Metallic Ratios are Defined by an Argument of a Normalized Complex Number

Szymon Łukaszyk

Łukaszyk Patent Attorneys, Głowackiego 8, 40-052 Katowice, Poland; szymon@patent.pl

Abstract: We show that metallic ratios for $k \in \mathbb{R}$ are defined by an argument of a normalized complex number, while for rational $k \neq \{0, \pm 2\}$, they are defined by Pythagorean triples. We further extend the concept of metallic ratios to metallic angles.

Keywords: metallic ratios; metallic angles; Pythagorean triples; emergent dimensionality; mathematical physics

---

1. Introduction

Each rectangle contains at least one square with an edge h equal to the shorter edge of the rectangle. If a rectangle contains k such squares and its edges $kh + d$ and h satisfy

$$M(k) := \frac{kh + d}{h} = \frac{h}{d},\tag{1}$$

they satisfy a metallic ratio; the golden ratio for $k = 1$, the silver ratio for $k = 2$, shown in Figure 1, the bronze ratio for $k = 3$, etc.
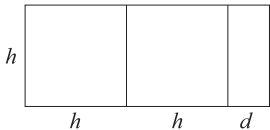
Figure 1. Silver ratio $M(2)_+ = (2h + d)/h = h/d$.

Solving the relation (1) for $M(k)$ leads to the quadratic equation

$$M(k)_\pm^2 - kM(k)_\pm - 1 = 0,\tag{2}$$

having roots

$$M(k)_\pm = \frac{k \pm \sqrt{k^2 + 4}}{2},\tag{3}$$

shown in Figure 2 for $k \in \mathbb{R}$.
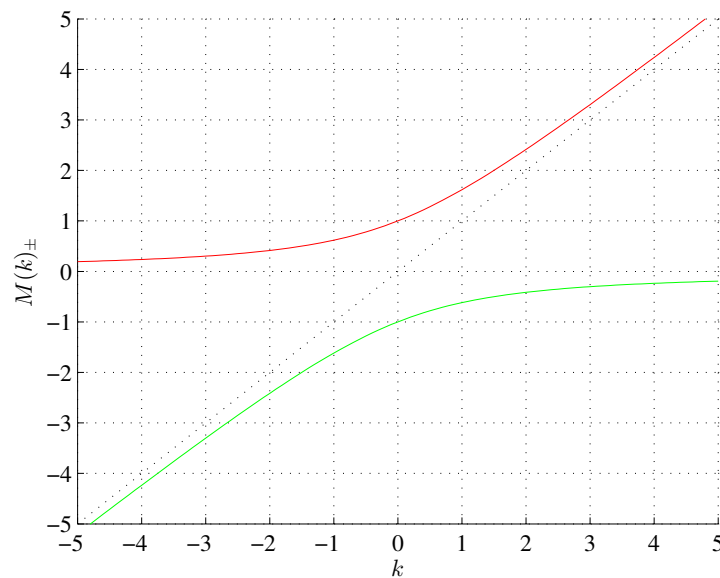
Figure 2. Metallic ratios: positive $M(k)_+$ (red), negative $M(k)_-$ (green) as continuous functions of $-5 \leq k \leq 5$.

Because the edge lengths of a metallic rectangle are assumed to be nonnegative, usually only the positive principal square root $M(k)_+$ of (2) is considered. However, distance nonnegativity (corresponding to the ontological principle of identity of indiscernibles) does not hold for the Ł–K metric [1], for example; such an axiomatization is misleading [2].

Metallic ratios (3) have interesting properties, such as

$$
\begin{aligned}
&1. \quad M(k)_- M(k)_+ = -1, \\
&2. \quad M(k)_- + M(k)_+ = k, \\
&3. \quad -M(-k)_- = M(k)_+, \quad \text{or} \\
&4. \quad M(k)_\pm = \pm e^{\mathrm{arcsinh}(\pm k/2)}.
\end{aligned}
\tag{4}
$$

Furthermore, as k goes to ifinity, the factor $+4$ in the square root becomes negligible, and $M(k)_\pm \approx \{k, 0\}$ for large k.

It was shown [3] that for $k \neq \{0, 2\}$ positive metallic ratios (3) can be expressed by primitive Pythagorean triples, as

$$
M(k)_+ = \cot\left(\frac{\theta}{4}\right),
\tag{5}
$$

and for $k \geq 3$

$$
k = 2\sqrt{\frac{c+b}{c-b}},
\tag{6}
$$

where θ is the angle between a longer cathetus b and hypotenuse c of a right triangle defined by a Pythagorean triple, as shown in Figure 3, whereas for $k = \{3, 4\}$ it is the angle between a hypotenuse and a shorter cathetus a ($\{M(3)_+, M(10)_+\}$ and $\{M(4)_+, M(6)_+\}$ are defined by the same Pythagorean triples, respectively, $(5, 12, 13)$ and $(3, 4, 5)$), and

$$
M(1)_+ = \cot\left(\frac{\pi - \theta_{(3,5)}}{4}\right).
\tag{7}
$$

For example the Pythagorean triple $(20, 21, 29)$ defines $M(5)_{5+}$, the Pythagorean triple $(3, 4, 5)$ defines $M(6)_+$, the Pythagorean triple $(28, 45, 53)$ defines $M(7)_+$, and so on.
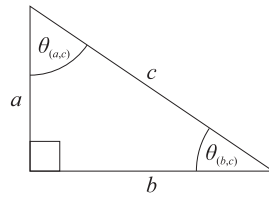
Figure 3. Right triangle showing a longer (b), shorter (a) hypotenuse, catheti (c) and angles $\theta = \theta_{(b,c)}$ and $\theta_{(a,c)}$.

2. Results

Theorem 1. *The metallic ratio $M(k)_\pm$ of $k \in \mathbb{R}$ is defined by an acute angle of a right triangle $0 < \theta < \pi/2$.*

Proof. We express the RHS of the equation (5) using half-angle formulas and substituting $\varphi := \theta/2$

$$
\begin{aligned}
\cot\left(\frac{\theta}{4}\right) = \cot\left(\frac{\varphi}{2}\right) &= \frac{1 + \cos\varphi}{\sin\varphi} = \frac{1 + \cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \\
&= \frac{1 + \operatorname{sgn}\left(\cos\left(\frac{\theta}{2}\right)\right)\sqrt{\frac{1+\cos\theta}{2}}}{\operatorname{sgn}\left(\sin\left(\frac{\theta}{2}\right)\right)\sqrt{\frac{1-\cos\theta}{2}}} = M(k)_+,
\end{aligned}
\tag{8}
$$

since $0 < \theta < \pi/2$ (we exclude degenerated triangles), so $\operatorname{sgn}(\sin(\theta/2)) = \operatorname{sgn}(\cos(\theta/2)) = 1$.

Multiplying the numerator and denominator of (8) by $\sqrt{(1 + \cos\theta)/2}$ and performing some basic algebraic manipulations, we arrive at the quadratic equation for $M(\theta)$

$$
\sin(\theta) M(\theta)^2 - 2\left[1 + \cos(\theta)\right] M(\theta) - \sin(\theta) = 0,
\tag{9}
$$

having roots

$$
M(\theta_+)_\pm = \frac{(1 + \cos(\theta_+)) \pm \sqrt{2(1 + \cos(\theta_+))}}{\sin(\theta_+)},
\tag{10}
$$

corresponding to the metallic ratios (3) for $0 < \theta_+ < \pi/2$. $\square$

We can extend the domain of Theorem 1 by analytic continuation to $0 < \theta_+ < \pi$ as $\operatorname{sgn}(\sin(\theta_+/2)) = \operatorname{sgn}(\cos(\theta_+/2)) = 1$ in this range. However, extending it further to $-\pi < \theta_- < 0$ we note that in this range $\operatorname{sgn}(\sin(\theta_-/2)) = -1$. Thus, the quadratic equation (9) becomes

$$
\sin(\theta_-) M(\theta_-)^2 + 2\left[1 + \cos(\theta_-)\right] M(\theta_-) - \sin(\theta_-) = 0,
\tag{11}
$$

and its roots are

$$
M(\theta_-)_\pm = \frac{-(1 + \cos(\theta_-)) \pm \sqrt{2(1 + \cos(\theta_-))}}{\sin(\theta_-)}.
\tag{12}
$$

Theorem 2. *The metallic ratio of $k \in \mathbb{R}$ is defined by an angle $-\pi < \theta \le \pi$.*

Proof. Equating relations (3) and (10) and solving for k gives

$$
k_+ = \frac{2(1 + \cos(\theta_+))}{\sin(\theta_+)},
\tag{13}
$$

for $0 < \theta_+ < \pi$. This identity can also be obtained directly from the second property (4) applied to the ratio (10). Solving the relation (13) for θ_+ yields

$$
\frac{k_+ + 2i}{k_+ - 2i} = e^{i\theta_+} = \cos\theta_+ + i\sin\theta_+ = \frac{a}{c} + \frac{b}{c}i := z(k_+),
\tag{14}
$$

Similarly, applying the second property (4) to the ratio (12) gives

$$k_- = \frac{-2\left(1 + \cos\left(\theta_-\right)\right)}{\sin\left(\theta_-\right)}, \tag{15}$$

for $-\pi < \theta_- < 0$. Solving the relation (15) for θ_- yields

$$\frac{k_- - 2i}{k_- + 2i} = e^{-i\theta_-} = \frac{a}{c} - \frac{b}{c}i := z(k_-) = \overline{z(k_+)}, \tag{16}$$

as a conjugate of the relation (14). The relations (14) and (16) remove the singularity of $\theta = l\pi, l \in \mathbb{Z}$ in the relations (13), (15), and $\lim_{k_+ \to \pm\infty} \arg(z(k_+)) = \lim_{k_+ \to \pm\infty} \arg(\overline{z(k_+)}) = 0$. □

Equations (14) and (16) relate $k_\pm \in \mathbb{R}$ which defines a metallic ratio (3) to the normalized complex number $z(k_+)$. The angles $\theta_+ = \arg(z(k_+))$ and $\theta_- = \arg(\overline{z(k_+)})$ are shown in Figure 4. There are two axes of symmetry.
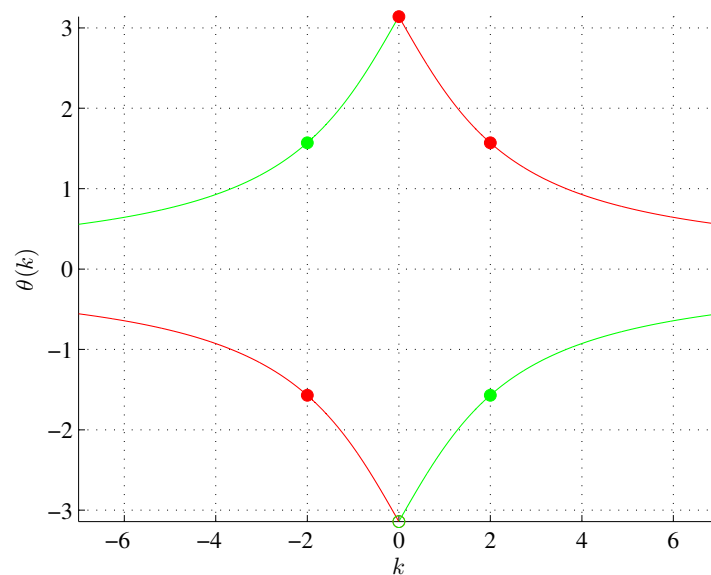
Figure 4. Phases of the complex number $z(k)$ (red, k_+) and its conjugate $\overline{z(k)}$ (green, k_-) for $-7 \le k \le 7$. $\theta(\pm 2) = \pm\pi/2$, $\theta(0) = \pi$.

In summary, metallic ratios as functions of θ are

$$M(\theta_\pm)_\pm = \frac{\pm\left(1 + \cos\left(\theta_\pm\right)\right) \pm \sqrt{2\left(1 + \cos\left(\theta_\pm\right)\right)}}{\sin\left(\theta_\pm\right)}, \tag{17}$$

where the first $\pm$ defines the range of θ and the second $\pm$ corresponds to positive or negative form of the ratio. Therefore, the first and second properties (4) hold for $M(\theta_\pm)_- M(\theta_\pm)_+ = -1$ and $M(\theta_\pm)_- + M(\theta_\pm)_+ = k_\pm$ but the third property (4) holds as $-M(-\theta_\pm)_\pm = M(\theta_\pm)_\pm$.

Figure 5 shows metallic ratios (10) and (12) as functions of $-1.2\pi \le \theta \le 1.2\pi$, $\theta_+ = \arg(z(k_+))$, and $\theta_- = \arg(\overline{z(k_+)})$.
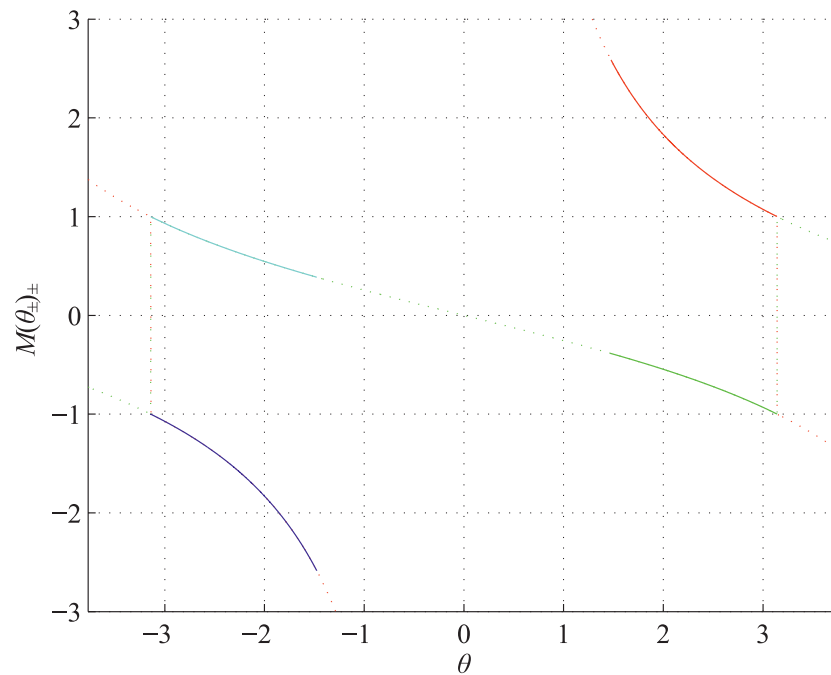
Figure 5. Metallic ratios: positive $M(\theta_+)_+$ (red), negative $M(\theta_+)_-$ (green) as a function of $-1.2\pi \leq \theta \leq 1.2\pi$ (dotted), and $\theta = \arg(z(k_+))$ (solid), and $\theta = \arg(\overline{z(k_+)})$ (positive: solid blue; negative: solid cyan) for $0 \leq k_+ \leq 2.2$.

Theorem 3. *For $k \neq \{0, \pm 2\}, k \in \mathbb{Q}$, the triple $\{a, b, c\}$ corresponding to the angle θ (14), (16) is a Pythagorean triple.*

Proof. Plugging rational $k := l/m, m \neq 0, l, m \in \mathbb{Z}$ into the relation (14) gives

$$\frac{l^2 - 4m^2}{l^2 + 4m^2} + \frac{4lm}{l^2 + 4m^2}i = \frac{a}{c} + \frac{b}{c}i, \tag{18}$$

and $a = l^2 - 4m^2, b = 4lm, c = l^2 + 4m^2, a, b, c \in \mathbb{Z}$ is a possible solution. It is easy to see that $a^2 + b^2 = c^2$. $k = 0$ implies $l = 0$ and $a = -4m^2, b = 0, c = 4m^2$ valid $\forall m \in \{\mathbb{R}, \mathbb{I}\}$. $k = \pm 2$ implies $l = \pm 2m$ and $a = 0, b = \pm 8m^2, c = 8m^2$ also valid $\forall m \in \{\mathbb{R}, \mathbb{I}\}$. $\square$

Table 1 shows the generalized Pythagorean triples that define the metallic ratios for $k = \{0.1, 0.2, \dots, 7\}$.

Table 1. Pythagorean triples associated with metallic ratios for rational $k = \{0.1, 0.2, \ldots, 7\}$.

k	a	b	c	k	a	b	c
0.1	-399	40	401	3.6	28	45	53
0.2	-99	20	101	3.7	969	1480	1769
0.3	-391	120	409	3.8	261	380	461
0.4	-12	5	13	3.9	1121	1560	1921
0.5	-15	8	17	4	3	4	5
0.6	-91	60	109	4.1	1281	1640	2081
0.7	-351	280	449	4.2	341	420	541
0.8	-21	20	29	4.3	1449	1720	2249
0.9	-319	360	481	4.4	48	55	73
1	-3	4	5	4.5	65	72	97
1.1	-279	440	521	4.6	429	460	629
1.2	-8	15	17	4.7	1809	1880	2609
1.3	-231	520	569	4.8	119	120	169
1.4	-51	140	149	4.9	2001	1960	2801
1.5	-7	24	25	5	21	20	29
1.6	-9	40	41	5.1	2201	2040	3001
1.7	-111	680	689	5.2	72	65	97
1.8	-19	180	181	5.3	2409	2120	3209
1.9	-39	760	761	5.4	629	540	829
2				5.5	105	88	137
2.1	41	840	841	5.6	171	140	221
2.2	21	220	221	5.7	2849	2280	3649
2.3	129	920	929	5.8	741	580	941
2.4	11	60	61	5.9	3081	2360	3881
2.5	9	40	41	6	4	3	5
2.6	69	260	269	6.1	3321	2440	4121
2.7	329	1080	1129	6.2	861	620	1061
2.8	12	35	37	6.3	3569	2520	4369
2.9	441	1160	1241	6.4	231	160	281
3	5	12	13	6.5	153	104	185
3.1	561	1240	1361	6.6	989	660	1189
3.2	39	80	89	6.7	4089	2680	4889
3.3	689	1320	1489	6.8	132	85	157
3.4	189	340	389	6.9	4361	2760	5161
3.5	33	56	65	7	45	28	53

For $k = \{-7, -6.9, \ldots, -0.1\}$ set $b \leftrightarrow -b$.

E.g. for $k = -7$, $\{45, 28, 53\} \leftrightarrow \{45, -28, 53\}$.

3. Metallic Angles

We can extend the concept of metallic ratios (1) to angles as

$$\frac{k(2\pi - \varphi) + \varphi}{2\pi - \varphi} = \frac{2\pi - \varphi}{\varphi},$$
(19)

where for $k = 1$ well known golden angle $\varphi(1)_- \approx 2.4$, shown in Figure 6, is obtained.
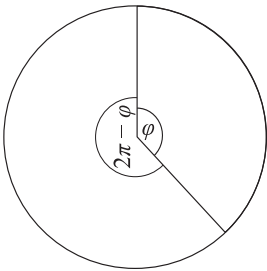
Figure 6. Golden angle $\varphi(1)_- = \pi(3 - \sqrt{5})$.

Solving the relation (19) for φ leads to the quadratic equation

$$k\varphi(k)_\pm^2 - 2\pi(k+2)\varphi(k)_\pm + 4\pi^2 = 0, \tag{20}$$

having roots

$$\varphi(k)_\pm = \frac{\pi}{k}\left(k + 2 \pm \sqrt{k^2 + 4}\right), \tag{21}$$

shown in Figure 7.
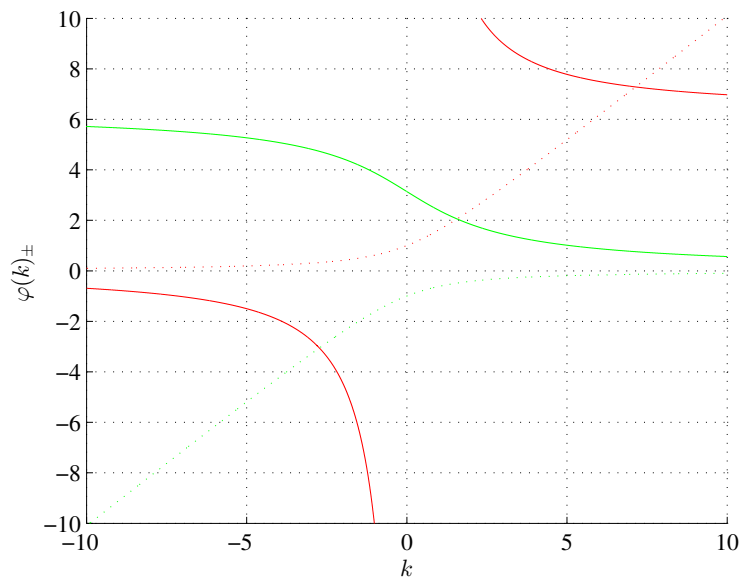
Figure 7. Metallic angles (solid) and ratios (dotted), positive (red), negative (green), as continuous functions of $-10 \le k \le 10$.

In this case, their products and sums

$$\varphi(k)_-\varphi(k)_+ = \frac{4\pi^2}{k}, \tag{22}$$
$$\varphi(k)_- + \varphi(k)_+ = 2\pi\frac{k+2}{k},$$

are dependent on k, where $\lim_{k\to\pm\infty}\varphi(k)_-\varphi(k)_+ = 0$ and $\lim_{k\to\pm\infty}\varphi(k)_- + \varphi(k)_+ = 2\pi$.

The positive metallic ratios (3) are equal to the positive metallic angles (21) for

$$\hat{k}_1 = \frac{4\pi(\pi+1)}{2\pi+1} \approx 7.1459, \tag{23}$$

where

$$M(\hat{k}_1)_+ = \varphi(\hat{k}_1)_+ = 2\pi + 1 \approx 7.2832, \tag{24}$$

the negative metallic ratios (3) are equal to the positive metallic angles (21) for

$$\hat{k}_2 = \frac{-1 + \sqrt{8\pi+1}}{4\pi} - \frac{\sqrt{8\pi+1}}{2} - \frac{1}{2} \approx -2.7288, \tag{25}$$

where

$$M(\hat{k}_2)_- = \varphi(\hat{k}_2)_+ = \frac{-\sqrt{8\pi+1}-1}{2} \approx -3.0560, \tag{26}$$

and the positive metallic ratios (3) are equal to the negative metallic angles (21) for

$$\hat{k}_3 = \frac{-1 - \sqrt{8\pi + 1}}{4\pi} + \frac{\sqrt{8\pi + 1}}{2} - \frac{1}{2} \approx 1.5696, \tag{27}$$

where

$$M(\hat{k}_3)_+ = \varphi(\hat{k}_3)_- = \frac{\sqrt{8\pi + 1} - 1}{2} \approx 2.0560. \tag{28}$$

Acknowledgments: I thank my wife Magdalena Bartocha for her unwavering motivation and my friend, Renata Sobajda, for her prayers.

References

1. Łukaszyk, S. A new concept of probability metric and its applications in approximation of scattered data sets. *Computational Mechanics* **2004**, *33*, 299–304. doi:10.1007/s00466-003-0532-2.
2. Łukaszyk, S.; Tomski, A. Omnidimensional Convex Polytopes. *Symmetry* **2023**, *15*, 755. doi:10.3390/sym15030755.
3. Rajput, C. Metallic Ratios in Primitive Pythagorean Triples : Metallic Means embedded in Pythagorean Triangles and other Right Triangles. *JOURNAL OF ADVANCES IN MATHEMATICS* **2021**, *20*, 312–344. doi:10.24297/jam.v20i.9088.