Article

Not peer-reviewed version

(m, n)-Prime Ideals of Commutative Rings

Hani Khashan * and Ece Yetkin Celikel

Posted Date: 5 January 2024

doi: 10.20944/preprints202401.0472.v1

Keywords: (m; n)-prime ideal; (m; n)-closed ideal; n-absorbing ideal; avoidance theorem

Article

(m,n)-Prime Ideals of Commutative Rings

Hani A. Khashan [1,*] and Ece Yetkin Celikel [2]

1 Department of Mathematics, Al al-Bayt University, Mafraq, Jordan
2 Department of Basic Sciences, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Turkey; ece.celikel@hku.edu.tr
* Correspondence: hakhashan@aabu.edu.jo

Abstract: Let R be a commutative ring with identity and m, n be positive integers. In this paper, we introduce the class of (m,n)-prime ideals which lies properly between the classes of prime and (m,n)-closed ideals. A proper ideal I of R is called (m,n)-prime if for $a,b \in R$, $a^m b \in I$ implies either $a^n \in I$ or $b \in I$. Several characterizations of this new class with many examples are given. Analougus to primary decomposition, we define the (m,n)-decomposition of ideals and show that every ideal in an n-Noetherian ring has an (m,n)-decomposition. Furthermore, the (m,n)-prime avoidance theorem is proved.

Keywords: (m; n)-prime ideal; (m; n)-closed ideal; n-absorbing ideal; avoidance theorem

1. Introduction

Throughout, all rings are assumed to be a commutative with identity. For such a ring R, by $Id(R)$, $N(R)$, $U(R)$ and $\dim(R)$, we denote the set of idempotent, the set of nilpotent, the set of unit elements and the Krull dimension of R. Moreover, for a proper ideal I of R, $\sqrt{I}$ denotes the prime radical of I.

As a more general concept than prime ideals, in 2011, Anderson and Badawi defined n-absorbing ideals, [1]. A proper ideal I of a ring R is called n-absorbing if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \cdots, a_{n+1} \in R$, then there are n of the a_i's whose product is in I. We also recall that a proper ideal I of a ring R is said to be semiprime if whenever $x^2 \in I$ for some $x \in R$, then $x \in I$. Generalizing this concept, for a positive integer n, a proper ideal I of a ring R is called semi-n-absorbing if for $x \in R$, $x^{n+1} \in I$ implies $x^n \in I$ [2]. It is clear that every n-absorbing ideal is semi n-absorbing. Let m and n be positive integers. Afterwards, in [2], the structure of (m,n)-closed ideals is first introduced. A proper ideal I of a ring R is called an (m,n)-closed ideal of R if whenever $a^m \in I$ for some $a \in I$, then $a^n \in I$. On the other hand, in 2020, Badawi and Yetkin Celikel introduced the concept of 1-absorbing primary ideals. According to [8], a proper ideal I of a ring R is said to be a 1-absorbing primary if for non-unit elements $a,b,c \in R$ such that $abc \in I$, then either $ab \in I$ or $c \in \sqrt{I}$. Following this paper, a subclass of 1-absorbing primary ideals is given in [15]. A proper ideal I of R is called 1-absorbing prime if for non-unit elements $a,b,c \in R$ with $abc \in I$, then either $ab \in I$ or $c \in I$. For the related papers, the reader may consult [4–8] and [13].

Motivated and inspired from the ideal notions above, in this paper, we introduce (m,n)-prime ideals which is a structure lies between a prime and primary ideals, i.e. Prime ideal $\Rightarrow$ (m,n)-prime ideal $\Rightarrow$ primary ideal. We call a proper ideal of a ring R a (m,n)-prime ideal where m,n are positive integers if for $a,b \in R$, $a^m b \in I$ implies either $a^n \in I$ or $b \in I$. In Section 2, we discuss all relationships among the ideal types listed above and the new one by supporting many examples (Remark 2 and Example 3). Furthermore, several characterizations of (m,n)-prime ideals of rings are given. We determine all (m,n)-prime ideals of some special rings such as integral domains and zero dimensional rings. Among many other results in this paper, a characterization for rings in which every ideal is (m,n)-prime is given (Theorem 11). Let I be an ideal of a ring R and n a positive integer. We define I to be of maximum length n if any ascending chain $I = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ of ideals of a ring R terminates and n is the largest integer such that $I_n = I_{n+1} = \cdots$. Moreover, a commutative ring R is called n-Noetherian if every ideal of R has a maximum length at most n. Analogous to primary ideal case, we introduce the (m,n)-decomposition of I which is an expression for I as a finite intersection of

 doi:10.20944/preprints202401.0472.v1

(m,n)-prime ideals. It is proved that every ideal in an n-Noetherian ring has (m,n)-decomposition (Theorem 23). In Section 3, we justify the behavior of (m,n)-prime ideals in localizations, quotient rings, finite direct product of rings, idealization rings and amalgamation rings. For an ideal I of R, we introduce the set $\Im(I) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : I \text{ is } (m,n)\text{-prime}\}$ and study some of its properties (Theorem 37). Analogous to prime avoidance theorem, the last section is devoted to state and prove the (m,n)-prime avoidance theorem (Theorem 40).

2. (m,n)-Prime Ideals

In this section, we give some basic properties of (m,n)-prime ideals and investigate (m,n)-prime ideals in several classes of rings. Especially, we determine the (m,n)-prime ideals of rings in which every power of a prime ideal is primary. Among many other results, we characterize rings in which every ideal is (m,n)-prime.

Definition 1. *Let I be a proper ideal of a ring R and m, n be positive integers. Then I is called a (m,n)-prime in R if for $a,b \in R$, $a^m b \in I$ implies either $a^n \in I$ or $b \in I$.*

It is clear that any (m,n)-prime ideal I in a ring R is both primary and (m,n)-closed. Hence, $P = \sqrt{I}$ is the smallest prime ideal of R containing I. In this case, we call I a P-(m,n)-prime ideal of R. Moreover, in the following remark, we justify the relationship between (m,n)-prime ideals and some other kinds of ideals.

Remark 2. *Let I be a proper ideal of R and m, n be positive integers.*

1. I is a prime ideal of R if and only if I is a $(1,1)$-prime ideal.
2. If I is a 1-absorbing prime (resp. if I is a prime) ideal of R, then I is a (m,n)-prime ideal for $n \geq 2$ (resp. for all n). Indeed, let $a,b \in R$ with $a^m b \in I$ and $b \notin I$. Then a is nonunit. If b is unit, then $a^m = a \cdot a^{m-2} \cdot a \in I$ and since I is 1-absorbing prime, we have $a^{m-1} = a \cdot a^{m-2} \in I$ or $a \in I$. Continue this process to get $a^2 \in I$ and so $a^n \in I$ for all $n \geq 2$, (if I is prime, then $a \in I$) as required. The converse is also true if I is a semi-prime (radical) ideal.
3. In general, we may find an n-absorbing ideal that is not (m,n)-prime for all integers m and n. For example, the ideal $18\mathbb{Z}$ is 3-absorbing in $\mathbb{Z}$ which is not (m,n)-prime for all integers m and n (as it is not primary).
4. If I is (m,n)-prime in R, then I is a semi n-absorbing ideal of R. Indeed, let $a \in R$ such that $a^{n+1} \in I$. Suppose $m \gneq n$ so that $a^m \in I$. Then $a^n \in I$ as I is (m,n)-closed in R. On the other hand, suppose $m \leq n$ and note that $a^m a^{n+1-m} \in I$. Then by assumption, either $a^n \in I$ or $a^{n+1-m} \in I$ and the result follows as $n+1-m \leq n$.
5. If I is (m,n)-prime in R, then it is (m',n')-prime where $n \leq n'$ and $m' \leq m$.
6. If I is a (m,n)-prime in R, then $(I:x)$ is (m,n)-prime ideal in R for all $x \in R \backslash I$.

We illustrate the place of the class of (m,n)-prime ideals for all positive integers m and n by the following diagram:

$$
\begin{array}{ccccc}
\text{semiprime} & \longrightarrow & \text{semi } n\text{-absorbing} & \longleftarrow & n\text{-absorbing} \\
\uparrow & \searrow & \uparrow & \swarrow & \\
\text{prime} & \longrightarrow & (m,n)\text{-closed} & \longrightarrow & \text{Quasi } (m,n)\text{-closed} \\
\downarrow & \searrow & \uparrow & & \\
\text{1-absorbing prime} & \longrightarrow & (m,n)\text{-prime} & &
\end{array}
$$

However, the arrows in the above diagram are irreversible as we can see in the following example.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

Example 3.

1. The ideal $I = 8\mathbb{Z}$ is a $(5,3)$-prime that is not prime in $\mathbb{Z}$. Indeed, let $a, b \in \mathbb{Z}$ such that $a^5 b \in I$. Then $ab \in 2\mathbb{Z}$ and so $a \in 2\mathbb{Z}$ or $b \in 2\mathbb{Z}$. if $a \in 2\mathbb{Z}$, then $a^3 \in I$. If $a \notin 2\mathbb{Z}$, then clearly we have $b \in 8\mathbb{Z} = I$.
2. The ideal $I = 16\mathbb{Z}$ is primary and clearly $(3,2)$-closed in $\mathbb{Z}$. However, I is not $(3,2)$-prime since for example, $2^3 \cdot 2 \in I$ but $2^2, 2 \notin I$.
3. The ideal $M = \langle X, Y \rangle$ is a maximal ideal of $F[x,y]$ where F is a field and so $M^2 = \langle X^2, XY, Y^2 \rangle$ is M-primary. On the other hand, M^2 is not $(2,1)$-prime in $F[x,y]$ since for example, $(X-Y)^2 \in M^2$ but $(X-Y) \notin M^2$.
4. In general, if $(I : x) \neq I$ and $(I : x)$ is a (m,n)-prime ideal in R for all $x \in R \backslash I$, then I need not be (m,n)-prime. Consider the ideal $I = \bar{8}\mathbb{Z}_{16}$ in the ring $\mathbb{Z}_{16}$. Then for all $x \in \mathbb{Z}_{16}$ such that $(I : x) \neq I$, we have $(I : x) = \bar{4}\mathbb{Z}_{16}$ or $\bar{2}\mathbb{Z}_{16}$ are clearly $(3,2)$-prime ideals of $\mathbb{Z}_{16}$. But, I is not $(3,2)$-prime as $\bar{2}^3.\bar{2} \in I$ where $\bar{2}^2, \bar{2} \notin I$.
5. Unlike the case of (m,n)-closed ideals, if $n \geq m$, then a proper ideal need not be (m,n)-prime. For example, the ideal $I = 32\mathbb{Z}$ is not $(3,4)$-prime in $\mathbb{Z}$ as $2^3.2^2 \in I$ but $2^4, 2^2 \notin I$.

In general, if I is an ideal of a ring R and n is a positive integer, then $P = \{a \in R : a^n \in I\}$ need not be an ideal of R. For example, consider the ideal $I = \langle X^2, Y^2 \rangle$ in the ring $R[X,Y]$. Then $X, Y \in \{f \in R[X,Y] : f^2 \in I\}$ but $X - Y \notin \{f \in R[X,Y] : f^2 \in I\}$ as $(X-Y)^2 \notin I$. However, for certain types of ideals I such as (m,n)-prime (in particular, radical) ideals, the set P is an ideal of R.

Lemma 4. *Let m and n be positive integers and I be a P-(m,n)-prime ideal of a ring R. Then $P = \{a \in R : a^n \in I\}$.*

Proof. Let $a \in P = \sqrt{I}$ and let k be the smallest positive integer such that $a^k \in I$. Now, $a \cdot a^{k-1} \in I$ implies $a^m \cdot a^{k-1} \in I$. Since I is (m,n)-prime and $a^{k-1} \notin I$, then $a^n \in I$ and so $\sqrt{I} \subseteq \{a \in R : a^n \in I\}$. The other containment is clear. □

Proposition 5. *Let m and n be positive integers and I be an ideal of a ring R. If $M = \{a \in R : a^n \in I\}$ is a maximal ideal of R, then I is an M-(m,n)-prime in R.*

Proof. It is clear that I is proper in R. Let $a^m b \in I$ for $a, b \in R$ such that $a^n \notin I$. Then $a \notin M$ and so $a^m \notin M$. Since M is maximal in R, then $M + Ra^m = R$ and so $1 = t + ra^m$ for some $t \in M$ and $r \in R$. Thus, $1 = 1^n = (t + ra^m)^n = t^n + sa^m$ for some $s \in R$. Hence, $b = b \cdot 1 = bt^n + bsa^m \in I$ and I is (m,n)-prime in R. Moreover, $\sqrt{I} = M$ by Lemma 4. □

Corollary 6. *Let m, n, k be positive integers. If $I = M^k$ for a maximal ideal M of R and $k \leq n$, then I is M-(m,n)-prime in R.*

Proof. Clearly, for $k \leq n$ we have $\left\{a \in R : a^n \in I = M^k\right\} = M$. Thus, I is M-(m,n)-prime in R by Proposition 5. □

However, if $\{a \in R : a^n \in I\}$ is a non-maximal prime ideal of R, then I need not be (m,n)-prime. Indeed, for a field F and the ideal $I = P^3$ of $R = F[x,y]/\langle x^2 y \rangle$ where $P = \langle \bar{x} \rangle$, we have $\{a \in R : a^3 \in I\} = P$ is a (non-maximal) prime ideal of R. But, I is not $(m,3)$-prime in $R = F[x,y]/\langle x^2 y \rangle$, see Example 15. Also, if $k \gneqq n$, then Corollary 6 may not be true, see Example 3.

Following [10], a proper ideal Q of a ring R is called uniformly primary, if there exists a positive integer k such that whenever $a, b \in R$ such that $ab \in Q$ and $b \notin Q$, then $a^k \in Q$. Moreover, a uniformly primary ideal Q has order n and write $o(Q) = n$ if n is the smallest positive integer for which the aforementioned property holds. While clearly every uniformly primary ideal is primary, the converse is not true. For example, in the ring $K[X_1, X_2, ...]$ where K is a field, the ideal $(\{X_i^2\}_{i=1}^{\infty}, \{X_1 X_i\}_{i=1}^{\infty})K[X_1, X_2, ...]$ is a primary ideal that is not uniformly primary, [10].

For positive integers m and n, if I is (m,n)-prime in R, then clearly I is uniformly primary. Moreover, the two concepts coincide if $o(I) \leq n$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

Proposition 7. *Let $\{m_i, n_i\}_{i=1}^{k}$ be positive integers and let $\{I_i\}_{i=1}^{k}$ be P-(m_i, n_i)-prime ideals of a ring R. Then $\bigcap_{i=1}^{k} I_i$ is a P-(m, n)-prime ideal of R for all $m \leq \min\{m_1, m_2, \cdots, m_k\}$ and $n \geq \max\{n_1, n_2, \cdots, n_k\}$.*

Proof. Suppose I_i is P-(m_i, n_i)-prime in R for all $i \in \{1, 2, \cdots, k\}$. Let $a^m b \in \bigcap_{i=1}^{k} I_i$ and $b \notin \bigcap_{i=1}^{k} I_i$ for $a, b \in R$. Then $b \notin I_j$ for some $j \in \{1, 2, \cdots, k\}$. Since $a^{m_j} b \in I_j$, then by assumption $a^{n_j} \in I_j$ and so $a \in P$. By Lemma 4, we have for all $i \in \{1, 2, \cdots, k\}$, $P = \{a \in R : a^{n_i} \in I_i\}$. Thus, $a^n \in \bigcap_{i=1}^{k} I_i$ as $n \geq \max\{n_1, n_2, \cdots, n_k\}$. Since also $\sqrt{\bigcap_{i=1}^{k} I_i} = \bigcap_{i=1}^{k} \sqrt{I_i} = P$, then $\bigcap_{i=1}^{k} I_i$ is a P-(m, n)-prime ideal of R. $\square$

Remark 8.

1. In general, if I and J are two (m, n)-prime ideals with $\sqrt{I} \neq \sqrt{J}$, then $I \cap J$ need not be (m, n)-prime. For example, the ideals $2\mathbb{Z}$ and $3\mathbb{Z}$ are (m, n)-prime ideal for all positive integers n and m (since they are prime), but $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not (m, n)-prime (since it is not primary).
2. If I and J are two P-(m, n)-prime ideals, then IJ or I^k $(k \leq m)$ need not be P-(m, n)-prime. For instance, consider the ring $R = \mathbb{Z} + pX\mathbb{Z}[X]$ where p is a prime integer and the ideal $P = pX\mathbb{Z}[X]$ of R. Since P is prime, it is P-(m, n)-prime for all m, n. However, P^k $(k \leq m)$ is not P-(m, n)-prime as $p^m X^m \in P^k$ but neither $p^n \in P^k$ nor $X^m \in P^k$.

Next, we give more characterizations of (m, n)-prime ideals.

Theorem 9. *Let I be a proper ideal of a ring R and let m and n be positive integers. Then the following statements are equivalent.*

1. I is a (m, n)-prime ideal of R.
2. $I = (I : a^m)$ for all $a \in R$ such that $a^n \notin I$.
3. If whenever $a \in R$ and K is an ideal of R with $a^m K \subseteq I$, then $a^n \in I$ or $K \subseteq I$.

Proof. (1)⇒(2) Let $a \in R$ such that $a^n \notin I$ and let $b \in (I : a^m)$. Then $a^m b \in I$ implies $b \in I$ as I is (m, n)-prime in R. Thus, $(I : a^m) \subseteq I$ and so $I = (I : a^m)$.

(2)⇒(3) Let $a \in R$ and K be an ideal of R with $a^m K \subseteq I$ and suppose $a^n \notin I$. Then by (2) $K \subseteq (I : a^m) = I$ as needed.

(3)⇒(1) It is straightforward. $\square$

In view of the above theorem, several equivalent characterizations of (m, n)-prime ideals of a principal ideal domain is given in the following.

Corollary 10. *Let R be a principal ideal domain and let m, n be positive integers. Then the following are equivalent.*

1. I is a (m, n)-prime ideal of R.
2. $I = (I : a^m)$ for all $a \in R$ such that $a^n \notin I$.
3. If $a \in R$ and K is an ideal of R with $a^m K \subseteq I$, then $a^n \in I$ or $K \subseteq I$.
4. If J and K are ideals of R with $J^m K \subseteq I$, then $J^n \subseteq I$ or $K \subseteq I$.
5. $I = (I : J^m)$ for all ideals J of R such that $J^n \nsubseteq I$.
6. If J is an ideal of R and $b \in R$ with $J^m b \subseteq I$, then $J^n \subseteq I$ or $b \in I$.

Proof. (1)⇒(2)⇒(3) Clear by Theorem 9.

(3)⇒(4) Since J is principal, $J =< a >$ for some $a \in R$. Hence, the claim is clear.

(4)⇒(5) is straightforward.

(5)⇒(6) Assume that $J^m b \subseteq I$ and $J^n \nsubseteq I$ Then $b \in (I : J^m) = I$ by (5), as needed.

(6)⇒(1) Let $a^m b \in I$ and $a^n \notin I$. Put $J =< a >$. Hence $J^m b$ and $J^n \nsubseteq I$ which imply by (6) that $b \in I$. Thus I is a (m, n)-prime ideal of R. $\square$

In the next theorem, we characterize rings in which every ideal is (m,n)-prime.

Theorem 11. *Let R be a ring and $m, n \in \mathbb{N}$. The following are equivalent.*

1. Every proper ideal of R is (m,n)-prime.
2. R has no non-trivial idempotents (for example, R is a quasi local ring or an integral domain), $\dim(R) = 0$ and $x^n = 0$ for all $x \in N(R)$.

Proof. (1) $\Rightarrow$ (2) Suppose that every proper ideal of R is a (m,n)-prime. Suppose there is an idempotent element $e \notin \{0,1\}$ in R. Since by assumption, $\langle 0 \rangle$ is (m,n)-prime in R, $e^m(1-e) = 0$ and $e \neq 1$, then $e = e^n = 0$, a contradiction. Therefore, R has no non-trivial idempotents. If $n < m$, then $\dim(R) = 0$ and $x^n = 0$ for all $x \in N(R)$ by [2, Theorem 2.14]. Suppose $n \geq m$ and $x \in N(R)$. Then $\langle x^{n+m} \rangle$ is a (m,n)-prime ideal of R and $x^m x^n \in \langle x^{n+m} \rangle$. Thus, $x^n \in \langle x^{n+m} \rangle$ and so $x^n = x^{n+m}y$ for some $y \in R$. Hence, $x^n(1 - x^m y) = 0$ and so $x^n = 0$ as $1 - x^m y \in U(R)$. Moreover, suppose in the case $n \geq m$ that $\dim(R) \gneqq 0$ and choose two prime ideals P_1 and P_2 in R such that $P_1 \subsetneq P_2$. If $x \in P_2 \backslash P_1$, then similar to the above argument, we get $x^n = x^{n+m}y$ and so $x^n(1 - x^m y) = 0 \subseteq P_1$. Thus, $1 - x^m y \in P_1 \subsetneq P_2$ as $x \notin P_1$. Since $x^m y \in P_2$, we conclude that $1 \in P_2$, a contradiction. Therefore, $\dim(R) = 0$ as required.

(2) $\Rightarrow$ (1) Let I be a proper ideal of R and let $a^m b \in I$ for $a, b \in R$ such that $b \notin I$. Since $\dim(R) = 0$, then R is π-regular and so $a = eu + c$ where $e \in Id(R)$, $u \in U(R)$ and $c \in N(R)$ by [14, Theorem 13]. Therefore, as $Id(R) = \{0,1\}$, we have either $a = c \in N(R)$ or $a = u + c \in U(R)$. In the first case, we conclude by assumption that $a^n = 0 \in I$. Otherwise, we have $b = (a^m)^{-1} a^m b \in I$. Therefore, every proper ideal of R is (m,n)-prime. □

Note that the condition "R has no non-trivial idempotents" in Theorem 11 can not be discarded. For example, the ring $\mathbb{Z}_6$ has non-trivial idempotents. Moreover, $\dim(\mathbb{Z}_6) = 0$ and $x^n = 0$ for all $x \in N(\mathbb{Z}_6)$ and $n \in \mathbb{N}$. However, the zero ideal of $\mathbb{Z}_6$ is not (m,n)-prime for any $m \in \mathbb{N}$ as it is not primary.

It is well-known that a field is characterized as a ring in which every proper ideal is prime ($(1,1)$-prime). Recall also that in a von Neumann regular ring every element is of the form ue for $u \in U(R)$ and $e \in Id(R)$. In the following corollary, we generalize this result.

Corollary 12. *Let R be a ring and $m \in \mathbb{N}$. Then every proper ideal of R is $(m,1)$-prime if and only if R is a field.*

Proof. If every proper ideal of R is $(m,1)$-prime, then by Theorem 11, R is a reduced zero dimensional ring and so von Neumann regular. Thus, every element of R is of the form ue for some $u \in U(R)$ and $e \in Id(R)$. Since also R has no non-trivial idempotents, then R is a field. The converse part is obvious. □

In the following theorem, we determine when the powers of a principal prime ideal of rings in which every power of a prime ideal is primary are (m,n)-prime.

Theorem 13. *Let R be a ring such that every power of a prime ideal is primary. Let m, n and k be positive integers and $I = \left\langle p^k \right\rangle$ where p is a prime element of R. Then I is a (m,n)-prime ideal of R if and only if $n \geq k$.*

Proof. Suppose $I = \left\langle p^k \right\rangle$ is a (m,n)-prime ideal of R. Suppose on contrary that $n \lneqq k$. If $k \leq m$, then $p^m \in I$ but $p^n \notin I$, a contradiction. If $k \gneqq m$, then $p^m p^{k-m} \in I$ but $p^n \notin I$ and $p^{k-m} \notin I$ which is also a contradiction. Therefore, $n \geq k$. Conversely, suppose $n \geq k$ and let $a, b \in R$ such that $a^m b \in I$ and $b \notin I$. Since by assumption I is primary, then $a^m \in \sqrt{I} = \langle p \rangle$. It follows that $a \in \langle p \rangle$ and so $a^n \in \langle p^n \rangle \subseteq \left\langle p^k \right\rangle = I$. Thus, I is a (m,n)-prime ideal of R. □

Corollary 14. *Let R be either an integral domain or a zero dimensional ring and m, n, k and I as in Theorem 13. Then I is a (m,n)-prime ideal of R if and only if $n \geq k$.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

If some power of a prime ideal of R is not primary, then Theorem 24 need not be true in general.

Example 15. *Consider the non integral domain $R = F[x,y]/\langle x^2y\rangle$ where F is any field. Then the ideal $P = \langle \bar{x}\rangle$ is prime in R as $\langle x\rangle$ is prime in $F[x,y]$ containing $\langle x^2y\rangle$. Now, we prove that $I = P^3$ is not primary in R. Indeed,we have $\overline{x^2}\overline{y} = \overline{0} \in I$ but $\overline{y} \notin \sqrt{I}$ as $y \notin \langle x\rangle$ in $F[x,y]$. If $\overline{x^2} \in I$ and $\varphi : F[x,y] \to R$ is the projection mapping, then $x^2 = \varphi^{-1}(\overline{x^2}) \in \varphi^{-1}(I) = \langle x^3, x^2y\rangle$ which is impossible. Thus, also $\overline{x^2} \notin I$ and $I = P^3$ is not primary in R. Hence, I is not (m,n)-prime in R for all positive integers m and n (and so in particular for all $n \geq k = 3$).*

In view of the above theorem and [2, Theorem 3.1], we have the following corollary.

Corollary 16. *Let R be an integral domain, m and n positive integers and $I = \left\langle p^k \right\rangle$ where p is a prime element of R and k is a positive integer. Then I is an (m,n)-closed ideal of R that is not a (m,n)-prime ideal of R if and only if the following hold.*

1. $n \lneqq k$.
2. $k = ma + r$, where $a, r \in \mathbb{N}$ such that $a \geq 0$ and $1 \leq r \leq n$, $a(m\ mod n) + r \leq n$, and if $a \neq 0$, then $m = n + c$ for an integer c with $1 \leq c \leq n - 1$.

Remark 17. *Let R be a ring such that every power of a prime ideal is primary (e.g. an integral domain or a zero dimensional) and m and n are positive integers. If $I = \left\langle p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} \right\rangle$ where $p_1, p_2, \cdots, p_t$ are non-associate prime elements of R and $k_1, k_2, \cdots, k_t$ are positive integers, then clearly, I is not primary in R. Thus, I is not (m,n)-prime in R.*

We note that [2, Theorem 3.4] and Remark 17 give plenty examples of (m,n)-closed ideals that are not (m,n)-prime.

Corollary 18. *Let R be a principal ideal domain, I a proper ideal of R and m and n positive integers. Then I is (m,n)-prime in R if and only if I is generated by a power less than or equal n of a prime element in R.*

Next, we define a new subclass of Noetherian rings.

Definition 19. *Let I be an ideal of a ring R. Then I is said to be of maximum length n if any ascending chain $I = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ of ideals of R terminates and n is the largest integer such that $I_n = I_{n+1} = \cdots$. Moreover, R is called n-Noetherian if every ideal of R has a maximum length at most n.*

Clearly, any n-Noetherian ring is Noetherian. But the converse need not be true as for example the Noetherian ring $\mathbb{Z}$ is not n-Noetherian for any positive integer n. Moreover, a 1-Noetherian ring is a field clearly as every ideal is prime.

If we consider the ideal $24\mathbb{Z}$ of the the ring $\mathbb{Z}$, then $24\mathbb{Z} \subsetneq 12\mathbb{Z} \subsetneq 6\mathbb{Z} \subsetneq 2\mathbb{Z} \subsetneq \mathbb{Z}$ is the chain of maximum length $n = 4$. In general, we have:

Example 20. *Let R be a principal ideal domain and $I = \left\langle p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} \right\rangle$ where $p_1, p_2, \cdots, p_t$ are non-associate prime elements R. Then I is of maximal length $k_1 + k_2 + \cdots + k_t$.*

Proof. We use mathematical induction on t. If $t = 1$, then $I = \left\langle p_1^{k_1} \right\rangle \subsetneq \left\langle p_1^{k_1 - 1} \right\rangle \subsetneq \cdots \subsetneq \langle p_1 \rangle \subsetneq R$ is the chain of maximum length $n = k_1$. Suppose the result is true for $t - 1$. Then

$$I = \left\langle p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} \right\rangle \subsetneq \left\langle p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t - 1} \right\rangle \subsetneq \left\langle p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t - 2} \right\rangle \subsetneq \cdots$$
$$\subsetneq \left\langle p_1^{k_1} p_2^{k_2} \cdots p_{t-1}^{k_{t-1}} \right\rangle \subsetneq_1 \cdots \subsetneq_{k_1 + k_2 + \cdots + k_{t-1}} R$$

is the chain of maximum length $n = k_1 + k_2 + \cdots + k_t$ as needed. □

 doi:10.20944/preprints202401.0472.v1

Thus, if $k = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ for distinct prime $p_1, p_2, \cdots, p_t$ elements, then the ring $\mathbb{Z}_k$ is n-Noetherian where $n = k_1 + k_2 + \cdots + k_t$.

Recall that an ideal I of a ring R is called irreducible if whenever $I = K \cap L$ for ideals K and L of R, then either $I = K$ or $I = L$. Next, we prove that for $m, n \in \mathbb{N}$, if I is an irreducible ideal of length n in a ring R, then I is (m, n)-prime in R.

Proposition 21. *Let m, n be positive integers and I be a proper ideal of R of maximum length n. If I is irreducible in R, then it is (m, n)-prime.*

Proof. Let $a, b \in R$ such that $a^m b \in I$. For each i consider the ideal $I_i = \{x \in R : a^i x \in I\}$. Then $I = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ and so $I_n = I_{n+1} = \cdots$ as I is of maximum length n. Thus, whenever $k \geq n$ and $a^k x \in I$, then $a^n x \in I$ for any $x \in R$. Now, let $Q = I + bR$ and $L = I + a^n R$. Then clearly $I \subseteq Q \cap L$. Let $y \in Q \cap L$, say, $y = x_1 + r_1 b = x_2 + r_2 a^n$ where $x_1, x_2 \in I$. Then $r_2 a^n - r_1 b \in I$ and so $r_2 a^{n+m} - r_1 b a^m \in I$. Since $a^m b \in I$, then $r_2 a^{n+m} \in I$ and so $r_2 a^n \in I$. Therefore, $y = x_2 + r_2 a^n \in I$ and so $Q \cap L \subseteq I$. Thus, $I = Q \cap L$ and by assumption, either $I = Q$ or $I = L$. If $I = Q$, then $b \in I$ and if $I = L$, then $a^n \in Q$ and so I is (m, n)-prime. □

Definition 22. *Let I be a proper ideal of a ring R and m, n positive integers. An (m, n)-decomposition of I is an expression for I as a finite intersection of (m, n)-prime ideals, say $I = \bigcap_{i=1}^{k} Q_i$ where Q_i is P_i-(m, n)-prime for all i. Moreover, such an (m, n)-decomposition of I is called minimal if*

1. $P_1, P_2, \cdots, P_k$ are different prime ideals of R, and
2. For all $j = 1, 2, \cdots, n$, we have $I \neq \bigcap_{\substack{i=1 \\ i \neq j}}^{k} Q_i$.

We say that I is (m, n)-decomposable in R precisely when it has an (m, n)-decomposition. By Proposition 7, the intersection of P-(m, n)-prime ideals is P-(m, n)-prime. Thus, similar to the case of primary decomposition of ideals, any (m, n)-decomposition of an ideal can be reduced to a minimal one.

Since any (m, n)-prime ideals is primary, then any (m, n)-decomposable ideal is decomposable. However, the converse is not true as for example, the ideal $72\mathbb{Z} = 2^3\mathbb{Z} \cap 3^2\mathbb{Z}$ is decomposable in $\mathbb{Z}$ but not $(3, 2)$-decomposable. Indeed, $2^3\mathbb{Z}$ is not $(3, 2)$-prime by Theorem 13 and any $(3, 2)$-prime ideal in $\mathbb{Z}$ is a power of a prime.

Let $I = \bigcap_{i=1}^{k} Q_i$ be a minimal primary decomposition of an ideal I of a ring R where $\sqrt{Q_i} = P_i$ for each $i = 1, 2, \cdots, k$. Recall that $\{P_1, P_2, \cdots, P_k\}$ is called the set of associated prime ideals of I (denoted by $ass(I)$) which is independent of the choice of minimal primary decomposition of I. Moreover, it is well-known that a prime ideal P of R is a minimal prime ideal of I if and only if P is a minimal member of $ass(I)$, [3].

Now, clearly any minimal (m, n)-decomposition of I is a minimal primary decomposition. Thus, if $I = \bigcap_{i=1}^{k} Q_i$ is any minimal (m, n)-decomposition of I where $\sqrt{Q_i} = P_i$ for each $i = 1, 2, \cdots, k$, then $ass(I) = \{P_1, P_2, \cdots, P_k\}$.

Theorem 23. *Let m, n be positive integers. If a ring R is n-Noetherian, then any ideal of R is (m, n)-decomposable.*

Proof. Suppose R is is n-Noetherian and let I be a proper ideal of R. Then I is of maximal length n. Since R is Noetherian, it is well-known that I is a finite intersection of irreducible ideals. Now, the result follows since every irreducible ideal is (m, n)-prime by Proposition 21. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

3. (m,n)-Prime Ideals in Extensions of Rings, Idealization and Amalgamation Rings

This section is devoted to justify the behavior of (m,n)-prime ideals in localizations, quotient rings, direct product of rings, idealization rings and amalgamation rings. Moreover, for an ideal I of a ring R, we study some properties of the set $\Im(I) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : I \text{ is } (m,n)\text{-prime}\}$.

Proposition 24. *Let $f : R_1 \to R_2$ be a ring homomorphism and m, n be positive integers.*

1. If J is a (m,n)-prime ideal of R_2, then $f^{-1}(J)$ is a (m,n)-prime ideal of R_1
2. If f is an epimorphism and I is a (m,n)-prime ideal containing $Ker\, f$, then $f(I)$ is a (m,n)-prime ideal of R_2.

Proof. (1) Let $a,b \in R_1$ such that $a^m b \in f^{-1}(J)$ and $b \notin f^{-1}(J)$. Then $f(a^m b) = f(a)^m f(b) \in J$ and $f(b) \notin J$ imply $f(a)^n = f(a^n) \in J$. Hence $a^n \in f^{-1}(J)$, as required.

(2) Let $a := f(x)$, $b := f(y) \in R_2$ such that $a^m b \in f(I)$ and $b \notin f(I)$.Then clearly we have $f(x^m y) \in f(I)$ and so $x^m y \in I$ as $Ker(f) \subseteq I$. Since I is (m,n)-prime, we conclude that $x^n \in I$ or $y \in I$. Therefore, $a^n = f(x^n) \in f(I)$ or $b = f(y) \in f(I)$. □

In view of Proposition 24, we have the following.

Corollary 25. *Let R be a ring and m,n positive integers. Then the following statements hold.*

1. If I is a (m,n)-prime ideal of an overring R' of R, then $I \cap R$ is a (m,n)-prime ideal of R.
2. If $I \subseteq J$ are be proper ideals of R, then J/I is a (m,n)-prime ideal of R/I if and only if J is a (m,n)-prime ideal of R.

Corollary 26. *Let I be a proper ideal of a ring R, X be an indeterminate and m, n be positive integers. Then the following statements hold.*

1. $< I,X >$ is a (m,n)-prime ideal of $R[X]$ if and only if I is a (m,n)-prime ideal of R.
2. If $I[X]$ is a (m,n)-prime ideal of $R[X]$, then I is a (m,n)-prime ideal of R.

Proof. (1) Keeping in mind the isomorphisms $R[X]/ < X >\cong R$ and $< I,X > / < X >\cong I$, we conclude by Corollary 25(2) that $< I,X >$ is a (m,n)-prime ideal of $R[X]$ if and only if I is a (m,n)-prime ideal of R.

(2) Clear by Corollary 25(1). □

In the following, $Z_I(R)$ denotes the set $\{x \in R : xy \in I \text{ for some } y \in R \backslash I\}$. Next, we discuss the relationship between (m,n)-prime ideals and their localizations.

Proposition 27. *Let I be a proper ideal of a ring R, S a multiplicatively closed subset of R such that $I \cap S = \emptyset$ and m, n be positive integers.*

1. If I is a P-(m,n)-prime ideal of R, then $S^{-1}I$ is an $S^{-1}P$-(m,n)-prime ideal of $S^{-1}R$.
2. If $S^{-1}I$ is a $\overline{P}$-(m,n)-prime ideal of $S^{-1}R$ and $S \cap Z_I(R) = \emptyset$, then I is a $(\overline{P} \cap R)$-(m,n)-prime ideal of R.

Proof. (1) Let $\left(\frac{a}{s_1}\right)^m \left(\frac{b}{s_2}\right) \in S^{-1}I$ for $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$. Then $(ua)^m b \in I$ for some $u \in S$ which implies either $(ua)^n \in I$ or $b \in I$. Hence, either $\left(\frac{a}{s_1}\right)^n = \frac{u^n a^n}{u^n s_1^n} \in S^{-1}I$ or $\frac{b}{s_2} \in S^{-1}I$. Now, since $\sqrt{I} = P$, then $\sqrt{S^{-1}I} = S^{-1}\sqrt{I} = S^{-1}P$.

(2) Let $a,b \in R$ with $a^m b \in I$. Then $\frac{a^m b}{1} = \left(\frac{a}{1}\right)^m \left(\frac{b}{1}\right) \in S^{-1}I$. Since $S^{-1}I$ is (m,n)-prime, we conclude either $\left(\frac{a}{1}\right)^n \in S^{-1}I$ or $\left(\frac{b}{1}\right) \in S^{-1}I$. Thus, there are some elements $u,v \in S$ such that $ua^n \in I$ or $vb \in I$. Our assumption yields $a^n \in I$ or $b \in I$. Moreover, as $\sqrt{I}$ is a prime ideal of R, we have $S^{-1}\sqrt{I} = \sqrt{S^{-1}I} = \overline{P}$ implies $\sqrt{I} = S^{-1}\sqrt{I} \cap R = \sqrt{S^{-1}I} \cap R = \overline{P} \cap R$. □

 doi:10.20944/preprints202401.0472.v1

Corollary 28. *Let I be a proper ideal of a ring R, P a prime ideal of R with $I \subseteq P$ and m, n positive integers. Then I is a Q-(m,n)-prime ideal of R if and only if I_P is a Q_P-(m,n)-prime ideal of R_P.*

Proof. $\Rightarrow$) Follows by Proposition 27(1).

$\Leftarrow$) Let $a, b \in R$ such that $a^m b \in I$. Consider the ideals $J_1 = \{r \in R : ra^n \in I\}$, $J_2 = \{r \in R : rb \in I\}$. Now, $(\frac{a}{1})^m(\frac{b}{1}) \in I_P$ implies $(\frac{a}{1})^n \in I_P$ or $(\frac{b}{1}) \in I_P$ as I is (m,n)-prime. Hence, there are $u, v \in R \backslash P$ such that $ua^n \in I$ or $vb \in I$. If $ua^n \in I$, then $J_1 \nsubseteq P$. Moreover, $J_1 \nsubseteq L$ for every prime ideal L such that $I \nsubseteq L$ as $I \subseteq J_1$. Thus, $J = R$ and $a^n \in I$. If $vb \in I$, then similarly, $J_2 = R$ and $b \in I$. Since also clearly $\sqrt{I_P} = Q_P$, then I is a Q_P-(m,n)-prime ideal of R. □

Let R be a ring and P a prime ideal of R. For a positive integer n, the kth symbolic power of P is the ideal $P^{(k)} = P^k R_P \cap R = \varphi^{-1}(P^k R_P)$ where $\varphi : R \to R_P$ is the natural canonical map. Thus, $P^{(k)} = \{a \in R : sa \in P^k \text{ for some } s \in R/P\}$. It is well-known that if P is prime, then $P^{(k)}$ is the smallest P-primary ideal containing P^k.

Corollary 29. *Let m, k be a positive integers and P be a prime ideal of a ring R. Then for all $n \geq k$, $P^{(k)}$ is the smallest P-(m,n)-prime ideal containing P^k.*

Proof. Since PR_P is maximal in R_P and $n \geq k$, then $P^k R_P = (PR_P)^k$ is a (m,n)-prime ideal of R_P for any positive integer m by Corollary 6. Thus, $P^{(k)} = P^k R_P \cap R$ is a (m,n)-prime ideal of R by Proposition 24(1).

Now, clearly $P^k \subseteq P^{(k)}$ since $1 \in R \setminus P$. Let J be another P-(m,n)-prime ideal with $P^k \subseteq J$ and let $r \in P^{(k)}$. Then $sr \in P^k$ for some $s \in R \setminus P$. Since $P^k \subseteq J$, then $sr \in J$, and so $s^m r \in J$. Hence, either $s \in P = \{x \in R : x^n \in J\}$ or $r \in J$ as J is P-(m,n)-prime. Since we chose $s \in R \setminus P$, then $r \in J$. Therefore, $P^{(k)} \subseteq J$ and $P^{(k)}$ is the smallest P-(m,n)-prime ideal containing P^k. □

Theorem 30. *Let $R_1, R_2, \cdots, R_k$ be rings, $R = R_1 \times R_2 \times \cdots \times R_k$ and $I_1, I_2, \cdots, I_k$ be ideals of $R_1, R_2, \cdots, R_k$, respectively. For any positive integers m and n, we have $I_1 \times I_2 \times \cdots \times I_k$ is a (m,n)-prime ideal of R if and only if there exists $i \in \{1, 2, \cdots, k\}$ such that I_i is a (m,n)-prime ideal of R_i and $I_j = R_j$ for all $j \neq i$.*

Proof. Suppose $I_1 \times I_2 \times \cdots \times I_k$ is a (m,n)-prime in R. Assume, say, I_1 and I_2 are proper and choose $a_1 \in I_1$ and $a_2 \in I_2$. Then $(a_1, 1, 0, ..., 0)^m(1, a_2, 0, ..., 0) \in I_1 \times I_2 \times \cdots \times I_k$ but neither $(a_1, 1, 0, ..., 0)^n \in I_1 \times I_2 \times \cdots \times I_k$ nor $(1, a_2, 0, ..., 0) \in I_1 \times I_2 \times \cdots \times I_k$. Thus, there is $i \in \{1, 2, \cdots, k\}$ such that $I_j = R_j$ for all $j \neq i$. Without loss of generality, assume $I_j = R_j$ for all $j \neq 1$. We show that I_1 is a (m,n)-prime ideal of R_1. Let $a, b \in R_1$ and $a^m b \in I_1$. Then $(a, 0, ..., 0)^m(b, 0, ..., 0) \in I_1 \times R_2 \times \cdots \times R_k$ which implies that $(a, 0, ..., 0)^n \in I_1 \times R_2 \times \cdots \times R_k$ or $(b, 0, ..., 0) \in I_1 \times R_2 \times \cdots \times R_k$. Thus $a^n \in I_1$ or $b \in I_1$ and I_1 is a (m,n)-prime ideal of R_1. Conversely, suppose, say, I_1 is a (m,n)-prime ideal of R_1 and $I_j = R_j$ for all $j \neq 1$. Suppose $(a_1, a_2, ..., a_k)^m(b_1, b_2, ..., b_k) \in I_1 \times R_2 \times \cdots \times R_k$ but $(b_1, b_2, ..., b_k) \notin I_1 \times R_2 \times \cdots \times R_k$. Then $a_1^m b_1 \in I_1$ and $b_1 \notin I_1$ imply that $a_1^n \in I$. Thus $(a_1, a_2, ..., a_k)^n \in I_1 \times R_2 \times \cdots \times R_k$, as needed. □

In particular, we have:

Corollary 31. *Let R_1 and R_2 be rings, $R = R_1 \times R_2$ and I, J be be ideals of R_1, R_2, respectively. For any positive integers m and n, we have $I \times J$ is a (m,n)-prime ideal of R if and only if one of the following statements is satisfied:*

1. I is a (m,n)-prime ideal of R_1 and $J = R_2$.
2. J is a (m,n)-prime ideal of R_2 and $I = R_1$.

Note that if I and J are (m,n)-prime ideals of R_1 and R_2, respectively, then I and J are proper and so $I \times J$ is never (m,n)-prime ideal in $R_1 \times R_2$.

Recall that the idealization of an R-module M denoted by $R(+)M$, is the commutative ring $R \times M$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

Proposition 32. *Let I be a proper ideal of a ring R, N be a proper submodule of an R-module M and m, n be positive integers. Then*

1. I is a (m, n)-prime ideal of R if and only if $I(+)M$ is a (m, n)-prime ideal of $R(+)M$.
2. If $I(+)N$ is a (m, n)-prime ideal of $R(+)M$, then I is a (m, n)-prime ideal of R.

Proof. (1) Let I be a (m, n)-prime ideal of R and $(a, x)^m(b, y) \in I(+)M$ for some $(a, x), (b, y) \in R(+)M$. Then $a^m b \in I$ which implies either $a^n \in I$ or $b \in I$. Hence, either $(a, x)^n \in I(+)M$ or $(b, y) \in I(+)M$. Conversely, if $a^m b \in I$ for some $a, b \in R$, then $(a, 0)^m(b, 0) \in I(+)M$ which implies $(a, 0)^n \in I(+)M$ or $(b, 0) \in I(+)M$, and so $a^n \in I$ or $b \in I$, we are done.

(2) Similar to the converse part of (1). □

We note that the converse of (2) of Proposition 32 is not true in general. For example, while $2\mathbb{Z}$ is a $(2, 1)$-prime ideal in $\mathbb{Z}$, the ideal $2\mathbb{Z}(+)2\mathbb{Z}$ is not so in $\mathbb{Z}(+)\mathbb{Z}$. Indeed, $(2, 1)^2 = (4, 4) \in 2\mathbb{Z}(+)2\mathbb{Z}$ but $(2, 1) \notin 2\mathbb{Z}(+)2\mathbb{Z}$.

Let R and S be two rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. As a subring of $R \times S$, the amalgamation of R and S along J with respect to f is defined by $R \bowtie^f J = (a, f(a) + j) : a \in R, j \in J\}$. If f is the identity homomorphism on R, then we get the amalgamated duplication of R along an ideal J, $R \bowtie J = \{(a, a + j) : a \in R, j \in J\}$. For more related definitions and several properties of this kind of rings, one can see [11]. If I is an ideal of R and K is an ideal of $f(R) + J$, then $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in K\}$ are ideals of $R \bowtie^f J$, [12].

For positive integers m and n, in the next result, we give a characterization about when the ideals $I \bowtie^f J$ and $\bar{K}^f$ are (m, n)-prime ideals of $R \bowtie^f J$.

Theorem 33. *Let R, S, f, J, I and K be as above. For positive integers m and n, we have:*

1. $I \bowtie^f J$ is a (m, n)-prime ideal of $R \bowtie^f J$ if and only if I is a (m, n)-prime ideal of R.
2. $\bar{K}^f$ is a (m, n)-prime ideal of $R \bowtie^f J$ if and only if K is a (m, n)-prime ideal of $f(R) + J$.

Proof. (1) Suppose $I \bowtie^f J$ is (m, n)-prime in $R \bowtie^f J$ and let $a, b \in R$ such that $a^m b \in I$. Then $(a, f(a))^m(b, f(b)) \in I \bowtie^f J$ and so either $(a, f(a))^n \in I \bowtie^f J$ or $(b, f(b)) \in I \bowtie^f J$. Thus, either $a^n \in I$ or $b \in I$ and I is (m, n)-prime in R. Conversely, suppose I is (m, n)-prime in R. Let $(a, f(a) + j_1), (b, f(b) + j_2) \in R \bowtie^f J$ such that $(a, f(a) + j_1)^m(b, f(b) + j_2) \in I \bowtie^f J$. Then $a^m b \in I$ and so either $a^n \in I$ or $b \in I$. It follows that $(a, f(a) + j_1)^n \in I \bowtie^f J$ or $(b, f(b) + j_2) \in I \bowtie^f J$ as needed.

(2) Suppose $\bar{K}^f$ is (m, n)-prime ideal in $R \bowtie^f J$. Let $f(a) + j_1, f(b) + j_2 \in f(R) + J$ such that $(f(a) + j_1)^m(f(b) + j_2) \in K$. Then $(a, f(a) + j_1)^m(b, f(b) + j_2) \in \bar{K}^f$ and hence by assumption, $(a, f(a) + j_1)^n \in \bar{K}^f$ or $(b, f(b) + j_2) \in \bar{K}^f$. It follows that $(f(a) + j_1)^n \in K$ or $(f(b) + j_2) \in K$. Conversely, suppose K is (m, n)-prime in $f(R) + J$. Suppose $(a, f(a) + j_1)^m(b, f(b) + j_2) \in \bar{K}^f$ for $(a, f(a) + j_1), (b, f(b) + j_2) \in R \bowtie^f J$. Then $(f(a) + j_1)^m(f(b) + j_2) \in K$ and so $(f(a) + j_1)^n \in K$ or $(f(b) + j_2) \in K$. Therefore, $(a, f(a) + j_1)^n \in \bar{K}^f$ or $(b, f(b) + j_2) \in \bar{K}^f$ and the result follows. □

In particular, we have:

Corollary 34. *Let I and J be an ideal of a ring R. Then $I \bowtie J$ is a (m, n)-prime ideal of $R \bowtie J$ if and only if I is a (m, n)-prime ideal of R.*

Lemma 35 ([9,12]). *Let $f : R \to S$ be a ring homomorphism and J be an ideal of S. Then*

1. $N(R \bowtie^f J) = \{(a, f(a) + j) : a \in N(R)), j \in N(S) \cap J\}$.
2. $\dim(R \bowtie^f J) = \max\{\dim(R), \dim(f(R) + J)\}$
3. $Id(R \bowtie^f J) = \{(a, f(a) + j) : a \in Id(R), f(a) + j \in Id(R) + J\}$.

Next, we use Lemma 35 and Theorem 11, to determine when every proper ideal of the amalgamation $R \bowtie^f J$ is (m, n)-prime.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

Theorem 36. *Let m, n be positive integers and R, S, f and J be as above where J is proper in S. Then every proper ideal of $R \bowtie^f J$ is (m,n)-prime if and only if the following statements hold*

1. Every proper ideal of R is (m,n)-prime.
2. Every proper ideal of $f(R)+J$ is (m,n)-prime.

Proof. Suppose every proper ideal of $R \bowtie^f J$ is (m,n)-prime. If there is a proper ideal I of R which is not (m,n)-prime, then $I \bowtie^f J$ is proper in $R \bowtie^f J$ which is not a (m,n)-prime ideal of $R \bowtie^f J$ by Theorem 33(1), a contradiction. Similarly, if K is a proper non (m,n)-prime ideal of $f(R)+J$, then $\bar{K}^f$ is a proper non (m,n)-prime ideal of $R \bowtie^f J$ by Theorem 33(2), which is also a contradiction.

Conversely, suppose (1) and (2) hold. Then $\dim(R) = \dim(f(R)+J) = 0$ by Theorem 11 and so $\dim(R \bowtie^f J) = \max\{\dim(R), \dim(f(R)+J)\} = 0$ by Lemma 35(2). Now, let $(a, f(a)+j) \in N(R \bowtie^f J)$. Then $a \in N(R)$ and $f(a)+j \in N(f(R)+J)$ by Lemma 35(1). Thus, $a^n = (f(a)+j)^n = 0$ by Theorem 11 and so $(a, f(a)+j)^n = (0,0)$. Again by Theorem 11, we have $Id(R) = \{0_R, 1_R\}$ and $Id(f(R)+J) = \{0_S, 1_S\}$. Since J is proper in S and by the definition of $R \bowtie^f J$, we have $(0_R, 1_S), (1_R, 0_S) \notin R \bowtie^f J$. By Lemma 35(3), $Id(R \bowtie^f J) = \{(0_R, 0_S), (1_R, 1_S)\}$. It follows that every proper ideal of $R \bowtie^f J$ is (m,n)-prime by Theorem 11. □

Following [2], for an ideal I of a ring R,

$$\Re(I) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : I \text{ is } (m,n)\text{-closed}\}$$

Similarly, we let

$$\Im(I) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : I \text{ is } (m,n)\text{-prime}\}$$

and assume $\Im(R) = \mathbb{N} \times \mathbb{N}$. It is clear that $\Im(I) \subseteq \Re(I)$ and this containment in general is proper as we have seen in Example 3. Moreover, we have $(1,1) \in \Im(I)$ if and only if I is prime.

For an ideal I of a ring R, the following are some properties concerning $\Im(I)$. Theses properties are analogous to those of (m,n)-closed ideals, [2, Theorem 4.1].

Theorem 37. *Let I and J be ideals of a ring R, and m, n, k and t be positive integers.*

1. If $(m,n) \in \Im(I)$, then $(m', n') \in \Im(I)$ for all positive integers m' and n' with $m' \leq m$ and $n' \geq n$.
2. If $(m,n) \in \Im(I)$, then $(km, tn) \in \Im(I)$ for all $t \geq k$.
3. If $(m,n) \in \Im(I)$ and $(n,k) \in \Re(I)$, then $(m,k) \in \Im(I)$.
4. $(m,n) \in \Im(I)$ if and only if $(m+1,n) \in \Im(I)$. Hence, $(m,n) \in \Im(I)$ if and only if $(t,n) \in \Im(I)$ for all $t \geq m$.
5. If I and J are proper, then $\Im(I \times J) = \phi$. If only one of I and J is proper, then $\Im(I \times J) = \Im(I) \cap \Im(J)$.

Proof. (1), (2) and (3): Clear.

(4) Suppose $(m,n) \in \Im(I)$ and let $a, b \in R$ such that $a^{m+1}b \in I$ and $b \notin I$. Then $(a^2)^m b \in I$ as $2m \geq m+1$. Since I is (m,n)-prime, then $a^{2n} \in I$. Thus, $a \in \sqrt{I}$ and so $a^n \in I$ by Lemma 4. The converse is clear by (1).

(5) If I and J are proper, then $\Im(I \times J) = \varnothing$ by Theorem 31. Suppose, say, $I \neq R$ and $J = R$. Then $\Im(I \times J) = \Im(I) \cap \Im(J)$ since $\Im(R) = \mathbb{N} \times \mathbb{N}$ and by using Corollary 31. □

The converse of (2) of Theorem 37 is not true in general. For example, the ideal $I = \left\langle p^k \right\rangle$ where p is a prime element of any integral domain R, is (k,k)-prime by Theorem 13. But, I is not $(1,1)$-prime as it is not prime in R.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

4. (m,n)-Prime Avoidance Theorem

In this section, we prove the (m,n)-prime avoidance theorem analogous to prime avoidance theorem. Recall that a covering $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ is said to be efficient if no I_k is superfluous. Also, $I = I_1 \cup I_2 \cup \cdots \cup I_n$ is an efficient union if none of the I_k may be excluded. Here, it is easy to see that a covering $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ automatically implies a union $I = (I \cap I_1) \cup (I \cap I_2) \cup \cdots \cup (I \cap I_n)$. First, we need to state a very useful lemma.

Lemma 38. *(McCoy) Let $I = I_1 \cup I_2 \cup \cdots \cup I_n$ be an efficient union of ideals where $n \gneqq 1$. Then $\bigcap\limits_{i \neq k} I_i = \bigcap\limits_{i=1}^{n} I_i$ for all k.*

Theorem 39. *Let $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ be an efficient covering of ideals $I_1, I_2, ..., I_n$ of R where $n \gneqq 2$. Suppose that $\sqrt{I_i} \nsubseteq \sqrt{(I_j : x)}$ for all $x \in R \backslash \sqrt{I_j}$ whenever $i \neq j$. Then no I_i $(1 \leq i \leq n)$ is a (m,n)-prime ideal of R for all $n \leq m$.*

Proof. Suppose on the contrary that I_k is a (m,n)-prime ideal of R for some $1 \leq k \leq n$. First, note that as $I \subseteq \bigcup\limits_{i=1}^{n} I_i$ is an efficient covering, then $I \subseteq \bigcup\limits_{i=1}^{n} (I_i \cap I)$ is also an efficient covering. It follows that

$$(*) \quad \left(\bigcap_{i \neq k} I_i \right) \cap I = \left(\bigcap_{i=1}^{n} I_i \right) \cap I \subseteq I_k \cap I$$

by Lemma 38. For all $x \in R \backslash \sqrt{I_k}$ and $i \neq k$, we have $\sqrt{I_i} \nsubseteq \sqrt{(I_k : x)}$ and so we can choose $a_i \in \sqrt{I_i} \backslash \sqrt{(I_k : x)}$. Then, there exists the least positive integer m_i such that $a_i^{m_i} \in I_i$ for each $i \neq k$. Write $a = a_1 a_2 \cdots a_{k-1}$, $b = a_{k+1} a_{k+2} \cdots a_n$ and $m = \max\{m_1, m_2, ..., m_{k-1}, m_{k+1}, ..., m_n\}$. Then $a^m b^m x \in \left(\bigcap\limits_{i \neq k} I_i \right) \cap I$.

In the rest of the proof, we show that $a^m b^m x \in \left(\left(\bigcap\limits_{i \neq k} I_i \right) \cap I \right) \backslash (I_k \cap I)$. For this purpose, assume on the opposite that $a^m b^m x \in I_k \cap I$. Then $a^m b^m \in (I_k : x) \subseteq \sqrt{(I_k : x)}$. Since $\sqrt{(I_k : x)}$ is a prime ideal by Proposition 4 (1) and (2), we get either $a = a_1 a_2 \cdots a_{k-1} \in \sqrt{(I_k : x)}$ or $b = a_{k+1} a_{k+2} \cdots a_n \in \sqrt{(I_k : x)}$. Again, since $\sqrt{(I_k : x)}$ is prime, $a_i \in \sqrt{(I_k : x)}$ for some $i \neq k$, a contradiction. Consequently, $a^m b^m x \notin (I_k \cap I)$, and so $a^m b^m x \in \left(\left(\bigcap\limits_{i \neq k} I_i \right) \cap I \right) \backslash (I_k \cap I)$ which contradicts $(*)$. Therefore, no I_i is a (m,n)-prime ideal for $1 \leq i \leq n$ and we are done. □

Theorem 40. *((m,n)-prime Avoidance Theorem) Let I, I_1, I_2, ..., I_n $(n \geq 2)$ be ideals of R such that at most two of $I_1, I_2, ..., I_n$ are not (m,n)-prime and $\sqrt{I_i} \nsubseteq \sqrt{(I_j : x)}$ for all $x \in R \backslash \sqrt{I_j}$ whenever $i \neq j$. If $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$, then $I \subseteq I_k$ for some $1 \leq k \leq n$.*

Proof. Assume that $I \nsubseteq I_k$ for all $1 \leq k \leq n$. Without loss of generality, we may assume that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ is an efficient covering of ideals of R as any covering can be reduced to an efficient one by omitting any unnecessary terms. It is well-known that a covering of an ideal by two ideals is never efficient. If $n \geq 3$, then no I_k is a (m,n)-prime ideal of R by Theorem 39. But our assumption implies that at most two of $I_1, I_2, ..., I_n$ are not (m,n)-prime. Thus, $I \subseteq I_k$ for some $1 \leq k \leq n$. □

Corollary 41. *Let I be a proper ideal of a ring R. If (m,n)-prime avoidance theorem holds for R, then the (m,n)-prime avoidance theorem holds for R/I.*

Proof. Let $J/I, I_1/I, I_2/I, ..., I_n/I$ $(n \geq 2)$ be ideals of R/I such that at most two of $I_1/I, I_2/I, ..., I_n/I$ are not (m,n)-prime and $J/I \subseteq (I_1/I) \cup (I_2/I) \cup \cdots \cup (I_n/I)$. Then, Corollary 25 implies that $J \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ and at most two of $I_1, I_2, ..., I_n$ are not (m,n)-prime. Suppose that $\sqrt{I_i/I} \nsubseteq \sqrt{(I_j/I : x + I)}$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2024 doi:10.20944/preprints202401.0472.v1

for all $x + I \in (R/I)\backslash\sqrt{(I_j/I)}$ whenever $i \neq j$. It is easy to verify that if $\sqrt{I_i} \subseteq \sqrt{(I_j : x)}$ for some $x \in R$, then $\sqrt{(I_i/I)} \subseteq \sqrt{(I_j/I : x + I)}$ for some $x + I \in R/I$. Also observe that if $x + I \in (R/I)\backslash\sqrt{(I_j/I)} = (R/I)\backslash(\sqrt{I_j}/I)$, then $x \in R\backslash\sqrt{I_j}$. Thus, by our assumption $\sqrt{(I_i/I)} \nsubseteq \sqrt{(I_j/I : x + I)}$ for all $x + I \in (R/I))\backslash\sqrt{(I_j/I)}$ whenever $i \neq j$. Hence, we conclude that $\sqrt{I_i} \nsubseteq \sqrt{(I_j : x)}$ for all $x \in R\backslash\sqrt{I_j}$ whenever $i \neq j$. Therefore, Theorem 40 implies $J \subseteq I_k$ for some $1 \leq k \leq n$.. Consequently, $J/I \subseteq I_k/I$ for some $1 \leq k \leq n$; so we are done. □

References

1. D. F. Anderson, A. Badawi, On n-absorbing ideals of commutative rings. Communications in Algebra, **39**(5) (2011), 1646-1672.
2. D. F. Anderson and A. Badawi, On (m,n)-closed ideals of commutative rings, J. Algebra Appl., **16**(1) (2017), 1750013 (21 pp).
3. M. F. Atiyah and I. G. Macdonald, Introduction to commutative algebra, Addison-Wesley, Reading, Massachusetts, 1969.
4. A. Badawi, On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc., **75** (2007), 417–429.
5. A. Badawi, Ü. Tekir and E. Yetkin, On 2-absorbing primary ideals in commutative rings, Bull. Korean Math. Soc., **51**(4) (2014), 1163–1173.
6. A. Badawi, n-absorbing ideals of commutative rings and recent progress on three conjectures: a survey, Rings, Polynomials, and Modules, (2017), 33-52. DOI: 10.1007/978-3-319-65874-2_3
7. A. Badawi, M. Issoual, N. Mahdou, On n-absorbing ideals and (m, n)-closed ideals in trivial ring extensions of commutative rings. Journal of Algebra and Its Applications, **18**(07) (2019) 1950123.
8. A. Badawi, E. Yetkin Celikel, On 1-absorbing primary ideals of commutative rings, Journal of Algebra and Applications, **19** (6) (2020), 2050111. DOI: 10.1142/S021949882050111X. 223-235.
9. M. Chhiti, N. Mahdou and M. Tamekkante, Clean property in amalgamated algebras along an ideal, Hacet. J. Math. Stat., **44**(1) (2015), 41-49.
10. J. A. Cox, A. J. Hetzel, Uniformly primary ideals. J. of Pure and Applied Algebra, **212** (2008), 1-8.
11. M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, J. Algebra Appl., **6**(**3**) (2007), 443–459.
12. M. D'Anna, C.A. Finocchiaro, and M. Fontana, Properties of chains of prime ideals in an amalgamated algebra along an ideal, J. Pure Appl. Algebra, **214** (2010), 1633-1641.
13. S. Koc, Ü. Tekir, E. Yıldız, On weakly 1-absorbing prime ideals, Ricerche di Matematica, (2021), 1-16.
14. K. N. Kyun and L. Yang, On right quasi-duo rings which are π-regular, Bull. Korean Math. Soc., **37** (2000) 217–227.
15. A. Yassine, M. J. Nikmehr, R. Nikandish, On 1-absorbing prime ideals of commutative rings, J. Algebra Appl., **20**(10) (2021), 2150175.