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Nonlocal Boundary Problem for a Loaded Equation of Mixed Type in a Special Domain

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Abstract: The purpose of the research is to investigate the unique solvability of a nonlocal boundary value problem for a loaded equation of parabolic-hyperbolic type in a special domain. Methods: Using representations of a general regular solution, the existence and uniqueness of the problem posed is proved. Results: A new method for proving the unique solvability of nonlocal boundary value problems for one loaded equation of mixed type in a special domain has been formed and developed. Conclusions: The obtained results make it possible to solve some problems of genetics, immunology, transonic gas dynamics and thermal physics, which are associated with displacement.

Keywords: loaded equation; nonlocal boundary value problem; mixed type equation; general solution representation; integral Volterra equation with shift; fractional derivatives

MSC: 35M10, 35M12

1. Introduction

Since the fifties of the last century, researchers have been intensively studying local and nonlocal problems for partial differential equations in simply connected regions. Currently, the range of such tasks is expanding in various directions. Among them, problems with displacement occupy a special place. This is explained by the fact that such problems cover various correct local boundary value problems, and many problems are reduced to them, for example, biological synergetics, genetics, immunology, transonic gas dynamics and thermal physics.

One of the most important classes of unloaded partial differential equations are second-order equations parabolo - hyperbolic and elliptic - hyperbolic types. An analogue of the Tricomi problem in a doubly connected domain for an equation of mixed type with one line of degeneracy was first posed and studied in A.V. Bitsadze [1, p.30], and for the Lavrentiev–Bitsadze equation - in the works of M. S. Salakhitdinov and A. K. Urinov [2].

Boundary value problems in classical domains for a loaded equation of hyperbolic type were studied in A.M.Nakhushev [3], V.M.Kaziev [4], A.Kh.Attaev [5], and the work of M.T.Dzhenaliev [6], M.Kh.Shkhankova [7] studied the local and nonlocal problem for a loaded equation of parabolic type. A.V. Borodin [8] studied the Dirichlet problem for a loaded equation of elliptic type. In the works of K.U.Khubiev [9], B.Islomov and D.M.Kuryazov [10], M.I.Ramazanov [11], analogues of the Tricomi and Gellerstedt problem for loaded hyperbolic-parabolic type equations were studied, and the Dirichlet problem for loaded equation with the Lavrentiev–Bitsadze operator in a rectangular domain were studied in the works of K.B. Sabitova and E.P. Melisheva [12-13]. Boundary value problems for an equation of hyperbolic type with a fractional derivative were also studied in works[30-33].

Local and non-local problems in simply connected domains for loaded equations of elliptic -hyperbolic and parabolic - hyperbolic types, when the loaded part contains a trace or derivative of the desired function, have been little studied. We note the works of B.I. Islomov and D.M. Kuryazov [14], B.I. Islomov and U.I. Boltaeva [15], [16], B.I. Islomov and Zh.A. Kholbekov [17], K.B. Sabitov [18], R.R. Ashurova and S.Z. Zhamalova [19], Yu.K. Sabitova [20], V.A. Eleeva [21]. This is due, first of all, to the lack of representation of a general solution for such equations; on the other hand, such problems are reduced to little-studied integral equations with a shift.

Currently, the range of such tasks is expanding in various special areas. However, boundary value problems for loaded equations of mixed type with an integral operator of fractional order in doubly connected domains have still not been sufficiently studied. Note that local and nonlocal boundary value problems for a loaded equation of elliptic -hyperbolic type in a doubly connected domain were studied in [22-25], in which the loaded part contains a differential operator or a trace of the desired function. The works [26-27] outline a technique for formulating correct boundary value problems with displacement for loaded second-order linear hyperbolic equations in special domains. It is shown that the correctness of such boundary value problems is significantly influenced by the loaded part of the equation under consideration.

Based on this, the present work is devoted to the formulation and study of a nonlocal boundary value problem for a loaded equation of parabolic - hyperbolic type in a special domain.

2. Statement of the problem C_μ

Let Ω_0 is the area bounded by the segments $A_jB_j, A_jK_j, B_jN_j, K_jN_j$ direct $y=0$, $x=(-1)^{j-1}$, $x=(-1)^{j-1}q$, $y=1$ at $y>0$. Ω is area limited by axis Ox segments $A_jB_j (j=1,2)$ and at $y<0$ characteristics $A_jC_1: x-(-1)^{j-1}y=(-1)^{j-1}$, $B_jC_2: x-(-1)^{j-1}y=(-1)^{j-1}q$ equations

$$0=\begin{cases} u_{xx}-u_y-\mu_j u_x(x,0), & C>0, \\ u_{xx}-u_{yy}-\mu_{j+2}u(\xi,0), & C<0, \end{cases} \tag{1}$$

emerging from points $A_j\left((-1)^{j-1};0\right)$ and $B_j\left((-1)^{j-1}q;0\right)$, intersecting at points $C_1(0;-1)$ and $C_2(0;-q)$, $\xi=x+y$. In equation (1) q, μ_j, $\mu_{j+2} (j=1,2)$ is given real numbers, and

$$0<q<1, \quad (-1)^j\mu_j>0, \quad (-1)^{j-1}\mu_{j+2}>0, \quad (j=1,2). \tag{2}$$

Let us introduce the following notation: $I=\{(x,y): x=0, -1<y<-q\}$, $J_1=\{(x,y): q<x<1, y=0\}$, $J_2=\{(x,y): -1<x<-q, y=0\}$, $D_j, E_j \in A_jC_1$,

$$B_jD_j: \ x+(-1)^{j-1}y=(-1)^{j-1}q, \ C_2E_j: \ x+(-1)^{j-1}y=(-1)^j q,$$

$$\left.\begin{matrix} A_jD_j \\ A_jE_j \end{matrix}\right\}: \ x-(-1)^{j-1}y=(-1)^{j-1}, (j=1,2).$$

Through Ω_j and Ω_{j+2}, Ω_5 respectively, denote the characteristic triangle and $A_jB_jD_j$ and quadrilaterals $B_jC_2E_jD_j$, $C_2E_1C_1E_2$, $(j=1,2)$.

$$\Omega_{5j}=\Omega_5\cap\left\{(-1)^j x<0, \ y<0\right\}, \ \Omega_{0j}=\Omega_0\cap\left\{(-1)^j x<0, \ y>0\right\}, \ \Omega_0=\Omega_{01}\cup\Omega_{02},$$

$$\Omega=\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4\cup\Omega_5\cup B_1D_1\cup C_2E_1\cup C_2E_2\cup B_2D_2, \ \Delta_j=\Omega_j\cup\Omega_{j+2}\cup\Omega_{5j},$$

$$\Delta=\Omega_0\cup\Omega\cup I\cup J_1\cup J_2, \Delta_{0j}=\Omega_{0j}\cup A_jB_j\cup K_jN_j, \ \Delta_j^*=\Omega_j\cup\Omega_{0j},$$

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$$\theta_1(x)=\theta_1\left(\frac{x+1}{2};\ \frac{x-1}{2}\right),\ \theta_2(x)=\theta_2\left(\frac{x-1}{2};\ -\frac{x+1}{2}\right), \tag{3}$$

$\theta_j(x)$ characteristic intersection points A_jD_j with a characteristic coming from the point $Z_j(x,0)\in J_j$, $(j=1,2)$.

Task C_μ. Find a function $u(x,y)$ that has the following properties:
1)$u\,(x,y)\in C(\bar{\Delta})\cap C^1(\Delta)$;
2)$u(x,y)\in C^{2,1}_{x,y}(\Delta_{0j})\cap C^{2,2}_{x,y}(\Delta_j)$ and satisfies equation (1) in the areas Δ_{0j} and Δ_j $\ (j=1,2)$;
3) $u(x,y)$ satisfies the boundary conditions:

$$u\,(x,y)|_{A_jK_j}=\varphi_j\,(y)\,,\qquad u\,(x,y)|_{B_jN_j}=g_j\,(y)\,,\quad 0\le y\le 1, \tag{4}$$

$$u\left[\theta_j\,(x)\right]+a_j\,(x)\,u\,(x,0)=b_j\,(x)\,,\quad (x,0)\in \bar{J}_j, \tag{5}$$

$$u(x,y)|_{B_jC_2}=\psi_j(x),\quad 0\le\left((-1)^{j-1}x\right)\le q,\ \ (j=1,2), \tag{6}$$

where are $\varphi_j(y),\ \ g_j(y),\ \ \psi_j(x),\ a_j(x),\ b_j(x)\ \ (j=1,2)$ is the given functions, and

$$\psi_1(0)=\psi_2(0),\quad g_j(0)=\psi_j((-1)^{j-1}q),\quad \varphi_j(0)(1+a_j(-1)^{j-1})=b_j((-1)^{j-1}), \tag{7}$$

$$\varphi_j\,(y)\,,\ \ g_j\,(y)\in C\,[0,1]\cap C^1(0,1), \tag{8}$$

$$a_j(x),\ b_j(x)\ \in C^1\left(\bar{J}_j\right)\cap C^3\left(J_j\right), \tag{9}$$

$$\psi_j(x)\in C\left(0\le\left((-1)^{j-1}x\right)\le q\right)\cap C^2\left(0<\left((-1)^{j-1}x\right)<q\right). \tag{10}$$

3. Investigation of the problem C_μ

For the study of the problem, C_μ the following tasks play an important role .
Task $C_{j\mu}$. Find a function $u(x,y)$ that has the following properties:
1) $u\,(x,y)\in C(\bar{\Delta}^*_j)\cap C^1(\Delta^*_j\cup J_1\cup J_2)$;
2) $u(x,y)\in C^{2,1}_{x,y}(\Omega_{0j})\cap C^{2,2}_{x,y}(\Omega_j)$ and satisfies equation (1) in the areas Ω_{0j} and Ω_j $\ (j=1,2)$;
3) $u(x,y)$ satisfies boundary conditions (4) and (5).

Theorem 1. *If conditions (2), (7)-(10) and*

$$(-1)^j\left(1+2a_j(x)\right)\ge 0,\quad a'_j(x)\ge 0,\ \ \forall x\in\bar{J}_j,\ \ (j=1,2), \tag{11}$$

then Ω there is a unique regular solution to the problem in the region $C_{j\mu}$.

Proof of Theorem 1 Solution of the Cauchy problem with conditions

$$u\,(x,-0)=\tau_j\,(x)\,,\quad (x,0)\in\bar{J}_j,\quad u_y\,(x,-0)=\nu_j\,(x)\,,\ \ (x,0)\in J_j,\ \ (j=1,2) \tag{12}$$

for equation (1) in the region Ω_j has the form:

$$u\,(x,y)=\frac{1}{2}\left[\tau_j\,(x+y)+\tau_j\,(x-y)\right]+\frac{1}{2}\int_{x-y}^{x+y}\nu_j\,(\xi)\,d\xi+$$

$$+\frac{\mu_{j+2}}{4}\int_{x-y}^{x+y}(x-y-\xi)\tau_j\,(\xi)\,d\xi,\ (j=1,2). \tag{13}$$

Substituting (13) at $j=1$ in (5) taking into account

$$u\left[\theta_1\,(x)\right]=u\left[\frac{x+1}{2};\ \frac{x-1}{2}\right]=\frac{1}{2}\left[\tau_1(x)+\tau_1(1)\right]-\frac{1}{2}\int_x^1\nu_1(t)dt-\frac{\mu_3}{4}\int_x^1(1-t)\tau_1\,(t)\,dt$$

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we have

$$\frac{1}{2}\left[\tau_1(x)+\tau_1(1)\right]-\frac{1}{2}\int_x^1 \nu_1(t)dt-\frac{\mu_3}{4}\int_x^1(1-t)\tau_1(t)\,dt+a_1(x)\tau_1(x)=b_1(x). \tag{14}$$

Differentiating (14) with respect to x, we obtain a functional relationship between $\tau_1(x)$ and $\nu_1(x)$ brought from the area Ω_1 to J_1:

$$\left[1+2a_1(x)\right]\tau_1'(x)+\left[\frac{\mu_3}{2}(1-x)+2a_1'(x)\right]\tau_1(x)+\nu_1(x)=2b_1'(x),\quad (x,0)\in J_1. \tag{15}$$

Similarly, $j=2$ using the solution to the Cauchy problem (13) with initial data (12) for equation (1) in the domain Ω_2 taking into account (3) and (5), we obtain the functional relationship between $\tau_2(x)$ and $\nu_2(x)$brought from the area Ω_2 to J_2:

$$\left[1+2a_2(x)\right]\tau_2'(x)+2a_2'(x)\tau_2(x)-\frac{\mu_4}{2}\int_{-1}^x \tau_2(t)dt-\nu_2(x)=2b_2'(x),\quad (x,0)\in J_2. \tag{16}$$

Consequently, as in [28 , pp. 40-47], due to conditions 1) - 2) of the problem $C_{j\mu}$, passing to the limit at $y\to+0$ in equation (1) taking into account (12), we obtain a functional relationship between $\tau_j'(x)$ and $\nu_j(x)$ brought from the region Ω_{0j} to J_j:

$$\tau_j'(x)-\mu_j\int_{(-1)^{j-1}}^x \tau_j'(t)\,dt=\tau_j'((-1)^{j-1})+\int_{(-1)^{j-1}}^x \nu_j(t)\,dt, \tag{17}$$

where $\tau_j'((-1)^{j-1})$ $(j=1,2)$ unknown is a constant to be determined.

Having excluded $\nu_j(x)$ from relations (15), (16) and (17), after some calculations we obtain an integral equation for $\tau_j'(x)$ $(j=1,2)$:

$$\tau_j'(x)-\int_{(-1)^{j-1}}^x M_j(x,t)\tau_j'(t)\,dt=\tau_j'((-1)^{j-1})+F_j(x),\quad (x,0)\in\bar{J}_j, \tag{18}$$

where

$$M_1(x,t)=\mu_1-(1+2a_1(x))-\int_x^t\left[2a_1'(z)+\frac{\mu_3(1-z)}{2}\right]dz, \tag{19}$$

$$M_2(x,t)=\mu_2+1+2a_2(x)+\int_t^x\left[2a_2'(z)-\frac{\mu_4(z-t)}{2}\right]dz, \tag{20}$$

$$F_1(x)=-\varphi_1(0)\int_x^1\left(\frac{\mu_3(1-t)}{2}+2a_1(t)\right)dt-2\int_x^1 b_1'(t)dt, \tag{21}$$

$$F_2(x)=\varphi_2(0)\int_{-1}^x\left[2a_2(t)-\frac{\mu_4(1+t)}{2}\right]dt-2\int_{-1}^x b_2'(t)dt. \tag{22}$$

By virtue of (2) and (9) from (19) - (22) it follows that

$$|M_1(x,t)|\le c_1 \text{ for any } \le x\le 1,\quad q\le t\le 1, \tag{23}$$

$$|M_2(x,t)|\le c_2 \quad\text{for any}\quad -1\le x\le -q,\quad -1\le t\le -q, \tag{24}$$

$$F_j(x)\in C\left(\bar{J}_j\right)\cap C^2\left(J_j\right),\quad (j=1,2). \tag{25}$$

Thus, by virtue of (23), (24) and (25), equation (18) is a Volterra integral equation of the second kind. According to Volterra's theory of integral equations [29, pp. 36-51], we conclude that the integral equation (18) is uniquely solvable in the class $C\left(\bar{J}_j\right) \cap C^2\left(J_j\right)$ and its solution is given by the formula

$$\tau_j'\left(x\right) = \tau_j'((-1)^{j-1}) + F_j(x) + \int_{(-1)^{j-1}}^{x} \left[\tau_j'((-1)^{j-1}) + F_j(t)\right] M_j^*(x,t)dt, \tag{26}$$

where is $M_j^*(x,t)$ $(j = 1,2)-$ the resolvent of the kernel $M_j(x,t)$.

Integrating (26) from x to 1 at $j = 1$(from -1 to x at $j = 2$) taking into account $\tau_1(1) = \varphi_1(0)$ $\left(\tau_2(-1) = \varphi_2(0)\right)$, we have

$$\tau_1\left(x\right) = \varphi_1(0) - \left[1 - x - \int_x^1 dt \int_t^1 M_1^*(t,z)dz\right] \tau_1'(1) - \int_x^1 F_1(t)dt +$$

$$+ \int_x^1 dt \int_t^1 M_1^*(t,z)F_1(z)dz, \qquad (x,0) \in \bar{J}_1, \tag{27}$$

$$\tau_2\left(x\right) = \varphi_2(0) + \left[1 + x + \int_{-1}^x dt \int_{-1}^t M_2^*(t,z)dz\right] \tau_2'(-1) + \int_{-1}^x F_2(t)dt -$$

$$- \int_{-1}^x dt \int_{-1}^t M_2^*(t,z)F_2(z)dz, \qquad (x,0) \in \bar{J}_2, \tag{28}$$

Now putting in (27) and (28) respectively $x = q$ and $x = -q$ taking into account $\tau_j((-1)^{j-1}q) = g_j\left(0\right)$ $(j = 1,2)$ we find the unknown constant $\tau_j'((-1)^{j-1})$:

$$\tau_j'((-1)^{j-1}) = \left[(-1)^{1-j}(1-q) - \int_q^1 dt \int_t^1 M_j^*((-1)^{j-1}t;(-1)^{j-1}z)dz\right]^{-1} \times$$

$$\times \left[\varphi_j(0) - g_j(0) - \int_q^1 F_j((-1)^{j-1}t)dt + \right.$$

$$\left. + \int_q^1 dt \int_t^1 M_j^*((-1)^{j-1}t;(-1)^{j-1}z)F_j((-1)^{j-1}z)dz\right], \quad (j = 1,2). \tag{29}$$

By virtue of (2), (11), from (19) and (20) it follows that $(-1)^{j-1}M_j(x,t) < 0,\ \forall\, x,t \in \bar{J}_j, (j = 1,2)$. Consequently, the resolvent of the kernel $M_j^*(x,t)$ is also negative, i.e. $(-1)^{j-1}M_j^*(x,t) < 0,\ \ \forall\, x,t \in \bar{J}_j, (j = 1,2)$. This means that the denominator of formula (29) for any $q \leq (-1)^{j-1}x \leq 1,\ \ q \leq (-1)^{j-1}t \leq 1$ does not vanish, i.e.

$$(-1)^{1-j}(1-q) - \int_q^1 dt \int_t^1 M_j^*((-1)^{j-1}t;(-1)^{j-1}z)dz > 0$$

.

By virtue of (23) - (25) from (27) and (28) taking into account (29) we conclude that

$$\tau_j(x) \in C^1\left(\bar{J}_j\right) \cap C^3\left(J_j\right), \ (j = 1,2). \tag{30}$$

Putting (30) into (15) and (16) taking into account (9), (30), we define the function $\nu_j\left(x\right)$ from class $\nu_j^-\left(x\right) \in C(\bar{J}_j) \cap C^1(J_j)$.

Thus, the solution to the problem $C_{j\mu}$ can be restored in the domain Ω_{0j} as a solution to the first boundary value problem for equation (1) [28, p. 99], and in the domains $\Omega_j(j = 1,2)$ as a solution to the Cauchy problem for equation (1) (see (13)).

Therefore, the task$C_{j\mu}$ is uniquely solvable.

Theorem 1 is proven.

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Now to restore the solution to the problem C_μ in the regions Ω_{j+2} and Ω_5 solve Goursat problems for equation (1).

Task Γ_j $(j = 1, 2)$. Find function $u(x, y)$ with the following properties:

1)$u(x, y) \in C(\bar{\Omega}_{j+2}) \cap C^1(\Omega_{j+2})$;

2) $u(x, y)$ is twice continuously differentiable solution of equation (1) in region Ω_{j+2} $(j = 1, 2)$;

3) $u(x, y)$ satisfies boundary conditions (6) and

$$u(x, y)|_{B_j D_j} = w_j(x), \qquad q \leq (-1)^{j-1} x \leq \frac{1+q}{2}, \tag{31}$$

where $w_j(x)$ $(j = 1, 2)$ are determined from

$$w_j\left(\frac{x + (-1)^{j-1} q}{2}\right) = \frac{1}{2}\left[\tau_j(x) + \tau_j((-1)^{j-1} q)\right] + \frac{(-1)^{j-1}}{2} \int\limits_x^{(-1)^{j-1} q} \nu_j(t)\, dt +$$

$$+ \frac{\mu_{j+2}}{4} (-1)^j \int\limits_x^{(-1)^{j-1} q} (t+q) \tau_j(t)\, dt + \frac{\mu_3 (1 + (-1)^{j-1})}{8} \int\limits_x^{q} (x+q) \tau_1(t)\, dt, \quad q \leq (-1)^{j-1} x \leq 1,$$

here $\nu_j(x)$ are $\tau_j(x)$ $(j = 1, 2)$ are known functions and they are determined respectively from (15), (16), and (27)-(29). $w_j(x)$ $(j = 1, 2)$ belongs to the class

$$w_j(x) \in C\left(q \leq (-1)^{j-1} x \leq \frac{1+q}{2}\right) \cap C^2\left(q < (-1)^{j-1} x < \frac{1+q}{2}\right), \tag{32}$$

and $w_j((-1)^{j-1} q) = \psi_j((-1)^{j-1} q)$.

Task Γ_3. Find a function $u(x, y)$ with the following properties:

1) $u(x, y) \in C(\bar{\Omega}_5) \cap C^1(\Omega_5)$, and the function $u_x(0, y)$ can go to infinity of order less than one at the ends of the interval I;

2) $u(x, y)$ is twice continuously differentiable solution of equation (1) in region Ω_{5j} $(j = 1, 2)$;

3) $u(x, y)$ satisfies the boundary conditions

$$u(x, y)|_{C_2 E_j} = h_j(y), \qquad -\frac{1+q}{2} \leq y \leq -q, \tag{33}$$

where $h_j(y)$ $(j = 1, 2)$ are determined from

$$h_1(y) = w_1(-y) - \frac{\mu_3 (y+q)}{2} \int_q^{-q} \tau_1(t) dt + \psi_1(0) - \psi_1(q),$$

$$h_2(y) = w_2(y) + \frac{\mu_4 q}{2} \int_{-q}^{2y+q} \tau_2(t) dt + \psi_2(0) - \psi_2(-q),$$

and belongs to the class

$$h_j(y) \in C\left[-\frac{1+q}{2}, -q\right] \cap C^2\left(-\frac{1+q}{2}, -q\right), \tag{34}$$

and $h_1(-q) = h_2(-q)$ $\Rightarrow \psi_1(0) = \psi_2(0)$. Here $\psi_j(x)$, $w_j(x)$, $\tau_j(x)$ is known functions and they are determined from (6), (12), (27), (28) and (29) respectively.

Theorem 2. ***Theorem 2.*** *If conditions (2), (10), (30), (32) are met, then Ω_{j+2} the solution to the problem Γ_j exists and is unique in the domain.*

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Proof of Theorem 2 The general solution to the equation (1) in the domain Ω_{j+2} has the form [30, p. 77]:

$$u\left(x,y\right)=\Phi_j\left(x+y\right)+G_j\left(x-y\right)+\frac{\mu_{j+2}(x-y-(-1)^{j-1}q)}{4}\int_{(-1)^{j-1}q}^{x+y}\tau_j\left(t\right)dt,\qquad (j=1,2),\tag{35}$$

where $\Phi_j(x),\ \ G_j(x)$ —are arbitrary twice continuously differentiable functions, and the function $\tau_j\left(x\right)\ \ (j=1,2)$ is determined from (27), (28) and (39).

Substituting (35) into (6) and (31) , we find

$$u\left(x,y\right)=\psi_j\left(\frac{x+y+(-1)^{j-1}q}{2}\right)+w_j\left(\frac{x-y+(-1)^{j-1}q}{2}\right)-\psi_j((-1)^{j-1}q)+$$

$$+\frac{\mu_{j+2}(x-y-(-1)^{j-1}q)}{4}\int_{(-1)^{j-1}q}^{x+y}\tau_j\left(t\right)dt,\qquad (j=1,2).\tag{36}$$

Taking into account the properties of the functions $w_j(x),\ \ \psi_j(x)$ and $\tau_j(x)$ from (36) it follows that the solution to the problem $\Gamma_j\ \ (j=1,2)$ exists, is unique and belongs to the class $u(x,y)\in C(\bar{\Omega}_{j+2})\cap C^2(\Omega_{j+2})$.

So the task $\Gamma_j\ (j=1,2)$ is uniquely solvable.

Theorem 2 is proven.

Theorem 3. ***Theorem 3.*** *If conditions (2), (34) are satisfied, then in the domain Ω_5 the solution to the problem Γ_3 exists and is unique.*

Proof of Theorem 3 To prove Theorem 3, the following problems play an important role :

Task $\Gamma_{3j}\ (j=1,2)$. Find in the region Ω_{5j} a solution $u(x,y)\in C(\bar{\Omega}_{5j})\cap C^1(\Omega_{5j}\bigcup I)\cap C^2(\Omega_{5j})$ to equation (1) satisfying conditions (33) and

$$u_j\left(0,y\right)=\tau_3\left(y\right),\qquad (0,y)\in\bar{I},\tag{37}$$

where $\tau_3\left(y\right)$ is a given function, and

$$\tau_3\left(y\right)\in C\left(\bar{I}\right)\cap C^2\left(I\right),\tau_3\left(-q\right)=h_1\left(-q\right)=h_2\left(-q\right).\tag{38}$$

Let's explore the problem $\Gamma_{3j}\ (j=1,2)$. The general solution to the equation (1) in the region Ω_{5j} has the form [15], [30, pp. 135-137]:

$$u\left(x,y\right)=\Phi_{j+2}\left(x+y\right)+G_{j+2}\left(x-y\right)+$$

$$+\frac{\mu_{j+2}(x-y-(-1)^{j-1}q)}{4}\int_{(-1)^{j-1}q}^{x+y}\tau_j\left(t\right)dt,\qquad (j=1,2),\tag{39}$$

where $\Phi_{j+2}(x),\ \ G_{j+2}(x)$ are arbitrary twice continuously differentiable functions, and the function $\tau_j\left(x\right)\ \ (j=1,2)$ is determined from (27), (28) and (29).

Then, substituting (39) into (33) and (37), we find a solution to the problem Γ_{31} for Γ_{32} equation (1) in the areas Ω_{51} and Ω_{52}:

$$u\left(x,y\right)=\tau_3\left(x+y\right)-h_1\left(\frac{x+y-q}{2}\right)+h_1\left(\frac{y-x-q}{2}\right)+\frac{\mu_3 x}{2}\int_{-q}^{x+y}\tau_1\left(t\right)dt\tag{40}$$

and

$$u\left(x,y\right)=\tau_3\left(y-x\right)+h_2\left(\frac{x+y-q}{2}\right)-h_2\left(\frac{y-x-q}{2}\right)+$$

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$$+\frac{\mu_4(x-y+q)}{4}\int_{-q}^{y-x}\tau_2\left(t\right)dt+\frac{\mu_4(x-y-q)}{4}\int_{-q}^{x+y}\tau_2\left(t\right)dt \tag{41}$$

respectively.

By virtue of (30), (34), (38) from (40) and (41), we conclude that the solution to the problem Γ_{3j} $(j=1,2)$exists, is unique, and belongs to the class $u(x,y)\in C(\bar{\Omega}_{5j})\cap C^1(\Omega_{5j}\bigcup I)\cap C^2(\Omega_{5j})$.

Now let's move on to exploring the problem Γ_3. Differentiating (40) and (41) with respect to x, then tending x to zero, we obtain the functional relationship between $\nu_3(y)$ and $\tau_3'(y)$, brought from the region Ω_{51} and Ω_{52} to I:

$$\nu_3(y)+(-1)^j\tau_3'(y)=F_{j+2}\left(y\right),\quad -1<y<-q,\quad (j=1,2), \tag{42}$$

where

$$\nu_3(y)\equiv u_x(-0,y)=u_x(+0,y),\ \ (0,y)\in I, \tag{43}$$

$$F_3\left(y\right)=-h_1'\left(\frac{y-q}{2}\right)+\frac{\mu_3}{2}\int_{-q}^{y}\tau_1(t)dt, F_4\left(y\right)=-h_2'\left(\frac{y-q}{2}\right)-\frac{\mu_4 q}{2}\tau_2(y)+\frac{\mu_4}{2}\int_{-q}^{y}\tau_2(t)dt. \tag{44}$$

Eliminating the function from (42) $\nu_3(y)$ taking (43) into account, we obtain

$$\tau_3'(y)=F_4(y)-F_3(y), \tag{45}$$

Integrating (45) from $-q$ up y to taking into account $\tau_3(-q)=h_1(-q)$, we have

$$\tau_3(y)=\int_{-q}^{y}\left[F_4(t)-F_3(t)\right]\,dt+h_1(-q). \tag{46}$$

By virtue of (30), (34) from (46) it follows that $\tau_3\left(y\right)\in C\left(\bar{I}\right)\cap C^2\left(I\right)$. Consequently, after defining the function, $\tau_3(y)$ the problem Γ_3 is reduced to the study of problems Γ_{31} and Γ_{32}, where $\tau_3(y)$ is determined by formula (46).

Hence we conclude that the task Γ_3 is uniquely solvable.

Theorem 3 has been proven.

This completes the study of the problem C_μ for equation (1).

4. Conclusions

This work is devoted to the formulation and study of a nonlocal boundary value problem for a loaded equation of parabolic-hyperbolic type in a special domain, as well as related results on the existence and behavior of solutions to the problem with integral gluing conditions. Thus, we can study various boundary value problems for loaded equations of generalized parabolic and hyperbolic types with fractional operators.

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