvious, the kernels of the integrals are the same, the only difference is in the space where the integration takes place and the way of calculating the Fourier transform - either in vector or in spinor space

$$W(J) = -\frac{1}{2}\iint \frac{d^4P}{(2\pi)^4}\overline{J_i(\mathbf{P})}\frac{\delta_{ij}}{{P_0}^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 - m^2}J_j(\mathbf{P})$$

$$W(J) = -\frac{1}{2}\iint \frac{d^4p}{(2\pi)^4}\overline{J_i(\mathbf{p})}\frac{8\delta_{ij}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}J_j(\mathbf{p})$$

The spinor space has the additional advantage that the integrand is factorized

$$W(J) = -\frac{1}{2}\iint \frac{d^4p}{(2\pi)^4} 8P_0 \frac{\overline{J_i(\mathbf{p})}}{(\overline{p_2}p_0 + \overline{p_3}p_1 - \overline{m})}\delta_{ij}\frac{J_j(\mathbf{p})}{(\overline{p_0}p_2 + \overline{p_1}p_3 - m)}$$

This factorization in momentum space looks like a consequence of a more fundamental property of factorization in coordinate space

$$\begin{aligned} W(J) &= -\frac{1}{2}\iint d^4x\, d^4y\, J_i(\mathbf{x})D_{ij}(\mathbf{x}-\mathbf{y})J_j(\mathbf{y}) \\ &= -\frac{1}{2}\iint d^4x\, d^4y\, J_i(\mathbf{x})\langle\Psi_n|\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{y})|\Psi_n\rangle_{ij}J_j(\mathbf{y}) \\ &= -\frac{1}{2}\iint d^4x\, d^4y\, \langle\Psi_n\left|J_i(\mathbf{x})\left(\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{y})\right)_{ij}J_j(\mathbf{y})\right|\Psi_n\rangle \\ &= -\frac{1}{2}\langle\Psi_n\left|\iint d^4x\, d^4y\, J_i(\mathbf{x})(\boldsymbol{a}(\mathbf{x})\boldsymbol{b}^{\mathrm{T}}(\mathbf{y}))_{ij}J_j(\mathbf{y})\right|\Psi_n\rangle \end{aligned}$$

We can assume that first it makes sense to perform integration separately on $\mathbf{x}$ and $\mathbf{y}$, and only then to perform multiplication

$$W(J) = -\frac{1}{2}\langle\Psi_n^*\left|\left(\int d^4x\, \boldsymbol{J}^{\boldsymbol{T}}(\boldsymbol{x})\,\boldsymbol{a}(\boldsymbol{x})\right)\left(\int d^4y\, \boldsymbol{b}^{T}(\boldsymbol{y})\,\boldsymbol{J}(\boldsymbol{y})\right)\right|\Psi_n\rangle$$

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Since earlier we have obtained an explicit representation of field operators in both vector and spinor space, we do not need to refer to the equation of motion and the Lagrangian density. Proceeding from these representations, we define the creation and annihilation operators, and from them we construct the matrix element as a function of relative coordinates.

It may seem artificial to add an additional multiplier to the denominator of the plane wave. But then we must remember where the plane wave itself came from, namely, that it is a solution of the homogeneous wave equation. But the wave cannot arise from nothing, it must have a source, that is, it must be a solution of the inhomogeneous wave equation. In the case of a source in the form of a delta function, this solution includes just such a multiplier in the denominator. Furthermore, the introduction into the denominator of a factor proportional to energy is often justified by the need to ensure the Lorentz invariance of the integral.

Using the found field operators and propagators, one can make corrections in the rules of construction of Feynman diagrams. In particular, one of 16 pseudospinors should be compared to the external lines of the fermion, and one of four pseudovectors should be compared to the external lines of the boson instead of the polarization vector. One should also correct the type of propagators of the fermion and the boson. All these corrections can be done even within the classical Feynman diagrams in vector space. Then the outer lines will be explicitly expressed through the momentum components, and the diagram will look the same in any reference frame. A complete transition from integration in vector space to integration in spinor space would be a more radical step, the consequences of which are yet to be studied.

In particular, when integrating over the momenta associated with the inner lines of Feynman diagrams in vector space, a divergence of the form $\int d^4P(1/P^4)$, for example, arises. In spinor space the denominator remains the same, but the numerator in the integral $\int d^4p(1/P^4)$ is of lower order, since the component of the momentum vector is a bilinear form of the component of the momentum spinor.

In light of the existence of operators capable of creating and annihilating fermions and bosons that are constrained to a particular point in space, a re-evaluation of the concept of elementary particle interaction becomes imperative. Without recourse to prior reasoning or justification, it can be assumed that the interaction is described by the multiplication of any set of such operators bound to the same point in space. This product, which also incorporates constant multipliers to account for the degree of interaction, is then integrated over the entire coordinate space. The result of this integration is a multiple integral over the momentum space, which in vector space often diverges, but in spinor space perhaps not.

Previously defined operators

$$\widetilde{\boldsymbol{b}}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} b_1^*(\mathbf{p})\overline{\mathbf{u1}}(\mathbf{p}) + ib_2^*(\mathbf{p})\overline{\mathbf{u3}}(\mathbf{p}) + id_2^*(\mathbf{p})\mathbf{u2}(\mathbf{p}) + d_1^*(\mathbf{p})\mathbf{u4}(\mathbf{p}) \\ +b_4^*(\mathbf{p})\overline{\mathbf{v1}}(\mathbf{p}) + ib_3^*(\mathbf{p})\overline{\mathbf{v3}}(\mathbf{p}) + id_3^*(\mathbf{p})\mathbf{v2}(\mathbf{p}) + d_4^*(\mathbf{p})\mathbf{v4}(\mathbf{p}) \end{bmatrix} \frac{e^{-i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}}{(\overline{p_0}p_2+\overline{p_1}p_3-m)}$$
$$\widetilde{\boldsymbol{a}}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}$$
$$\begin{bmatrix} d_1(\mathbf{p})\mathbf{u1}(\mathbf{p}) + id_2(\mathbf{p})\mathbf{u3}(\mathbf{p}) + ib_2(\mathbf{p})\overline{\mathbf{u2}}(\mathbf{p}) + b_1(\mathbf{p})\overline{\mathbf{u4}}(\mathbf{p}) \\ +d_4(\mathbf{p})\mathbf{v1}(\mathbf{p}) + id_3(\mathbf{p})\mathbf{v3}(\mathbf{p}) + ib_3(\mathbf{p})\overline{\mathbf{v2}}(\mathbf{p}) + b_4(\mathbf{p})\overline{\mathbf{v4}}(\mathbf{p}) \end{bmatrix} \frac{e^{i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}}{(\overline{p_2}p_0+\overline{p_3}p_1-\overline{m})}$$

create or annihilate the fermion at a precisely defined point of space, while the momentum of the fermion is not defined. Taking into account the symmetry between the coordinate and momentum operators in quantum mechanics, we can postulate similar operators for a fermion with exactly definite momentum and indefinite coordinate.

$$\widetilde{\boldsymbol{b}}(\mathbf{p}) = \int d^4x$$
$$\begin{bmatrix} b_1^*(\mathbf{x})\overline{\mathbf{u1}}(\mathbf{x}) + ib_2^*(\mathbf{x})\overline{\mathbf{u3}}(\mathbf{x}) + id_2^*(\mathbf{x})\mathbf{u2}(\mathbf{x}) + d_1^*(\mathbf{x})\mathbf{u4}(\mathbf{x}) \\ +b_4^*(\mathbf{x})\overline{\mathbf{v1}}(\mathbf{x}) + ib_3^*(\mathbf{x})\overline{\mathbf{v3}}(\mathbf{x}) + id_3^*(\mathbf{x})\mathbf{v2}(\mathbf{x}) + d_4^*(\mathbf{x})\mathbf{v4}(\mathbf{x}) \end{bmatrix} \frac{e^{-i(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})})}}{(\overline{x_0}x_2+\overline{x_1}x_3-l)}$$
$$\widetilde{\boldsymbol{a}}(\mathbf{p}) = \int d^4x$$

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$$\begin{bmatrix} d_1(\mathbf{x})\mathbf{u1}(\mathbf{x}) + id_2(\mathbf{x})\mathbf{u3}(\mathbf{x}) + ib_2(\mathbf{x})\overline{\mathbf{u2}}(\mathbf{x}) + \ b_1(\mathbf{x})\overline{\mathbf{u4}}(\mathbf{x}) \\ +d_4(\mathbf{x})\mathbf{v1}(\mathbf{x}) + id_3(\mathbf{x})\mathbf{v3}(\mathbf{x}) + ib_3(\mathbf{x})\overline{\mathbf{v2}}(\mathbf{x}) + b_4(\mathbf{x})\overline{\mathbf{v4}}(\mathbf{x}) \end{bmatrix} \frac{e^{-i\left(\overline{p_0}x_2+\overline{p_1}x_3+\overline{p_2}x_0+\overline{p_3}x_1+\overline{(\mathbf{p},\mathbf{x})}\right)}}{\left(\overline{x_2}x_0+\overline{x_3}x_1-\overline{l}\right)}$$

where l is the invariant interval of the spinor coordinate space. Now the difference between an electron and a positron is determined by the sign of the interval.

The simplest recipe for the description of the electron-positron field seems to be the use in its equation of motion of the matrix, which we obtained in the construction of the anticommuting field operator, namely

$$S^R(\mathbf{p}) = \begin{pmatrix} \overline{p_3} \\ -\overline{p_2} \\ p_0 \\ p_1 \end{pmatrix} (p_1, -p_0, \overline{p_2}, \overline{p_3}) - \begin{pmatrix} -p_0 \\ -p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix} (\overline{p_2}, \overline{p_3}, p_1, -p_0) +$$

$$+ \begin{pmatrix} p_0 \\ p_1 \\ -\overline{p_3} \\ \overline{p_2} \end{pmatrix} (\overline{p_2}, \overline{p_3}, -p_1, p_0) - \begin{pmatrix} -\overline{p_3} \\ \overline{p_2} \\ p_0 \\ p_1 \end{pmatrix} (-p_1, p_0, \overline{p_2}, \overline{p_3}) =$$

which after the substitutions transforms into a differential operator

$$-S^R = \begin{pmatrix} \partial_1 \\ -\partial_0 \\ \overline{\partial_2} \\ \overline{\partial_3} \end{pmatrix} (\overline{\partial_3}, -\overline{\partial_2}, \partial_0, \partial_1) - \begin{pmatrix} \overline{-\partial_2} \\ -\partial_1 \\ -\partial_1 \\ \partial_0 \end{pmatrix} (\partial_0, \partial_1, \partial_1, \overline{-\partial_2}) +$$

$$+ \begin{pmatrix} \overline{\partial_2} \\ \overline{\partial_3} \\ -\partial_1 \\ \partial_0 \end{pmatrix} (\overline{p_2}, \partial_1, -\overline{\partial_3}, \overline{\partial_2}) - \begin{pmatrix} -\partial_1 \\ \overline{p_2} \\ \overline{\partial_2} \\ \overline{\partial_3} \end{pmatrix} (\overline{-\partial_3}, \overline{\partial_2}, \partial_0, \partial_1)$$

Earlier we showed that

$$S^R(\mathbf{p}) = 2 \begin{pmatrix} \overline{p_2}p_0 + \overline{p_3}p_1 & 0 & 0 & 0 \\ 0 & \overline{p_2}p_0 + \overline{p_3}p_1 & 0 & 0 \\ 0 & 0 & \overline{p_2}p_0 + \overline{p_3}p_1 & 0 \\ 0 & 0 & 0 & \overline{p_2}p_0 + \overline{p_3}p_1 \end{pmatrix} =$$

$$= 2 \begin{pmatrix} \overline{m} & 0 & 0 & 0 \\ 0 & \overline{m} & 0 & 0 \\ 0 & 0 & \overline{m} & 0 \\ 0 & 0 & 0 & \overline{m} \end{pmatrix} = 2\overline{m}I$$

Thus, the equation of motion has the form

$$\left(\overline{S^R} + 2mI\right)\boldsymbol{\varphi}(\mathbf{x}) = 0$$

After transition to the momentum space by means of integral transformation

$$\boldsymbol{\varphi}(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, \boldsymbol{\varphi}(\mathbf{p}) e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

we get the equation of motion

$$\left(\overline{S^R}(\mathbf{p}) - 2mI\right)\boldsymbol{\varphi}(\mathbf{p}) = 0$$

$$\left(\overline{(\overline{p_2}p_0 + \overline{p_3}p_1)} - m\right)I\boldsymbol{\varphi}(\mathbf{p}) = 0$$

The Green's function is a solution of the inhomogeneous equation

$$\left(\overline{S^R} + 2mI\right)D^R(\mathbf{x}) = I\delta(\mathbf{x})$$

$$D^R(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4}\, D^R(\mathbf{p}) e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

$$\left(\overline{S^R}(\mathbf{p}) - 2mI\right)D^R(\mathbf{p}) = I$$

$$\left(\overline{(\overline{p_2}p_0 + \overline{p_3}p_1)} - m\right)D^R(\mathbf{p}) = I$$

$$D^R(\mathbf{p}) = \frac{I}{\overline{(\overline{p_2}p_0 + \overline{p_3}p_1)} - m}$$

$$D^R(\mathbf{x}) = \int \frac{d^4p}{(2\pi)^4} \frac{I}{\overline{(\overline{p_2}p_0 + \overline{p_3}p_1)} - m} e^{i\left((\mathbf{p},\mathbf{x})+\overline{(\mathbf{p},\mathbf{x})}\right)}$$

Let us note that the diagonality of the left part of the free field equation

$$\overline{(\overline{p_2}p_0 + \overline{p_3}p_1)}I\boldsymbol{\varphi}(\mathbf{p}) = mI\boldsymbol{\varphi}(\mathbf{p})$$

allows to take into account in a simple way the influence of the external electromagnetic potential expressed in the spinor form

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$$\overline{\left(\overline{(p_2 - a_2)}(p_0 - a_0) + \overline{(p_3 - a_3)}(p_1 - a_1)\right)} I\boldsymbol{\varphi}(\mathbf{p}) = mI\boldsymbol{\varphi}(\mathbf{p})$$

Due to the diagonality of the equation, the addition of an external field does not break its invariance, unlike in the vector space. It turns out that at the spinor level the fields interact additively. There arises a strange assumption that since the matrix is diagonal, there is no need to zero out the non-diagonal elements to ensure mass invariance, hence Newton's law is not relevant in spinor space, and the components of the momentum spinor always commute.

Replacing the momentum components by derivatives gives us an analogue of the Schrödinger equation

$$\left((i\overline{\partial_0} - a_2)\overline{(\imath\overline{\partial_2} - a_0)} + (i\overline{\partial_1} - a_3)\overline{(\imath\overline{\partial_3} - a_1)}\right) I\boldsymbol{\varphi}(\mathbf{p}) = mI\boldsymbol{\varphi}(\mathbf{p})$$

However, we must not overlook the fact that we arrived at this equation by assuming that the particle is free and therefore equal to zero, for example, a matrix of the form

$$K^V(\mathbf{p}) = \begin{pmatrix} 0 & 0 & [p_2\overline{p_2} - \overline{p_2}p_2] & [p_2\overline{p_3} - \overline{p_3}p_2] \\ 0 & 0 & [p_3\overline{p_2} - \overline{p_2}p_3] & [p_3\overline{p_3} - \overline{p_3}p_3] \\ [p_0\overline{p_0} - \overline{p_0}p_0] & [p_0\overline{p_1} - \overline{p_1}p_0] & 0 & 0 \\ [p_1\overline{p_0} - \overline{p_0}p_1] & [p_1\overline{p_1} - \overline{p_1}p_1] & 0 & 0 \end{pmatrix}$$

from the initial composition of the matrix

$$S^V(\mathbf{p}) = K^V(\mathbf{p}) +$$

$$\begin{pmatrix} 0 \\ 0 \\ p_0 \\ p_1 \end{pmatrix}(\overline{p_0}, \overline{p_1}, 0,0) + \begin{pmatrix} \overline{p_1} \\ -\overline{p_0} \\ 0 \\ 0 \end{pmatrix}(0,0, p_1, -p_0) + \begin{pmatrix} 0 \\ 0 \\ \overline{p_3} \\ -\overline{p_2} \end{pmatrix}(p_3, -p_2, 0,0) + \begin{pmatrix} p_2 \\ p_3 \\ 0 \\ 0 \end{pmatrix}(0,0, \overline{p_2}, \overline{p_3})$$

If a particle is in an electromagnetic field, then after substitution

$$p_1 \to i\overline{\partial_3} - a_1 \quad p_0 \to i\overline{\partial_2} + a_0 \quad p_3 \to i\overline{\partial_1} + a_3 \quad p_2 \to i\overline{\partial_0} + a_2$$
$$\overline{p_1} \to i\partial_3 - \overline{a_1} \quad \overline{p_0} \to i\partial_2 + \overline{a_0} \quad \overline{p_3} \to i\partial_1 + \overline{a_3} \quad \overline{p_2} \to i\partial_0 + \overline{a_2}$$

the matrix $K^V(\mathbf{p})$ is generally not equal to zero. However, in developing the principle of absolute mass invariance, we must impose the condition that it is equal to zero, which will lead to an analogue of Newton's law in spinor space. For the matrix $S^R(\mathbf{p})$, there is also a corresponding commutator matrix $K^V(\mathbf{p})$, which is zero for a free particle, and in the presence of a field, this condition must be imposed separately after substitutions.

The theory outlined in the article allows us to answer the question how the fermion field changes under the action of Lorentz transformations on the coordinates. Exactly, if we move to another frame of reference by rotations and boosts, the coordinate spinor changes. As a consequence, the momentum spinor changes, the components of which are the coefficients of the expansion on the new coordinates, and the momentum spinor undergoes exactly the same transformation as the coordinates, so that the phases of all plane waves in spinor space do not change. The components of the new momentum spinor are substituted into the 16 spinors describing the fermion field. Thus, there is no any uniform law of transformation of a spinor of the fermionic field, each of 16 spinors corresponding to the particles forming it, is transformed in its own way.

4. Momentum Tensors for Maxwell's and Einstein's Equations

We need to take a closer look at Maxwell's inhomogeneous equations. Let's repeat the relations that we will analyze

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$P_{\mu\nu} \equiv P_\mu P_\nu - P_\nu P_\mu = -\partial_\mu P_\nu = \partial_\mu P_\nu - \partial_\nu P_\mu$$

$$P_{\mu\nu} = \begin{pmatrix} 0 & -\partial_0 P_1 & -\partial_0 P_2 & -\partial_0 P_3 \\ \partial_0 P_1 & 0 & -\partial_1 P_2 & -\partial_1 P_3 \\ \partial_0 P_2 & \partial_1 P_2 & 0 & -\partial_2 P_3 \\ \partial_0 P_3 & \partial_1 P_3 & \partial_2 P_3 & 0 \end{pmatrix}$$

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$$P_{\mu\nu} = \begin{pmatrix} 0 & -\partial_0 P_1 & -\partial_0 P_2 & -\partial_0 P_3 \\ \partial_0 P_1 & 0 & \partial_1 P_2 - \partial_2 P_1 & \partial_1 P_3 - \partial_3 P_1 \\ \partial_0 P_2 & \partial_2 P_1 - \partial_1 P_2 & 0 & \partial_2 P_3 - \partial_3 P_2 \\ \partial_0 P_3 & \partial_3 P_1 - \partial_1 P_3 & \partial_3 P_2 - \partial_2 P_3 & 0 \end{pmatrix}$$

On the one hand, the requirement of invariance of mass leads us to the equation

$$P_{\mu\nu} + eF_{\mu\nu} = 0$$

from which we can obtain

$$\partial_\mu P^{\mu\nu} + e\partial_\mu F^{\mu\nu} = 0$$

On the other hand, we have the universally recognized Maxwell's inhomogeneous equations

$$\partial_\mu F^{\mu\nu} = j^\nu$$

In the results, we are forced to recognize the validity of the relationship

$$\partial_\mu P^{\mu\nu} = -ej^\nu$$

On the left side is the vector obtained as a result of the contraction of the tensor consisting of the derivatives of the momentum vector. This tensor contains information about the internal rotations of the particle, for example, the spin of an electron. After contraction, all this information is explicitly absent. In addition, when contracting electromagnetic tensor to obtain the generally accepted four independent equations, the assumption is made that the derivatives with respect to different coordinate components commute, and, in addition, Lorentz gauge is applied. As a result, generality s violated and a too special type of equations is obtained. On the right-hand side is the current vector, which can be interpreted as the momentum vector multiplied by the charge. This view of current arose historically, since current was considered as the movement of charges. Quantities with the dimension of charge appear twice here, since current density also has this dimension. This is logical, since the left side is proportional to force, and it depends both on the charge generating the potential and on the charge of the particle on which the force acts. There is a clear analogy with Coulomb's law, which includes the product of interacting charges.

It can be argued that the expression

$$\partial_\mu P^{\mu\nu} + e\partial_\mu F^{\mu\nu} = 0$$

is a theoretically sound formulation of inhomogeneous Maxwell's equations, which follows from the requirement of mass invariance, while the expression

$$\partial_\mu F^{\mu\nu} = j^\nu$$

is an empirically established relationship, where the right-hand side contains a surrogate vector that historically arose from practical considerations.

By analogy, we can recall that in Einstein's equation, the Riemann tensor also contracts to the Ricci tensor with a loss of information. It can be assumed that by simultaneously contracting all the terms in the original equation, it is possible to strictly derive Einstein's inhomogeneous equation, as is the case with Maxwell's inhomogeneous equations.

The expression we propose looks like the definition of a current vector by a given momentum tensor $P^{\mu\nu}$. But why do we contract the tensor and reduce it to a vector? After all, when performing contraction, some of the information contained in the tensor is lost. For example, any information about the internal rotations of a particle described by the momentum rotor is definitely lost. One of the reasons we are forced to do this is that, in the early days of electrodynamics, current was described exclusively as a vector, and not as a tensor. Therefore, in order to write the equations, it was necessary to convert the left-hand side into a vector using a contraction, which is also combined with taking derivatives. With the development of quantum electrodynamics, this approach was reinforced by the fact that for an electron, the Dirac spinors describing it are combined in such a way that the current again turns out to be a vector. As a result, the system using the current vector turns out to be closed.

But if electric current is the movement of electrons, and electrons have spin, then spin does not disappear anywhere in electric current. Having spin, electrons are a source of magnetic fields, and when contracted into the current vector, this property of the source magnetic field source disappears, and there are no magnetic field sources on the right side of Maxwell's equations. This means that describing current as a vector is insufficient; we would like to define current as a tensor formed from the original spinor of the electron. The following definition is proposed

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$$T_{\mu\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\left[\frac{1}{2}\overline{p_\alpha}p_\beta\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\right]$$

where we use the following six matrices - commutators of gamma matrices in the Weyl basis

$$\sigma_{\mu\nu} = \frac{i}{2}\gamma_0^V[\gamma_\mu^V, \gamma_\nu^V]$$

$$\sigma_{01} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \sigma_{02} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \sigma_{03} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\sigma_{12} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \sigma_{13} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \sigma_{23} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This current tensor is antisymmetric and has zero trace

$$T = \begin{pmatrix} 0 & t_1 & t_2 & t_3 \\ -t_1 & 0 & t_4 & t_5 \\ -t_2 & -t_4 & 0 & t_6 \\ -t_3 & -t_5 & -t_6 & 0 \end{pmatrix}$$

here we have denoted, for example,

$$t_4 \equiv t_{12}$$

Direct calculations show that when the original spinor is rotated 360 degrees around any of the axes, the current tensor created from it becomes identical to itself, i.e., its spin is equal to one.

The Lorentz-transformed spinor yields a tensor that can be obtained in parallel from the original tensor by applying the corresponding Lorentz transformation matrix in Minkowski vector space.

$$(\Lambda T\Lambda^T)_{\mu\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\left[\frac{1}{2}\overline{(Np)_\alpha}(Np)_\beta\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\right]$$

The value $T_{\mu\nu}T^{\mu\nu}$ is invariant to rotations and boosts, and if the mass is a real or imaginary quantity, then the following equality holds, in which the right-hand side is positive in the first case and negative in the second

$$T_{\mu\nu}T^{\mu\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3} T_{\alpha\beta}\, T_{\gamma\delta}\mathrm{g}_{\alpha\gamma}\mathrm{g}_{\beta\delta} = 2m^2$$

Using the 16 spinor at our disposal, we can define 16 pseudotensors of the form, for example

$$J_{\mu\nu}^{(\mathbf{u1})} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\left[\frac{1}{2}\overline{p_\alpha^{(\mathbf{u1})}}p_\beta^{(\mathbf{u1})}\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\right]$$

If the spinor mass is not zero, then each of these pseudotensors is equal to one representative out of four pairs of pseudotensors; in a pair, they differ in their common sign.

$$\begin{pmatrix} 0 & j_1 & -j_2 & -j_3 \\ -j_1 & 0 & j_4 & j_5 \\ j_2 & -j_4 & 0 & -j_6 \\ j_3 & -j_5 & j_6 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -j_1 & j_2 & j_3 \\ j_1 & 0 & -j_4 & -j_5 \\ -j_2 & -j_4 & 0 & j_6 \\ -j_3 & j_5 & -j_6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -j_1 & -j_2 & j_3 \\ j_1 & 0 & -j_4 & j_5 \\ j_2 & j_4 & 0 & j_6 \\ -j_3 & -j_5 & -j_6 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & j_1 & j_2 & -j_3 \\ -j_1 & 0 & j_4 & -j_5 \\ -j_2 & -j_4 & 0 & -j_6 \\ j_3 & j_5 & j_6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -j_1 & j_2 & j_3 \\ j_1 & 0 & -j_4 & -j_5 \\ -j_2 & -j_4 & 0 & j_6 \\ -j_3 & j_5 & -j_6 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & j_1 & -j_2 & -j_3 \\ -j_1 & 0 & j_4 & j_5 \\ j_2 & j_4 & 0 & -j_6 \\ j_3 & -j_5 & j_6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & j_1 & -j_2 & -j_3 \\ -j_1 & 0 & -j_4 & -j_5 \\ j_2 & j_4 & 0 & j_6 \\ j_3 & j_5 & -j_6 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -j_1 & j_2 & j_3 \\ j_1 & 0 & j_4 & j_5 \\ -j_2 & -j_4 & 0 & -j_6 \\ -j_3 & -j_5 & j_6 & 0 \end{pmatrix}$$

If the mass of the spinor is zero, then the tensor obtained from the spinor has the form

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$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & t_1 & t_2 \\ 0 & -t_1 & 0 & t_3 \\ 0 & -t_2 & -t_3 & 0 \end{pmatrix}$$

and 16 pseudotensors are reduced to possible polarizations in the form of two pairs of pseudotensors with opposite common signs

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & j_1 & j_2 \\ 0 & -j_1 & 0 & -j_3 \\ 0 & -j_2 & j_3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -j_1 & -j_2 \\ 0 & j_1 & 0 & j_3 \\ 0 & j_2 & -j_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -j_1 & j_2 \\ 0 & j_1 & 0 & j_3 \\ 0 & -j_2 & -j_3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & j_1 & -j_2 \\ 0 & -j_1 & 0 & -j_3 \\ 0 & j_2 & j_3 & 0 \end{pmatrix}$$

The structure of the current tensor at nonzero mass coincides with the structure of the electromagnetic tensor (antisymmetry, zero trace), which suggests writing the equation

$$eF_{\mu\nu} = T_{\mu\nu}$$

Like Maxwell's equations, this equation establishes a connection between a massless field and its sources in the form of massive particles, exhaustively characterizing their mutual influence. This equation does not contain a contraction, which leads to a loss of information in the inhomogeneous Maxwell's equations. However, there is nothing to prevent us from taking derivatives and contracting both sides of this equation.

We can also write the relation for momentum

$$P_{\mu\nu} = T_{\mu\nu}$$

This relation has a different meaning and is rather not an equation, but an identity that connects the external manifestation of current in Minkowski space with its internal structure at the spinor level. These are two representation of the energy-momentum tensor of a charged particle.

Now, we can propose the following definition of the Lagrangian density for electrodynamics

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + eF_{\mu\nu}T^{\mu\nu} + T_{\mu\nu}T^{\mu\nu}$$

The quantity $\mathcal{L}$ is invariant under rotations and under boosts.

Let us formulate a conclusion from the calculations performed. The interaction of an electromagnetic field at an arbitrarily moving point in space with the field of a charged particle or, more strictly, with a field possessing an electric charge, is most accurately described not by Maxwell's equations, part of the information in which is lost due to the contraction of the electrodynamics tensor, but by an equation with the direct participation of this tensor.

$$eF_{\mu\nu} = P_{\mu\nu} = T_{\mu\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\left[\frac{1}{2}\overline{p_\alpha}p_\beta\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\right]$$

Here, the tensor $P_{\mu\nu}$ contains the derivatives of the momentum vector components in Minkowski space, which, in quantum mechanics, are interpreted as mixed partial derivatives of the wave function in the vector coordinate representation. The tensor $T_{\mu\nu}$ contains the components of the momentum spinor, which, in turn, are interpreted as derivatives of the wave function in the spinor coordinate representation.

It should be emphasized that this equation is valid only when the current tensor is not equal to zero, i.e., when there is a field with an electric charge at the point in space under consideration. If the point is in a vacuum, then the equation means that the electromagnetic field strength is zero. However, an electromagnetic field can exist in a vacuum, so for a vacuum it is necessary to use the contraction of the left-hand side together with the derivatives, i.e., Maxwell's equations, which allow for a non-zero field even when the right-hand side is zero.

Let us consider constructing an explicit expression for the tensor with spin two to propose a model for the gravitational field carrier. A tensor formed by the direct production of vectors does not give the desired result, because its spin remains equal to one. Let us apply the method used to construct the current tensor and define the coordinate and momentum tensors

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$$\chi_{\mu\nu\xi\tau} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3}\left[\frac{1}{4}\overline{x_\alpha}x_\beta\overline{x_\gamma}x_\delta\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\left(\sigma_{\xi\tau}\right)_{\gamma\delta}\right]$$

$$\pi_{\mu\nu\xi\tau} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3}\left[\frac{1}{4}\overline{p_\alpha}p_\beta\overline{p_\gamma}p_\delta\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\left(\sigma_{\xi\tau}\right)_{\gamma\delta}\right]$$

We can rotate the initial spinor, and the tensor constructed from it is rotated together with it. By direct calculations we can check that when we rotate the spinor around any of the axes by 720 degrees, it turns into itself, the vector turns into itself when we rotate by 360 degrees, and 180 degrees is enough to translate our tensor into itself, i.e. the tensor defined by us has spin two.

For a tensor created from a transformed spinor

$$\rho_{\mu\nu\xi\tau} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3}\left[\frac{1}{4}\overline{(N\mathbf{p})}_\alpha p(N\mathbf{p})_\beta\overline{p(N\mathbf{p})_\gamma}(N\mathbf{p})_\delta\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\left(\sigma_{\xi\tau}\right)_{\gamma\delta}\right]$$

the transformation law using Lorentz matrices applies

$$\rho_{\mu\nu\xi\tau} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3}\pi_{\alpha\beta\gamma\delta}\,\Lambda_{\mu\alpha}\Lambda_{\nu\beta}\Lambda_{\xi\gamma}\Lambda_{\tau\delta}$$

Note that from the 16 pseudospinors that make up the complete basis of states, we can form 256 pseudotensors of the form

$$\pi_{\mu\nu\xi\tau}^{(\mathbf{u1})(\overline{\mathbf{u3}})} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3}\left[\frac{1}{4}\overline{p_\alpha^{(\mathbf{u1})}}p_\beta^{(\mathbf{u1})}\overline{p_\gamma^{(\overline{\mathbf{u3}})}}p_\delta^{(\overline{\mathbf{u3}})}\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\left(\sigma_{\xi\tau}\right)_{\gamma\delta}\right]$$

The tensor $\pi_{\mu\nu\xi\tau}$ is antisymmetric with respect to the first two indices and antisymmetric with respect to the last two indices. It is symmetric when the first pair of indices is permuted with the second pair. As is well known, the Riemann tensor has the same properties. The Bianchi identity also holds for the Riemann tensor. Let us check whether the Bianchi identity holds for our tensor. Direct calculation shows that for a given spinor $\mathbf{p}$, a surprising equality holds when any index is fixed and the other three are cyclically permuted, for example

$$\pi_{0123} + \pi_{2013} + \pi_{1203} = B$$

where

$$B = -Re(\overline{p_0}p_2 + \overline{p_1}p_3)Im(\overline{p_0}p_2 + \overline{p_1}p_3) = -Re(m)Im(m)$$

Thus, if the mass is valid, for example, then Bianchi's identity for such a combination, where all indices are different, is satisfied. If there are at least two identical indices, then the identity is satisfied for any spinor with any mass. The value B for any combination is invariant under rotations and boosts.

As a result, it turns out that if the mass of a particle is real, imaginary, or simply zero, then the proposed tensor has the same properties as the Riemann tensor, so that we can write down a component-by-component equation

$$R_{\mu\nu\xi\tau} = \pi_{\mu\nu\xi\tau}$$

On the left is the tensor describing space, on the right is the tensor describing energy and momentum, just as Einstein wanted. This equation can be contracted in various ways, yielding, for example, the Ricci tensor or scalar curvature on the left-hand side. At the same time, there is room for physical interpretation of the simultaneously contracted right-hand side. We know that the left-hand side of Einstein's equation consists of a combination of Riemann's tensor, Ricci's tensor, and scalar curvature.

If we perform the same tensor contraction on the right-hand side, we obtain a second-rank tensor with the same properties as the Ricci tensor. It has spin two and can be interpreted as an energy-momentum tensor and equated to the Ricci tensor, yielding the gravitational field equation

$$R_{\mu\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\pi_{\alpha\mu\beta\nu}\,g_{\alpha\beta}$$

However, since some information is lost when contracting the Riemann tensor, it may be possible to find a way to solve the original uncontracted equation for some fairly simple cases. As in

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the case of electromagnetic interaction, the original uncontracted equation is valuable for its universal orientation. With a rigidly fixed momentum tensor, this equation uniquely determines the field; with a rigidly specified field, it uniquely determines the nature of motion. In reality, neither is rigidly fixed; the equation takes into account mutual influence in dynamics, and no deviations from equality are possible, except for the influence of external forces of a non-gravitational nature, if they are introduced into the equation in the format of the same tensor with four indices.

Direct calculation shows that the contraction, analogous to scalar curvature, is nonzero only for nonzero mass of the initial spinor. If the spinor mass has a real value, the curvature is positive; if it is imaginary, the curvature is negative. Real mass creates positive curvature and attracts other masses. What would happen if imaginary mass existed and created negative curvature?

$$R = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3} \pi_{\alpha\beta\gamma\delta}\, g_{\alpha\gamma} g_{\beta\delta} = \pm 2m^2$$

We can write Einstein's equation in our notation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \pi_{\alpha\mu\beta\nu}\, g_{\alpha\beta} - g_{\mu\nu} m^2$$

The advantage of this formulation is that the divergence of the left-hand side is zero due to Bianchi's differential identity. This means that the divergence of the right-hand side is also zero, which implies conservation of energy-momentum, one of the most important requirements for a gravitational field equation. Now that this requirement is satisfied, we can reduce the second terms in both parts at any moment due to their equality, especially since g_μν is the metric tensor of the local Minkowski space at the point for which the equation is written.

We assumed the proportionality coefficient for the right-hand sides in all of the above equations to be equal to one. Its correct value can be determined by analyzing Newton's approximation for slow motion in a week stationary field (G - Newton's constant)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = (-8\pi G)\left(\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \pi_{\alpha\mu\beta\nu}\, g_{\alpha\beta} - g_{\mu\nu} m^2\right)$$

In conclusion, we note that direct calculation shows that the above tensors of current and gravitational field source are related by a simple relationship for any values of the indices.

$$\pi_{\alpha\beta\gamma\delta} = T_{\alpha\beta} T_{\gamma\delta}$$

however, this equality follows from the definition of these tensors.

Taking into account the above interpretation of Maxwell's equations and Einstein's equation

$$eF_{\mu\nu} = T_{\mu\nu}$$
$$R_{\mu\nu\xi\tau} = (-8\pi G) T_{\mu\nu} T_{\xi\tau}$$
$$eF_{\mu\nu} - T_{\mu\nu} = 0$$
$$R_{\mu\nu\xi\tau} - (-8\pi G) T_{\mu\nu} T_{\xi\tau} = 0$$

we can make an assumption about how they are combined. At the location of a charged particle (or, in the interpretation of field theory, at the point where there is a field with a charge), there must be a relationship connecting three types of components: gravity, electromagnetic field, and matter. Gravity and matter are described by a fourth-rank tensor, and the electromagnetic field by a second-rank tensor, from which a fourth-rank tensor can also be constructed. The logic is that gravity is an external force for Maxwell's equation, so its force acts against electromagnetic forces, and matter is the object of application of these opposing forces, so that gravity and the electromagnetic field enter with opposite signs. If, on the contrary, we consider the interaction of gravity and matter to be primary and electromagnetic forces to be external, then again it is logical that they act against the force of gravity

$$R_{\mu\nu\xi\tau} + (-8\pi G) eF_{\mu\nu} F_{\xi\tau} - (-8\pi G) T_{\mu\nu} T_{\xi\tau} = 0$$
$$R_{\mu\nu\xi\tau} = (-8\pi G) T_{\mu\nu} T_{\xi\tau} - (-8\pi G) eF_{\mu\nu} F_{\xi\tau}$$

Incidentally, we note that Bianchi's identities hold for the tensors $F_{\alpha\beta} F_{\gamma\delta}$ and $P_{\alpha\beta} P_{\gamma\delta}$.

At any point in space, all three fields – gravitational, electromagnetic and matter – are rigidly linked by this relationship. An external addition to one of the fields accelerates a particle of matter

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and changes another field, while the mechanical acceleration of a particle changes both fields. Under certain conditions, waves arise in the fields and propagate in space where there is no matter; the fields in these waves are linked by a relationship from which matter has been removed.

$$R_{\mu\nu\xi\tau} = 8\pi GeF_{\mu\nu}F_{\xi\tau}$$

This relationship indicates that gravitational and electromagnetic fields cannot exist independently of each other. It also solves the problem that the zero equality of the Riemann tensor or the electromagnetic field tensor means the absence of the corresponding fields. This relationship indicates that the Riemann tensor is never zero because there is always an electromagnetic field tensor and vice versa.

It should be noted that there is another interpretation of Einstein's equation with the Riemann tensor instead of the Ricci tensor. It consists in the fact that such an equation is applicable only when the right-hand side is nonzero, when there is a field with mass at the point of space under consideration. If the point is in a vacuum, then the equation means that the gravitational field strength is zero. But a field can also exist in a vacuum, so for a vacuum it is necessary to use the contraction of the left-hand side, i.e., the Einstein tensor, whose equality to zero does not mean that the Riemann tensor is equal to zero. Here we see an analogy with the equations of the electromagnetic field for a vacuum. But there is a radical difference: Maxwell's equations are obtained not simply by contraction, but by contraction with derivatives, while Einstein's equation performs contraction without taking derivatives.

It must be acknowledged that Maxwell's and Einstein's equations are not universal; a homogeneous equation cannot be obtained simply by setting the right-hand side of a inhomogeneous equation to zero; the left-hand side must also be changed.

The most general approach to describing electromagnetic and gravitational phenomena is as follows. The coincidence of the left-hand sides of Einstein's homogeneous and inhomogeneous equations is a random coincidence. The homogeneous equation is an identity reflecting the properties of the Riemann curvature tensor. One of Bianchi's identities can equally well serve as the homogeneous equation. The inhomogeneous equation is a contraction of the original complete equation, which includes the Riemann tensor, and its subsequent contractions lead to the loss of some information about the interaction between matter and the field. This situation is even more evident in Maxwell's equations for the electromagnetic field. Here, the left-hand side of the homogeneous and inhomogeneous equations is completely different. Maxwell's homogeneous equations are identities describing the properties of the electromagnetic field tensor, while the inhomogeneous equations are contractions that hide some of the information from the original equation with the full electromagnetic field tensor. The wave equation with second-order derivatives for the electromagnetic potential is a separate equation, which is the Klein-Gordon equation for a zero-mass field. The inhomogeneous Maxwell equation in this hierarchy is Newton's law, which ensures that the Klein-Gordon equation is satisfied for a particle in the presence of a field. Similarly, Einstein's homogeneous equation is an identity that reflects the properties of the Riemann tensor and describes the behavior of a free field. Einstein's inhomogeneous equation is a contraction of the complete equation for the Riemann tensor, and it plays the role of Newton's law for gravity. And for some reason, the Klein-Gordon equation for gravity is not used at all.

For comparison with canonical Einstein's equation, we will combine all terms so that Riemann's tensor becomes Ricci's tensor and scalar curvature.

$$T_{\alpha\mu}T^{\alpha}{}_{\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} T_{\alpha\mu}T_{\beta\nu}\, g_{\alpha\beta}$$

$$F_{\alpha\mu}F^{\alpha}{}_{\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} F_{\alpha\mu}F_{\beta\nu}\, g_{\alpha\beta}$$

$$T_{\alpha\beta}T^{\alpha\beta} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3} T_{\alpha\beta}T_{\gamma\delta}\, g_{\alpha\gamma}g_{\beta\delta}$$

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$$F_{\alpha\beta}F^{\alpha\beta} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3} F_{\alpha\beta}F_{\gamma\delta}\,g_{\alpha\gamma}g_{\beta\delta}$$

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + 8\pi G\left(\left(T_{\alpha\mu}T^{\alpha}{}_{\nu} - \frac{1}{2}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta}\right) - e\left(F_{\alpha\mu}F^{\alpha}{}_{\nu} - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)\right) = 0$$

For vacuum, we obtain

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi Ge\left(F_{\alpha\mu}F^{\alpha}{}_{\nu} - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

The electromagnetic part of this equation can be compared with the conventional expression for the energy-momentum tensor of an electromagnetic field, it differs by a coefficient of ¼.

$$T^{\mu\nu}_{\text{эм}} = \frac{1}{4\pi}(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$$

As noted above, without prejudice to the conservation of energy-momentum, all scalar terms can be removed from the general equation and the equation can be solved using only the Ricci tensor. However, in the absence of scalar terms, there is no point in switching from the Riemann tensor to the Ricci tensor with a loss of information during contraction; it is logical to solve the equation

$$R_{\mu\nu\xi\tau} = -8\pi G T_{\mu\nu}T_{\xi\tau} + 8\pi Ge F_{\mu\nu}F_{\xi\tau}$$

$$R_{\mu\nu\xi\tau} = -8\pi G P_{\mu\nu}P_{\xi\tau} + 8\pi Ge F_{\mu\nu}F_{\xi\tau}$$

Terms describing strong and weak interactions can be added to this equation by representing them as fourth-rank tensors with spatial indices. The tensors describing the Yang-Mills field strengths will enter the equation as terms where a refers to the symmetry group of the corresponding field, and g is its coupling constant.

$$8\pi G g F^{a}_{\mu\nu}F^{a}_{\xi\tau}$$

The matter field does not necessarily participate in interactions with each of the fields included in the sum. For example, at the point under consideration, there may be a quark field and an electron field. The quark participates in strong interactions, while the electron does not, but both interact with the electromagnetic field. Ultimately, it must be remembered that the equation consists of mixed partial derivatives of different fields with respect to vector or spinor coordinates. In turn, any derivative is a commutator of the coordinate function with the component of the momentum vector or spinor.

The logic behind deriving Maxwell's and Einstein's non-homogeneous equations using the principle of mass invariance is as follows. The initial equation for a free particle, invariant to Lorentz transformations ($\eta^{\mu\nu}$ is the metric tensor of Minkowski space)

$$\eta^{\mu\nu}(S_\mu P_\mu)(S_\nu P_\nu) = m^2$$

taking into account the anticommutativity of matrices composed of Pauli matrices

$$S_\mu S_\nu + S_\nu S_\mu = 2\delta_{\mu\nu}$$

leads to the Klein-Gordon equation

$$\eta^{\mu\nu}P_\mu P_\nu = m^2$$

under the condition of commutativity of momentum components

$$P_\mu P_\nu - P_\nu P_\mu = 0$$

With the same success, instead of S_μ, one can use matrices $(\gamma_0^V\gamma_\mu^V)$.

When adding a vector field

$$\eta^{\mu\nu}S_\mu(P_\mu + A_\mu)S_\nu(P_\nu + A_\nu) = m^2$$

we again obtain the Klein–Gordon equation for a particle in a field (at zero mass, we obtain the Klein–Gordon wave equation for an electromagnetic field in a vacuum)

$$\eta^{\mu\nu}(P_\mu + A_\mu)(P_\nu + A_\nu) = m^2$$

only if the commutativity condition is satisfied, which in the quantum interpretation corresponds to the commutativity of the covariant derivative. This condition leads to the equation

$$P_{\mu\nu} \equiv P_\mu P_\nu - P_\nu P_\mu = P_\mu A_\nu - P_\nu A_\mu \equiv F_{\mu\nu}$$

which, if the particle has a charge, can be reduced to the inhomogeneous Maxwell equation, taking derivatives along the way.

Let us write down the generalized equation for a free particle, which is also invariant to Lorentz transformations:

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$$\eta^{\mu\nu}\eta^{\xi\tau}S_\mu S_\nu S_\xi S_\tau P_\mu P_\nu P_\xi P_\tau = m^4$$

Let us add the vector field

$$\eta^{\mu\nu}\eta^{\xi\tau}S_\mu S_\nu S_\xi S_\tau(P_\mu + A_\mu)(P_\nu + A_\nu)(P_\xi + A_\xi)(P_\tau + A_\tau) = m^4$$

$$\left(\eta^{\mu\nu}S_\mu S_\nu(P_\mu + A_\mu)(P_\nu + A_\nu)\right)\left(\eta^{\xi\tau}S_\xi S_\tau(P_\xi + A_\xi)(P_\tau + A_\tau)\right) = m^4$$

This equation yields the Klein–Gordon equation for a particle in a gravitational field (at zero mass, we obtain the Klein–Gordon wave equation for a gravitational field in a vacuum)

$$\left(\eta^{\mu\nu}(P_\mu + A_\mu)(P_\nu + A_\nu)\right)\left(\eta^{\xi\tau}(P_\xi + A_\xi)(P_\tau + A_\tau)\right) = m^4$$

$$\eta^{\mu\nu}\eta^{\xi\tau}(P_\mu + A_\mu)(P_\nu + A_\nu)(P_\xi + A_\xi)(P_\tau + A_\tau) = m^4$$

under the following commutativity condition

$$P_{\mu\nu} \equiv P_\mu P_\nu - P_\nu P_\mu = P_\mu A_\nu - P_\nu A_\mu \equiv F_{\mu\nu}$$

$$P_{\xi\tau} \equiv P_\xi P_\tau - P_\nu P_\xi = P_\xi A_\tau - P_\nu A_\xi \equiv F_{\xi\tau}$$

$$P_{\mu\nu\xi\tau} \equiv P_{\mu\nu}P_{\xi\tau} = (P_\mu P_\nu - P_\nu P_\mu)(P_\xi P_\tau - P_\nu P_\xi) =$$

$$= (P_\mu A_\nu - P_\nu A_\mu)(P_\xi A_\tau - P_\nu A_\xi) = F_{\mu\nu}F_{\xi\tau} \equiv R_{\mu\nu\xi\tau}$$

which is Newton's law and at the same time is a non-homogeneous gravitational field equation, from which, using contraction and replacing the momentum components with derivatives with respect to the coordinates, we obtain Einstein's non-homogeneous equation.

$$P_\mu A_\nu P_\xi A_\tau \to (\partial_\mu A_\nu)(\partial_\xi A_\tau) + A_\nu(\partial_\mu \partial_\xi A_\tau)$$

We propose to identify the tensor $R_{\mu\nu\xi\tau}$ with the Riemann curvature tensor, defined in the standard way through the second derivatives of the space metric. At least, these tensors, as shown above, have the same symmetry properties. It is also necessary to take into account the previously derived relations

$$P_{\mu\nu\xi\tau} = P_{\mu\nu}P_{\xi\tau} = T_{\mu\nu}T_{\xi\tau} = \pi_{\mu\nu\xi\tau}$$

In order to find the commutativity condition for the general case of mass to the fourth power, we used the fact that the corresponding invariant equation can be represented as a product of invariant equations for the square of mass. And we had already found the commutativity condition for them. Thus, we avoided the need to use a tensor field of gravity with two indices in the derivation. But we can return to this field by finding the solution to Einstein's equation with the Riemann tensor, which we arrived at by our indirect method.

Although for simplicity we used the same notation for vector fields when deriving the equations for electrodynamics and gravity, we do not claim that in both cases it is the same field. In the case of gravity, this field can be considered auxiliary and we can work exclusively with the space curvature tensor. However, in the special case where only gravitational and electromagnetic fields are present in a vacuum, these vector fields are equated with a certain coefficient in Einstein's equation, although they may have different physical natures. The electromagnetic field forms the energy-momentum tensor on the right-hand side of the equation.

Conclusions

An alternative approach to analyze relativistic and quantum effects inherent in charged particles in the presence of an electromagnetic field is proposed. Two ways of describing the electron behavior in the electromagnetic field are considered: by means of the vector equation, which is based on the plane wave model for a free electron, and the spinor equation, which is based on the representation of the electron as a plane wave in spinor space. For both equations, which are valid for a free particle, their applicability to an arbitrary physical situation is postulated, in particular to describe the behavior of a particle in the presence of an electromagnetic field. The presented equations are intended to fulfill the same role as the Schrödinger equation and the Dirac equation. At the same time, in our opinion, the spinor equations more accurately describe the details of the interaction between fields and particles.

According to the results of consideration the following picture emerges. Physical processes occur in the spinor coordinate space, which is perceived by us as a vector Minkowski coordinate space, the coordinates of which are equal to bilinear combinations of spinor coordinates. In the spinor space

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propagate plane waves whose phase is equal to the scalar product of the momentum and coordinate spinors. As amplitudes of plane waves in fermions are linear combinations of 16 pseudospinors, whose components are components of the momentum spinor with changing signs of some of them. The sign of the charge of a fermion is determined by the sign of its mass, which is calculated from its momentum and invariant to Lorentz transformations. Scalar, vector and tensor fields also propagate as plane waves in spinor space, but the coefficients of these waves are scalars, vectors and tensors formed by simple combinations on the basis of Pauli matrices of 16 fermionic pseudospinors. In this case only four pseudovectors (corresponding to four polarizations) of the photon and only two pseudotensors (corresponding to two polarizations) of the graviton are different. The spinor field has spin equal to 1/2, the spin of scalar fields is automatically zero, that of vector fields is unitary, and that of tensor fields is double.

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